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AND QUANDLE HOMOLOGY**

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We give a construction of quandle cocycles from group cocycles, especially, for any integer $p \geq 3$, quandle cocycles of the dihedral quandle R_p from group cocycles of the cyclic group \mathbb{Z}/p . We show that a group 3-cocycle of \mathbb{Z}/p gives rise to a nontrivial quandle 3-cocycle of R_p . When p is an odd prime, since $\dim_{\mathbb{F}_p} H_Q^3(R_p; \mathbb{F}_p) = 1$, our 3-cocycle is a constant multiple of the Mochizuki 3-cocycle up to coboundary. Dually, we construct a group cycle represented by a cyclic branched covering branched along a knot K from the quandle cycle associated with a colored diagram of K .

1. Introduction

A quandle, which was introduced by Joyce [1982], is an algebraic object whose axioms are motivated by knot theory and conjugation in a group. Carter, Jelsovsky, Kamada, Langford and Saito [Carter et al. 2003] introduced a quandle homology theory, and they defined the quandle cocycle invariants for classical knots and surface knots. The quandle homology is defined as the homology of a chain complex generated by cubes whose edges are labeled by elements of a quandle. On the other hand, the group homology is defined as the homology of a chain complex generated by tetrahedra whose edges are labeled by elements of a group. So it is natural to ask for a relation between quandle homology and group homology. This question also arises from the fact that the quandle cocycle invariants were defined as an analogue of the Dijkgraaf–Witten invariants, which are defined using group cocycles.

In [Inoue and Kabaya 2010], we defined a simplicial version of quandle homology and constructed a homomorphism from the usual quandle homology to the simplicial quandle homology. The important point is that this homomorphism gives a triangulation of a knot complement in algebraic fashion. This enables us to relate the quandle homology to the topology of knot complements.

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In this paper, we apply the results of [Inoue and Kabaya 2010] to construct quandle cocycles from group cocycles. First, we demonstrate how to give a quandle cocycle of the dihedral quandle R_p from a group cocycle of the cyclic group \mathbb{Z}/p for any integer $p \geq 3$ in Section 7. We show that a generator of $H^3(\mathbb{Z}/p; \mathbb{Z}/p)$ gives rise to a nontrivial quandle 3-cocycle of $H_Q^3(R_p; \mathbb{Z}/p)$. When p is an odd prime, since $\dim_{\mathbb{F}_p} H_Q^3(R_p; \mathbb{F}_p) = 1$, our quandle 3-cocycle is equal to a constant multiple of the Mochizuki 3-cocycle [2003] up to coboundary.

Then we generalize the construction to wider classes of quandles. Let G be a group and let h be an element of G ; then the set $\text{Conj}(h) = \{g^{-1}hg \mid g \in G\}$ forms a quandle by conjugation. (It is known from [Joyce 1982] that any faithful homogeneous quandle has such a presentation.) When some obstruction in second cohomology vanishes, we construct a quandle cocycle of $\text{Conj}(h)$ from a group cocycle of G .

Dually, we relate the quandle cycle associated with an arc and region coloring (shadow coloring) of a knot K to a group cycle represented by a cyclic branched covering branched along K . Let D be a diagram of K . We can define the notion of arc and region colorings of D by a quandle $\text{Conj}(h)$. A pair of an arc and a region coloring is called a *shadow coloring*. We can associate a cycle of a quandle homology group to a shadow coloring of D . Using the homomorphism constructed in [Inoue and Kabaya 2010], we construct a group cycle of G represented by a cyclic branched covering branched along K . This reveals a close relationship between the shadow cocycle invariant of a knot and the Dijkgraaf–Witten invariant of the cyclic branched cover.

Hatakenaka and Nosaka [2012] defined an invariant of 3-manifolds called the *4-fold symmetric quandle homotopy invariant*, based on the fact that any 3-manifold can be represented as a 4-fold simple branched covering of S^3 along a link. As an application, they showed that the shadow cocycle invariant of a link for the Mochizuki 3-cocycle is equal to a scalar multiple of the Dijkgraaf–Witten invariant of the double branched cover along the link.

This paper is organized as follows. In Section 2, we recall the definition of group homology and show how to represent a group cycle by a triangulation with a labeling of its 1-simplices. We give a presentation of the fundamental group of a cyclic branched covering branched along a knot in Section 3, which is independent from the other sections. In Section 4, we construct a group cycle represented by a cyclic branched cover. We recall the definition of quandles and their homology theory in Section 5. We review some results from [Inoue and Kabaya 2010] in Section 6 and apply them to construct quandle cocycles of the dihedral quandle R_p in Section 7. The reader interested in the form of the 3-cocycle should consult (7-5) (and (7-4)). We generalize the construction to wider classes of quandles in Section 8. In Section 9, we construct a group cycle represented by a cyclic branched

covering from the quandle cycle associated with a shadow coloring. The reader who is only interested in the construction of quandle cocycles from group cocycles may skip Sections 2B 2C 3, and 4,

2. Group homology

In this section, we collect basic facts on group homology, starting with a review of definitions; see [Brown 1982] for details. The material discussed in Sections 2B and 2C was developed in [Neumann 2004].

2A. Group homology. Let G be a group. Let $C_n(G)$ be the free $\mathbb{Z}[G]$ -module generated by n -tuples $[g_1 | \dots | g_n]$ of elements of G . Define the boundary map $\partial : C_n(G) \rightarrow C_{n-1}(G)$ by

$$\begin{aligned} \partial([g_1 | \dots | g_n]) &= g_1[g_2 | \dots | g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1 | \dots | g_i g_{i+1} | \dots | g_n] + (-1)^n [g_1 | \dots | g_{n-1}]. \end{aligned}$$

We remark that the chain complex $\{\dots \rightarrow C_1(G) \rightarrow C_0(G) \rightarrow \mathbb{Z} \rightarrow 0\}$ is acyclic, where $C_0(G) \cong \mathbb{Z}[G] \rightarrow \mathbb{Z}$ is the augmentation map. So the chain complex $C_*(G)$ gives a free resolution of \mathbb{Z} . Let M be a right $\mathbb{Z}[G]$ -module. The homology of $C_n(G; M) = M \otimes_{\mathbb{Z}[G]} C_n(G)$ is called the *group homology* of M and denoted by $H_n(G; M)$. In other words, $H_n(G; M) = \text{Tor}_n^{\mathbb{Z}[G]}(M, \mathbb{Z})$.

Let $C'_n(G)$ be the free \mathbb{Z} -module generated by $(g_0, \dots, g_n) \in G^{n+1}$. Then $C'_n(G)$ is a left $\mathbb{Z}[G]$ -module by the action $g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$. Define the boundary operator of $C'_n(G)$ by

$$\partial(g_0, \dots, g_n) = \sum_{i=0}^n (-1)^i (g_0, \dots, \widehat{g_i}, \dots, g_n).$$

$C_*(G)$ and $C'_*(G)$ are isomorphic as chain complexes. In fact, the following correspondence gives an isomorphism:

$$[g_1 | g_2 | \dots | g_n] \leftrightarrow (1, g_1, g_1 g_2, \dots, g_1 \cdots g_n),$$

or, equivalently,

$$g_0 [g_0^{-1} g_1 | g_1^{-1} g_2 | \dots | g_{n-1}^{-1} g_n] \leftrightarrow (g_0, \dots, g_n).$$

The notation that uses (g_0, \dots, g_n) is called *homogeneous*, and the one that uses $[g_1 | \dots | g_n]$ is called *inhomogeneous*.

Factoring out $C_n(G)$ by the degenerate subcomplex generated by $[g_1 | \dots | g_n]$ such that $g_i = 1$ for some i , we obtain the *normalized* chain complex and its homology group. It is known that the group homology using the normalized

chain complex coincides with the homology using the unnormalized one. In the homogeneous notation, we factor out $C'_n(G)$ by the subcomplex generated by (g_0, \dots, g_n) such that $g_i = g_{i+1}$ for some i to define the normalized chain complex.

For a left $\mathbb{Z}[G]$ -module N , the group cohomology $H^n(G; N)$ is defined as the cohomology of the cochain complex $C^n(G; N) = \text{Hom}_{\mathbb{Z}[G]}(C_n(G), N)$. Let A be an abelian group. A cocycle of $C^n(G; A)$ in the homogeneous notation is a function $f : G^{n+1} \rightarrow A$ satisfying the following conditions:

- (1) $\sum_{i=0}^{n+1} (-1)^i f(x_0, \dots, \widehat{x}_i, \dots, x_{n+1}) = 0$,
- (2) $f(gx_0, \dots, gx_n) = f(x_0, \dots, x_n)$ for any $g \in G$ (left invariance).

If f also satisfies

- (3) $f(x_0, \dots, x_n) = 0$ if $x_i = x_{i+1}$ for some i ,

then f is a normalized n -cocycle. We can show that any n -cocycle is cohomologous to a normalized n -cocycle.

2B. Cycles represented by triangulations. Let Δ be an n -dimensional simplex. We label the vertices of Δ by $0, 1, \dots, n$. A face of Δ is presented by a subset of $\{0, 1, \dots, n\}$. Let $\langle i_0, \dots, i_k \rangle$ be the face spanned by $i_0, \dots, i_k \in \{0, \dots, n\}$, with a vertex ordering given by i_0, \dots, i_k . Any face inherits a vertex ordering from the vertex ordering of Δ , that is, $\langle i_0, \dots, i_k \rangle$ with $i_0 < i_1 < \dots < i_k$.

Definition 2.1. Let T be a CW-complex obtained by gluing a finite number of n -dimensional simplices along their $(n - 1)$ -dimensional faces in pairs by simplicial homeomorphisms. We denote the k -skeleton of T by $T^{(k)}$. We assume that the gluing maps preserve the vertex orderings of the faces. Then $T - T^{(n-3)}$ is homeomorphic to a topological n -manifold (not orientable in general). When $T - T^{(n-3)}$ is oriented, we call T an *ordered n -cycle*.

Consider an n -cycle σ of $C_n(G; \mathbb{Z})$. Then σ is represented by a sum

$$\sum_j \epsilon_j [g_{j1} | \dots | g_{jn}],$$

where $\epsilon_j = \pm 1$ and $g_{jk} \in G$. For each $[g_{j1} | \dots | g_{jn}]$, take an n -simplex Δ_j . Then label the edge $\langle i_1 i_2 \rangle$ of Δ_j by $g_{ji_1} g_{j(i_1+1)} \dots g_{ji_2}$ for $i_1 < i_2$. In particular, we have

$$\langle 0, 1 \rangle \leftrightarrow g_{j1}, \quad \langle 1, 2 \rangle \leftrightarrow g_{j2}, \quad \dots, \quad \langle n - 1, n \rangle \leftrightarrow g_{jn}.$$

We denote the label of $\langle i_1, i_2 \rangle$ by $\lambda \langle i_1, i_2 \rangle$. For $i_1 > i_2$, label the oriented edge $\langle i_1, i_2 \rangle$ by $\lambda \langle i_2, i_1 \rangle^{-1}$. For any 2-dimensional face $\langle i_0, i_1, i_2 \rangle$, we have

$$\lambda \langle i_0, i_1 \rangle \lambda \langle i_1, i_2 \rangle = \lambda \langle i_0, i_2 \rangle.$$

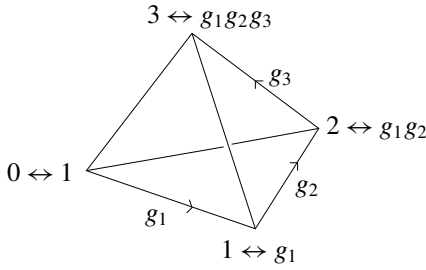


Figure 1. A labeling of a simplex.

