Pacific Journal of Mathematics

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IVAN MATIĆ

Volume 259 No. 2 October 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 259, No. 2, 2012
dx.doi.org/10.2140/pjm.2012.259.445

THETA LIFTS OF STRONGLY POSITIVE DISCRETE SERIES: THE CASE OF $(\widetilde{\mathrm{Sp}}(n), O(V))$

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Let F denote a nonarchimedean local field of characteristic zero with odd residual characteristic. Using the results of Gan and Savin, in this paper we determine the first occurrence indices and theta lifts of strongly positive discrete series representations of metaplectic groups over F in terms of our recent classification of this class of representations. Also, we determine the first occurrence indices of some strongly positive representations of odd orthogonal groups.

1. Introduction

One of the main issues in the local theta correspondence is a precise determination of the theta lifts of irreducible representations. This problem is by now completely solved for cuspidal representations [Mœglin et al. 1987, Théorème principal] and for discrete series for the dual pair $(\mathrm{Sp}(n), O(V))$ [Muić 2004, Theorems 4.2 and 4.3]. Muić used an inductive procedure to investigate certain embeddings of theta lifts of discrete series representations so as to obtain explicit information about the structure of these lifts and to derive the first occurrence indices.

The description given there is based on the classification of discrete series of the classical groups given in [Mœglin 2002; Mœglin and Tadić 2002], which relies on certain conjectures called the *basic assumption* (we emphasize that Arthur has recently announced a proof of his conjectures about the stable transfer coming from the twisted endoscopy, which should imply the basic assumption). On the other hand, we have recently classified the strongly positive discrete series of metaplectic groups, and our classification uses no hypothesis and can be applied much more generally. It is natural to try to relate this classification to the determination of the lifts of those representations. Thus, it is the purpose of this paper to determine the first occurrence indices of the strongly positive discrete series for the dual pair $(\widetilde{\mathrm{Sp}}(n), O(V))$, where $\widetilde{\mathrm{Sp}}(n)$ is the universal cover of $\mathrm{Sp}(n)$, and to obtain as much information about the structure of theta lifts of such representations as possible.

Muić [2008] has obtained some fundamental results on the structure of theta

MSC2010: primary 22E35, 22E50; secondary 11F27.
Keywords: theta correspondence, metaplectic groups, strongly positive representations.

lifts of discrete series without using the Mœglin–Tadić classification. Although very powerful, the methods used there could not provide an explicit description of the first occurrence indices. Nevertheless, his results have recently been rewritten by Gan and Savin [2012] for the dual pair $(\widetilde{\mathrm{Sp}}(n), O(V))$ over a nonarchimedean field of characteristic zero with odd residual characteristic. Another crucial result of their paper is a natural correspondence between irreducible representations on a certain level of metaplectic and odd orthogonal towers, which partially generalizes results of Waldspurger [1984; 1991].

These results are of much importance for us, because they allow us to start our investigation of the first occurrence index with the lift that is a discrete series representation at a quite low level of the tower. The disadvantage of this approach is that it prevents us from determining both first occurrence indices when lifting from the metaplectic tower. So we determine just the lower one.

We do not adopt the methods used in [Muić 2004], choosing rather to describe theta lifts of strongly positive discrete series directly from their cuspidal supports. The advantage of using this method lies in the fact that the structure of the obtained theta lifts can be explicitly described in a purely combinatorial way.

We now describe the contents of this paper. The next section presents some preliminaries, while in Section 3, we summarize without proofs the relevant material on the strongly positive discrete series. In that section we also obtain some useful embeddings of the general discrete series representations. Section 4 provides a detailed exposition of the results about Howe correspondence, which will be used through the paper. Section 5 is the technical heart of the paper, containing several results regarding the theta lifts of irreducible representations.

In Section 6, we state and prove our main results about the lifts of strongly positive irreducible representations of the metaplectic groups, using case-by-case consideration. In Section 7, we determine the first occurrence indices of certain strongly positive representations of the odd orthogonal groups. The observed cases happen to be quite similar in both directions, so the proofs made in the sixth section help us shorten those in the seventh one.

However, for the sake of completeness and to avoid possible confusion, we discuss the details of the lifts of representations of the metaplectic groups and those of the orthogonal ones in separate sections.

2. Notations and preliminaries

Let F be a nonarchimedean local field of characteristic zero with odd residual characteristic.

For a reductive group G, let $\mathrm{Irr}(G)$ stand for the set of isomorphism classes of irreducible admissible (genuine) representations of G.

First we discuss the groups that we consider.

Let V_0 be an anisotropic quadratic space over F of odd dimension. Then its dimension can only be 1 or 3. For more details about the invariants of this space, such as the quadratic character χ_{V_0} related to the quadratic form on V_0, we refer the reader to [Kudla 1986] and [Kudla and Rallis 2005]. In each step we add a hyperbolic plane and obtain an enlarged quadratic space, a tower of quadratic spaces, and a tower of corresponding orthogonal groups. In the case when r hyperbolic planes are added to the anisotropic space, the enlarged quadratic space will be denoted by V_r, while a corresponding orthogonal group will be denoted by $O(V_r)$. Set $m_r = (1/2)\dim V_r$.

To a fixed quadratic character χ_{V_0}, one can attach two odd orthogonal towers, one with $\dim V_0 = 1$ ($+$-tower) and the other with $\dim V_0 = 3$ ($-$-tower), as in Chapter V of [Kudla 1996]. In that case, for corresponding orthogonal groups of the spaces obtained by adding r hyperbolic planes, we write $O(V_r^+)$ and $O(V_r^-)$.

Let $S_1(n)$ be the Grothendieck group of the category of all admissible representations of finite length of $O(V_n)$ (that is, a free abelian group over the set of all irreducible representations of $O(V_n)$), and define $S_1 = \bigoplus_{n\geq 0} S_1(n)$.

Let $\widetilde{\mathrm{Sp}}(n)$ be the metaplectic group of rank n, the unique nontrivial two-fold central extension of symplectic group $\mathrm{Sp}(n, F)$. In other words, the following holds:

$$1 \to \mu_2 \to \widetilde{\mathrm{Sp}}(n) \to \mathrm{Sp}(n, F) \to 1,$$

where $\mu_2 = \{1, -1\}$. The multiplication in $\widetilde{\mathrm{Sp}}(n)$ (which is as a set given by $\mathrm{Sp}(n, F) \times \mu_2$) is given by Rao's cocycle [Ranga Rao 1993]. More details on the structural theory of metaplectic groups can be found in [Hanzer and Muić 2010], [Kudla 1996], and [Ranga Rao 1993].

In this paper we are interested only in genuine representations of $\widetilde{\mathrm{Sp}}(n)$ (that is, those that do not factor through μ_2). So, let $S_2(n)$ be the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{\mathrm{Sp}}(n)$ and define $S_2 = \bigoplus_{n\geq 0} S_2(n)$.

Let $\widetilde{\mathrm{GL}}(n, F)$ be a double cover of $\mathrm{GL}(n, F)$, where the multiplication is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1\epsilon_2(\det g_1, \det g_2)_F).$$

Here $\epsilon_i \in \mu_2$, $i = 1, 2$, and $(\cdot\,, \cdot)_F$ denotes the Hilbert symbol of the field F.

The pair $(\mathrm{Sp}(n), O(V_r))$ is a reductive dual pair in $\mathrm{Sp}(n \cdot \dim V_r)$. Since the dimension of the space V_r is odd, the theta correspondence relates the representations of the metaplectic group $\widetilde{\mathrm{Sp}}(n)$ and those of the orthogonal group $O(V_r)$. We use the abbreviation $n_1 = n \cdot \dim V_r$. Let $\omega_{n_1,\psi}$ be the Weil representation of $\widetilde{\mathrm{Sp}}(n_1)$ depending on the nontrivial additive character ψ, and let $\omega_{n,r}$ denote the pull-back of that representation to the pair $(\widetilde{\mathrm{Sp}}(n), O(V_r))$.

Here and subsequently, ψ denotes a nontrivial additive character of F. Further, we fix a character $\chi_{V,\psi}$ of $\widetilde{\mathrm{GL}}(n, F)$ given by

$$\chi_{V,\psi}(g, \epsilon) = \chi_V(\det g)\epsilon\gamma(\det g, \tfrac{1}{2}\psi)^{-1}.$$

Here γ denotes the Weil invariant, while χ_V is a character related to the quadratic form on $O(V_r)$. We write $\alpha = \chi^2_{V,\psi}$ and observe that α is a quadratic character on $\mathrm{GL}(n, F)$.

Let

$$\mathcal{R}^{\mathrm{gen}} = \bigoplus_{n\geq 0} \mathcal{R}^{\mathrm{gen}}(n),$$

where $\mathcal{R}^{\mathrm{gen}}(n)$ denotes the Grothendieck group of smooth genuine representations of finite length of $\widetilde{\mathrm{GL}}(n, F)$. Similarly, define

$$\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}(n),$$

where $\mathcal{R}(n)$ denotes the Grothendieck group of smooth genuine representations of finite length of $\mathrm{GL}(n, F)$.

To simplify the notation, in the sequel we write

$$\mathcal{R}' = \begin{cases} \mathcal{R} & \text{in the orthogonal case,} \\ \mathcal{R}^{\mathrm{gen}} & \text{in the metaplectic case,} \end{cases}$$

and

$$S' = \begin{cases} S_1 & \text{in the orthogonal case,} \\ S_2 & \text{in the metaplectic case.} \end{cases}$$

By ν we denote the character of $\mathrm{GL}(n, F)$ defined by $|\det|_F$.

An irreducible representation $\sigma \in S'$ is called strongly positive if for each representation $\nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{\mathrm{cusp}}$, where $\rho_i \in \mathcal{R}'$, $i = 1, 2, \ldots, k$ are irreducible cuspidal unitary representations, $\sigma_{\mathrm{cusp}} \in S'$ is an irreducible cuspidal representation, and $s_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$ such that

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{\mathrm{cusp}},$$

we have $s_i > 0$ for each i.

Irreducible strongly positive representations are called strongly positive discrete series.

If $\rho \in \mathcal{R}'(m)$ is an irreducible unitary cuspidal representation, we say that $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ is a segment, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$; and we abbreviate $\{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ as $[\nu^a\rho, \nu^{a+k}\rho]$. We denote by $\delta(\Delta)$ the unique irreducible subrepresentation of $\nu^{a+k}\rho \times \nu^{a+k-1}\rho \times \cdots \times \nu^a\rho$. This $\delta(\Delta)$ is an essentially square-integrable representation attached to the segment Δ.

