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ENERGY IDENTITY FOR THE MAPS FROM A SURFACE WITH TENSION FIELD BOUNDED IN L^p

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Let M be a closed Riemannian surface and u_n a sequence of maps from M to Riemannian manifold N satisfying

$$\sup_n \big(\|\nabla u_n\|_{L^2(M)} + \|\tau(u_n)\|_{L^p(M)} \big) \le \Lambda$$

for some $p > 1$, where $\tau(u_n)$ is the tension field of the mapping u_n.

For a general target manifold N, if $p \ge \frac{6}{5}$, we prove the energy identity and the neckless property during blowing up.

1. Introduction

Let (M, g) be a closed Riemannian manifold and (N, h) be a Riemannian manifold without boundary. For a mapping u from M to N in $W^{1,2}(M, N)$, the energy density of u is defined by

$$e(u) = \tfrac{1}{2}|du|^2 = \mathrm{Trace}_g u^* h,$$

where u^*h is the pull-back of the metric tensor h.

The energy of the mapping u is defined as

$$E(u) = \int_M e(u)\, dV,$$

where dV is the volume element of (M, g).

A map $u \in C^1(M, N)$ is called harmonic if it is a critical point of the energy E.

By the Nash embedding theorem we know that (N, h) can be isometrically into a Euclidean space $\mathbb{R}^K$ with some positive integer K. Then (N, h) may be considered as a submanifold of $\mathbb{R}^K$ with the metric induced from the Euclidean metric. Thus a map $u \in C^1(M, N)$ can be considered as a map of $C^1(M, \mathbb{R}^K)$ whose image lies in N. In this sense we can get the Euler–Lagrange equation

$$\triangle u = A(u)(du, du).$$

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The tension field $\tau(u)$ is defined by

$$\tau(u) = \triangle_M u - A(u)(du, du),$$

where $A(u)(du, du)$ is the second fundamental form of N in $\mathbb{R}^K$. So u being harmonic means that $\tau(u) = 0$.

The harmonic mappings are of special interest when M is a Riemann surface. Consider a sequence of mappings u_n from Riemann surface M to N with bounded energies. It is clear that u_n converges weakly to u in $W^{1,2}(M, N)$ for some u in $W^{1,2}(M, N)$. But in general, it may not converge strongly in $W^{1,2}(M, N)$. When $\tau(u_n) = 0$, that is, when u_n are all harmonic, Parker [1996] proved that the lost energy is exactly the sum of some harmonic spheres, which are defined as harmonic mappings from S^2 to N. This result is called the energy identity. Also he proved that the images of these harmonic spheres and $u(M)$ are connected, that is, there is no neck during blowing up.

When $\tau(u_n)$ is bounded in L^2, the energy identity was proved in [Qing 1995] for the sphere, and in [Ding and Tian 1995] and [Wang 1996] for a general target manifold. Qing and Tian [1997] proved there is no neck during blowing up. For the heat flow of harmonic mappings, the results can also be found in [Topping 2004a; 2004b]. When the target manifold is a sphere, we proved the energy identity in [Li and Zhu 2011] for a sequence of mappings with tension fields bounded in $L \ln^+ L$, using good observations from [Lin and Wang 2002]. On the other hand, in the same paper we constructed a sequence of mappings with tension fields bounded in $L \ln^+ L$ such that there is a positive neck during blowing up. In [Zhu 2012] the neckless property during blowing up was proved for a sequence of maps u_n with

$$\lim_{\delta \to 0} \sup_n \sup_{B(x,\delta) \subset D_1} \|\tau(u_n)\|_{L \ln^+ L(B(x,\delta))} = 0.$$

In this paper we prove the energy identity and neckless property during blowing up of a sequence of maps u_n with $\tau(u_n)$ bounded in L^p for some $p \geq \frac{6}{5}$, for a general target manifold.

When $\tau(u_n)$ is bounded in L^p for some $p > 1$, the small energy regularity proved in [Ding and Tian 1995] implies that u_n converges strongly in $W^{1,2}(M, N)$ outside a finite set of points. For simplicity of exposition, it is no matter to assume that M is the unit disk $D_1 = D(0, 1)$ and there is only one singular point at 0.

In this paper we prove the following theorem.

Theorem 1. *Let $\{u_n\}$ be a sequence of mappings from D_1 to N in $W^{1,2}(D_1, N)$ with tension field $\tau(u_n)$. If*

(a) $\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$ *for some* $p \geq \frac{6}{5}$,

(b) $u_n \to u$ *strongly in* $W^{1,2}(D_1 \setminus \{0\}, \mathbb{R}^K)$ *as* $n \to \infty$, *then there exists a subsequence of* $\{u_n\}$ (*we still denote it by* $\{u_n\}$) *and some non-negative integer* k *so that for any* $i = 1, \dots, k$, *there exist points* x_n^i, *positive numbers* r_n^i *and a nonconstant harmonic sphere* w^i (*which we view as a map from* $\mathbb{R}^2 \cup \{\infty\} \to N$) *such that*:

(1) $x_n^i \to 0, r_n^i \to 0$ *as* $n \to \infty$.