Rewriting them in the homogeneous notation, we assign labels to the vertices of Δ_j as

$$0 \leftrightarrow 1, \quad 1 \leftrightarrow g_j1, \quad 2 \leftrightarrow g_j1g_j2, \quad \dots, \quad n \leftrightarrow g_j1 \dots g_jn,$$

up to the left action of G (Figure 1).

Since $\partial\sigma = 0$, $(n - 1)$ -dimensional faces cancel in pairs. Gluing the Δ_j 's along their faces according to such pairings, we obtain an n -cycle T . At any $(n - 1)$ -simplex of T , there exist exactly two adjacent n -simplices. The labelings of the $(n - 1)$ -simplex derived from these two n -simplices coincide. Thus we have a well-defined labeling of 1-simplices $\lambda : \{\text{oriented 1-simplices of } T\} \rightarrow G$ satisfying:

- (1) $\lambda\langle i_0, i_1 \rangle \lambda\langle i_1, i_2 \rangle = \lambda\langle i_0, i_2 \rangle$ for any 2-dimensional face $\langle i_0, i_1, i_2 \rangle$,
- (2) $\lambda\langle i_1, i_0 \rangle = \lambda\langle i_0, i_1 \rangle^{-1}$.

We call a labeling of 1-simplices satisfying the conditions (1) and (2) a G -valued 1-cocycle. Orient Δ_j positively if $\epsilon_j = 1$ and negatively if $\epsilon_j = -1$. Since these orientations agree on face pairings, we thus have an orientation on T . Therefore T is an ordered n -cycle with a G -valued 1-cocycle λ . In general, T may not be connected, but we assume that T is connected because we can treat each connected component separately in our arguments. Conversely, any ordered n -cycle T with a G -valued 1-cocycle λ represents an n -cycle of $C_n(G; \mathbb{Z})$.

2C. Group cycles and representations. Suppose a cycle $\sigma \in C_n(G; \mathbb{Z})$ is represented by an ordered n -cycle T with a G -valued 1-cocycle λ . Then σ induces a homomorphism from $\pi_1(T)$ to G as follows. Let \tilde{T} be the universal covering of T and let $p : \tilde{T} \rightarrow T$ be the covering map. Then the simplices of T lift to simplices of \tilde{T} , and each lift has an induced vertex ordering compatible with adjacent n -simplices. The G -valued 1-cocycle λ of T induces a G -valued 1-cocycle of \tilde{T} . Consider a *fundamental domain* of T , that is, a contractible subcomplex D of \tilde{T} such that

- $\tilde{T} = \bigcup_{\gamma \in \pi_1(M)} \gamma D$,
- $D \cap \gamma D = (\text{lower-dimensional simplices}), \text{ for any } \gamma \neq 1$,

where we regard $\pi_1(T)$ as the group of deck transformations. By definition, the number of n -simplices in D coincides with the number of n -simplices in T ; in particular, both are finite. We fix a base point \ast in the interior of D . Each $(n-1)$ -simplex on ∂D is glued to another $(n-1)$ -simplex on ∂D . We denote such a pair of faces by F_i^\pm . Let x_i be a path in T that starts at $\ast = p(\ast)$, traverses $p(F_i)$ in the direction from F_i^+ to F_i^- , and ends at \ast . These paths form a system of generators of the fundamental group $\pi_1(T, \ast)$. The relations are given at any $(n-2)$ -simplices, around which there are a finite number of $p(F_i)$'s.

Fix an n -simplex Δ in D and a labeling of vertices of Δ derived from λ . Then n -simplices adjacent to Δ inherit labelings of vertices from λ . In this way, all vertices of D are labeled by elements of G . Now consider the labeling of vertices of F_i^+ and F_i^- . Since these reduce to the same labeling of edges, they coincide up to left multiplication. Therefore there exists an element of G that sends the labeling of vertices of F_i^- to the one of F_i^+ . Denote the element by $\rho(x_i)$. This ρ induces a homomorphism $\rho : \pi_1(T, \ast) \rightarrow G$.

Conversely, if we have an ordered n -cycle T and a homomorphism

$$\rho : \pi_1(T, \ast) \rightarrow G,$$

we can construct a G -valued 1-cocycle λ and then a cycle of $C_n(G; \mathbb{Z})$ up to boundary as follows. Since ρ induces a map

$$T \rightarrow K(\pi_1(T, \ast), 1) \rightarrow BG,$$

we obtain a labeling of 1-simplices λ of T . This gives rise to a G -valued 1-cocycle λ and a homology class in $H_n(G; \mathbb{Z})$. The G -valued 1-cocycle λ is well-defined up to the coboundary action. A map $\mu : \{0\text{-simplices of } T\} \rightarrow G$ acts on a G -valued 1-cocycle λ as a coboundary action by

$$\langle i_1, i_2 \rangle \mapsto \mu(i_1)^{-1} \lambda \langle i_1, i_2 \rangle \mu(i_2).$$

We can show that the homology class obtained from λ does not change under the coboundary action. For any $g \in G$, the cycle corresponding to the representation $g^{-1} \rho g$ is obtained from λ by the coboundary action by $\mu \equiv g$. As a result, the homology class obtained from ρ depends only on the conjugacy class of ρ .

For a closed oriented n -manifold M and a representation $\rho : \pi_1(M) \rightarrow G$, we have a homology class defined by the image of the fundamental class $[M]$ under the map $H_n(M) \rightarrow H_n(K(\pi_1(M), 1)) \rightarrow H_n(G; \mathbb{Z})$. When M is homeomorphic to an ordered n -cycle T , the homology class is represented by a G -valued 1-cocycle λ of T associated with ρ . In this situation, we say that the homology class defined by M and ρ is *represented* by T and λ .

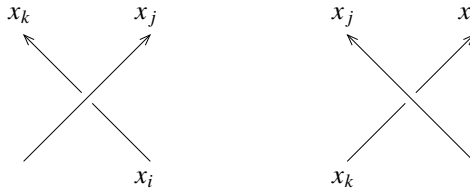


Figure 2. The relation is given by $x_k = x_j^{-1}x_i x_j$.

3. Cyclic branched covering

In this section, we give a presentation of the fundamental group of a cyclic branched covering from the Wirtinger presentation.

3A. Presentation of the fundamental group of the branched cover. Let K be a knot in S^3 and let D be a diagram of K . Then $\pi_1(S^3 - K)$ is presented by generators and relations called the Wirtinger presentation. Let x_1, \dots, x_n be the generators of the Wirtinger presentation that correspond to the arcs of D . Each crossing (Figure 2) gives rise to a relation $x_k = x_j^{-1}x_i x_j$.

For any integer $l > 1$, let C_l be the l -fold cyclic covering of K , that is, the manifold corresponding to the kernel of

$$\pi_1(S^3 - K) \rightarrow H_1(S^3 - K) \cong \mathbb{Z} \rightarrow \mathbb{Z}/l.$$

Putting back the knot K to C_l , we obtain the l -fold cyclic branched covering \widehat{C}_l of K .

Proposition 3.1. $\pi_1(C_l)$ has the following presentation:

Generators: $x_{i,s}$ (for $i = 1, 2, \dots, n$ and $s = 0, 1, \dots, l - 1$),

Relations: $x_{k,s} = x_{j,s-1}^{-1}x_{i,s-1}x_{j,s}$ (for each crossing and $s = 0, 1, \dots, l - 1$),

$$x_{1,0} = x_{1,1} = \dots = x_{1,l-2} = 1.$$

The inclusion map $\pi_1(C_l) \rightarrow \pi_1(S^3 - K)$ is given by

$$x_{i,s} \mapsto x_1^{s-1}x_i x_1^{-s},$$

if we take appropriate base points. By adding a relation $x_{1,l-1} = 1$, we obtain a presentation of $\pi_1(\widehat{C}_l)$.

A method for obtaining a presentation of the fundamental group of a branched covering is given in [Rolfsen 1976]. But we give a proof here because some techniques are used to construct a group cycle represented by \widehat{C}_l later.

Proof. First, we construct a handle decomposition of the knot complement associated with the Wirtinger presentation (Figure 3). Then we lift the handle decomposition

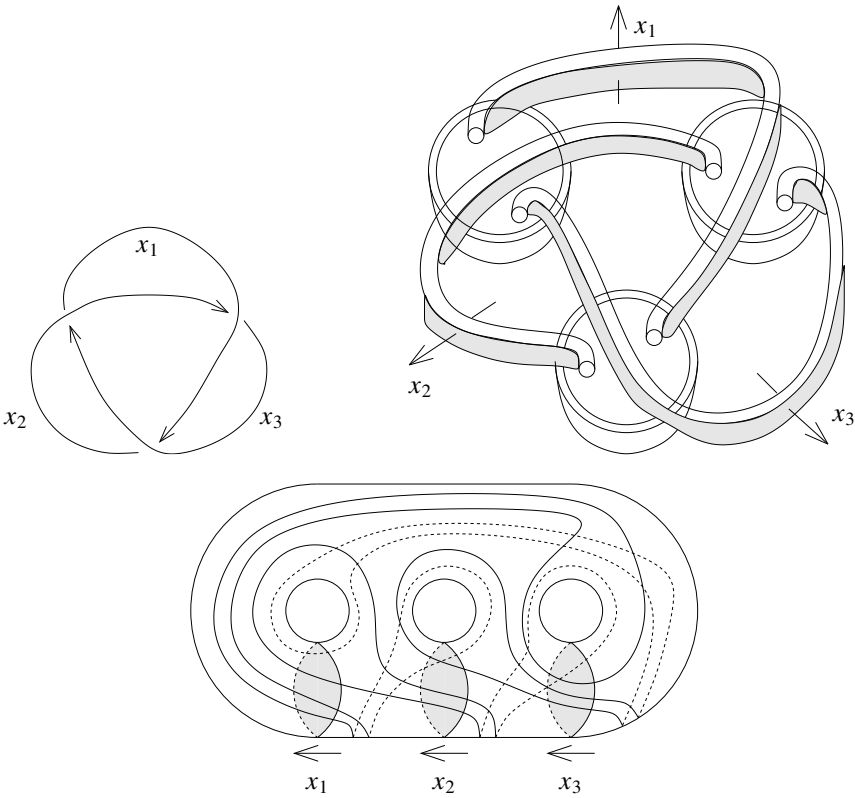


Figure 3. A handle decomposition of a knot complement.

to a handle decomposition of C_l (Figure 4). After that, we read the relations given by attaching 2-handles.

Let $N(K)$ be a regular neighborhood of K . We give a handle decomposition of $S^3 - N(K)$. We represent S^3 by the one-point compactification of

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Let $B_+ = \{z \geq 0\} \cup \{\infty\}$ and $B_- = \{z \leq 0\} \cup \{\infty\}$. We denote the equatorial sphere of S^3 by $S_0 = \{z = 0\} \cup \{\infty\}$. Put K in a position such that the projection to S_0 has only double points. Let n be the number of crossings of this projection. We deform K in the z -direction so that K intersects S_0 at $2n$ points and each of $B_{\pm} \cap K$ consists of n arcs. We call $B_+ \cap K$ over-crossing arcs and $B_- \cap K$ under-crossing arcs. Index the arcs of $B_+ \cap K$ by x_i ($i = 1, 2, \dots, n$).

Now $B_+ - N(K)$ is homeomorphic to a handlebody of genus n (Figure 3). Projecting the over-crossing arcs to S_0 , we obtain a meridian disk system of the handlebody. We denote the meridian disk corresponding to x_i by D_i . For each under-crossing arc, attach a 2-handle $D^2 \times D^1$ along $\partial D^2 \times D^1$ to $B_+ - N(K)$

(Figure 3). Then the resulting manifold is homeomorphic to $(S^3 - N(K)) - B^3$, where B^3 is a 3-ball. Attaching B^3 along the boundary to $\partial((S^3 - N(K)) - B^3)$, the resulting manifold is homeomorphic to $S^3 - N(K)$. So we have a handle decomposition of $S^3 - N(K)$ into one 0-handle, n 1-handles, n 2-handles and one 3-handle. In this handle decomposition, 1-handles correspond to the Wirtinger generators and 2-handles to the Wirtinger relations. We denote the set of i -handles by h^i and $X^{(i)} = h^0 \cup \dots \cup h^i$.