For every irreducible cuspidal representation $\rho \in \mathcal{R}'(m)$, there exists a unique $e(\rho) \in \mathbb{R}$ such that the representation $\nu^{-e(\rho)}\rho$ is a unitary cuspidal representation. From now on, let $e\big([\nu^a\rho, \nu^b\rho]\big) = (a+b)/2$.

For an ordered partition $s = (n_1, n_2, \ldots, n_j)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $\mathrm{Sp}(n, F)$ (consisting of block upper-triangular matrices) whose Levi factor equals

$$\mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \cdots \times \mathrm{GL}(n_j) \times \mathrm{Sp}(n - |s|, F),$$

where $|s| = m = \sum_{i=1}^{j} n_i$. Then the standard parabolic subgroup $\widetilde{P_s}$ of $\widetilde{\mathrm{Sp}}(n)$ is the preimage of P_s in $\widetilde{\mathrm{Sp}}(n)$. We have the analogous notation for the Levi subgroups of the metaplectic groups, which are described in more detail in Section 2.2 of [Hanzer and Muić 2010]. The standard parabolic subgroups (containing the upper triangular Borel subgroup) of $O(V_r)$ have an analogous description to the standard parabolic subgroups of $\mathrm{Sp}(n, F)$. If $\widetilde{P_s}$ is a standard parabolic subgroup of $\widetilde{\mathrm{Sp}}(n)$ described above, or P_s is a similar standard parabolic subgroup of $O(V_r)$, the normalized Jacquet module of a smooth representation σ of $\widetilde{\mathrm{Sp}}(n)$ (resp. $O(V_r)$) with respect to $\widetilde{P_s}$ (resp. P_s) is denoted by $R_{\widetilde{P_s}}(\sigma)$ (resp. $R_{P_s}(\sigma)$). From now on, $R_{P_1}(\pi)(\chi)$ (or $R_{\widetilde{P_1}}(\pi)(\chi)$) stands for the isotypic component of $R_{P_1}(\pi)$ along the generalized character χ.

Also, in dealing with Jacquet modules of $\omega_{n,r}$, we use the shorthand $R_{P_1}(\omega_{n,r})$ (resp. $R_{\widetilde{P_1}}(\omega_{n,r})$) for $R_{\widetilde{\mathrm{Sp}}(n)\times P_1}(\omega_{n,r})$ (resp. $R_{\widetilde{P_1}\times O(V_m)}(\omega_{n,r})$), following the notation of [Hanzer and Muić 2011].

For any irreducible representation $\pi \in S'(n)$, there exist an ordered partition $s = (n_1, n_2, \ldots, n_j)$ of some $m \leq n$, cuspidal representations $\rho_i \in \mathrm{Irr}(\mathcal{R}'(n_i))$, and $\pi_{\mathrm{cusp}} \in S'(n-|s|)$ such that π is an irreducible subquotient of the induced representation $\rho_1 \times \rho_2 \times \cdots \times \rho_j \rtimes \pi_{\mathrm{cusp}}$. In this situation, we write $[\pi] = [\rho_1, \rho_2, \ldots, \rho_j; \pi_{\mathrm{cusp}}]$, following the notation used in [Kudla 1996].

Let $\sigma \in S'(n)$ denote an irreducible representation. To simplify notation, set $P'_s = P_s$ in the orthogonal case and $P'_s = \widetilde{P_s}$ in the metaplectic one. We define $\mu^*(\sigma) \in \mathcal{R}' \otimes S'$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \mathrm{s.s.}(P'_{(k)}(\sigma)),$$

where s.s. denotes the semisimplification. We extend μ^* linearly to the whole of S'.

In the following lemma, we recall a useful formula for calculations with Jacquet modules, which is valid in both the orthogonal and metaplectic cases [Tadić 1995; Hanzer and Muić 2010]. Set $\alpha' = \alpha$ in the metaplectic case, while in the orthogonal case α' denotes a trivial character.

Lemma 2.1. *Let $\rho \in \mathcal{R}'$ be an irreducible cuspidal representation and let $a, b \in \mathbb{R}$ be such that $a + b \in \mathbb{Z}_{\geq 0}$. Let $\sigma \in S'$ be an admissible representation of finite length.*

Write $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. *Then the following holds*:

$$\mu^*\big(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma\big) = \sum_{i=-a-1}^{b} \sum_{j=i}^{b} \sum_{\pi,\sigma'} \delta\big([\nu^{-i}\alpha'\tilde{\rho}, \nu^{a}\alpha'\tilde{\rho}]\big) \times \delta\big([\nu^{j+1}\rho, \nu^b\rho]\big) \times \pi \otimes \delta\big([\nu^{i+1}\rho, \nu^j\rho]\big) \rtimes \sigma'. \tag{1}$$

We omit $\delta\big([\nu^x\rho, \nu^y\rho]\big)$ *if* $x > y$.

We take a moment to recall the formulation of the second Frobenius isomorphism.

Generally, for some reductive group G', its parabolic subgroup P' with the Levi subgroup M', and its opposite parabolic subgroup $\overline{P}'$, the second Frobenius isomorphism is

$$\mathrm{Hom}_{G'}\big(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi\big) \cong \mathrm{Hom}_{M'}\big(\pi, R_{\overline{P}'}(\Pi)\big),$$

for some smooth representation π (resp. Π) of the group M' (resp. G'). We denote the space of the representation π by V_π.

The above isomorphism can be explicitly described in the following way: let Ψ denote the embedding

$$\Psi : V_\pi \hookrightarrow R_{\overline{P}'}\big(\mathrm{Ind}_{M'}^{G'}(V_\pi)\big),$$

which corresponds to the open cell $P'\overline{P}'$ in G' [Bernstein 1987]. Now, for some $T \in \mathrm{Hom}_{G'}\big(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi\big)$, compose Ψ with the corresponding mapping

$$T_{\overline{P}'} : R_{\overline{P}'}\big(\mathrm{Ind}_{M'}^{G'}(\pi)\big) \to R_{\overline{P}'}(\Pi).$$

3. Embeddings of discrete series

In this section we recall the classification of strongly positive discrete series and obtain further embeddings of general discrete series that will be used later.

In the following theorem, we gather the results obtained in Section 5 of [Matić 2011]. The arguments used there rely on Jacquet module methods, and build up in an essentially combinatorial way from the cuspidal reducibility values. Moreover, the underlying combinatorics are essentially the same for classical groups. Thus, our classification is valid for both metaplectic and orthogonal groups.

Theorem 3.1. *We define a collection of pairs* (Jord, σ'), *where* σ' *is an irreducible cuspidal representation of some* $S'(n_{\sigma'})$ *and* Jord *has the following form*: Jord $= \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$, *where*:

- $\{\rho_1, \rho_2, \dots, \rho_k\}$ *is a* (*possibly empty*) *set of mutually nonisomorphic irreducible self-dual cuspidal representations of some* $\mathcal{R}'(m_1), \mathcal{R}'(m_2), \dots, \mathcal{R}'(m_k)$ *such that* $\nu^{a_{\rho_i}}\rho_i \rtimes \sigma'$ *reduces for* $a_{\rho_i} > 0$ (*this defines* a_{ρ_i}).
- $k_i = \lceil a_{\rho_i} \rceil$, *the smallest integer that is not smaller than* a_{ρ_i}.

- *For each $i = 1, \dots, k$, the sequence $b_1^{(i)}, \dots, b_{k_i}^{(i)}$ consists of real numbers such that $a_{\rho_i} - b_j^{(i)}$ is an integer, for $j = 1, 2, \dots, k_i$ and $-1 < b_1^{(i)} < b_2^{(i)} < \cdots < b_{k_i}^{(i)}$.*

There is a bijective correspondence between the set of all irreducible strongly positive representations in S' and the set of all pairs (Jord, σ').

We describe this correspondence more precisely. The pair corresponding to an irreducible strongly positive representation $\sigma \in S'$ is denoted by $(\mathrm{Jord}(\sigma), \sigma'(\sigma))$.

Suppose that cuspidal support of σ is contained in the set

$$\{\nu^x \rho_1, \dots, \nu^x \rho_k, \sigma_{\mathrm{cusp}} : x \in \mathbb{R}\},$$

with k minimal (here ρ_i denotes an irreducible cuspidal self-dual representation of some $\mathcal{R}'(n_{\rho_i})$).

Let $a_{\rho_i} > 0$, $i = 1, 2, \dots, k$ denote the unique positive $s \in \mathbb{R}$ such that the representation $\nu^s \rho_i \rtimes \sigma_{\mathrm{cusp}}$ reduces. Set $k_i = \lceil a_{\rho_i} \rceil$. For each $i = 1, 2, \dots, k$, there exists a unique increasing sequence of real numbers $b_1^{(i)}, b_2^{(i)}, \dots, b_{k_i}^{(i)}$, where $a_{\rho_i} - b_j^{(i)}$ is an integer, for $j = 1, 2, \dots, k_i$ and $b_1^{(i)} > -1$, such that σ is the unique irreducible subrepresentation of the induced representation

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \sigma_{\mathrm{cusp}}.$$

Now, $\mathrm{Jord}(\sigma) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$ *and* $\sigma'(\sigma) = \sigma_{\mathrm{cusp}}$.

We note that results of [Arthur 2011] should imply that every a_{ρ_i} in the previous theorem is half integral.

This classification implies some interesting properties of strongly positive discrete series, which are listed in the next two lemmas.

Lemma 3.2 [Matić 2012, Lemma 3.5]. *Let $\sigma \in S'$ be a strongly positive discrete series. Then σ is uniquely determined by* $[\sigma]$.

The next result follows rather straightforwardly from the classification above:

Lemma 3.3. *Let $\sigma \in S'$ denote a strongly positive discrete series and suppose that $\nu^x \rho$ appears in $[\sigma]$, where $\rho \in \mathcal{R}'$ is an irreducible unitarizable cuspidal representation and $|x| \leq 1$. Then the representation $\nu^x \rho$ appears in $[\sigma]$ with multiplicity one. Also, if $\nu^y \rho$ appears in $[\sigma]$ for some $y \neq x$, then $|y| > 1$.*

Proof. It is enough to prove the lemma for $x \geq 0$, since otherwise the same conclusion can be drawn for $|x|$.

We write σ as the unique irreducible subrepresentation of the induced representation of the form

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \sigma_{\mathrm{cusp}}.$$

Obviously, ρ is isomorphic to ρ_l for some $l \in \{1, 2, \dots, k\}$.

By the assumption of the lemma, there is some $j \in \{1, 2, \dots, k_l\}$ such that $a_{\rho_l} - k_l + j \leq x \leq b_j^{(l)}$. Strong positivity of σ implies $x > 0$. Since $a_{\rho_l} - k_l + j > 1$ for $j \geq 2$, it follows that $\nu^x \rho$ appears in the segment

$$\left[\nu^{a_{\rho_l} - k_l + 1} \rho_l, \nu^{b_1^{(l)}} \rho_l\right]$$

and $\nu^x \rho$ does not appear in $\left[\nu^{a_{\rho_l} - k_l + j} \rho_l, \nu^{b_j^{(l)}} \rho_l\right]$, for $j \geq 2$. Further, using $x - 1 \leq 0$, we obtain $x = a_{\rho_l} - k_l + 1$.