(2) $\lim_{n\to\infty}\left(\frac{r_n^i}{r_n^j} + \frac{r_n^j}{r_n^i} + \frac{|x_n^i - x_n^j|}{r_n^i + r_n^j}\right) = \infty$ *for any* $i \neq j$.

(3) w^i *is the weak limit or strong limit of* $u_n(x_n^i + r_n^i x)$ *in* $W^{1,2}_{Loc}(\mathbb{R}^2, N)$.

(4) ***Energy identity***: *We have*

$$\lim_{n\to\infty} E(u_n, D_1) = E(u, D_1) + \sum_{i=1}^{k} E(w^i). \tag{1-1}$$

(5) ***Neckless property***: *The image* $u(D_1) \cup \bigcup_{i=1}^k w^i(\mathbb{R}^2)$ *is a connected set.*

This paper is organized as follows. In Section 2 we state some basic lemmas and some standard arguments in the blow-up analysis.

In Section 3 and Section 4 we prove Theorem 1. In the proof, we use delicate analysis on the difference between normal energy and tangential energy. The energy identity is proved in Section 3 and the neckless property is proved in Section 4.

Throughout this paper, the letter C denotes a positive constant that depends only on p, Λ and the target manifold N and may vary in different places. We also don't distinguish between a sequence and one of its subsequences.

2. Some basic lemmas and standard arguments

We recall the regular theory for a mapping with small energy on the unit disk and tension field in L^p $(p > 1)$.

Lemma 2. *Let* $\bar{u}$ *be the mean value of* u *on the disk* $D_{1/2}$. *There exists a positive constant* ϵ_N *that depends only on the target manifold such that if* $E(u, D_1) \le \epsilon_N^2$ *then*

$$\|u - \bar{u}\|_{W^{2,p}(D_{1/2})} \le C\big(\|\nabla u\|_{L^2(D_1)} + \|\tau(u)\|_p\big), \tag{2-1}$$

where $p > 1$.

As a consequence of (2-1) *and the Sobolev embedding* $W^{2,p}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$, *we have*

$$\|u\|_{Osc(D_{1/2})} = \sup_{x,y\in D_{1/2}} |u(x) - u(y)| \le C\big(\|\nabla u\|_{L^2(D_1)} + \|\tau(u)\|_p\big). \tag{2-2}$$

Remarks.

- In [Ding and Tian 1995] this lemma is proved for the mean value of u on the unit disk. Note that
$$\left|\frac{\int_{D_1} u(x)\,dx}{|D_1|}-\frac{\int_{D_{1/2}} u(x)\,dx}{|D_{1/2}|}\right|\le C\|\nabla u\|_{L^2(D_1)}.$$
So we can use the mean value of u on $D_{1/2}$ in this lemma.
- Suppose we have a sequence of mappings u_n from the unit disk D_1 to N with $\|u_n\|_{W^{1,2}(D_1)}+\|\tau(u_n)\|_{L^p(D_1)}\le\Lambda$ for some $p>1$.

  A point $x\in D_1$ is called an energy concentration point (blow-up point) if for any r such that $D(x,r)\subset D_1$, we have
$$\sup_n E(u_n,D(x,r))>\epsilon_N^2,$$
where ϵ_N is given in this lemma. If $x\in D_1$ isn't an energy concentration point, we can find a positive number δ such that
$$E(u_n,D(x,\delta))\le\epsilon_N^2\quad\text{for all } n.$$
Then it follows from Lemma 2 that we have a uniformly $W^{2,p}(D(x,\delta/2))$-bound for u_n. Because $W^{2,p}$ is compactly embedded in $W^{1,2}$, there is a subsequence of u_n (still denoted by u_n) and $u\in W^{2,p}(D(x,\delta/2))$ such that
$$\lim_{n\to\infty}u_n=u\quad\text{in } W^{1,2}(D(x,\delta/2)).$$
So u_n converges to u strongly in $W^{1,2}(D_1)$ outside a finite set of points.

Under the assumptions of our theorem, by the standard blow-up argument, that is by repeatedly rescaling u_n in a suitable way, we can obtain some nonnegative integer k so that for any $i=1,\dots,k$, there exist a point x_n^i, a positive number r_n^i and a nonconstant harmonic sphere w^i satisfying (1), (2) and (3) of Theorem 1. By the standard induction argument in [Ding and Tian 1995] we only need to prove the theorem in the case where there is only one bubble.