Next we consider the preimage of $X^{(1)}$ in C_l . Cut $X^{(1)}$ along the meridian disks D_i , and denote the resulting manifold by B , which is homeomorphic to a 3-ball. Let $*$ be a point in $B \subset S^3 - N(K)$. We take a loop in $X^{(1)}$ that starts at $*$, intersects the meridian disk D_i , and ends at $*$. Orient the loop so that it corresponds to the generator of $H_1(S^3 - K)$. By abuse of notation, we also denote this loop by x_i . Take a lift $B_0 \subset C_l$ of B and denote the preimage of $*$ in B_0 by $\tilde{*}$. Then there exists a unique lift \tilde{x}_i of the loop x_i starting at $\tilde{*}$. Since x_i corresponds to the generator of $H_1(S^3 - K)$, \tilde{x}_i ends at another lift of B , which we denote by B_1 . Similarly, the lift of x_i starting at B_1 ends at another lift, which we denote by B_2 . Continuing this, we see that $B_l = B_0$ and all lifts of B will appear. Therefore the preimage of $X^{(1)}$ is decomposed into l 3-balls $B_0 \cup B_1 \cup \dots \cup B_{l-1}$. The intersection of B_s and B_{s+1} consists of n disks, each of which is a lift of a meridian disk D_i . We denote this lifted disk by $D_{i,s}$ (Figure 4). It is easy to check that

$$\{D_{1,l-1}\} \cup \{D_{i,s} : i = 2, \dots, n; s = 0, \dots, l-1\}$$

forms a meridian disk system of the preimage of $X^{(1)}$. Denote by $\tilde{x}_{i,s}$ the lift of x_i starting at B_s and ending at B_{s+1} .

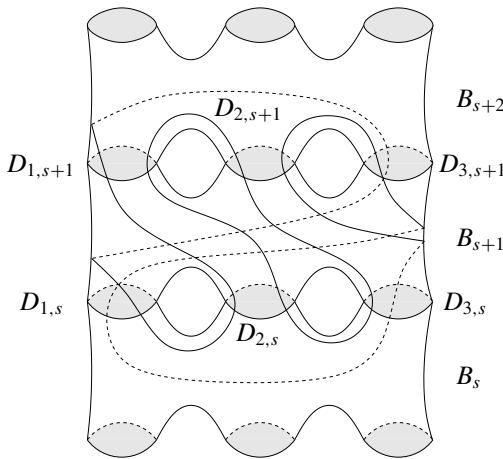


Figure 4. A handle decomposition of C_l .

The generator of $\pi_1(C_l, \tilde{*})$ corresponding to the meridian disk $D_{i,s}$ is given by

$$(3-1) \quad \tilde{x}_{1,0}\tilde{x}_{1,1} \dots \tilde{x}_{1,s-1}\tilde{x}_{i,s}\tilde{x}_{1,s}^{-1}\tilde{x}_{1,s-1}^{-1} \dots \tilde{x}_{1,0}^{-1}.$$

We denote this element by $x_{i,s}$. To give a simple presentation of $\pi_1(C_l, \tilde{*})$, we add extra generators $x_{1,0}, x_{1,1}, \dots, x_{1,l-2}$ corresponding to $\tilde{x}_{1,0}, \tilde{x}_{1,1}, \dots, \tilde{x}_{1,l-2}$, respectively, and relations $x_{1,0} = x_{1,1} = \dots = x_{1,l-2} = 1$.

Finally, we consider the relations given by the lifts of 2-handles. We see that the relation $x_k = x_j^{-1}x_i x_j$ lifts to

$$x_{k,s} = x_{j,s-1}^{-1}x_{i,s-1}x_{j,s} \quad (s = 0, 1, \dots, l-1);$$

see [Figure 4](#).

Since the generator $x_{i,s}$ is represented by (3-1), the inclusion map $\pi_1(C_l, \tilde{*}) \rightarrow \pi_1(S^3 - K, *)$ is given by $x_{i,s} \mapsto x_1^{s-1}x_s x_1^{-s}$. This proves the second statement.

By adding a 2-handle to C_l along $\tilde{x}_{1,0}\tilde{x}_{1,1} \dots \tilde{x}_{1,l-1}$ and capping off the resulting sphere, we obtain a manifold homeomorphic to the cyclic branched covering \widehat{C}_l . Therefore a presentation of the cyclic branched covering is obtained by adding a relation $x_{1,l-1} = 1$. □

4. Cycle represented by cyclic branched covering

For a representation $\rho : \pi_1(S^3 - K) \rightarrow G$, we have the restriction map $\rho|_{\pi_1(C_l)} : \pi_1(C_l) \rightarrow G$ given by

$$(4-1) \quad \rho|_{\pi_1(C_l)}(x_{i,s}) = \rho(x_1)^{s-1}\rho(x_i)\rho(x_1)^{-s}.$$

If $\rho(x_1)^l = 1$, it reduces to a representation $\widehat{\rho} : \pi_1(\widehat{C}_l) \rightarrow G$ and there is a group cycle given by \widehat{C}_l and $\widehat{\rho}$. In this section, we construct an explicit ordered 3-cycle and its G -valued 1-cocycle representing the homology class given by \widehat{C}_l and $\widehat{\rho}$. First we give a triangulation of $S^3 - N(K)$ associated with the Wirtinger presentation.

Let K be a knot and fix a diagram of K . As in the proof of [Proposition 3.1](#), we define B_{\pm} and give a Heegaard splitting of $S^3 - N(K)$ with the meridian disks D_i . Cutting the handlebody $B_+ - N(K)$ along the meridian disks D_i , the result is a ball with $2n$ 2-cells on the boundary. We denote the resulting 3-ball by B and the pair of 2-cells corresponding to D_i by F_i^+ and F_i^- so that the Wirtinger generator corresponding to x_i runs from F_i^+ to F_i^- ([Figure 5\(A\)](#)). Now consider the attaching regions of the two handles corresponding to under-crossing arcs, which consist of annuli $S^1 \times D^1 (\subset D^2 \times D^1)$ on $\partial(B_+ - N(K))$. Each annulus is divided by the D_i 's into four rectangles on ∂B . We define a graph on ∂B with vertices consisting of F_i^{\pm} and edges consisting of these rectangles ([Figure 5\(B\)](#)). Each vertex of this graph has valency at least four. We can make all vertices of the graph into trivalent vertices ([Figure 5\(C\)](#)) by adding extra 1-handles and 2-handles (stabilizations of

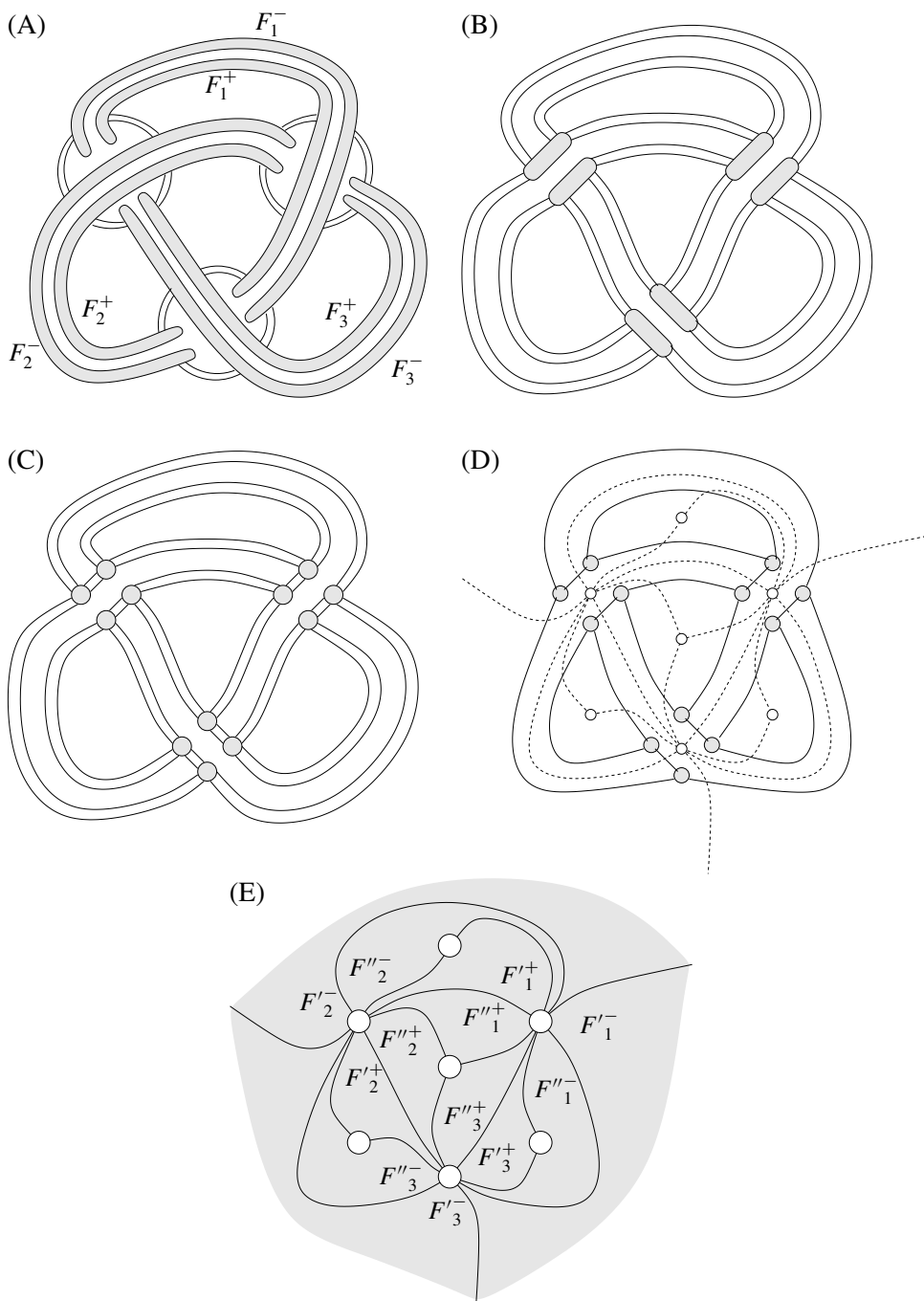


Figure 5. A polyhedral decomposition of $S^3 - N(K)$. We are looking from inside of B_+ . The white rectangles in (A) and (B) correspond to the attaching regions of the 2-handles.

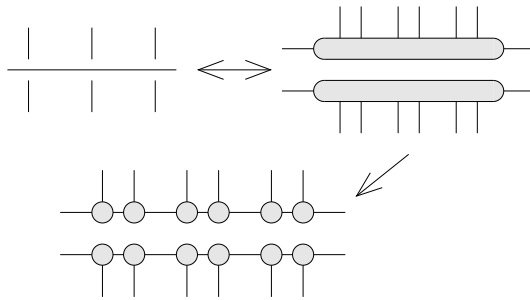


Figure 6. Stabilizations of the handle decomposition.

the Heegaard splitting; see Figure 6). We denote the new vertices, which originally belonged to F_i^\pm , by $F_i'^\pm, F_i''^\pm, \dots$. The dual of the graph gives a triangulation of ∂B (Figures 5(D) and 5(E)). By abuse of notation, we denote the triangles dual to the vertices $F_i'^\pm, F_i''^\pm, \dots$ by the same symbols. Taking a cone from an interior point of B , we obtain a triangulation T of B into $4n$ tetrahedra. Regluing the triangles $F_i'^+, F_i''^+, \dots$ to $F_i'^-, F_i''^-, \dots$, we obtain a triangulation of $S^3 - N(K)$ into $4n$ tetrahedra, which was explained in [Weeks 2005]. We remark that this is not a triangulation in the usual sense: it is not a simplicial complex, and moreover, the link of some 0-simplex is not homeomorphic to the 2-sphere. Actually there exist only three 0-simplices; one is the cone point in B (the north pole in [Weeks 2005]), the second is the 0-simplex corresponding to the complementary regions of the diagram (the south pole), and the last one is a 0-simplex whose small neighborhood is homeomorphic to the cone over the torus $\partial N(K)$.