Consequently, $\nu^x \rho$ appears in $[\sigma]$ with multiplicity one.

The inequality $|y| > 1$ for $y \neq x$ such that $\nu^y \rho$ appears in $[\sigma]$ is a consequence of the fact that $|y| - x$ is a positive integer and $x > 0$. $\square$

The principal significance of the following lemma is that it allows us to obtain certain embeddings of general discrete series.

Lemma 3.4. *Suppose that $\pi \in S'(n)$ is an irreducible representation that is not in the discrete series. Then there exists an embedding of the form*

$$\pi \hookrightarrow \delta\big([\nu^a \rho, \nu^b \rho]\big) \rtimes \pi',$$

where $a + b \leq 0$ and $\rho \in \mathcal{R}'$ and $\pi' \in S'$ are irreducible representations.

Proof. We adopt the approach from Section 3 of [Matić 2011], which was motivated by [Muić 2006]. Suppose that

$$\pi \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \pi_{\text{cusp}}$$

is an embedding of the representation π contradicting Casselman's square-integrability criterion (whose metaplectic version is written in [Ban and Jantzen 2009]), $\rho_i \in \mathcal{R}'$ is an irreducible cuspidal representation for $i \in \{1, 2, \dots, k\}$, and $\pi_{\text{cusp}} \in S'(n')$ is an irreducible cuspidal representation. Further, we consider all possible embeddings of the form

$$\pi \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_m) \rtimes \pi_{\text{cusp}},$$

contradicting the square-integrability criterion, where $\Delta_1 + \Delta_2 + \cdots + \Delta_m = \{\rho_1, \rho_2, \dots, \rho_k\}$, viewed as the equality of multisets. Clearly, $e(\Delta_i) \leq 0$ for some $i \in \{1, 2, \dots, m\}$. The set of all such embeddings is obviously finite and nonempty.

Each $\delta(\Delta_i)$ is an irreducible representation of some $\mathcal{R}'(n_i)$ (this defines n_i), for $i = 1, 2, \dots, m$. To every such embedding we attach an $(n - n')$-tuple

$$\big(e(\Delta_1), \dots, e(\Delta_1), e(\Delta_2), \dots, e(\Delta_2), \dots, e(\Delta_m), \dots, e(\Delta_m)\big) \in \mathbb{R}^{n-n'},$$

where $e(\Delta_i)$ appears n_i times.

Denote by

$$\pi \hookrightarrow \delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \pi_{\text{cusp}}$$

the minimal such embedding with respect to the lexicographic ordering on $\mathbb{R}^{n-n'}$. In the same way as in the proof of Theorem 3.3 of [Matić 2011], we conclude $e(\Delta'_1) \le e(\Delta'_2) \le \cdots \le e(\Delta'_{m'})$. This gives $e(\Delta'_1) \le 0$. Now Lemma 3.2 of [Mœglin and Tadić 2002] finishes the proof. □

We are ready to describe useful embeddings of general discrete series (this parallels the result of Lemma 3.1 of [Mœglin 2002]).

Theorem 3.5. *Let $\sigma \in S'(n)$ denote a discrete series representation. Then there exists an embedding of the form*

$$\sigma \hookrightarrow \delta\big([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1]\big) \times \delta\big([\nu^{a_2}\rho_2, \nu^{b_2}\rho_2]\big) \times \cdots \times \delta\big([\nu^{a_k}\rho_k, \nu^{b_k}\rho_k]\big) \rtimes \sigma_{\mathrm{sp}},$$

where $a_i \le 0$ and $a_i + b_i > 0$, and $\rho_i \in \mathcal{R}'$ is an irreducible representation for $i = 1, 2, \ldots, k$, while $\sigma_{\mathrm{sp}} \in S'$ is a strongly positive discrete series (*we allow $k = 0$*).

Proof. If σ is a strongly positive discrete series, then $k = 0$ and $\sigma \simeq \sigma_{\mathrm{sp}}$. Thus, we may suppose that σ is not strongly positive.

Again, we start with an embedding of the representation σ of the form

$$\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma_{\mathrm{cusp}},$$

where each $\rho_i \in \mathcal{R}'$ is an irreducible cuspidal representation and $\sigma_{\mathrm{cusp}} \in S'(n')$ is a partial cuspidal support of σ, and consider all possible embeddings of the form

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_m) \rtimes \sigma_{\mathrm{cusp}},$$

where $\Delta_1 + \Delta_2 + \cdots + \Delta_m = \{\rho_1, \rho_2, \ldots, \rho_l\}$, viewed as the equality of multisets. In the same way as in the proof of the previous lemma, to every such embedding we attach an element of $\mathbb{R}^{n-n'}$ and denote by

$$\sigma \hookrightarrow \delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}} \tag{2}$$

the minimal such embedding with respect to the lexicographic ordering on $\mathbb{R}^{n-n'}$. Analysis similar to that in the proof of Theorem 3.3 of [Matić 2011] shows $e(\Delta'_1) \le e(\Delta'_2) \le \cdots \le e(\Delta'_{m'})$.

Write each element of the multiset $\{\rho_1, \rho_2, \ldots, \rho_l\}$ in form $\rho_i = \nu^{a_i}\rho_{i,u}$, where $\rho_{i,u}$ is an irreducible unitary cuspidal representation. Define

$$a = \min\{a_i : 1 \le i \le l\}.$$

The assumption that σ is not strongly positive yields $a \le 0$. Suppose that $\nu^a\rho$ appears in the segment Δ'_i, with i minimal (for appropriate ρ). Then $\Delta'_i = [\nu^a\rho, \nu^b\rho]$, for some b.

If the segment Δ'_i is not connected in the sense of Zelevinsky with any of the segments $\Delta'_1, \ldots, \Delta'_{i-1}$, we obtain the embedding

$$\sigma \hookrightarrow \delta(\Delta'_i) \times \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}}.$$

Suppose that there is some segment Δ'_j, $1 \le j \le i-1$, such that the segments Δ'_i and Δ'_j are connected in the sense of Zelevinsky. We choose the largest such j and denote it by j again. Also, we write $\Delta'_j = [\nu^{a'}\rho, \nu^{b'}\rho]$. The intertwining operator $\delta(\Delta'_j) \times \delta(\Delta'_i) \to \delta(\Delta'_i) \times \delta(\Delta'_j)$ gives the maps

$$\begin{aligned}\sigma &\hookrightarrow \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_j) \times \delta(\Delta'_i) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}} \\ &\to \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_i) \times \delta(\Delta'_j) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}}.\end{aligned}$$

Observe that the kernel of the previous intertwining operator equals

$$\delta(\Delta'_1) \times \cdots \times \delta\big([\nu^{a}\rho, \nu^{b'}\rho]\big) \times \delta\big([\nu^{a'}\rho, \nu^{b}\rho]\big) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}}.$$

Since $e(\Delta'_j) \le e(\Delta'_i)$, the inequality $a < a'$ implies $e\big([\nu^{a}\rho, \nu^{b'}\rho]\big) < e(\Delta'_j)$. Thus, the minimality of the embedding (2) shows that σ is not contained in the kernel of the observed intertwining operator, which gives

$$\sigma \hookrightarrow \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_i) \times \delta(\Delta'_j) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}}.$$

Repeated application of the above procedure enables us to obtain the embedding

$$\sigma \hookrightarrow \delta(\Delta'_i) \times \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_{m'}) \rtimes \sigma_{\mathrm{cusp}}.$$

Lemma 3.2 of [Mœglin and Tadić 2002] implies that there is some irreducible representation σ_1 such that $\sigma \hookrightarrow \delta\big([\nu^{a}\rho, \nu^{b}\rho]\big) \rtimes \sigma_1$. Square-integrability of σ shows $a+b > 0$. We claim that σ_1 is a discrete series representation.

Suppose, on the contrary, that σ_1 is not in the discrete series. Then the previous lemma shows that it can be written as a subrepresentation of the induced representation of the form $\delta\big([\nu^{x}\rho', \nu^{y}\rho']\big) \rtimes \sigma'_1$, where $x+y \le 0$. Thus, $\sigma \hookrightarrow \delta\big([\nu^{a}\rho, \nu^{b}\rho]\big) \times \delta\big([\nu^{x}\rho', \nu^{y}\rho']\big) \rtimes \sigma'_1$. Square-integrability of the representations σ shows that the segments $[\nu^{a}\rho, \nu^{b}\rho]$ and $[\nu^{x}\rho', \nu^{y}\rho']$ are connected in the sense of Zelevinsky, and consequently $\sigma \hookrightarrow \delta\big([\nu^{a}\rho, \nu^{y}\rho]\big) \times \delta\big([\nu^{x}\rho', \nu^{b}\rho']\big) \rtimes \sigma'_1$.

The choice of a shows that $a \le x$, which leads to $a+y \le x+y \le 0$; that is, $e\big([\nu^{a}\rho, \nu^{y}\rho]\big) \le 0$, contradicting the square-integrability of σ. In this way we have proved that σ_1 is also a discrete series representation.

We continue in this fashion to obtain that either σ_1 is strongly positive or it can be written as a subrepresentation of the induced representation of the form $\delta\big([\nu^{a'}\rho', \nu^{b'}\rho']\big) \rtimes \sigma_2$, where $a' \le 0$ and $\sigma_2 \in S'$ is a discrete series representation. Repeating this procedure, after a finite number of steps we obtain the claim of the theorem. □

4. Howe's correspondence and results of Gan and Savin and of Kudla

In this section we review some results about Howe correspondence.

For an irreducible genuine smooth representation $\sigma \in S_2(n)$, let $\Theta(\sigma, r)$ be a smooth representation of $O(V_r)$, given as the full lift of σ to the r-level of the orthogonal tower, that is, the biggest quotient of $\omega_{n,r}$ on which $\widetilde{\mathrm{Sp}}(n)$ acts as a multiple of σ. As a representation of $\widetilde{\mathrm{Sp}}(n) \times O(V_r)$ it has a form $\sigma \otimes \Theta(\sigma, r)$. We write $\Theta^+(\sigma, r)$ (resp. $\Theta^-(\sigma, r)$) for the lift on the $+$-tower (resp. $-$-tower), when emphasizing the tower.

Similarly, if τ is an irreducible representation of $O(V_r)$, then one has its full lift $\Theta(\tau, n)$, which is a smooth representation of $\widetilde{\mathrm{Sp}}(n)$.