In that case we can assume that w is the strong limit of the sequence $u_n(x_n+r_nx)$ in $W^{1,2}_{Loc}(\mathbb{R}^2)$. We may assume that $x_n=0$. Set $w_n(x)=u_n(r_nx)$.

As
$$\lim_{\delta\to0}\lim_{n\to\infty}E(u_n,D_1\setminus D_\delta)=E(u,D_1),$$
the energy identity is equivalent to

$$\lim_{\delta\to0}\lim_{n\to\infty}\lim_{R\to\infty}E(u_n,D_\delta\setminus D_{r_nR})=0. \tag{2-3}$$

To prove the sets $u(D_1)$ and $w(\mathbb{R}^2\cup\infty)$ are connected, it is enough to show that

$$\lim_{\delta\to0}\lim_{n\to\infty}\lim_{R\to\infty}\sup_{x,y\in D_\delta\setminus D_{r_nR}}|u_n(x)-u_n(y)|=0. \tag{2-4}$$

3. Energy identity

In this section, we prove the energy identity for a general target manifold when $p \geq \frac{6}{5}$.

Assume that there is only one bubble w which is the strong limit of $u_n(r_n \cdot)$ in $W^{1,2}_{Loc}(\mathbb{R}^2)$. Let ϵ_N be the constant in Lemma 2. By the standard argument of blow-up analysis we can assume that, for any n,

$$E(u_n, D_{r_n}) = \sup_{\substack{r \leq r_n \\ D(x,r) \subseteq D_1}} E(u_n, D(x,r)) = \tfrac{1}{4}\epsilon_N^2. \tag{3-1}$$

Lemma 3 [Ding and Tian 1995]. *If $\tau(u_n)$ is bounded in L^p for some $p > 1$, then the tangential energy on the neck domain is zero, that is,*

$$\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{n \to \infty} \int_{D_\delta \setminus D_{r_n R}} |x|^{-2} |\partial_\theta u|^2 \, dx = 0. \tag{3-2}$$

Proof. The proof is the same as in [Ding and Tian 1995], so we only sketch it.

For any $\epsilon > 0$, take δ, R such that, for any n,

$$E(u, D_{4\delta}) + E(w, \mathbb{R}^2 \setminus D_R) + \delta^{4(p-1)/p} < \epsilon^2.$$

We may suppose that $r_n R = 2^{-j_n}, \delta = 2^{-j_0}$. When n is big enough we have, for any $j_0 \leq j \leq j_n$,

$$E(u_n, D_{2^{1-j}} \setminus D_{2^{-j}}) < \epsilon^2.$$

For any j, set

$$h_n(2^{-j}) = \frac{1}{2\pi} \int_{S^1} u_n(2^{-j}, \theta) \, d\theta$$

and

$$h_n(t) = h_n(2^{-j}) + \big(h_n(2^{1-j}) - h_n(2^{-j})\big) \frac{\ln(2^j t)}{\ln 2}, \quad t \in [2^{-j}, 2^{1-j}].$$

It is easy to check that

$$\frac{d^2 h_n(t)}{dt^2} + \frac{1}{t} \frac{d h_n(t)}{dt} = 0, \quad t \in [2^{-j}, 2^{1-j}].$$

Consider $h_n(x) = h_n(|x|)$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^K$, then $\triangle h_n = 0$ in $\mathbb{R}^2$. Setting $P_j = D_{2^{1-j}} \setminus D_{2^{-j}}$ we have

$$\triangle(u_n - h_n) = \triangle u_n - \triangle h_n = \triangle u_n = A(u_n) + \tau(u_n), \quad x \in P_j. \tag{3-3}$$

Taking the inner product of this equation with $u_n - h_n$ and integrating over P_j, we get that

$$\int_{P_j} |\nabla(u_n - h_n)|^2 dx = -\int_{P_j} (u_n - h_n)(A(u_n) + \tau(u_n)) \, dx + \int_{\partial P_j} (u_n - h_n)(u_n - h_n)_r \, ds.$$

Note that by definition, $h_n(2^{-j})$ is the mean value of $\{2^{-j}\} \times S^1$ and $(h_n)_r$ is independent of θ. So the integral of $(u_n - h_n)(h_n)_r$ on ∂P_j vanishes.