Using this triangulation, we construct a triangulation of the cyclic branched covering \widehat{C}_l . Let B_0, B_1, \dots, B_{l-1} be l copies of B and let T_s be the triangulation of B_s we have constructed. We denote the triangles $F_i'^\pm, F_i''^\pm, \dots$ on ∂B_s by $F_{i,s}'^\pm, F_{i,s}''^\pm, \dots$ respectively. By abuse of notation, we regard $F_{i,s}'^\pm, F_{i,s}''^\pm, \dots$ simply as $F_{i,s}^\pm$. Glue the T_s 's along their boundary triangles by the identification maps

$$F_{i,s}^- \rightarrow F_{i,s-1}^+ \quad (i = 1, 2, \dots, n, \quad s = 0, 1, \dots, l - 1).$$

Denote this triangulation by \widehat{T} . We define an ordering of each tetrahedron by assigning 0 to the interior vertex of B_s , 1 to the vertex corresponding to the complementary region of the diagram, 2 to the vertex corresponding to the under-crossing arc, and 3 to the vertex corresponding to the over-crossing arc, respectively (Figure 7). The orderings are compatible under the gluing maps. So \widehat{T} is an ordered 3-cycle. Here \widehat{T} is a triangulation of \widehat{C}_l except near the 0-simplex corresponding to K , whose small neighborhood is homeomorphic to a cone over a torus. We can resolve this singularity by inserting a suspension of a $2l$ -gon around K (Figure 8). As a

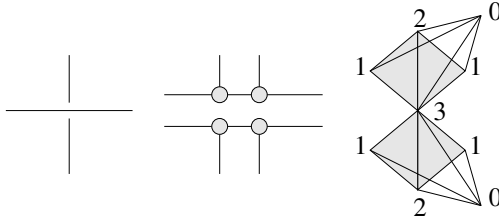


Figure 7. The orderings of the tetrahedra of B .

result, we obtain an ordered 3-cycle homeomorphic to \widehat{C}_l . (This procedure is called “blowing up” at a 0-simplex in [Neumann 2004].)

We construct a group cycle given by \widehat{C}_l and a representation $\widehat{\rho} : \pi_1(\widehat{C}_l) \rightarrow G$ using the triangulation \widehat{T} . Here $\widehat{\rho}$ is given by the set $\{\widehat{\rho}(x_{i,s})\} \subset G$ satisfying

$$\begin{aligned} \widehat{\rho}(x_{k,s}) &= \widehat{\rho}(x_{j,s-1})^{-1} \widehat{\rho}(x_{i,s-1}) \widehat{\rho}(x_{j,s}), \\ \widehat{\rho}(x_{1,0}) &= \widehat{\rho}(x_{1,1}) = \widehat{\rho}(x_{1,2}) = \cdots = \widehat{\rho}(x_{1,l-1}) = 1, \end{aligned}$$

with i, j, k as in Figure 2 and $s = 0, 1, \dots, l - 1$. Give a labeling of vertices on T_s for each s . Let (g_1, g_2, g_3) be the labeling of vertices on $F_{i,s+1}^-$ and let (g'_1, g'_2, g'_3) be the labeling of vertices on $F_{i,s}^+$. If these are related by

$$\widehat{\rho}(x_{i,s})(g_1, g_2, g_3) = (g'_1, g'_2, g'_3),$$

then we obtain a G -valued 1-cocycle λ on \widehat{T} by gluing T_s using $\widehat{\rho}$. To obtain a group cycle given by \widehat{C}_l and $\widehat{\rho}$, we insert a suspension over a $2l$ -gon to \widehat{T} at the 0-simplex corresponding to K . We give an ordering on the vertices of each tetrahedron of the suspension compatible with the ordering on the $2l$ -gon; for example, order the central 0-simplex maximal. These tetrahedra inherit a vertex labeling by G on the boundary faces of the suspension. We assign any labeling at the central 0-simplex of the suspension. Then upper and lower tetrahedra have the same labelings with different orientations. So these cancel out in pairs and give no contribution to the group cycle represented by \widehat{C}_l and $\widehat{\rho}$. Therefore the homology class given by \widehat{T} and λ represents the homology class given by \widehat{C}_l and $\widehat{\rho}$.

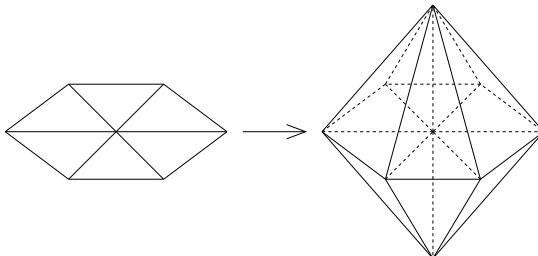


Figure 8

5. Quandle homology

In this section, we review the definitions of quandles, rack (co)homology and quandle (co)homology. Our treatment of quandle (or rack) homology follows that of [Etingof and Graña 2003]. In Sections 5B and 5C, we recall the notions of colorings and quandle cocycle invariants defined in [Carter et al. 2003].

5A. Quandle and quandle homology. A quandle X is a set with a binary operation $*$ satisfying the following axioms:

- (Q1) $x * x = x$ for any $x \in X$,
- (Q2) the map $*y : X \rightarrow X$ defined by $x \mapsto x * y$ is a bijection for any $y \in X$, and
- (Q3) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

We denote the inverse of $*y$ by $*^{-1}y$. For a quandle X , we define the associated group G_X by

$$G_X = \langle x \in X \mid y^{-1}xy = x * y \ (x, y \in X) \rangle.$$

A quandle X has a right G_X -action in the following way. Let $g = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ be an element of G_X where $x_i \in X$ and $\epsilon_i = \pm 1$. Define

$$x * g = (\dots ((x *^{\epsilon_1} x_1) *^{\epsilon_2} x_2) \dots) *^{\epsilon_n} x_n.$$

One can easily check that this is a right action of G_X on X . So the free abelian group $\mathbb{Z}[X]$ generated by X is a right $\mathbb{Z}[G_X]$ -module.

Let $C_n^R(X)$ be the free left $\mathbb{Z}[G_X]$ -module generated by X^n . We define the boundary map $C_n^R(X) \rightarrow C_{n-1}^R(X)$ by

$$\begin{aligned} \partial(x_1, x_2, \dots, x_n) &= \sum_{i=1}^n (-1)^i ((x_1, \dots, \widehat{x}_i, \dots, x_n) - x_i(x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)). \end{aligned}$$

Figure 9 shows a graphical picture of the boundary map. Let $C_n^D(X)$ be the $\mathbb{Z}[G_X]$ -submodule of $C_n^R(X)$ generated by (x_1, \dots, x_n) with $x_i = x_{i+1}$ for some i . Now $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X) = C_n^R(X)/C_n^D(X)$. For a right $\mathbb{Z}[G_X]$ -module M , we define the rack homology of M by the homology of $C_n^R(X; M) = M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ and denote it by $H_n^R(X; M)$. We also define the quandle homology of M by the homology of $C_n^Q(X; M) = M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$ and denote it by $H_n^Q(X; M)$. The homology $H_n^Q(X; \mathbb{Z})$, where \mathbb{Z} is the trivial $\mathbb{Z}[G_X]$ -module, is equal to the usual quandle homology $H_n^Q(X)$. Let Y be a set with a right G_X -action. For any abelian group A , the abelian group $A[Y]$ freely generated by Y over A is a right $\mathbb{Z}[G_X]$ -module. The homology group $H_n^Q(X; A[Y])$ is usually denoted by $H_n^Q(X; A)_Y$ [Kamada 2007].

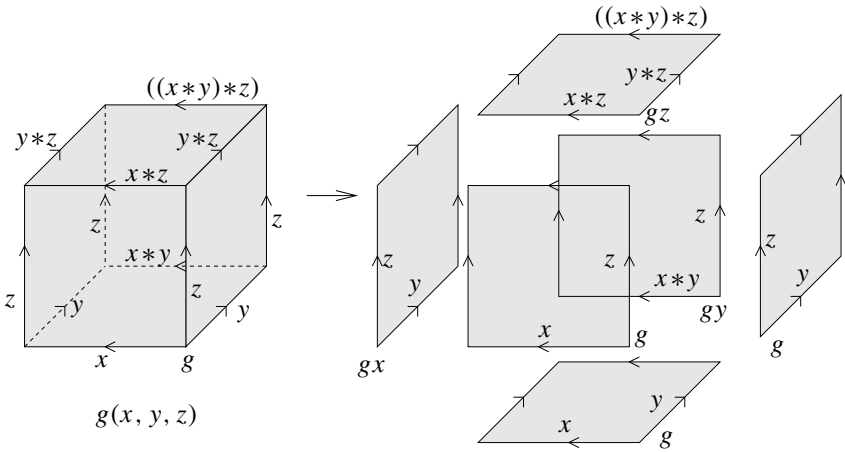


Figure 9. The boundary map $\partial(g(x, y, z)) = -(g(y, z) - gx(y, z)) + (g(x, z) - gy(x*y, z)) - (g(x, y) - gz(x*z, y*z))$. Here $x, y, z \in X$ and $g \in G_X$. Edges are labeled by elements of X and vertices are labeled by elements of G_X .

Let N be a left $\mathbb{Z}[G_X]$ -module. We define the rack cohomology $H_R^n(X; N)$ by the cohomology of $C_R^n(X; N) = \text{Hom}_{\mathbb{Z}[G_X]}(C_n^R(X), N)$. The quandle cohomology $H_Q^n(X; N)$ is defined in a similar way. For a set Y with a right G_X -action and an abelian group A , we let $\text{Func}(Y, A)$ be the left $\mathbb{Z}[G_X]$ -module generated by functions $\phi : Y \rightarrow A$, where the action is defined by $(g\phi)(y) = \phi(yg)$ for $y \in Y$ and $g \in G_X$. The cohomology group $H_Q^n(X; \text{Func}(Y, A))$ is usually denoted by $H_Q^n(X; A)_Y$ [Kamada 2007].

5B. Shadow coloring and associated quandle cycle. Let X be a quandle. Let L be an oriented link in S^3 and let D be a diagram of L . An arc coloring of D is an assignment of elements of X to arcs of D satisfying the following relation at each crossing:

$$\begin{array}{c} | \quad x * y \\ \uparrow \quad | \\ \quad \quad \rightarrow y \\ \quad \quad | \\ \quad \quad x \end{array}$$

where $x, y \in X$. By the Wirtinger presentation of the knot complement, an arc coloring determines a representation of $\pi_1(S^3 - L)$ into the associated group G_X . This is obtained by sending each meridian to its color.

Let Y be a set with a right G_X action. A region coloring of D is an assignment of elements of Y to regions of D satisfying the relation

$$\begin{array}{c} r \cdot x \\ \uparrow \quad | \\ \quad \quad \rightarrow x \\ \quad \quad r \end{array}$$

for any pair of adjacent regions, where $r \in Y$ and $x \in X$. A pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ is called a *shadow coloring*. If we fix a color of a region of D , then colors of other regions are uniquely determined. Therefore there always exists a region coloring compatible with a given arc coloring.