In the following theorem we summarize some basic results about the theta correspondence.

Theorem 4.1 [Kudla 1996; Mœglin et al. 1987]. *Let σ denote an irreducible genuine representation of $\widetilde{\mathrm{Sp}}(n)$. Then there exists an integer $r \geq 0$ such that $\Theta(\sigma, r) \neq 0$. The smallest such r is called the first occurrence index of σ in the orthogonal tower. Also, $\Theta(\sigma, r') \neq 0$ for $r' \geq r$.*

The representation $\Theta(\sigma, r)$ is either zero or it has finite length. If the residual characteristic of field F is other than 2, then $\Theta(\sigma, r)$ is either zero or it has a unique irreducible quotient. Following [Muić 2004], we write $\sigma(r)$ for this unique irreducible quotient.

The analogous statements hold for $\Theta(\tau, n)$ if τ is an irreducible representation of $O(V_r)$.

Now we state the results of Gan and Savin [2012, Section 6 and Theorem 8.1] that serve as a cornerstone for our determination of lifts of the strongly positive discrete series.

Theorem 4.2. *Let F be a nonarchimedean local field of characteristic 0 with odd residual characteristic. For each nontrivial additive character ψ of F, there is an injection*

$$\Theta_\psi : \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n)) \to \mathrm{Irr}(O(V_n^+)) \sqcup \mathrm{Irr}(O(V_{n-1}^-))$$

given by the theta correspondence (with respect to ψ). Suppose that $\sigma \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n))$ and $\tau \in \mathrm{Irr}(O(V))$ correspond under Θ_ψ. Then σ is a discrete series representation if and only if τ is a discrete series representation.

Let σ_{cusp} denote an irreducible cuspidal genuine representation of $\widetilde{\mathrm{Sp}}(n')$. We write $\Theta(\sigma, r)$ for the smooth isotypic component of σ in $\omega_{n,r}$. Since σ_{cusp} is cuspidal, for the smallest r' such that $\Theta(\sigma_{\mathrm{cusp}}, r') \neq 0$, we have that $\Theta(\sigma_{\mathrm{cusp}}, r')$ is an irreducible cuspidal representation of $O(V_{r'})$; we denote it by τ_{cusp}.

Let $\rho \in \mathcal{R}$ be an irreducible cuspidal self-contragredient representation. Results of Silberger [1980] (in the orthogonal case) and of Hanzer and Muić [2011] (in the metaplectic case) show that there exist unique nonnegative real numbers s_1 and s_2 such that the induced representations $\nu^{s_1}\rho \rtimes \tau_{\mathrm{cusp}}$ and $\nu^{s_2}\chi_{V,\psi}\rho \rtimes \sigma_{\mathrm{cusp}}$ reduce.

If ρ is not a trivial character of $F^{\times}$, then $s_1 = s_2$. Otherwise, the representation $\nu^{s_1} \rtimes \tau_{\text{cusp}}$ reduces for $s_1 = |n' - m_{r'}|$, while the representation $\nu^{s_2}\chi_{V,\psi} \rtimes \sigma_{\text{cusp}}$ reduces for $s_2 = |m_{r'} - n' - 1|$, where $m_{r'} = (1/2)\dim V_{r'}$.

We take a moment to state the results from Section 2 of [Kudla 1986], which happen to be crucial for our investigation.

Theorem 4.3. *Let $\tau \in S_1(r)$ denote an irreducible representation and suppose $[\tau] = [\rho_1, \rho_2, \ldots, \rho_k; \tau_{\text{cusp}}]$, with $\tau_{\text{cusp}} \in S_1(r')$ being an irreducible cuspidal representation. Let $\sigma_{\text{cusp}} = \tau(n')$ be the first nonzero lift of the representation τ_{cusp} and observe that $\sigma_{\text{cusp}} \in S_2(n')$ is an irreducible cuspidal representation. Let σ denote an irreducible quotient of $\Theta(\tau, n)$. We have the following possibilities*:

- *If $n \geq n' + r - r'$, then*

$$[\sigma] = \big[\chi_{V,\psi}\nu^{m_r-n}, \chi_{V,\psi}\nu^{m_r-n+1}, \ldots, \chi_{V,\psi}\nu^{m_{r'}-n'-1}, \\ \chi_{V,\psi}\rho_1, \chi_{V,\psi}\rho_2, \ldots, \chi_{V,\psi}\rho_k; \sigma_{\text{cusp}}\big].$$

- *If $n < n' + r - r'$, set $t = r - r' - n + n'$. Then there exist $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, k\}$ such that $\rho_{i_j} = \nu^{m_r-n-j}$ for $j = 1, 2, \ldots, t$ and*

$$[\sigma] = \big[\chi_{V,\psi}\rho_1, \ldots, \widehat{\chi_{V,\psi}\rho_{i_1}}, \ldots, \widehat{\chi_{V,\psi}\rho_{i_t}}, \ldots, \chi_{V,\psi}\rho_k; \sigma_{\text{cusp}}\big],$$

where $\widehat{\chi_{V,\psi}\rho_i}$ means that we omit $\chi_{V,\psi}\rho_i$.

Similarly, let $\sigma \in S_2(n)$ denote an irreducible representation and suppose $[\sigma] = [\chi_{V,\psi}\rho_1, \chi_{V,\psi}\rho_2, \ldots, \chi_{V,\psi}\rho_k; \sigma_{\text{cusp}}]$, with $\sigma_{\text{cusp}} \in S_2(n')$ being an irreducible cuspidal representation. Let $\tau_{\text{cusp}} = \sigma(r')$ be the first nonzero lift of the representation σ_{cusp}, and observe that $\tau_{\text{cusp}} \in S_1(r')$ is an irreducible cuspidal representation. Let τ denote an irreducible quotient of $\Theta(\sigma, r)$. We have the following possibilities:

- *If $r \geq r' + n - n'$, then*

$$[\tau] = [\nu^{m_r-n-1}, \nu^{m_r-n-2}, \ldots, \nu^{m_{r'}-n'}, \rho_1, \rho_2, \ldots, \rho_k; \tau_{\text{cusp}}].$$

- *If $r < r' + n - n'$, set $t = r' - n' + n - r$. Then there exist $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, k\}$ such that $\rho_{i_j} = \nu^{m_r-n+j-1}$ for $j = 1, 2, \ldots, t$ and*

$$[\tau] = [\rho_1, \ldots, \widehat{\rho_{i_1}}, \ldots, \widehat{\rho_{i_t}}, \ldots, \rho_k; \tau_{\text{cusp}}],$$

where $\widehat{\rho_i}$ means that we omit ρ_i.

The next theorem that we need is Kudla's filtration of Jacquet modules of the oscillatory representation:

Theorem 4.4 [Kudla 1986, Theorem 2.8]. *Let $\omega_{n,r}$ denote the oscillatory representation of the group $\widetilde{\mathrm{Sp}}(n) \times O(V_r)$ corresponding to the nontrivial additive character ψ.*

- *Let P_j denote the standard maximal parabolic subgroup of $O(V_r)$. Then Jacquet module $R_{P_j}(\omega_{n,r})$ has $\widetilde{\mathrm{Sp}}(n) \times M_j$-invariant filtration given by I_{jk}, $0 \le k \le j$, where*
$$I_{jk} \simeq \mathrm{Ind}_{P_{jk} \times \widetilde{P}_k \times O(V_{r-j})}^{\widetilde{\mathrm{Sp}}(n) \times M_j} (\gamma_{jk} \otimes \Sigma'_k \otimes \omega_{n-k,r-j}).$$
Here, P_{jk} is a standard parabolic subgroup of $\mathrm{GL}(j, F)$ corresponding to the partition $(j-k, k)$, γ_{jk} is a character of $\mathrm{GL}(j-k, F) \times \widetilde{\mathrm{GL}}(k, F)$ given by
$$\gamma_{jk}(g_1, g_2) = \nu^{-(m_r - n - (j-k+1)/2)}(g_1) \chi_{V,\psi}(g_2),$$
and Σ'_k is a twist of the standard representation of $\mathrm{GL}(k, F) \times \mathrm{GL}(k, F)$ on the space of smooth locally constant compactly supported complex-valued functions $C_c^\infty(\mathrm{GL}(k, F))$:
$$\Sigma'_k(g_1, g_2) f(g) = \nu^{-(m_{r-j} + (k-1)/2)} \nu^{m_{r-j} + (k-1)/2} f(g_1^{-1} g g_2).$$
In particular, a quotient I_{j0} equals $\nu^{-(m_r - n - (j+1)/2)} \otimes \omega_{n,r-j}$ and a subrepresentation I_{jj} equals
$$\mathrm{Ind}_{\mathrm{GL}(j,F) \times \widetilde{P}_j \times O(V_{r-j})}^{\widetilde{\mathrm{Sp}}(n) \times M_j} (\chi_{V,\psi} \otimes \Sigma'_j \otimes \omega_{n-j,r-j}).$$
- *Let $\widetilde{P}_j$ denote the standard maximal parabolic subgroup of $\widetilde{\mathrm{Sp}}(n)$. Then Jacquet module $R_{\widetilde{P}_j}(\omega_{n,r})$ has $\widetilde{M}_j \times O(V_r)$-invariant filtration given by J_{jk}, $0 \le k \le j$, where*
$$J_{jk} \simeq \mathrm{Ind}_{\widetilde{P}_{jk} \times P_k \times \widetilde{\mathrm{Sp}}(n-j)}^{\widetilde{M}_j \times O(V_r)} (\beta_{jk} \otimes \Sigma'_k \otimes \omega_{n-j,r-k}).$$
Here, $\widetilde{P}_{jk}$ is a standard parabolic subgroup of $\widetilde{\mathrm{GL}}(j, F)$ corresponding to the partition $(j-k, k)$, β_{jk} is a character of $\widetilde{\mathrm{GL}}(j-k, F) \times \widetilde{\mathrm{GL}}(k, F)$ given by
$$\beta_{jk}(g_1, g_2) = \big(\chi_{V,\psi} \nu^{m_r - n - (j-k-1)/2}\big)(g_1) \chi_{V,\psi}(g_2),$$
and Σ'_k is a twist of the standard representation of $\mathrm{GL}(k, F) \times \mathrm{GL}(k, F)$ on the space of smooth locally constant compactly supported complex-valued functions $C_c^\infty(\mathrm{GL}(k, F))$:
$$\Sigma'_k(g_1, g_2) f(g) = \nu^{m_r + (k+1)/2} \nu^{-(m_r + (k+1)/2)} f(g_1^{-1} g g_2).$$
In particular, a quotient J_{j0} equals $\chi_{V,\psi} \nu^{m_r - n + (j-1)/2} \otimes \omega_{n-j,r}$ and a subrepresentation J_{jj} equals $\mathrm{Ind}_{\widetilde{\mathrm{GL}}(j,F) \times P_j \times \widetilde{\mathrm{Sp}}(n-j)}^{\widetilde{M}_j \times O(V_r)} (\chi_{V,\psi} \otimes \Sigma'_j \otimes \omega_{n-j,r-j})$.