When $j_0 < j < j_n$, by Lemma 2 we have

$$\begin{aligned}\|u_n - h_n\|_{C^0(P_j)} &\le \|u_n - h_n(2^{-j})\|_{C^0(P_j)} + \|u_n - h_n(2^{1-j})\|_{C^0(P_j)} \\ &\le 2\|u_n\|_{Osc(P_j)} \\ &\le C\big(\|\nabla u_n\|_{L^2(P_{j-1}\cup P_j \cup P_{j+1})} + 2^{2(1-p)j/p}\|\tau(u_n)\|_p\big) \\ &\le C\big(\epsilon + 2^{-2(p-1)j/p}\big) \\ &\le C\big(\epsilon + \delta^{2(p-1)/p}\big) \le C\epsilon.\end{aligned}$$

Summing over j for $j_0 < j < j_n$ gives

$$\begin{aligned}\text{(3-4)}\quad &\int_{D_\delta \setminus D_{2r_n R}} |\nabla(u_n - h_n)|^2 dx \\ &= \sum_{j_0<j<j_n} \int_{P_j} |\nabla(u_n - h_n)|^2\, dx \\ &\le \sum_{j_0<j<j_n} \int_{P_j} |u_n - h_n|\big(|A(u_n)| + |\tau(u_n)|\big)\, dx \\ &\qquad + \sum_{j_0<j<j_n} \int_{\partial P_j} (u_n - h_n)(u_n - h_n)_r\, ds \\ &\le C\epsilon\bigg(\int_{D_{2\delta}\setminus D_{2r_n R}} \big(|\nabla u_n|^2 + |\tau(u_n)|\big)\, dx + \int_{\partial D_{2\delta}\cup \partial D_{2r_n R}} |\nabla u_n|\, ds\bigg) \\ &\le C\epsilon\bigg(\int_{D_{2\delta}\setminus D_{2r_n R}} |\nabla u_n|^2\, dx + \delta^{2(p-1)/p} + \epsilon\bigg) \le C\epsilon.\end{aligned}$$

Here we use the inequality

$$\int_{\partial D_{2\delta}\cup \partial D_{2r_n R}} |\nabla u_n|\, ds \le C\epsilon,$$

which can be derived from the Sobolev trace embedding theorem.

As $h_n(x)$ is independent of θ, it can be shown that

$$\int_{D_{2\delta}\setminus D_{2r_n R}} |x|^{-2} |\partial_\theta u_n|^2\, dx \le \int_{D_{2\delta}\setminus D_{2r_n R}} |\nabla(u_n - h_n)|^2\, dx \le C\epsilon,$$

so this lemma is proved. □

It is left to show that the normal energy on the neck domain also equals to zero. We need the following equality.

Lemma 4 (Pohozaev equality [Lin and Wang 1998, Lemma 2.4, page 374]). *Let u be a solution to*

$$\triangle u + A(u)(du, du) = \tau(u).$$

Then

$$\int_{\partial D_t}\big(|\partial_r u|^2 - r^{-2}|\partial_\theta u|^2\big)\, ds = \frac{2}{t}\int_{D_t} \tau\cdot(x\nabla u)\, dx. \tag{3-5}$$

As a direct corollary, by integrating over $[0,\delta]$, *we have*

$$\int_{D_\delta}\big(|\partial_r u|^2 - r^{-2}|\partial_\theta u|^2\big)\, dx = \int_0^\delta \frac{2}{t}\int_{D_t} \tau\cdot(x\nabla u)\, dx dt. \tag{3-6}$$

Proof. Multiplying both sides of the equation by $x\nabla u$ and integrating over D_t, we get

$$\int_{D_t}|\nabla u|^2\, dx - t\int_{\partial D_t}|\partial_r u|^2\, ds + \frac{1}{2}\int_{D_t} x\nabla|\nabla u|^2\, dx = -\int_{D_t}\tau\cdot(x\nabla u)\, dx.$$

Note that

$$\frac{1}{2}\int_{D_t} x\nabla|\nabla u|^2\, dx = -\int_{D_t}|\nabla u|^2\, dx + \frac{t}{2}\int_{\partial D_t}|\nabla u|^2\, ds.$$

Hence,

$$\int_{\partial D_t}\big(|\partial_r u|^2 - \tfrac{1}{2}|\nabla u|^2\big)\, ds = \frac{1}{t}\int_{D_t}\tau\cdot(x\nabla u)\, dx.$$

As $|\nabla u|^2 = |\partial_r u|^2 + r^{-2}|\partial_\theta u|^2$, we have proved this lemma. □

Now we use this equality to estimate the normal energy on the neck domain. We prove the following lemma.

Lemma 5. *If* $\tau(u_n)$ *is bounded in* L^p *for some* $p \geq \frac{6}{5}$, *then for* δ *small enough we have*

$$\left|\int_{D_\delta}\big(|\partial_r u_n|^2 - |x|^{-2}|\partial_\theta u|^2\big)\, dx\right| \leq C\delta^{(p-1)/p},$$

where C *depends on* p, Λ, *the target manifold* N *and the bubble* w.