Define a cycle $C(\mathcal{S})$ of $C_2^Q(X; \mathbb{Z}[Y])$ for a shadow coloring \mathcal{S} . Put $+r \otimes (x, y)$ for a positive crossing and $-r \otimes (x, y)$ for a negative crossing colored by



respectively. Then define

$$C(\mathcal{S}) = \sum_{c: \text{crossing}} \epsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[Y]),$$

where $\epsilon_c = \pm 1$. We can easily check the following:

Proposition 5.1 [Inoue and Kabaya 2010]. *$C(\mathcal{S})$ is a cycle and the homology class $[C(\mathcal{S})]$ is invariant under Reidemeister moves. Moreover it does not depend on the choice of the region coloring if the action of G_X on Y is transitive.*

So the homology class $[C(\mathcal{S})]$ is an invariant of the arc coloring \mathcal{A} in many cases. There are two important sets with right G_X -action: one is when Y consists of one point $\{*\}$ and the other is when $Y = X$. Eisermann [2003; 2007] showed that the cycle $[C(\mathcal{S})]$ for $Y = \{*\}$ is essentially described by the monodromy of some representation of the knot group along the longitude. So we concentrate on the invariant $[C(\mathcal{S})]$ in the case of $Y = X$ from now on.

5C. Quandle cocycle invariant. Let X be a quandle with $|X| < \infty$. Let A be an abelian group and f be a cocycle of $H_Q^2(X; \text{Func}(X, A))$. We define the (*shadow*) *quandle cocycle invariant* by

$$\frac{1}{|X|} \sum_{\mathcal{S}: \text{shadow colorings}} \langle f, C(\mathcal{S}) \rangle \in \mathbb{Z}[A].$$

Here the sum is finite because there are only a finite number of shadow colorings of D . This is an invariant of oriented knots by Proposition 5.1. If G_X acts on X transitively (that is, X is connected), $\langle f, C(\mathcal{S}) \rangle$ does not depend on the choice of a region coloring \mathcal{R} by Proposition 5.1, and thus

$$\sum_{\substack{\mathcal{S}=(\mathcal{A}, \mathcal{R}), \\ \mathcal{A}: \text{arc coloring}}} \langle f, C(\mathcal{S}) \rangle$$

coincides with the quandle cocycle invariant, where \mathcal{R} is a region coloring compatible with \mathcal{A} .

We can regard $f \in C_Q^n(X; A)$ as an element of $C_Q^{n-1}(X; \text{Func}(X, A))$ by

$$f(x_1, x_2, \dots, x_{n-1})(r) = f(r, x_1, x_2, \dots, x_{n-1}).$$

This gives a homomorphism $H_Q^n(X; A) \rightarrow H_Q^{n-1}(X; \text{Func}(X, A))$. Therefore a quandle 3-cocycle $f \in H_Q^3(X; A)$ gives rise to a quandle cocycle invariant. Explicitly, the cocycle invariant has the form

$$\sum_{\substack{\mathcal{J}=(\mathcal{A}, \mathcal{R}), \\ \mathcal{A}: \text{arc coloring}}} \sum_{c: \text{crossing}} \epsilon_c f(r_c, x_c, y_c) \in \mathbb{Z}[A],$$

where $x_c, y_c \in X$ are given by \mathcal{A} and $r_c \in X$ are given by \mathcal{R} .

6. $H_n^\Delta(X; \mathbb{Z})$ and the map $H_n^R(X; \mathbb{Z}[X]) \rightarrow H_{n+1}^\Delta(X; \mathbb{Z})$

Let X be a quandle. Let $C_n^\Delta(X) = \text{span}_{\mathbb{Z}}\{(x_0, \dots, x_n) \mid x_i \in X\}$. We define the boundary operator of $C_n^\Delta(X)$ by

$$\partial(x_0, \dots, x_n) = \sum_{i=0}^n (-1)^i (x_0, \dots, \widehat{x}_i, \dots, x_n).$$

Since X has a right action of G_X , the chain complex $C_n^\Delta(X)$ has a right action of G_X by $(x_0, \dots, x_n) * g = (x_0 * g, \dots, x_n * g)$. Let M be a left $\mathbb{Z}[G_X]$ -module. We denote the homology of $C_n^\Delta(X) \otimes_{\mathbb{Z}[G_X]} M$ by $H_n^\Delta(X; M)$. For any abelian group A , we can also define the cohomology group $H_\Delta^n(X; A)$ in a similar way.

Let I_n be the set consisting of maps $\iota: \{1, 2, \dots, n\} \rightarrow \{0, 1\}$. We let $|\iota|$ denote the cardinality of the set $\{i \mid \iota(i) = 1, 1 \leq i \leq n\}$. For each generator $r \otimes (x_1, x_2, \dots, x_n)$ of $C_n^R(X; \mathbb{Z}[X])$, where $r, x_1, \dots, x_n \in X$, we define

$$\begin{aligned} r(\iota) &= r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)}) \in X, \\ x(\iota, i) &= x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}) \in X, \end{aligned}$$

for any $\iota \in I_n$. Fix an element $q \in X$. For each $n \geq 1$, we define a homomorphism

$$\varphi: C_n^R(X; \mathbb{Z}[X]) \rightarrow C_{n+1}^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$$

by

$$(6-1) \quad \varphi(r \otimes (x_1, x_2, \dots, x_n)) = \sum_{\iota \in I_n} (-1)^{|\iota|} (q, r(\iota), x(\iota, 1), x(\iota, 2), \dots, x(\iota, n)).$$

For example, in the case $n = 2$ (Figure 10),

$$\begin{aligned} &\varphi(r \otimes (x, y)) \\ &= (q, r, x, y) - (q, r * x, x, y) - (q, r * y, x * y, y) + (q, (r * x) * y, x * y, y), \end{aligned}$$

then f is a cocycle of $H_{\Delta}^k(X; A)$ and $\varphi^* f$ is a cocycle of $H_Q^{k-1}(X; \text{Func}(X, A))$. Moreover, $\varphi^* f$ can be regarded as a cocycle in $H_Q^k(X; A)$ by

$$(\varphi^* f)(r, x_1, \dots, x_{k-1}) = (\varphi^* f)(x_1, \dots, x_{k-1})(r).$$

We will construct functions satisfying these three conditions from group cocycles.

7. Cocycles of dihedral quandles

For any integer $p > 2$, let R_p denote the cyclic group \mathbb{Z}/p with quandle operation defined by $x * y = 2y - x \pmod p$. Actually this operation satisfies the quandle axioms. The quandle R_p is called the *dihedral quandle*. In this section, we construct quandle cocycles of R_p from group cocycles of $G = \mathbb{Z}/p$. In the next section, we will propose a general construction of quandle cocycles from group cocycles.

7A. Group cohomology of cyclic groups. Let G be the cyclic group \mathbb{Z}/p (where p is an integer greater than 2). The first cohomology $H^1(G; \mathbb{Z}/p) = \text{Hom}(\mathbb{Z}/p, \mathbb{Z}/p)$ is generated by the 1-cocycle b_1 defined by

$$b_1(x) = x.$$

The connecting homomorphism $\delta : H^1(G, \mathbb{Z}/p) \rightarrow H^2(G; \mathbb{Z})$ of the long exact sequence corresponding to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$ maps b_1 to a generator of $H^2(G; \mathbb{Z})$, and the reduction $H^2(G; \mathbb{Z}) \rightarrow H^2(G; \mathbb{Z}/p)$ maps it to a generator b_2 of $H^2(G; \mathbb{Z}/p)$. Explicitly, we have

$$(7-1) \quad b_2(x, y) = \frac{1}{p}(\bar{y} - \overline{\bar{x} + y} + \bar{x}) = \begin{cases} 1 & \text{if } \bar{x} + \bar{y} \geq p, \\ 0 & \text{otherwise,} \end{cases}$$

where \bar{x} is an integer $0 \leq \bar{x} < p$ such that $\bar{x} \equiv x \pmod p$. Cup products of b_1 's and b_2 's are also cocycles. Moreover, when p is an odd prime, it is known that any element of $H^*(G; \mathbb{Z}/p)$ can be presented by a cup product of b_1 's and b_2 's; see, for example, [Benson 1991, Proposition 3.5.5]. We remark that b_1 and b_2 and their products are normalized cocycles.

7B. Cocycle of R_p . For an integer $p > 2$, let f be a normalized k -cocycle of $H^k(G, \mathbb{Z}/p)$. Regarding R_p as $G = \mathbb{Z}/p$, we obtain a map $f : (R_p)^{k+1} \rightarrow \mathbb{Z}/p$ satisfying

$$(1) \quad \sum_{i=0}^{k+1} (-1)^i f(x_0, \dots, \widehat{x}_i, \dots, x_{k+1}) = 0,$$

$$(3) \quad f(x_0, \dots, x_k) = 0 \text{ if } x_i = x_{i+1} \text{ for some } i,$$

by using homogeneous notation (Section 2A). If f also satisfies the condition

$$(2) \quad f(x_0 * y, \dots, x_k * y) = f(x_0, \dots, x_k) \text{ for any } y \in R_p,$$

then f gives rise to a quandle k -cocycle of $H^k_Q(R_p; \mathbb{Z}/p)$ by the construction of Section 6. Define $\tilde{f} : (R_p)^{k+1} \rightarrow \mathbb{Z}/p$ by

$$(7-2) \quad \tilde{f}(x_0, \dots, x_k) = f(x_0, \dots, x_k) + f(-x_0, \dots, -x_k).$$

Then \tilde{f} satisfies condition (2) by the left invariance of the homogeneous group cocycle. It is easy to check that \tilde{f} also satisfies conditions (1) and (3). So we obtain a quandle k -cocycle.

We give an explicit presentation of the 3-cocycle arising from $b_1 b_2 \in H^3(G; \mathbb{Z}/p)$. Let

$$d(x, y) = b_2(x, y) - b_2(-x, -y);$$

then d is a 2-cocycle. (We can check that d is cohomologous to $2b_2$.) Then by the defining equation (7-2), $\widetilde{b_1 b_2}$ is given by

$$[x|y|z] \mapsto x \cdot d(y, z).$$

By definition we have

$$(7-3) \quad d(-x, -y) = -d(x, y)$$

and

$$(7-4) \quad d(x, y) = \begin{cases} 1 & \text{if } \bar{x} + \bar{y} > p, \\ -1 & \text{if } \bar{x} + \bar{y} < p, x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the cocycle d can be easily understood geometrically. Identify $i \in \mathbb{Z}/p$ with the complex number ζ^i , where $\zeta = \exp(2\pi\sqrt{-1}/p)$. Then $d(x, y) = -1$ if $(0, x, x + y)$ is counterclockwise, $d(x, y) = +1$ if $(0, x, x + y)$ is clockwise, and $d(x, y) = 0$ if $(0, x, x + y)$ is degenerate (Figure 11). This interpretation and Equation (7-3) make various calculations easy.

Proposition 7.1. *The quandle 3-cocycle arising from $b_1 b_2 \in H^3(G; \mathbb{Z}/p)$ has the following presentation:*

$$(7-5) \quad (x, y, z) \mapsto 2z(d(y - x, z - y) + d(y - x, y - z)), \quad (x, y, z \in R_p).$$

This is a nontrivial quandle 3-cocycle of R_p with \mathbb{Z}/p coefficients.

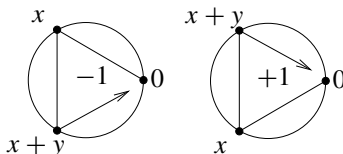


Figure 11. The value of $d(x, y)$.