5. Some technical results on lifts

The purpose of this section is to state and prove many technical results that will be of particular importance in the following sections.

An elementary but useful criterion for pushing down the lifts of irreducible representations is established by the following two propositions.

Proposition 5.1. *Let $\tau \in S_1(r)$ be an irreducible representation.*

(1) *Suppose that $\Theta(\tau, n) \neq 0$. Then $R_{\widetilde{P}_1}(\Theta(\tau, n+1))(\chi_{V,\psi}\nu^{m_r-(n+1)}) \neq 0$.*

(2) *Suppose that $R_{P_1}(\tau)(\nu^{m_r-(n+1)}) = 0$. Then $\Theta(\tau, n) \neq 0$ if and only if*

$$R_{\widetilde{P}_1}(\Theta(\tau, n+1))(\chi_{V,\psi}\nu^{m_r-(n+1)}) \neq 0.$$

Proof. The proof follows the same lines as that of Theorem 4.5 of [Hanzer and Muić 2011].

Assume that $\Theta(\tau, n) \neq 0$. Then there exists an epimorphism $\omega_{n,r} \to \tau \otimes \Theta(\tau, n)$. Kudla's filtration gives the epimorphisms

$$R_{\widetilde{P}_1}(\omega_{n+1,r}) \to \chi_{V,\psi}\nu^{m_r-(n+1)} \otimes \omega_{n,r} \to \chi_{V,\psi}\nu^{m_r-(n+1)} \otimes \tau \otimes \Theta(\tau, n).$$

Using Frobenius reciprocity, we get a nontrivial intertwining

$$\Theta(\tau, n+1) \to \chi_{V,\psi}\nu^{m_r-(n+1)} \rtimes \Theta(\tau, n).$$

This obviously proves the first statement of the proposition.

It remains to prove sufficiency in the second statement. The condition

$$R_{\widetilde{P}_1}(\Theta(\tau, n+1))(\chi_{V,\psi}\nu^{m_r-(n+1)}) \neq 0$$

gives $\Theta(\tau, n+1) \neq 0$, which gives an epimorphism $\omega_{n+1,r} \to \tau \otimes \Theta(\tau, n+1)$. Applying Jacquet modules, we get an epimorphism

$$R_{\widetilde{P}_1}(\omega_{n+1,r}) \to \tau \otimes \chi_{V,\psi}\nu^{m_r-(n+1)} \otimes \sigma'$$

for some irreducible representation $\sigma' \in S_1(n)$. If we suppose that the restriction of this epimorphism to a subrepresentation J_{11} is nonzero, second Frobenius reciprocity gives a nonzero intertwining map

$$\chi_{V,\psi} \otimes \Sigma_1' \otimes \omega_{n,r-1} \to \widetilde{R_{P_1}(\widetilde{\tau})} \otimes \chi_{V,\psi}\nu^{m_r-(n+1)} \otimes \sigma'.$$

From this intertwining, we deduce $\tau \hookrightarrow \nu^{m_r-(n+1)} \rtimes \tau'$ for some irreducible representation $\tau' \in S_2(r-1)$, contradicting the assumption of the proposition. Consequently, there exists a nonzero intertwining $J_{10} \to \tau \otimes \chi_{V,\psi}\nu^{m_r-(n+1)} \otimes \sigma'$, which gives $\Theta(\tau, n) \neq 0$. □

We omit the proof of the next proposition, since it is completely analogous to the proof of the previous one.

Proposition 5.2. *Let $\sigma \in S_2(n)$ be an irreducible representation.*

(1) *Suppose that $\Theta(\sigma, r) \neq 0$. Then $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}) \neq 0$.*

(2) *Suppose that* $R_{\widetilde{P}_1}(\sigma)(\chi_{V,\psi}\nu^{-(m_{r+1}-n-1)}) = 0$. *Then* $\Theta(\sigma, r) \neq 0$ *if and only if* $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}) \neq 0$.

Now we prove an important result regarding the square-integrability of the lifts of strongly positive discrete series. In particular, this result gives an alternative and essentially combinatorial proof of a special case of the results of [Muić 2008].

Proposition 5.3. *Let* $\sigma \in S_2(n)$ *denote a strongly positive discrete series. Suppose that* $\Theta(\sigma, r) \neq 0$, *for some* r *such that* $m_r \leq n + \frac{1}{2}$, *and that*

$$R_{\widetilde{P}_1}(\sigma)(\chi_{V,\psi}\nu^{-(m_k-n-1)}) = 0$$

for $k \geq r+1$. *Then* $\sigma(r)$ *is a discrete series representation.*

Proof. We prove this proposition by downwards induction on r, starting with an r such that $m_r = n + \frac{1}{2}$. If $m_r = n + \frac{1}{2}$, Theorem 4.2 shows our claim. Thus, suppose that the claim holds for some $r+1$ such that $m_{r+1} \leq n + \frac{1}{2}$. We prove it for r.

It may be easily concluded from the proof of Proposition 5.1 (in the same way as in the proof of Lemma 5.1 of [Muić 2004]) that there is a nonzero intertwining $\sigma(r) \hookrightarrow \nu^{-(m_r-n-1)} \rtimes \sigma(r-1)$.

Note that in our case, $m_r < n+\frac{1}{2}$, which implies $-(m_r-n-1) \geq \frac{3}{2}$. Now, suppose that $\sigma(r-1)$ is not a discrete series representation. According to Lemma 3.4, there is an embedding $\sigma(r-1) \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'$, where $a+b \leq 0$. Obviously, $a \leq 0$.

Since $m_r-n-1 \leq -\frac{3}{2}$, the strong positivity of the representation σ and Lemma 3.3 together with Theorem 4.3 imply there is at most one $x \in \mathbb{R}$, $0 < |x| \leq 1$ such that $\nu^x\rho$ appears in $[\sigma(r-1)]$. Therefore, $b \leq 0$ and the representation $\nu^{-(m_r-n-1)} \times \nu^b\rho$ is irreducible and isomorphic to $\nu^b\rho \times \nu^{-(m_r-n-1)}$.

We thus get the embeddings and isomorphisms

$$\begin{aligned}\sigma(r) &\hookrightarrow \nu^{-(m_r-n-1)} \rtimes \sigma(r-1) \hookrightarrow \nu^{-(m_r-n-1)} \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma' \\ &\hookrightarrow \nu^{-(m_r-n-1)} \times \nu^b\rho \times \delta([\nu^a\rho, \nu^{b-1}\rho]) \rtimes \sigma' \\ &\simeq \nu^b\rho \times \nu^{-(m_r-n-1)} \times \delta([\nu^a\rho, \nu^{b-1}\rho]) \rtimes \sigma',\end{aligned}$$

contradicting square-integrability of $\sigma(r)$. This proves the proposition. □

In pretty much the same way one can also prove:

Corollary 5.4. *Let* $\tau \in S_1(r)$ *denote a strongly positive discrete series. Suppose that* $\Theta(\tau, n) \neq 0$, *for some* n *such that* $m_r \geq n + \frac{1}{2}$. *Then* $\tau(n)$ *is a discrete series representation.*

The last two propositions of this section contain rather important results on the transfer of certain embeddings by the theta lifts. We omit the proofs, since these results can be obtained in a completely analogous way as in [Muić 2004, Remark 5.2], that is, by precise examination of the filtration of Jacquet modules quoted in Theorem 4.4.

Proposition 5.5. *Suppose that the representation* $\sigma \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n))$ *may be written as an irreducible subrepresentation of the induced representation of the form* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'$, *where* ρ *is an irreducible cuspidal genuine representation*, $\sigma' \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n'))$, *and* $b - a \geq 0$. *Let* $\Theta(\sigma, r) \neq 0$. *Then one of the following holds*:

- *There is an irreducible representation* τ *of some* $O(V_{r'})$ *such that* $\sigma(r)$ *is a subrepresentation of* $\delta([\nu^a\chi_{V,\psi}^{-1}\rho, \nu^b\chi_{V,\psi}^{-1}\rho]) \rtimes \tau$.
- *There is an irreducible representation* τ *of some* $O(V_{r'})$ *such that* $\sigma(r)$ *is a subrepresentation of* $\delta([\nu^{a+1}\chi_{V,\psi}^{-1}\rho, \nu^b\chi_{V,\psi}^{-1}\rho]) \rtimes \tau$.

The latter situation is impossible unless $(a, \rho) = (m_r - n, \chi_{V,\psi})$.

Proposition 5.6. *Suppose that the representation* $\tau \in \mathrm{Irr}(O(V_r))$ *may be written as an irreducible subrepresentation of the induced representation of the form* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau'$, *where* ρ *is an irreducible cuspidal representation and* $\tau' \in \mathrm{Irr}(O(V_{r'}))$ *and* $b - a \geq 0$. *Let* $\Theta(\tau, n) \neq 0$. *Then one of the following holds*:

- *There is an irreducible representation* σ *of some* $\widetilde{\mathrm{Sp}}(n')$ *such that* $\tau(n)$ *is a subrepresentation of* $\delta([\nu^a\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma$.
- *There is an irreducible representation* σ *of some* $\widetilde{\mathrm{Sp}}(n')$ *such that* $\tau(n)$ *is a subrepresentation of* $\delta([\nu^{a+1}\chi_{V,\psi}\rho, \nu^b\chi_{V,\psi}\rho]) \rtimes \sigma$.

The latter situation is impossible unless $(a, \rho) = (n - m_r + 1, 1_{F^\times})$.

6. Lifts of strongly positive discrete series of the metaplectic groups

In this section we determine the structure of certain lifts of the strongly positive discrete series of the metaplectic groups. We also obtain precise information about the first occurrence of strongly positive discrete series in the orthogonal tower, depending on its cuspidal support.

Let $\sigma \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n))$ denote a strongly positive discrete series. According to the classification given in Theorem 3.1, we may write σ as a unique irreducible subrepresentation of the induced representation

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{k_i}\delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i])\right) \rtimes \sigma_{\mathrm{cusp}}, \tag{3}$$

with k minimal and k_i minimal for $i = 1, 2, \ldots, k$, where

$$\sigma_{\mathrm{cusp}} \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n'))$$

is an irreducible cuspidal representation and ρ_i an irreducible cuspidal representation of $\mathrm{GL}(n_{\rho_i}, F)$ (this defines n_{ρ_i}) for $i = 1, 2, \ldots, k$. We note that the minimality of k and k_i for $i = 1, 2, \ldots, k$ implies that there are no empty segments in (3).