Proof. Take $\psi \in C_0^\infty(D_2)$ satisfying $\psi = 1$ in D_1, then

$$\triangle(\psi u_n) = \psi A(u_n)(du_n, du_n) + \psi\tau_n + 2\nabla\psi\nabla u_n + u_n\triangle\psi.$$

Set $g_n = \psi A(u_n)(du_n, du_n) + \psi\tau_n + 2\nabla\psi\nabla u_n + u_n\triangle\psi$. When $|x| < 1$,

$$\partial_i u_n(x) = R_i * g_n(x) = \int \frac{x_i - y_i}{|x-y|^2} g_n(y)\, dy.$$

Let Φ_n be the Newtonian potential of $\psi\tau_n$, then $\triangle\Phi_n = \psi\tau_n$. The corresponding Pohozaev equality is

$$\int_{D_\delta}\big(|\partial_r \Phi_n|^2 - r^{-2}|\partial_\theta \Phi_n|^2\big)\, dx = \int_0^\delta \frac{2}{t}\int_{D_t}\psi\tau_n\cdot(x\nabla\Phi_n)\, dx dt. \tag{3-7}$$

Here

$$\partial_i \Phi_n(x) = R_i * (\psi \tau_n)(x) = \int \frac{x_i - y_i}{|x-y|^2} (\psi \tau_n)(y)\, dy.$$

As τ_n is bounded in L^p $(p > 1)$, we have

$$\int_{D_\delta} |\nabla \Phi_n|^2\, dx \le C\delta^{4(p-1)/p} \|\nabla \Phi_n\|^2_{2p/(2-p)} \le C\delta^{4(p-1)/p} \|\tau_n\|_p^2 \le C\delta^{4(p-1)/p}.$$

By (3-7), it can be shown that for any $\delta > 0$,

$$\left| \int_0^\delta \frac{1}{t} \int_{D_t} \psi \tau_n \cdot (x \nabla \Phi_n)\, dx dt \right| \le \int_{D_\delta} |\nabla \Phi_n|^2\, dx \le C\delta^{4(p-1)/p}. \tag{3-8}$$

For δ small enough, we have

$$\begin{aligned} &\left| \int_{D_\delta} \big(|\partial_r u_n|^2 - r^{-2} |\partial_\theta u_n|^2 \big)\, dx \right| \\ &= \left| \int_0^\delta \frac{2}{t} \int_{D_t} \tau_n \cdot (x \nabla u_n)\, dx dt \right| \\ &\le 2 \left| \int_0^\delta \frac{1}{t} \int_{D_t} \tau_n \cdot (x \nabla \Phi_n)\, dx dt \right| + 2 \int_0^\delta \frac{1}{t} \int_{D_t} |x \tau_n|\, |\nabla (u_n - \Phi_n)(x)|\, dx dt \\ &\le C\delta^{4(p-1)/p} + 2 \int_{D_\delta} \big| x \tau_n \big| \big| \nabla (u_n - \Phi_n)(x) \big| \left(\int_{|x|}^\delta \frac{1}{t} dt \right) dx \\ &\le C\delta^{4(p-1)/p} + 2 \int_{D_\delta} |\tau_n|\, |\nabla (u_n - \Phi_n)(x)|\, |x| \ln \frac{1}{|x|}\, dx. \end{aligned} \tag{3-9}$$

For any $j > 0$, set $\varphi_j(x) = \psi\big(\frac{x}{2^{2-j}\delta}\big) - \psi\big(\frac{x}{2^{-2-j}\delta}\big)$. When $2^{-j}\delta \le |x| < 2^{1-j}\delta$, we obtain

$$\begin{aligned} |\partial_i (u_n - \Phi_n)(x)| &= \left| \int \frac{x_i - y_i}{|x-y|^2} (g_n(y) - \psi \tau_n(y))\, dy \right| \\ &\le \int \frac{|\psi A(u_n)(du_n, du_n) + 2 \nabla \psi \nabla u_n + u_n \triangle \psi|(y)}{|x-y|} dy \\ &\le \int \frac{|\psi A(u_n)(y)|}{|x-y|} dy + C \int_{1<|y|<2} (|\nabla u_n| + |u_n|)(y)\, dy \\ &\le \int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy + \int \frac{|(\psi - \varphi_j) A(u_n)(y)|}{|x-y|} dy + C \\ &\le \int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy + \frac{\int |A(u_n)(y)|\, dy}{|x|} + C \\ &\le \int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy + \frac{C}{|x|}. \end{aligned} \tag{3-10}$$

When $\delta > 0$ is small enough and n is big enough, for any $j > 0$, we claim that

$$\|\varphi_j A(u_n)\|_{p/(2-p)} \le C(2^{-j}\delta)^{-4(p-1)/p}, \tag{3-11}$$

where the constant C depends only on p, Λ, the bubble w and the target manifold N.