Proof. In (6-1), since the map φ_* does not depend on the choice of $q \in R_p$, we let $q = 0$. Then we have

$$\begin{aligned} \varphi(x, y, z) &= (0, x, y, z) - (0, x * y, y, z) \\ &\quad - (0, x * z, y * z, z) + (0, (x * y) * z, y * z, z) \\ &= (0, x, y, z) - (0, 2y - x, y, z) \\ &\quad - (0, 2z - x, 2z - y, z) + (0, 2z - 2y + x, 2z - y, z), \end{aligned}$$

for $x, y, z \in R_p$. Rewriting in inhomogeneous notation, this is equal to

$$\begin{aligned} [x \mid y - x \mid z - y] - [2y - x \mid x - y \mid z - y] \\ - [2z - x \mid x - y \mid y - z] + [2z - 2y + x \mid y - x \mid y - z]. \end{aligned}$$

The evaluation of $\widetilde{b_1 b_2}$ on this chain is

$$\begin{aligned} x \cdot d(y - x, z - y) - (2y - x)d(x - y, z - y) \\ - (2z - x)d(x - y, y - z) + (2z - 2y + x)d(y - x, y - z) \\ = x \cdot d(y - x, z - y) + (2y - x)d(y - x, y - z) \\ + (2z - x)d(y - x, z - y) + (2z - 2y + x)d(y - x, y - z) \\ = 2z \cdot d(y - x, z - y) + 2z \cdot d(y - x, y - z). \end{aligned}$$

We will see that this cocycle is nontrivial because the evaluation on the cycle given by a shadow coloring \mathcal{S} of the $(2, p)$ -torus link (Figure 12) is nonzero. Color two arcs by $x, y \in R_p$ as in Figure 12. Then other arcs must be colored by $(i + 1)y - ix$ by the relations at the crossings. Let r be the color of the central region. Then we have

$$C(\mathcal{S}) = \sum_{i=0}^{p-1} r \otimes (iy - (i - 1)x, (i + 1)y - ix).$$

We assume that $r = 0$ and $x = 0$. Evaluation of the cocycle on $C(\mathcal{S})$ yields

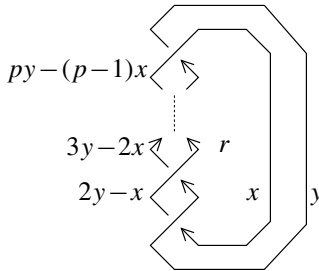


Figure 12. A shadow coloring of the $(2, p)$ -torus knot by R_p (for any $x, y, r \in R_p$).

$$\begin{aligned}
& \sum_{i=0}^{p-1} 2(i+1)y(d(iy, y) + d(iy, -y)) \\
&= 2y \sum_{i=0}^{p-1} (i+1)(d(iy, y) - d(-iy, y)) \quad (\text{by (7-3)}) \\
&= 2y \sum_{i=0}^{p-1} ((i+1)d(iy, y) + (i-1)d(iy, y)) = 4y \sum_{i=0}^{p-1} i \cdot d(iy, y).
\end{aligned}$$

By the next lemma, this is equal to $-4y^2 \pmod p$. □

Lemma 7.2. *Let $p > 2$ be an integer. For $0 < y < p$, we have*

$$\sum_{i=0}^{p-1} i \cdot d(iy, y) = \begin{cases} -y & p: \text{ odd,} \\ \frac{p}{2} - y & p: \text{ even.} \end{cases}$$

When $y = 0$, the left-hand side is 0.

Proof. When $y = 0$, this is straightforward since $d(x, 0) = 0$ for any $0 \leq x < p$. When y is a unit in \mathbb{Z}/p , by (7-4) we have

$$\begin{aligned}
\sum_{i=0}^{p-1} i \cdot d(iy, y) &= \sum_{i=0}^{p-1} \frac{i}{y} \cdot d(i, y) \\
&= \frac{1}{y} (-1 - 2 - \dots - (p-y-1) + (p-y+1) + (p-y+2) + \dots + (p-1)) \\
&= \frac{1}{y} \left(\frac{(p-y-1)(y-p)}{2} + \frac{(y-1)(2p-y)}{2} \right) \\
&= \frac{1}{y} \left(-y^2 - \frac{p}{2} + 2py - \frac{p^2}{2} \right) \equiv -y - \frac{p}{2} \cdot \frac{1}{y}.
\end{aligned}$$

This is equal to $-y$ when p is odd. When p is even, $-p/2 \cdot 1/y \equiv p/2 \pmod p$ since $1/y$ is a unit in \mathbb{Z}/p .

When y is not a unit in \mathbb{Z}/p , let c be the greatest common divisor of y and p . Then

$$\sum_{i=0}^{p-1} i \cdot d(iy, y) = c \sum_{j=0}^{p/c-1} \frac{j}{y/c} \cdot d(jc, y) = \frac{c}{y/c} \sum_{j=0}^{p/c-1} j \cdot d(j, y/c).$$

Since y/c is a unit in $\mathbb{Z}/(p/c)$, this reduces to the previous case. □

Since 2 is divisible in \mathbb{Z}/p when p is odd, we have:

Corollary 7.3. *When $p > 2$ is an odd number,*

$$(7-6) \quad (x, y, z) \mapsto z(d(y-x, z-y) + d(y-x, y-z))$$

is a nontrivial quandle 3-cocycle of R_p with \mathbb{Z}/p coefficients.

When p is prime, it is known that $\dim_{\mathbb{F}_p} H_Q^3(R_p; \mathbb{F}_p) = 1$. Therefore our cocycle is a constant multiple of the Mochizuki 3-cocycle [2003]. We remark that when p is prime, $\dim_{\mathbb{F}_p} H_Q^n(R_p; \mathbb{F}_p)$ was calculated for any n by Nosaka [2009], who gave a system of generators of $H_Q^n(R_p; \mathbb{F}_p)$.

When p is an odd integer, Nosaka [2010] showed that $H_3^Q(R_p; \mathbb{Z}) \cong \mathbb{Z}/p$. Since $H_2^Q(R_p; \mathbb{Z})$ is zero, we have $H_Q^3(R_p; \mathbb{Z}/p) \cong \mathbb{Z}/p$. This means that there exists a nontrivial quandle 3-cocycle of R_p with \mathbb{Z}/p coefficients.

8. General construction

In this section, we generalize the construction in the previous section to wider classes of quandles. We construct a quandle cocycle of a faithful homogeneous quandle X from a group cocycle of $\text{Aut}(X)$ when an obstruction living in the second cohomology of $\text{Aut}(X)$ vanishes.

8A. Let G be a group. Fix an element $h \in G$. Let $\text{Conj}(h) = \{g^{-1}hg \mid g \in G\}$. Now $\text{Conj}(h)$ has a quandle operation by $x * y = y^{-1}xy$. In this section, we construct a quandle cocycle of $\text{Conj}(h)$ from a group cocycle of G . First we shall show that this class of quandles is not so special.

Let X be a quandle. We denote the group of the quandle automorphisms of X by $\text{Aut}(X)$. We consider an automorphism that acts on X from the right. For $x \in X$, let $S(x)$ be the map that sends y to $y * x$. By the axioms (Q2) and (Q3), $S(x)$ is a quandle automorphism. X is called *faithful* if $S : X \rightarrow \text{Aut}(X)$ is injective. A quandle X is *homogeneous* if $\text{Aut}(X)$ acts on X transitively. The following lemma was essentially shown in Theorem 7.1 of [Joyce 1982], but we include a proof for completeness.

Lemma 8.1. *Every faithful homogeneous quandle X is represented by $\text{Conj}(h)$ with some group G and $h \in G$.*

Proof. For $x \in X$ and $g \in \text{Aut}(X)$, we have $S(xg) = g^{-1}S(x)g$. In fact, $(y)S(xg) = y * (xg) = (yg^{-1} * x)g = (y)g^{-1}S(x)g$ for any $y \in X$.

Let $G = \text{Aut}(X)$ and fix an element $x_0 \in X$. Put $h = S(x_0)$. Because X is homogeneous, for any $x \in X$, there exists $g \in G$ such that $x = x_0g$. So we have $S(x) = S(x_0g) = g^{-1}hg$, that is, $S(x) \in \text{Conj}(h)$. Therefore we obtain a homomorphism $S : X \rightarrow \text{Conj}(h)$. This is surjective since $g^{-1}S(x_0)g = S(x_0g)$, and injective since X is faithful. □

Let $Z(h) = \{g \in G \mid gh = hg\}$ be the centralizer of h in G .

Lemma 8.2. *A map*

$$\begin{array}{ccc} \text{Conj}(h) & \rightarrow & Z(h) \setminus G \\ \cup & & \cup \\ g^{-1}hg & \mapsto & Z(h)g \end{array}$$

is well-defined and bijective.

Proof. Let $g_1^{-1}hg_1 = g_2^{-1}hg_2$. Then $(g_1g_2^{-1})^{-1}h(g_1g_2^{-1}) = h$, so $g_1g_2^{-1} \in Z(h)$ and $g_1 \in Z(h)g_2$. This means that g_1 and g_2 belong to the same right coset. Therefore the map is well-defined. By a similar argument, we can show the injectivity. Surjectivity is trivial by definition. \square

Now we study the quandle structure on $Z(h)\backslash G$ and construct a section of the projection $\pi : G \rightarrow Z(h)\backslash G$. The quandle operation on $\text{Conj}(h)$ induces a quandle operation on $Z(h)\backslash G$:

$$\begin{aligned} Z(h)g_1 * Z(h)g_2 &\leftrightarrow (g_1^{-1}hg_1) * (g_2^{-1}hg_2) = (g_2^{-1}hg_2)^{-1}(g_1^{-1}hg_1)(g_2^{-1}hg_2) \\ &= (g_1g_2^{-1}hg_2)^{-1}h(g_1g_2^{-1}hg_2) \leftrightarrow Z(h)g_1(g_2^{-1}hg_2). \end{aligned}$$

This quandle operation on $Z(h)\backslash G$ lifts to a quandle operation $\tilde{*}$ on G by

$$(8-1) \quad g_1 \tilde{*} g_2 = h^{-1}g_1(g_2^{-1}hg_2), \quad (g_1, g_2 \in G).$$

We can easily check that $\tilde{*}$ satisfies the quandle axioms (the inverse operation is given by $g_1 \tilde{*}^{-1} g_2 = hg_1g_2^{-1}h^{-1}g_2$) and that the projection map $\pi : G \rightarrow Z(h)\backslash G$ is a quandle homomorphism. We remark that the quandle operation given by (8-1) has been already studied by Joyce [1982] and Eisermann [2003].

Let $s : Z(h)\backslash G \rightarrow G$ be a section of π , that is, a map (not a homomorphism) satisfying $\pi \circ s = \text{id}$. Since $s(x * y)$ and $s(x) \tilde{*} s(y)$ are in the same coset in $Z(h)\backslash G$, there exists an element $c(x, y) \in Z(h)$ satisfying

$$s(x) \tilde{*} s(y) = c(x, y)s(x * y).$$

Lemma 8.3. *If $Z(h)$ is an abelian group, $c : Z(h)\backslash G \times Z(h)\backslash G \rightarrow Z(h)$ is a quandle 2-cocycle. If the cocycle c is cohomologous to zero, we can change the section s to satisfy $s(x * y) = s(x) \tilde{*} s(y)$.*

Proof. For $c_1, c_2 \in Z(h)$ and $g_1, g_2 \in G$, we have

$$(c_1g_1) \tilde{*} (c_2g_2) = h^{-1}c_1g_1g_2^{-1}c_2^{-1}hc_2g_2 = c_1h^{-1}g_1g_2^{-1}hg_2 = c_1(g_1 \tilde{*} g_2).$$

Therefore,

$$\begin{aligned} (8-2) \quad (s(x) \tilde{*} s(y)) \tilde{*} s(z) &= (c(x, y)s(x * y)) \tilde{*} s(z) \\ &= c(x, y)(s(x * y) \tilde{*} s(z)) \\ &= c(x, y)c(x * y, z)((x * y) * z) \end{aligned}$$

and

$$\begin{aligned} (8-3) \quad (s(x) \tilde{*} s(z)) \tilde{*} (s(y) \tilde{*} s(z)) &= (c(x, z)s(x * z)) \tilde{*} (c(y, z)s(y * z)) \\ &= c(x, z)c(x * z, y * z)s((x * z) * (y * z)) \end{aligned}$$

for any $x, y, z \in Z(h) \setminus G$. Comparing (8-2) and (8-3), we have

$$c(x, z)c(x * y, z)^{-1}c(x, y)^{-1}c(x * z, y * z) = 1.$$

By $s(x) \tilde{*} s(x) = s(x)$, we also have $c(x, x) = 1$.