Theorem 4.2 shows that either $\Theta^+(\sigma, n) \neq 0$ or $\Theta^-(\sigma, n-1) \neq 0$.

The following theorem describes the first occurrence indices of the strongly positive discrete series of the metaplectic group.

Theorem 6.1. *Let $\sigma \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n))$ be a strongly positive discrete series. If*

$$\Theta^+(\sigma, n) \neq 0,$$

let $(\epsilon, r) = (+, n)$; otherwise let $(\epsilon, r) = (-, n-1)$. Suppose that $\sigma_{\mathrm{cusp}} \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n'))$ is a partial cuspidal support of σ and $\tau_{\mathrm{cusp}} \in \mathrm{Irr}(O(V^\epsilon_{r'}))$ is the first nonzero lift of σ_{cusp}. Further, set $M = \{|x| : \chi_{V,\psi}\nu^x$ appears in $[\sigma]\}$ and denote by $a_{\min}$ the minimal element of M. If $M = \varnothing$, let $a_{\min} = n' - \frac{1}{2}\dim V^\epsilon_{r'} + 2$.

If $a_{\min} = \frac{1}{2}$ or $n' = r' + \frac{1}{2}(\dim V^\epsilon_0 - 1)$, then the first occurrence index of σ is r. Otherwise, the first occurrence index of σ is $r - a_{\min} + \frac{3}{2}$.

The rest of this section is devoted to the proof of Theorem 6.1. The proof is divided into several cases depending on the structure of the cuspidal support of σ and on the first nonzero lift of σ_{cusp}.

In this section, m_r denotes $\frac{1}{2}\dim V^\epsilon_r = n + \frac{1}{2}$ and $\sigma(l)$ denotes the unique irreducible quotient of the representation $\Theta^\epsilon(\sigma, l)$.

Observe that Proposition 5.2 implies that the representation $\sigma(l)$ is not a discrete series representation for $l > r$.

There are two main cases that we consider.

Suppose the representation $\chi_{V,\psi}\nu^{1/2}$ does not appear in $[\sigma]$. Since $m_r - n = \frac{1}{2}$, Theorem 4.3 yields $n' \geq r' + \frac{1}{2}(\dim(V^\epsilon_0) - 1)$. We have two possibilities:

- $n' = r' + \frac{1}{2}(\dim(V^\epsilon_0) - 1)$:

In this case, both representations $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ and $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduce for $s = \frac{1}{2}$. Therefore, by Theorem 3.1, there is no representation of the form $\chi_{,\psi}\nu^s$ appearing in $[\sigma]$. Further, Theorem 3.5 of [Hanzer and Muić 2011] implies that the representation $\chi_{V,\psi}\nu^s\rho_i \rtimes \sigma_{\mathrm{cusp}}$ reduces if and only if the representation $\nu^s\rho_i \rtimes \tau_{\mathrm{cusp}}$ reduces.

One of the main results of [Gan and Savin 2012] states that $\sigma(r)$ is a discrete series representation. Applying Equation (2), we obtain the embedding

$$\sigma(r) \hookrightarrow \delta([\nu^{a_1}\rho'_1, \nu^{b_1}\rho'_1]) \times \delta([\nu^{a_2}\rho'_2, \nu^{b_2}\rho'_2]) \times \cdots \times \delta([\nu^{a_l}\rho'_l, \nu^{b_l}\rho'_l]) \rtimes \tau_{\mathrm{sp}},$$

where $a_i \leq 0$ and $\rho'_i \in \{\rho_1, \rho_2, \ldots, \rho_k\}$ for $i = 1, 2, \ldots, l$, and $\tau_{\mathrm{sp}} \in \mathrm{Irr}(O(V^\epsilon_{r'}))$ is a strongly positive discrete series for some r'.

Since the representation σ is strongly positive, Theorem 4.3 and Lemma 3.3 show that for every $i \in \{1, 2, \ldots, k\}$, there is at most one representation of the form $\nu^x \rho_i$ that appears in $[\sigma(r)]$ with $0 \leq |x| < 1$. In the same way as in the proof of Proposition 5.3, we deduce $\sigma(r) \simeq \tau_{\mathrm{sp}}$, that is, $\sigma(r)$ is a strongly positive representation.

It is now easy to see, using Lemma 3.2, that $\sigma(r)$ is a unique irreducible subrepresentation of the induced representation

$$\left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

Suppose that $\Theta^{\epsilon}(\sigma, r-1) \neq 0$. Then Proposition 5.2 implies $R_{P_1}(\Theta^{\epsilon}(\sigma, r))(\nu^{1/2}) \neq 0$, which is impossible. Thus, r is the first occurrence index of σ.

- $n' > r' + \frac{1}{2}(\dim(V_0^{\epsilon}) - 1)$.

In this case, the representation $\chi_{V,\psi} \nu^s \rtimes \sigma_{\mathrm{cusp}}$ reduces for $s = n' - m_{r'} + 1$, and the representation $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduces for $s = n' - m_{r'}$.

Observe that $[\sigma(r)]$ is obtained from $[\sigma]$ by multiplying by $\chi_{V,\psi}^{-1}$ all representations of the form $\chi_{V,\psi} \nu^x \rho_i$ appearing in $[\sigma]$, adding the representations $\nu^{-1/2}$, $\nu^{-3/2}, \dots, \nu^{m_{r'}-n'}$, and replacing σ_{cusp} with τ_{cusp}.

There are two possible cases that we consider:

(1) Some representation of the form $\chi_{V,\psi} \nu^s$, $s \in \mathbb{R}$ appears in $[\sigma]$: We may suppose that ρ_1 is a trivial representation. Note that $a_{\rho_1} - k_1 + 1$ is strictly greater than $\frac{1}{2}$ and that a_{ρ_1} equals $n' - m_{r'} + 1$.

For simplicity of notation, let a_j stand for $a_{\rho_1} - k_1 + j$, for $j = 1, 2, \dots, k_1$. Again, we know that $\sigma(r)$ is a discrete series representation. Inspecting its cuspidal support more precisely, it is not hard to see that it has to be strongly positive. Using Lemma 3.2, we get that $\sigma(r)$ can be obtained as the unique irreducible subrepresentation of

$$\nu^{1/2} \times \nu^{3/2} \times \cdots \times \nu^{a_1-2} \times \left(\prod_{j=1}^{k_1} \delta\big([\nu^{a_j-1}, \nu^{b_j^{(1)}}]\big) \right) \\ \times \left(\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

If $a_1 \geq \frac{5}{2}$, Theorem 5.3 of [Matić 2012] implies $R_{P_1}(\sigma(r))(\nu^{1/2}) \neq 0$. If $a_1 = \frac{3}{2}$, the same result shows that $R_{P_1}(\sigma(r))(\nu^{1/2}) = 0$ (since $b_1^{(1)} \geq a_1 > \frac{1}{2}$). Using Proposition 5.2, we conclude that $\Theta^{\epsilon}(\sigma, r-1) \neq 0$ if $a_1 \geq \frac{5}{2}$, and $\Theta^{\epsilon}(\sigma, r-1) = 0$ otherwise.

If $a_1 \geq \frac{5}{2}$, combining the square-integrability of $\sigma(r-1)$ (by Proposition 5.3) with the fact that $[\sigma(r-1)]$ is obtained from $[\sigma(r)]$ by subtracting $\nu^{1/2}$, we get that $\sigma(r-1)$ is a strongly positive discrete series that can be realized as a unique irreducible subrepresentation of

$$\nu^{3/2} \times \nu^{5/2} \times \cdots \times \nu^{a_1-2} \times \left(\prod_{j=1}^{k_1} \delta\big([\nu^{a_j-1}, \nu^{b_j^{(1)}}]\big) \right) \times \left(\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

Proceeding with the same analysis as above, we obtain that $\Theta^{\epsilon}(\sigma, r-l) \neq 0$ for $l = 1, 2, \ldots, r - a_1 + \frac{3}{2}$ and that $\sigma(r-l)$ is a strongly positive discrete series that can be realized as a unique irreducible subrepresentation of

$$\nu^{l+1/2} \times \nu^{l+3/2} \times \cdots \times \nu^{a_1-2} \times \left(\prod_{j=1}^{k_1} \delta\big([\nu^{a_j-1}, \nu^{b_j^{(1)}}]\big) \right) \times \left(\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

Further, it is easy to check that the first occurrence index of σ equals $r - a_1 + \frac{3}{2}$.

(2) There is no representation of the form $\chi_{V,\psi} \nu^s$, $s \in \mathbb{R}$ appearing in $[\sigma]$: As in the previous case, we conclude that $\sigma(r)$ is a strongly positive discrete series. An easy computation shows that $\sigma(r)$ is a unique irreducible subrepresentation of the induced representation

$$\nu^{1/2} \times \nu^{3/2} \times \cdots \times \nu^{n'-m_{r'}} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

Now Theorem 5.3 of [Matić 2012] shows that $R_{P_1}(\sigma(r))(\nu^{1/2}) \neq 0$. Because $R_{\widetilde{P}_1}(\sigma)(\chi_{V,\psi} \nu^{1/2}) = 0$, part (2) of Proposition 5.2 implies $\Theta^{\epsilon}(\sigma, r-1) \neq 0$.

Note that $[\sigma(r-1)]$ and $[\sigma(r)]$ differ by $\nu^{1/2}$. Proposition 5.3 now shows that $\sigma(r-1)$ is a discrete series representation, and we again conclude that it must be strongly positive. Thus, $\sigma(r-1)$ is a unique irreducible subrepresentation of the induced representation

$$\nu^{3/2} \times \nu^{5/2} \times \cdots \times \nu^{n'-m_{r'}} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j} \rho_i, \nu^{b_j^{(i)}} \rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

If $n' - m_{r'} > \frac{1}{2}$, in the same way as above we deduce $\Theta^{\epsilon}(\sigma, r-2) \neq 0$. We continue in this fashion, obtaining that $\Theta^{\epsilon}(\sigma, r-j) \neq 0$ for $j = 1, 2, \ldots, n' - m_{r'} + \frac{1}{2}$, and that $\sigma(r-j)$ is a strongly positive discrete series that can be characterized as the unique irreducible subrepresentation of

$$\nu^{j+(1/2)} \times \nu^{j+(3/2)} \times \cdots \times \nu^{n'-m_{r'}} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

From Proposition 5.2, we conclude that the first occurrence index of σ equals

$$r - n' + m_{r'} - \tfrac{1}{2} = r - (n' - \tfrac{1}{2}\dim V_{r'}^{\epsilon} + 2) + \tfrac{3}{2}.$$

Second, suppose that the representation $\chi_{V,\psi}\nu^{1/2}$ appears in $[\sigma]$. There is no loss of generality in assuming that ρ_1 is a trivial representation. We have to examine the following three possibilities:

- $n' = r' + \frac{1}{2}(\dim(V_0^{\epsilon}) - 1)$:

Observe that in this case both representations $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ and $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduce for $s = \frac{1}{2}$. Obviously, Theorem 3.1 implies $k_1 = 1$.