Take $\delta > 0$ and $R(w)$ that depends on w such that

$$E(u, D_{8\delta}) \le \tfrac{1}{8}\epsilon_N^2 \quad \text{and} \quad E(w, \mathbb{R}^2 \setminus D_{R(w)}) \le \tfrac{1}{8}\epsilon_N^2.$$

The standard blow-up analysis (see [Ding and Tian 1995]) shows that for any j with $8r_n R(w) \le 2^{-j}\delta$ and n big enough, we have

$$E(u_n, D_{2^{4-j}\delta} \setminus D_{2^{-3-j}\delta}) \le \tfrac{1}{3}\epsilon_N^2.$$

By (3-1), when $2^{-j}\delta < r_n/16$, we get

$$E(u_n, D_{2^{4-j}\delta} \setminus D_{2^{-3-j}\delta}) \le \tfrac{1}{4}\epsilon_N^2.$$

So when $2^{-j}\delta < r_n/16$ or $2^{-j}\delta \ge 8r_n R(w)$, by Lemma 2, we see that

$$\begin{aligned}
\|\varphi_j A(u_n)\|_{p/(2-p)} &\le C\|\nabla u_n\|^2_{L^{2p/(2-p)}(D_{2^{3-j}\delta}\setminus D_{2^{-2-j}\delta})}\\
&\le C\|u_n - \bar{u}_{n,j}\|^2_{W^{2,p}(D_{2^{3-j}\delta}\setminus D_{2^{-2-j}\delta})}\\
&\le C\big[(2^{-j}\delta)^{-4\frac{p-1}{p}}\|\nabla u_n\|^2_{L^2(D_{2^{4-j}\delta}\setminus D_{2^{-4-j}\delta})} + \|\tau(u_n)\|_p^2\big]\\
&\le C(2^{-j}\delta)^{-4\frac{p-1}{p}},
\end{aligned}$$

where $\bar{u}_{n,j}$ is the mean of u_n on $D_{2^{3-j}\delta} \setminus D_{2^{-2-j}\delta}$.

On the other hand, when $r_n/16 \le 2^{-j}\delta \le 8r_n R(w)$, we can find no more than $CR(w)^2$ balls with radius $r_n/2$ to cover $D_{2^{3-j}\delta} \setminus D_{2^{-2-j}\delta}$, that is,

$$D_{2^{3-j}\delta} \setminus D_{2^{-2-j}\delta} \subset \bigcup_{i=1}^{m} D(y_i, \tfrac{1}{2}r_n).$$

Set $B_i = D(y_i, \frac{1}{2}r_n)$ and $2B_i = D(y_i, r_n)$. By (3-1), for any i with $i \le m$ we have

$$E(u_n, 2B_i) \le \tfrac{1}{4}\epsilon_N^2.$$

Using Lemma 2 we obtain

$$\begin{aligned}
\|\varphi_j A(u_n)\|_{p/(2-p)} &\le C\|\nabla u_n\|^2_{L^{2p/(2-p)}(D_{2^{3-j}\delta}\setminus D_{2^{-2-j}\delta})}\\
&\le C\bigg(\sum_{i=1}^{m}\|\nabla u_n\|^{2p/(2-p)}_{L^{2p/(2-p)}(B_i)}\bigg)^{(2-p)/p}\\
&\le C\sum_{i=1}^{m}\|\nabla u_n\|^2_{L^{2p/(2-p)}(B_i)}
\end{aligned}$$

$$\le C \sum_{i=1}^{m} \|u_n - \bar{u}_{n,i}\|^2_{W^{2,p}(B_i)}$$
$$\le C \sum_{i=1}^{m} \big((r_n)^{-4(p-1)/p} \|\nabla u_n\|^2_{L^2(2B_i)} + \|\tau(u_n)\|_p^2 \big)$$
$$\le Cm\big((2^{-j}\delta)^{-4(p-1)/p} + 1 \big)$$
$$\le C(2^{-j}\delta)^{-4(p-1)/p},$$

where $\bar{u}_{n,i}$ is the mean of u_n over B_i and the constant C depends only on p, Λ, the bubble w and the target manifold N. So we have proved (3-11).