If c is cohomologous to zero; then there exists a map $b : Z(h) \setminus G \rightarrow Z(h)$ satisfying $c(x, y) = b(x)b(x * y)^{-1}$. Put $s'(x) = b(x)^{-1}s(x)$, then s' satisfies $s'(x) \tilde{*} s'(y) = s'(x * y)$. \square

Remark 8.4. The 2-cocycle c has already appeared in [Eisermann 2003] in a similar context.

Example 8.5. Let G be the dihedral group

$$D_{2p} = \langle h, x \mid h^2 = x^p = hxhx = 1 \rangle,$$

where p is an odd number greater than 2. Then we have $Z(h) = \{1, h\}$ and $\text{Conj}(h) = \{x^{-i}hx^i \mid i = 0, 1, \dots, p-1\} = \{hx^{2i} \mid i = 0, \dots, p-1\}$. We can identify $x^{-i}hx^i \in \text{Conj}(h)$ with $i \in R_p = \{0, 1, 2, \dots, p-1\}$. Define a section $s : Z(h) \setminus G \rightarrow G$ by

$$\begin{array}{ccccc} \text{Conj}(h) & \cong & Z(h) \setminus G & \xrightarrow{s} & G \\ \Psi & & \Psi & & \Psi \\ x^{-i}hx^i & \leftrightarrow & Z(h)x^i & \mapsto & hx^i \end{array}$$

Then we have

$$\begin{aligned} s(Z(h)x^i * Z(h)x^j) &= s(Z(h)x^{2j-i}) = hx^{2j-i} \\ &= h^{-1}(hx^i)(x^{-j}hx^j) = s(Z(h)x^i) \tilde{*} s(Z(h)x^j). \end{aligned}$$

Therefore $c(x, y) = 0$ for any $x, y \in R_p$.

8B. Let G be a group. Fix $h \in G$ with $h^l = 1$ ($l > 1$). In the following we assume:

Assumption 8.6. $Z(h)$ is abelian and the 2-cocycle corresponding to $G \rightarrow Z(h) \setminus G$ is cohomologous to zero.

Under this assumption, we can take a section $s : Z(h) \setminus G \rightarrow G$ satisfying

$$s(x * y) = s(x) \tilde{*} s(y),$$

by Lemma 8.3. Let $f : G^{k+1} \rightarrow A$ be a normalized group k -cocycle of G in the homogeneous notation, where A is an abelian group. Then f satisfies

- (1) $\sum_{i=0}^{k+1} (-1)^i f(x_0, \dots, \widehat{x}_i, \dots, x_{k+1}) = 0$,
- (2) $f(gx_0, \dots, gx_k) = f(x_0, \dots, x_k)$ for any $g \in G$ (left invariance), and
- (3) $f(x_0, \dots, x_k) = 0$ if $x_i = x_{i+1}$ for some i .

Since it is convenient to use a right-invariant function in the following construction, we modify condition (2) by replacing $f(x_0, \dots, x_k)$ with $f(x_0^{-1}, \dots, x_k^{-1})$:

$$(2') \quad f(x_0g, \dots, x_kg) = f(x_0, \dots, x_k) \text{ for any } g \in G \text{ (right invariance).}$$

Define $\tilde{f} : \text{Conj}(h)^{k+1} \rightarrow A$ by

$$\tilde{f}(x_0, \dots, x_k) = \sum_{i=0}^{l-1} f(h^i s(x_0), \dots, h^i s(x_k)),$$

for $x_0, \dots, x_k \in \text{Conj}(h)$.

Proposition 8.7. *The function \tilde{f} satisfies conditions (1), (2) and (3) of Section 6. Therefore \tilde{f} gives rise to a k -cocycle of $H_{\Delta}^k(\text{Conj}(h); A)$.*

Proof. It is clear that (1) and (3) are satisfied from the conditions on a normalized group cocycle in homogeneous notation. We only have to check the second property.

$$\begin{aligned} & \tilde{f}(x_0 * y, \dots, x_k * y) \\ &= \sum_{i=0}^{l-1} f(h^i s(x_0 * y), \dots, h^i s(x_k * y)) \\ &= \sum_{i=0}^{l-1} f(h^i s(x_0) \tilde{*} s(y), \dots, h^i s(x_k) \tilde{*} s(y)) \\ &= \sum_{i=0}^{l-1} f(h^{i-1} s(x_0)(s(y)^{-1} h s(y)), \dots, h^{i-1} s(x_k)(s(y)^{-1} h s(y))) \\ &= \sum_{i=0}^{l-1} f(h^{i-1} s(x_0), \dots, h^{i-1} s(x_k)) \quad (\text{right invariance}) \\ &= \tilde{f}(x_0, \dots, x_k). \quad \square \end{aligned}$$

Combining with the arguments of Section 6, we have:

Corollary 8.8. *If $Z(h)$ is abelian and the 2-cocycle corresponding to $G \rightarrow Z(h) \setminus G$ is cohomologous to zero, then there is a homomorphism*

$$H^n(G; A) \rightarrow H^n_Q(\text{Conj}(h); A)$$

for any abelian group A .

8C. We return to the case of R_p discussed in the previous section. We assume that p is an odd integer greater than 2. Let G be the dihedral group

$$D_{2p} = \langle h, x \mid h^2 = x^p = hxhx = 1 \rangle.$$

Consider the short exact sequence

$$(8-4) \quad 0 \rightarrow \mathbb{Z}/p \rightarrow D_{2p} \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

We regard $\mathbb{Z}/2$ as $\{1, h\}$ by taking coset representatives in D_{2p} . Then $\mathbb{Z}/2$ acts on \mathbb{Z}/p by $h(x^i) = x^{-i}$. This induces the restriction map

$$H^*(D_{2p}; \mathbb{Z}/p) \rightarrow H^*(\mathbb{Z}/p; \mathbb{Z}/p)^{\mathbb{Z}/2}.$$

We can show that this homomorphism is an isomorphism [Brown 1982, Proposition III.10.4]. To obtain a group cocycle of D_{2p} from a group cocycle of \mathbb{Z}/p , we need the inverse map, which is called the *transfer*. The transfer map is described as follows (see also [Brown 1982]). Let r be the map $D_{2p} \rightarrow \mathbb{Z}/p$ defined by $r(x^i) = x^i$ and $r(hx^i) = x^{-i}$. For a cocycle f of $H^n(\mathbb{Z}/p; \mathbb{Z}/p)^{\mathbb{Z}/2}$, the image f' of the transfer is given by

$$f'(x_0, \dots, x_n) = f(rx_0, \dots, rx_n) + f(hrx_0, \dots, hrx_n)$$

in homogeneous notation. When restricted to the image of $s : Z(h) \setminus G \rightarrow G$, this is equal to the map defined in (7-2). Applying our construction for this group cocycle, we obtain a quandle 3-cocycle of R_p , which is twice the cocycle constructed in the previous section.

8D. We end this section by giving another homomorphism from $H^n(G; A)$ to $H^n_Q(\text{Conj}(h); A)$ arising from a more general context.

Let X be a quandle and let M be a right $\mathbb{Z}[G_X]$ -module. We can construct a map from the rack homology $H_n^R(X; M)$ to the group homology $H_n(G_X; M)$. The following lemma is well-known; for example, see Lemma 7.4 of Chapter I of [Brown 1982].

Lemma 8.9. *Let $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a chain complex where the P_i are projective (for example, free). Let $\dots \rightarrow C_1 \rightarrow C_0 \rightarrow N \rightarrow 0$ be an acyclic complex. Any homomorphism $M \rightarrow N$ can be extended to a chain map from $\{P_*\}$ to $\{C_*\}$. Moreover, such a chain map is unique up to chain homotopy.*

So there exists a unique chain map from $C_*^R(X)$ to $C_*(G_X)$ up to homotopy. This map induces $M \otimes_{\mathbb{Z}[G_X]} C_*^R(X) \rightarrow M \otimes_{\mathbb{Z}[G_X]} C_*(G_X)$ and then $H_n^R(X; M) \rightarrow H_n(G_X; M)$. Using normalized chains in group homology, we can also construct a map $H_n^O(X; M) \rightarrow H_n(G_X; M)$. We give an explicit chain map. Let (x_1, \dots, x_n) be a generator of $C_n^R(X)$. We define a map ψ by

$$\psi((x_1, \dots, x_n)) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) [y_{\sigma,1} | \dots | y_{\sigma,i} | \dots | y_{\sigma,n}],$$

where $y_{\sigma,i} \in X$ is defined for a permutation σ and $i \in \{1, \dots, n\}$ as follows. Let

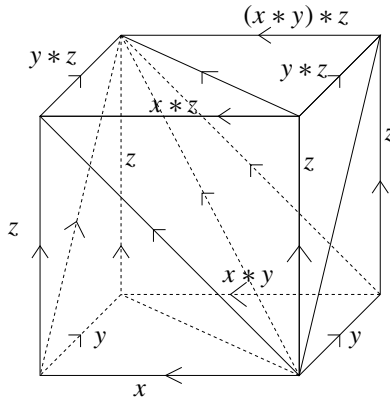


Figure 13

$j_1, \dots, j_i < i$ be the maximal set of numbers satisfying

$$\sigma(i) < \sigma(j_1) < \sigma(j_2) < \dots < \sigma(j_i).$$

Then define

$$y_{\sigma,i} = x_{\sigma(i)} * (x_{\sigma(j_1)} x_{\sigma(j_2)} \cdots x_{\sigma(j_i)}).$$

A graphical picture of this map is given in [Figure 13](#). For example, when $n = 3$,

$$\begin{aligned} \psi((x, y, z)) = & [x|y|z] - [x|z|y * z] \\ & + [y|z|(x * y) * z] - [y|x * y|z] + [z|x * z|y * z] - [z|y * z|(x * y) * z], \end{aligned}$$

for $(x, y, z) \in C_3^R(X)$. Dually, we also have a map $H^n(G_X; M) \rightarrow H_Q^n(X; M)$.

We apply this map for $\text{Conj}(h)$. Since there exists a natural homomorphism from the associated group $G_{\text{Conj}(h)}$ to G , we have a homomorphism

$$H^n(G; A) \rightarrow H^n(G_{\text{Conj}(h)}; A) \rightarrow H_Q^n(\text{Conj}(h); A).$$

Fenn, Rourke and Sanderson [[Fenn et al. 1995](#)] defined the *rack space* BX . Since $\pi_1(BX)$ is isomorphic to G_X , there exists a unique map, up to homotopy, from BX to the Eilenberg–MacLane space $K(G_X, 1)$ that induces the isomorphism between their fundamental groups. This map induces a homomorphism $H^n(G_X; M) \rightarrow H_Q^n(X; M)$, which is equal to the map we have constructed in this subsection. Clauwens [[2011, Proposition 25](#)] showed that this map vanishes under some conditions on X and M . In particular, when p is odd prime,

$$H^n(D_{2p}; \mathbb{Z}/p) \rightarrow H_Q^n(R_p; \mathbb{Z}/p)$$

vanishes for $n > 0$.

9. Quandle cycle and branched cover

In this section, we study the dual of the previous construction. We show that the cycle $C(\mathcal{S})$ associated with a shadow coloring $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ of a knot K gives rise to a group cycle represented by a cyclic branched covering along K and the representation induced from the arc coloring \mathcal{A} .