Observe that $[\sigma(r)]$ is obtained from $[\sigma]$ simply by replacing σ_{cusp} with τ_{cusp} and multiplying all $\widetilde{\mathrm{GL}}$-members of $[\sigma]$ by $\chi_{V,\psi}^{-1}$; consequently, the discrete series $\sigma(r)$ may be realized as the unique irreducible subrepresentation of

$$\delta\big([\nu^{1/2}, \nu^{b_1^{(1)}}]\big) \times \left(\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta\big([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]\big) \right) \rtimes \tau_{\mathrm{cusp}}.$$

We note that for each $i \in \{1, 2, \ldots, k\}$, there is at most one $x \in \mathbb{R}$, $0 \le |x| \le 1$ such that $\nu^x\rho_i$ appears in $[\sigma(r)]$, and thus τ has to be strongly positive.

Obviously, $R_{P_1}(\sigma(r))(\nu^{1/2}) \neq 0$ if and only if $b_1^{(1)} = \frac{1}{2}$.

If $b_1^{(1)} > \frac{1}{2}$, using Proposition 5.2, we directly conclude that $\Theta^{\epsilon}(\sigma, r-1) = 0$. Suppose that $b_1^{(1)} = \frac{1}{2}$. If $\Theta^{\epsilon}(\sigma, r-1) \neq 0$, we get that $\nu^{1/2}$ does not appear in $[\sigma(r-1)]$, contradicting Proposition 5.5 (we are in the first case there). Thus, r is the first occurrence index of σ.

- $n' < r' + \frac{1}{2}(\dim(V_0^{\epsilon}) - 1)$:

In this case, the representation $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n' - 1$ and the representation $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n'$.

According to Theorem 4.3, $[\sigma(r)]$ is obtained from $[\sigma]$ by multiplying all $\widetilde{\mathrm{GL}}$-members of $[\sigma]$ by $\chi_{V,\psi}^{-1}$, subtracting the representations $\nu^{1/2}, \nu^{3/2}, \ldots, \nu^{m_{r'}-n'-1}$, and replacing σ_{cusp} with τ_{cusp}. In the same way as before, we conclude that $\sigma(r)$ is a strongly positive discrete series that is characterized as a unique irreducible subrepresentation of

$$\delta([\nu^{3/2}, \nu^{b_1^{(1)}}]) \times \delta([\nu^{5/2}, \nu^{b_2^{(1)}}]) \times \cdots \times \delta([\nu^{m_{r'}-n'}, \nu^{b_{k_1}^{(1)}}]) \\ \times \Big(\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]) \Big) \rtimes \tau_{\text{cusp}}.$$

Since $\nu^{1/2}$ does not appear in $[\sigma(r)]$, it follows that r is the first occurrence index of σ.

- $n' > r' + \frac{1}{2}(\dim(V_0^{\epsilon}) - 1)$:

Now the representation $\chi_{V,\psi}\nu^s \rtimes \sigma_{\text{cusp}}$ reduces for $s = n' - m_{r'} + 1$, and the representation $\nu^s \rtimes \tau_{\text{cusp}}$ reduces for $s = n' - m_{r'}$.

Theorem 4.3 now shows that $[\sigma(r)]$ is obtained from $[\sigma]$ by multiplying all $\widetilde{\mathrm{GL}}$-members of $[\sigma]$ by $\chi_{V,\psi}^{-1}$, adding the representations $\nu^{-1/2}, \nu^{-3/2}, \ldots, \nu^{m_{r'}-n'}$, and replacing σ_{cusp} with τ_{cusp}.

From Theorem 4.2, we know that the representation $\sigma(r)$ is in the discrete series. But $\nu^{1/2}$ appears in $[\sigma(r)]$ with multiplicity two, and consequently $\sigma(r)$ can't be a strongly positive representation (by Lemma 3.3).

In what follows, we use Theorem 3.5 to describe discrete series $\sigma(r)$ as precisely as we can. So, we write $\sigma(r)$ as a subrepresentation of the induced representation of the form

$$\delta([\nu^{a_1'}\rho_1', \nu^{b_1'}\rho_1']) \times \delta([\nu^{a_2'}\rho_2', \nu^{b_2'}\rho_2']) \times \cdots \times \delta([\nu^{a_l'}\rho_l', \nu^{b_l'}\rho_l']) \rtimes \tau_{\text{sp}},$$

where $\rho_i' \in \{\rho_1, \rho_2, \ldots, \rho_k\}$, $a_i' \le 0$, and $a_i' + b_i' > 0$ for $i = 1, 2, \ldots, l$. Further, τ_{sp} is an irreducible strongly positive representation such that $[\tau_{\text{sp}}]$ is contained in $[\sigma(r)]$. Hence, at least one of the representations $\nu^{1/2}$ and $\nu^{-1/2}$ has to appear in some segment $[\nu^{a_i'}\rho_i', \nu^{b_i'}\rho_i']$, $i \in \{1, 2, \ldots, l\}$. Since $a_i' \le 0$ and $b_i' > 0$, both these representations appear in this segment.

Our next claim is that $l = 1$. Suppose, on the contrary, that $l > 1$.

Then there is some $j \in \{1, 2, \ldots, l\}$, $j \neq i$ such that $\nu^{1/2} \notin [\nu^{a_j'}\rho_j', \nu^{b_j'}\rho_j']$. But the union of the segments $[\nu^{a_i'}, \nu^{b_i'}]$ and $[\nu^{a_j'}\rho_j', \nu^{b_j'}\rho_j']$ is contained in $[\sigma(r)]$, so there is at most one x, $0 \le |x| \le 1$ such that $\nu^x\rho_j'$ appears in $[\nu^{a_j'}\rho_j', \nu^{b_j'}\rho_j']$. This contradicts the fact that the ends of segment $[\nu^{a_j'}\rho_j', \nu^{b_j'}\rho_j']$ satisfy $a_j' \le 0$ and $b_j' > 0$. Thus, $l = 1$ and $\rho_1' \cong 1_{F^\times}$.

In this way we obtain the following embedding:

$$\sigma(r) \hookrightarrow \delta([\nu^{a_1'}, \nu^{b_1'}]) \rtimes \tau_{\text{sp}}.$$

Since $a_1' \le 0$, using Proposition 5.6 we obtain a contradiction with the strong positivity of σ. Therefore, this case is impossible and Theorem 6.1 is proved.

The results obtained closely parallel those contained in Theorem 4.2 of [Muić 2004] for the dual pair $(\mathrm{Sp}(n), O(V))$.

7. Lifts of strongly positive discrete series of the orthogonal groups

The purpose of this section is to determine the first occurrence indices of strongly positive discrete series of the odd orthogonal groups that appear in the correspondence given by Theorem 4.2, and to provide a description of the lifts of such representations in the metaplectic tower.

Thus, we let $\tau \in \mathrm{Irr}(O(V_r))$ denote a strongly positive discrete series such that $\Theta(\tau, m_r - \frac{1}{2}) \neq 0$, and realize it as a unique irreducible subrepresentation of the induced representation of the form

$$\Bigl(\prod_{i=1}^{k}\prod_{j=1}^{k_i} \delta\bigl([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i]\bigr)\Bigr) \rtimes \tau_{\mathrm{cusp}},$$

with k minimal and k_i minimal for $i = 1, 2, \ldots, k$, where $\tau_{\mathrm{cusp}} \in \mathrm{Irr}(O(V_{r'}))$ is a cuspidal representation and ρ_i an irreducible cuspidal representation of $\mathrm{GL}(n_{\rho_i}, F)$ (this defines n_{ρ_i}) for $i = 1, 2, \ldots, k$.

Note that Proposition 5.1 yields that the representation $\tau(l)$ is not a discrete series representation for $l > m_r - \frac{1}{2}$.

In the following theorem, we describe the first occurrence indices of certain strongly positive discrete series of the odd orthogonal groups.

Theorem 7.1. *Let $\tau \in \mathrm{Irr}(O(V_r))$ be a strongly positive discrete series with a nonzero lift on the $(m_r - \frac{1}{2})$-th level of the metaplectic tower. Suppose that $\tau_{\mathrm{cusp}} \in \mathrm{Irr}(O(V_{r'}))$ is a partial cuspidal support of τ and that $\sigma_{\mathrm{cusp}} \in \mathrm{Irr}(\widetilde{\mathrm{Sp}}(n'))$ is the first nonzero lift of τ_{cusp}. Let $n = m_r - \frac{1}{2}$. Further, define $M = \{|x| : \nu^x$ appears in $[\tau]\}$ and denote by $a_{\min}$ the minimal element of M. If $M = \emptyset$, let $a_{\min} = m_{r'} - n' + 1$.*

If $a_{\min} = \frac{1}{2}$ or $r' = n' - \frac{1}{2}(\dim(V_0) - 1)$, then the first occurrence index of τ is n. Otherwise, the first occurrence index of τ is $n - a_{\min} + \frac{3}{2}$.

The remaining part of this section is devoted to the proof this theorem.

Again, we have two main cases to discuss.

First, assume that $\nu^{1/2}$ does not appear in $[\tau]$. This implies

$$r' \geq n' - \tfrac{1}{2}(\dim(V_0) - 1).$$

This leaves us two possibilities:

- $r' = n' - \frac{1}{2}(\dim(V_0) - 1)$:

In this case, both representations $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ and $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduce for $s = \frac{1}{2}$. From the classification of strongly positive discrete series, elaborated in Section 2, we deduce that there are no representations of the form ν^s appearing in $[\tau]$.

Applying Theorem 4.2, we obtain that $\tau(n)$ is a discrete series representation, and in the same way as before, we may conclude that it is strongly positive. This yields the embedding

$$\tau(n) \hookrightarrow \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i]) \right) \rtimes \sigma_{\mathrm{cusp}}.$$

Proposition 5.1 implies $\Theta(\tau, n-1) = 0$, so n is the first occurrence index of τ.

- $r' > n' - \frac{1}{2}(\dim(V_0) - 1)$:

In this case, the representation $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n'$ and the representation $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n' - 1$.

Theorem 4.3 shows that $[\tau(n)]$ is obtained from $[\tau]$ by multiplying all elements of $\mathcal{R}$ appearing in $[\tau]$ by $\chi_{V,\psi}$, adding the representations $\chi_{V,\psi}\nu^{1/2}$, $\chi_{V,\psi}\nu^{3/2}, \dots,$ $\chi_{V,\psi}\nu^{m_{r'}-n'-1}$, and replacing τ_{cusp} with σ_{cusp}.