By (3-10) and (3-11), when $p > 1$ we get

$$\int_{D_\delta} |\tau_n|\, |\nabla(u_n - \Phi_n)(x)|\, |x| \ln \frac{1}{|x|}\, dx \tag{3-12}$$
$$\le \sum_{j=1}^{\infty} \int_{2^{-j}\delta<|x|<2^{1-j}\delta} |\tau_n|\, |\nabla(u_n - \Phi_n)(x)|\, |x| \ln \frac{1}{|x|}\, dx$$
$$\le C \sum_{j=1}^{\infty} \int_{2^{-j}\delta<|x|<2^{1-j}\delta} |\tau_n| \bigg(\frac{1}{|x|} + \int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy \bigg) |x| \ln \frac{1}{|x|}\, dx$$
$$\le C \Bigg(\int_{D_\delta} |\tau_n| \ln \frac{1}{|x|}\, dx + \sum_{j=1}^{\infty} \int_{2^{-j}\delta<|x|<2^{1-j}\delta} |\tau_n| \bigg(\int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy \bigg) |x| \ln \frac{1}{|x|}\, dx \Bigg)$$
$$\le C \Bigg(\Big\| \ln \frac{1}{|\cdot|} \Big\|_{L^{p/(p-1)}(D_\delta)} + \sum_{j=1}^{\infty} 2^{-j}\delta \ln \frac{2^j}{\delta} \Big\| \int \frac{|\varphi_j A(u_n)(y)|}{|\cdot - y|} dy \Big\|_{\frac{p}{p-1}} \Bigg) \times \|\tau_n\|_p$$
$$\le C \Bigg(\delta^2 \Big(\ln \frac{1}{\delta} \Big)^{1/(p-1)} + \sum_{j=1}^{\infty} 2^{-j}\delta \ln \frac{2^j}{\delta} \|\varphi_j A(u_n)\|_{2p/(3p-2)} \Bigg).$$

Here we use the fact that the fraction integral operator $I(f) = \frac{1}{|\cdot|} * f$ is bounded from $L^q(\mathbb{R}^2)$ to $L^{2q/(2-q)}(\mathbb{R}^2)$ for $1 < q < 2$.

When $p \ge \frac{6}{5}$, that is, when $2p/(3p-2) \le p/(2-p)$, by (3-11) we have

$$\|\varphi_j A(u_n)\|_{\frac{2p}{3p-2}} \le C(2^{-j}\delta)^{\frac{5p-6}{p}} \|\varphi_j A(u_n)\|_{\frac{p}{2-p}} \tag{3-13}$$
$$\le C(2^{-j}\delta)^{\frac{5p-6}{p} - \frac{4(p-1)}{p}} \le C(2^{-j}\delta)^{-\frac{2-p}{p}}.$$

From (3-12) and (3-13) we get

$$\begin{aligned}(3\text{-}14)\quad & \int_{D_\delta} |\tau_n|\,|\nabla(u_n-\Phi_n)(x)|\,|x|\ln\frac{1}{|x|}\,dx \\ &\le C\left(\delta^2\left(\ln\frac{1}{\delta}\right)^{\frac{1}{p-1}} + \sum_{j=1}^{\infty} 2^{-j}\delta\ln\frac{2^j}{\delta}\|\varphi_j A(u_n)\|_{\frac{2p}{3p-2}}\right) \\ &\le C\left(\delta + \sum_{j=1}^{\infty} 2^{-j}\delta\ln\frac{2^j}{\delta}(2^{-j}\delta)^{-\frac{2-p}{p}}\right) \\ &\le C\left(\delta+\delta^{\frac{2(p-1)}{p}}\ln\frac{1}{\delta}\right) \le C\delta^{\frac{p-1}{p}}.\end{aligned}$$

It is clear that (3-9) and (3-14) imply that

$$(3\text{-}15)\qquad \left|\int_{D_\delta}\left(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\right)dx\right| \le C\delta^{(p-1)/p}.$$

This concludes the proof. □

Now we use these lemmas to prove the energy identity. Note that w is harmonic. From Lemma 4 we see that $\int_{D_R}(|\partial_r w|^2 - r^{-2}|\partial_\theta w|^2)dx = 0$ for any $R > 0$. It is easy to see that

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}\left|\int_{D_{r_n R}}\left(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\right)dx\right| &= \lim_{R\to\infty}\left|\int_{D_R}\left(|\partial_r w|^2 - r^{-2}|\partial_\theta w|^2\right)dx\right| \\ &= 0.\end{aligned}$$

Letting $\delta\to 0$ in (3-15), we obtain

$$\begin{aligned}(3\text{-}16)\quad \lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}&\left|\int_{D_\delta\setminus D_{r_n R}}\left(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\right)dx\right| \\ &\le \lim_{\delta\to 0}\lim_{n\to\infty}\left|\int_{D_\delta}\left(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\right)dx\right| \\ &\qquad + \lim_{R\to\infty}\lim_{n\to\infty}\left|\int_{D_{r_n R}}\left(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\right)dx\right| \\ &= 0.\end{aligned}$$

Using Lemma 3 we obtain that the normal energy also vanishes on the neck domain, so the energy identity is proved.

4. Neckless property

In this section we use the method in [Qing and Tian 1997] to prove the neckless property during blowing up.