9A. Let X be a quandle. Let D be a diagram of a knot K . For a shadow coloring $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ of D whose arcs and regions are colored by X , define $\mathcal{A} * a$ and $\mathcal{R} * a$ for $a \in X$ by

$$(\mathcal{A} * a)(x) = \mathcal{A}(x) * a, \quad (\mathcal{R} * a)(r) = \mathcal{R}(r) * a \quad (\text{for any arc } x \text{ and region } r).$$

By the axiom (Q3), $\mathcal{S} * a = (\mathcal{A} * a, \mathcal{R} * a)$ is also a shadow coloring.

In the following, we assume that $X = \text{Conj}(h)$ for some group G and that h is in G and satisfies Assumption 8.6. As in the previous section, let $s : \text{Conj}(h) \cong Z(h) \backslash G \rightarrow G$ be a section satisfying $s(a * b) = s(a) \tilde{*} s(b)$ for $a, b \in X$. Let

$$\iota : C_n^\Delta(X) \rightarrow C_n(G; \mathbb{Z}) : (a_0, \dots, a_n) \mapsto (s(a_0), \dots, s(a_n)).$$

Composing with $\varphi : C_n^Q(X; \mathbb{Z}[X]) \rightarrow C_{n+1}^\Delta(X)$, we have a map

$$C_2^Q(X; \mathbb{Z}[X]) \xrightarrow{\varphi} C_3^\Delta(X) \xrightarrow{\iota} C_3(G; \mathbb{Z}).$$

For a shadow coloring \mathcal{S} , $\iota\varphi(C(\mathcal{S}))$ is not a cycle of $C_3(G; \mathbb{Z})$ in general. But we can show:

Theorem 9.1. *Let $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ be a shadow coloring of a diagram D of a knot K by $\text{Conj}(h)$. Let $a \in \text{Conj}(h)$ be the color of an arc of D . If $h^l = 1$, then*

$$(9-1) \quad \iota\varphi(C(\mathcal{S})) + \iota\varphi(C(\mathcal{S} * a)) + \iota\varphi(C(\mathcal{S} * a^2)) + \dots + \iota\varphi(C(\mathcal{S} * a^{l-1})) \in C_3(G; \mathbb{Z})$$

is a group cycle represented by the l -fold cyclic branched covering \widehat{C}_l along the knot K and the representation $\pi_1(\widehat{C}_l) \rightarrow G$ induced from the arc coloring \mathcal{A} .

Proof. Let x_1, \dots, x_n be the arcs of the diagram D such that $a = \mathcal{A}(x_1)$. We denote the color $\mathcal{A}(x_i)$ by a_i ($a_1 = a$). The arc coloring \mathcal{A} induces a representation $\rho : \pi_1(S^3 - K) \rightarrow G$. Using $s : Z(h) \backslash G \rightarrow G$, ρ is given by

$$\rho(x_i) = s(a_i)^{-1} h s(a_i) \in \text{Conj}(h) \subset G.$$

We have

$$\begin{aligned} s(b * a_i) &= h^{-1} s(b) s(a_i)^{-1} h s(a_i) = h^{-1} s(b) \rho(x_i) \in G, \\ s(b *^{-1} a_i) &= h s(b) s(a_i)^{-1} h^{-1} s(a_i) = h s(b) \rho(x_i)^{-1} \in G, \end{aligned}$$

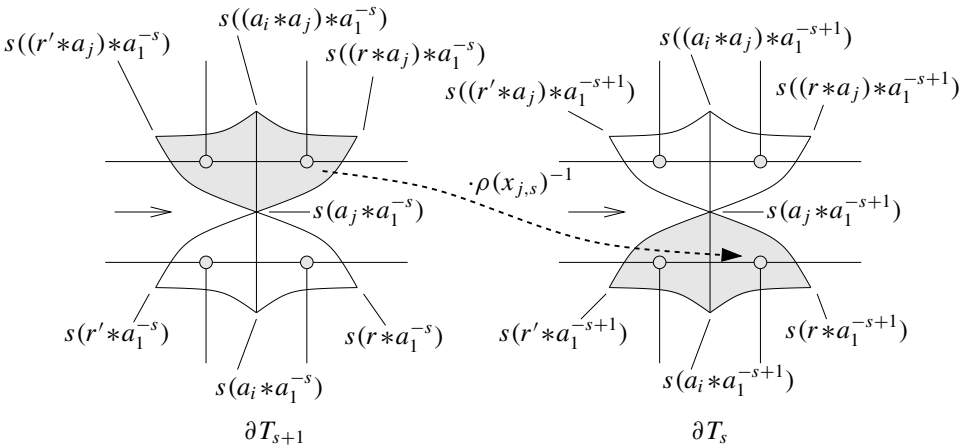


Figure 14. A labeling at a crossing. ($r' = r * \pm 1 a_i$.)

for any $b \in \text{Conj}(h)$. So we have

$$s((b * a_i) * a_1^{-s}) = h^{s-1} s(b) \rho(x_i) \rho(x_1)^{-s},$$

$$s(b * a_1^{-s+1}) = h^{s-1} s(b) \rho(x_1)^{-s+1}.$$

Since

$$\rho(x_{i,s}) = \rho(x_1^{s-1} x_i x_1^{-s}) = \rho(x_1)^{s-1} \rho(x_i) \rho(x_1)^{-s}$$

by (4-1), we have

$$(9-2) \quad s(b * a_1^{-s+1}) = s((b * a_i) * a_1^{-s}) \rho(x_{i,s})^{-1}.$$

Let T_s ($s = 0, 1, \dots, l-1$) be copies of the triangulation of a 3-ball with a vertex ordering on each tetrahedron constructed in Section 4 (Figures 5 and 7). Then we define a labeling of vertices of T_s by $\iota\varphi(C(\mathcal{G} * a_1^{-s+1}))$ (Figure 14). The vertices of the face $F'_{j,s+1} \subset T_{s+1}$ are labeled by

$$(s((r * a_j) * a_1^{-s}), \quad s((a_i * a_j) * a_1^{-s}), \quad s(a_j * a_1^{-s})),$$

and the vertices of the face $F'_{j,s} \subset T_s$ are labeled by

$$(s(r * a_1^{-s+1}), \quad s(a_i * a_1^{-s+1}), \quad s(a_j * a_1^{-s+1})),$$

where r is the color of the region as indicated in Figure 14. By (9-2), the labelings of the face $F'_{j,s+1}$ and $F'_{j,s}$ are related by

$$(s(r * a_1^{-s+1}), \quad s(a_i * a_1^{-s+1}), \quad s(a_j * a_1^{-s+1}))$$

$$= (s((r * a_j) * a_1^{-s}), \quad s((a_i * a_j) * a_1^{-s}), \quad s(a_j * a_1^{-s})) \rho(x_{j,s})^{-1}.$$

Therefore this gives a G -valued 1-cocycle on \widehat{T} as constructed in Section 4. Thus the chain given by (9-1) is a cycle represented by the cyclic branched cover \widehat{C}_l and the representation induced from the arc coloring \mathcal{A} . \square

9B. We end this section by comparing the shadow cocycle invariant of the $(2, p)$ -torus knot (p : odd) for the 3-cocycle obtained in Section 7 and the Dijkgraaf–Witten invariant of the lens space $L(p, 1)$.

Let G be a finite group and α be a cocycle of $H^3(G; A)$, where A is an abelian group. Dijkgraaf and Witten defined an invariant of closed oriented 3-manifolds for each α . For an oriented closed manifold M , it is defined by

$$\sum_{\rho: \pi_1(M) \rightarrow G} \langle \rho^* \alpha, [M] \rangle \in \mathbb{Z}[A],$$

where $\rho^* \alpha$ is the pull-back of α by the classifying map $M \rightarrow BG$ corresponding to ρ . Since $L(p, 1)$ is a double branched cover along the $(2, p)$ -torus knot, the cycle obtained from a shadow coloring by R_p gives rise to a group 3-cycle of D_{2p} represented by $L(p, 1)$ by Theorem 9.1. Since every representation of $\pi_1(L(p, 1)) \cong \mathbb{Z}/p$ into D_{2p} reduces to a representation into \mathbb{Z}/p , it is natural to ask whether the shadow cocycle invariant for our quandle 3-cocycle coincides with the Dijkgraaf–Witten invariant for $b_1 b_2 \in H^3(\mathbb{Z}/p; \mathbb{Z}/p)$. As we remarked in the introduction, these invariants coincide up to some constant by the result of Hatakenaka and Nosaka [2012] when p is prime.

Let \mathcal{S} be the shadow coloring indicated in Figure 12. Since the homology class $[C(\mathcal{S})]$ does not change under the action of R_p on the shadow coloring \mathcal{S} (see Lemma 4.5 of [Inoue and Kabaya 2010]), we assume that $x = 0$. As computed in Section 7B, the evaluation of the $C(\mathcal{S})$ at the 3-cocycle derived from $b_1 b_2$ is equal to $-4y^2 \pmod p$. Therefore the shadow cocycle invariant is

$$p \sum_{y=0}^{p-1} t^{-4y^2} = p \sum_{y=0}^{p-1} t^{-y^2} \in \mathbb{Z}[t]/(t^p - 1) \cong \mathbb{Z}[\mathbb{Z}/p].$$

We compute the Dijkgraaf–Witten invariant of the lens space $L(p, q)$ for the group 3-cocycle $b_1 b_2 \in H^3(\mathbb{Z}/p; \mathbb{Z}/p)$. Although this was computed in [Murakami et al. 1992], we give a proof based on a triangulation. We represent $L(p, q)$ by an ordered 3-cycle and give a labeling of 1-simplices as indicated in Figure 15. Here the triple a, b and c must satisfy $a^p = 1$ and $b = a^q bc$. Using multiplicative notation, the evaluation of $b_1 b_2$ on this cycle is

$$\sum_{i=0}^{p-1} b_1 b_2([a|ia + b| - qa]) = \sum_{i=0}^{p-1} a \cdot b_2(ia + b, -qa) = -qa^2.$$

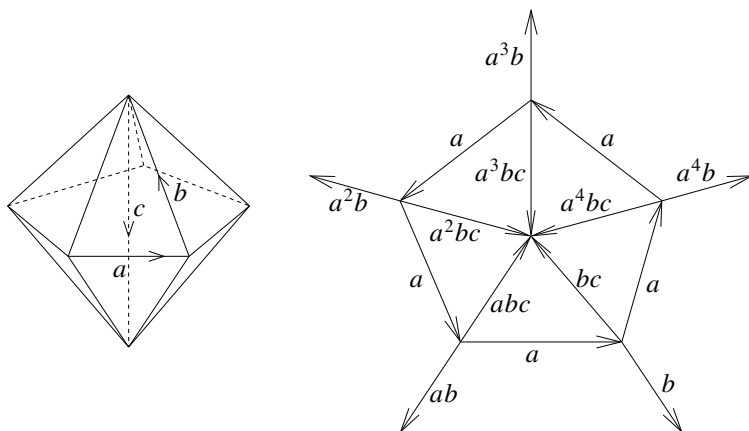


Figure 15. A triangulation of $L(5, q)$ and a labeling where $a^5 = 1$ and $b = a^q bc$.

The second equality follows from the fact that $ia + b$ runs over \mathbb{Z}/p and from (7-1) when a is a unit, and also when a is not a unit by a similar argument. Therefore the Dijkgraaf–Witten invariant of $L(p, q)$ is equal to

$$\sum_{a=0}^{p-1} t^{-qa^2} \in \mathbb{Z}[t]/(t^p - 1) \cong \mathbb{Z}[\mathbb{Z}/p].$$

Therefore this coincides with the shadow cocycle invariant up to a constant.

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