There are two main cases to consider:

(1) There is no representation of the form ν^s appearing in $[\tau]$, for $s \in \mathbb{R}$: As before, we conclude that $\tau(n)$ is a strongly positive discrete series that is a unique irreducible subrepresentation of

$$\chi_{V,\psi}\nu^{1/2} \times \chi_{V,\psi}\nu^{3/2} \times \cdots \times \chi_{V,\psi}\nu^{m_{r'}-n'-1} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i]) \right) \rtimes \sigma_{\mathrm{cusp}}.$$

Theorem 5.3 of [Matić 2012] implies $R_{P_1}(\tau(n))(\chi_{V,\psi}\nu^{1/2}) \neq 0$. Since

$$R_{P_1}(\tau)(\nu^{1/2}) = 0,$$

part (2) of Proposition 5.1 shows $\Theta(\sigma, n-1) \neq 0$.

From Corollary 5.4, we obtain that $\tau(n-l)$ is a discrete series representation for each $l > 0$ such that $\Theta(\tau, n-l) \neq 0$. In the same way as above, we see that it must be strongly positive.

Since $[\tau(n-l)]$ is obtained from $[\tau(n)]$ by subtraction of the representations $\chi_{V,\psi}\nu^{1/2}$, $\chi_{V,\psi}\nu^{3/2}, \dots, \chi_{V,\psi}\nu^{(2l-1)/2}$, for $l \in \{1, 2, \dots, m_{r'} - n' - \frac{1}{2}\}$, it is not hard to see, using Proposition 5.1, that $\Theta(\tau, n-l) \neq 0$ for $l \in \{1, 2, \dots, m_{r'} - n' - \frac{1}{2}\}$. Furthermore, $\tau(n-l)$ is a unique irreducible subrepresentation of the induced representation

$$\chi_{V,\psi}\nu^{(2l+1)/2} \times \chi_{V,\psi}\nu^{(2l+3)/2} \times \cdots \times \chi_{V,\psi}\nu^{m_{r'}-n'-1} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i]) \right) \rtimes \sigma_{\mathrm{cusp}},$$

for $l \in \{1, 2, \dots, m_{r'} - n' - \frac{1}{2}\}$.

There is no representation of the form $\chi_{V,\psi}\nu^s$ appearing in $[\tau(n-m_{r'}+n'+\frac{1}{2})]$, so Proposition 5.1 shows that the first occurrence index of τ equals $n-m_{r'}+n'+\frac{1}{2}$.

(2) There is some representation of the form ν^s appearing in $[\tau]$: We may suppose that ρ_1 is a trivial representation. Obviously, $a_{\rho_1}-k_1+1$ is strictly greater than $\frac{1}{2}$ and a_{ρ_1} equals $m_{r'}-n'$.

For brevity, let a_j stand for $a_{\rho_1}-k_1+j$, for $j=1,2,\dots,k_1$. Since $\chi_{V,\psi}\nu^{1/2}$ appears in $[\tau(n)]$ with multiplicity one, it follows that $\tau(n_1)$ is a strongly positive representation for each $n_1\leq n$ such that $\Theta(\tau,n_1)\neq 0$.

Also, $\tau(n)$ is the unique irreducible subrepresentation of

$$\chi_{V,\psi}\nu^{1/2}\times\chi_{V,\psi}\nu^{3/2}\times\cdots\times\chi_{V,\psi}\nu^{a_1-2}\times\left(\prod_{j=1}^{k_1}\delta\big([\chi_{V,\psi}\nu^{a_j-1},\chi_{V,\psi}\nu^{b_j^{(1)}}]\big)\right)\\ \times\left(\prod_{i=2}^{k}\prod_{j=1}^{k_i}\delta\big([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i,\chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i]\big)\right)\rtimes\sigma_{\text{cusp}}.$$

Arguing in the same way as in the analogous situation in the metaplectic case, we deduce that $\Theta(\tau,n-l)\neq 0$ for $l\in\{1,2,\dots,a_1-\frac{3}{2}\}$ and that $n-a_1+\frac{3}{2}$ is the first occurrence index of τ. Further, $\tau(n-l)$ is a unique irreducible representation of the induced representation

$$\chi_{V,\psi}\nu^{l+1/2}\times\chi_{V,\psi}\nu^{l+3/2}\times\cdots\times\chi_{V,\psi}\nu^{a_1-2}\times\left(\prod_{j=1}^{k_1}\delta\big([\chi_{V,\psi}\nu^{a_j-1},\chi_{V,\psi}\nu^{b_j^{(1)}}]\big)\right)\\ \times\left(\prod_{i=2}^{k}\prod_{j=1}^{k_i}\delta\big([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i,\chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i]\big)\right)\rtimes\sigma_{\text{cusp}},$$

for $l\in\{1,2,\dots,a_1-\frac{3}{2}\}$.

It remains to consider the case when the representation $\nu^{1/2}$ appears in $[\tau]$. Without loss of generality, we may suppose that ρ_1 is a trivial character. Similarly to the previous section, we have to examine three possibilities.

- $r'=n'-\frac{1}{2}(\dim(V_0)-1)$:

The specificity of this case is that both induced representations $\nu^s\rtimes\tau_{\text{cusp}}$ and $\chi_{V,\psi}\nu^s\rtimes\sigma_{\text{cusp}}$ reduce for $s=\frac{1}{2}$. On account of Theorem 3.1, we have $k_1=1$ and $a_{\rho_1}=\frac{1}{2}$.

Furthermore, $[\tau(n)]$ is obtained from $[\tau]$ by replacing τ_{cusp} with σ_{cusp} and multiplying all other members of $[\tau]$ by $\chi_{V,\psi}$.

From the equality of cuspidal reducibilities for τ_{cusp} and σ_{cusp}, it may be concluded that $\tau(n)$ is the strongly positive discrete series that is a unique irreducible subrepresentation of

$$\delta([\chi_{V,\psi}\nu^{1/2}, \chi_{V,\psi}\nu^{b_1^{(1)}}]) \times \left(\prod_{i=2}^{k}\prod_{j=1}^{k_i}\delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\rho_i])\right) \rtimes \sigma_{\mathrm{cusp}}.$$

Suppose that the lift $\Theta(\tau, n-1)$ is nonzero. Then Proposition 5.1, enhanced by Theorem 5.3 of [Matić 2012], implies $b_1^{(1)} = \frac{1}{2}$. From Theorem 4.3, it follows that there is no representation $\chi_{V,\psi}\nu^{1/2}$ appearing in $[\tau(n-1)]$, contrary to Proposition 5.6.

It follows that n is the first occurrence index of τ.

- $r' < n' - \frac{1}{2}(\dim(V_0) - 1)$:

The induced representation $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduces for $s = n' - m_{r'}$ and the induced representation $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ reduces for $s = n' - m_{r'} + 1$. According to Theorem 4.3, $[\tau(n)]$ is obtained from $[\tau]$ by replacing τ_{cusp} with σ_{cusp}, multiplying GL-members of $[\tau]$ by $\chi_{V,\psi}$, and then subtracting the representations $\chi_{V,\psi}\nu^{1/2}, \chi_{V,\psi}\nu^{3/2}, \dots, \chi_{V,\psi}\nu^{n'-m_{r'}}$.

The strong positivity of the representation τ and the above discussion show that for each $i \in \{1, 2, \dots, k\}$, there is at most one x, $|x| \leq 1$ such that $\chi_{V,\psi}\nu^x$ appears in $[\tau(n)]$. Since $\tau(n)$ is in the discrete series, from Theorem 3.5 we see that it is strongly positive.

An easy computation shows that $\tau(n)$ is a unique irreducible subrepresentation of the induced representation

$$\begin{aligned}\delta([\chi_{V,\psi}\nu^{3/2}, \chi_{V,\psi}\nu^{b_1^{(1)}}]) \times \delta([\chi_{V,\psi}\nu^{5/2}, \chi_{V,\psi}\nu^{b_2^{(1)}}]) \times \cdots \\ \times \delta([\chi_{V,\psi}\nu^{n'-m_{r'}+1}, \chi_{V,\psi}\nu^{b_{k_1}^{(1)}}]) \\ \times \left(\prod_{i=2}^{k}\prod_{j=1}^{k_i}\delta([\chi_{V,\psi}\nu^{a_{\rho_i}-k_i+j}\rho_i, \chi_{V,\psi}\nu^{b_j^{(i)}}\chi_{V,\psi}\rho_i])\right) \rtimes \sigma_{\mathrm{cusp}}.\end{aligned}$$

That n is the first occurrence index of τ follows directly from Proposition 5.1.

- $r' > n' - \frac{1}{2}(\dim(V_0) - 1)$:

The induced representation $\nu^s \rtimes \tau_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n'$, and the representation $\chi_{V,\psi}\nu^s \rtimes \sigma_{\mathrm{cusp}}$ reduces for $s = m_{r'} - n' - 1$. The representation $\chi_{V,\psi}\nu^{1/2}$ appears in $[\tau(n)]$ with multiplicity two, since $[\tau(n)]$ is obtained from $[\tau]$ by replacing τ_{cusp} with σ_{cusp}, multiplying other members of $[\tau]$ by $\chi_{V,\psi}$, and adding $\chi_{V,\psi}\nu^{1/2}, \chi_{V,\psi}\nu^{3/2}, \dots, \chi_{V,\psi}\nu^{m_{r'}-n'-1}$.

According to Lemma 3.3, $\tau(n)$ is not a strongly positive discrete series, but the results in [Gan and Savin 2012] show that it is a discrete series representation.

Applying Theorem 3.5 and analysis similar to that in the last case considered in the previous section, we write $\tau(n)$ as an irreducible subrepresentation of the induced representation of the form

$$\delta\big([\chi_{V,\psi}\nu^a, \chi_{V,\psi}\nu^b]\big) \rtimes \sigma_{\mathrm{sp}},$$

where $a \leq 0$, $a+b > 0$, and $\sigma_{\mathrm{sp}} \in S_2$ is a strongly positive discrete series.

Using Proposition 5.5, we obtain an embedding that contradicts the strong positivity of τ. Consequently, this case is not possible.

This completes the proof of Theorem 7.1.

Acknowledgements

The author would like to thank Goran Muić for suggesting this problem. The author would also like to thank the referee for helping to improve the style of the presentation.

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Received March 8, 2011. Revised July 14, 2012.

IVAN MATIĆ
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSIJEK
TRG LJUDEVITA GAJA 6
31000 OSIJEK
CROATIA
imatic@mathos.hr

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 259 No. 2 October 2012

0030-8730(201210)259:2;1-6