For any $\epsilon > 0$, take δ, R such that

$$E(u, D_{4\delta}) + E(w, \mathbb{R}^2 \setminus D_R) + \delta^{4(p-1)/p} < \epsilon^2.$$

Suppose $r_n R = 2^{-j_n}, \delta = 2^{-j_0}$. When n is big enough, the standard blow-up analysis shows that for any $j_0 \le j \le j_n$,

$$E(u_n, D_{2^{1-j}} \setminus D_{2^{-j}}) < \epsilon^2.$$

For any $j_0 < j < j_n$, set $L_j = \min\{j - j_0, j_n - j\}$. Now we estimate the norm $\|\nabla u_n\|_{L^2(P_j)}$. Set $P_{j,t} = D_{2^{t-j}} \setminus D_{2^{-t-j}}$ and take $h_{n,j,t}$ similar to h_n in the last section, but

$$h_{n,j,t}(2^{\pm t-j}) = \frac{1}{2\pi} \int_{S^1} u_n(2^{\pm t-j}, \theta)\, d\theta.$$

By an argument similar to the one used in deriving (3-4), we have, for $0 < t \le L_j$,

$$\text{(4-1)} \quad \int_{P_{j,t}} r^{-2} |\partial_\theta u_n|^2\, dx \le C\epsilon \left(\int_{P_{j,t}} |\nabla u_n|^2\, dx + (2^{t-j})^{\frac{2(p-1)}{p}} \right) + \int_{\partial P_{j,t}} |u_n - h_{n,j,t}|\, |\nabla u_n|\, ds.$$

Set $f_j(t) = \int_{P_{j,t}} |\nabla u_n|^2 dx$, a simple computation shows that

$$f_j'(t) = \ln 2 \left(2^{t-j} \int_{\{2^{t-j}\}\times S^1} |\nabla u_n|^2\, ds + 2^{-t-j} \int_{\{2^{-t-j}\}\times S^1} |\nabla u_n|^2\, ds \right).$$

Combining that $h_{n,j,t}$ is independent of θ and $h_{n,j,t}$ is the mean value of u_n at the two components of $\partial P_{j,t}$ with the Poincaré inequality yields that

$$\begin{aligned}
\int_{\partial P_{j,t}} &|u_n - h_{n,j,t}|\, |\nabla u_n|\, ds \\
&= \int_{\{2^{t-j}\}\times S^1} |u_n - h_{n,j,t}|\, |\nabla u_n|\, ds + \int_{\{2^{-t-j}\}\times S^1} |u_n - h_{n,j,t}|\, |\nabla u_n|\, ds \\
&\le \left(\int_{\{2^{t-j}\}\times S^1} |u_n - h_{n,j,t}|^2\, ds \right)^{\frac12} \left(\int_{\{2^{t-j}\}\times S^1} |\nabla u_n|^2\, ds \right)^{\frac12} \\
&\qquad + \left(\int_{\{2^{-t-j}\}\times S^1} |u_n - h_{n,j,t}|^2\, ds \right)^{\frac12} \left(\int_{\{2^{-t-j}\}\times S^1} |\nabla u_n|^2\, ds \right)^{\frac12} \\
&\le C \left(2^{t-j} \int_{\{2^{t-j}\}\times S^1} |\nabla u_n|^2\, ds + 2^{-t-j} \int_{\{2^{-t-j}\}\times S^1} |\nabla u_n|^2\, ds \right) \\
&\le C f_j'(t).
\end{aligned}$$

On the other hand, by a similar argument as we made to obtain (3-15), we get

$$\left|\int_{P_{j,t}} \big(|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2\big)\,dx\right| \le C\Big((2^{t-j})^{\frac{p-1}{p}} + (2^{-t-j})^{\frac{p-1}{p}}\Big) \le C(2^{t-j})^{\frac{p-1}{p}}. \tag{4-2}$$

Since $|\nabla u|^2 = |\partial_r u|^2 + r^{-2}|\partial_\theta u|^2 = 2r^{-2}|\partial_\theta u|^2 + (|\partial_r u|^2 - r^{-2}|\partial_\theta u|^2)$, by (4-1) and (4-2) we have

$$\begin{aligned} f_j(t) &\le 2\int_{P_{j,t}} r^{-2}|\partial_\theta u_n|\,dx + \left|\int_{P_{j,t}} (|\partial_r u_n|^2 - r^{-2}|\partial_\theta u_n|^2)\,dx\right| \\ &\le C\epsilon\Big(f_j(t) + (2^{t-j})^{\frac{2(p-1)}{p}}\Big) + Cf_j'(t) + C(2^{t-j})^{\frac{p-1}{p}} \\ &\le C\Big(\epsilon f_j(t) + 2^{-\frac{(p-1)j}{p}}2^{\frac{(p-1)t}{p}} + f_j'(t)\Big). \end{aligned}$$

Take ϵ small enough and set $\epsilon_p = \frac{p-1}{p}\ln 2$, then for some positive constant C big enough we get

$$f_j'(t) - \frac{1}{C}f_j(t) + Ce^{-\epsilon_p j}e^{\epsilon_p t} \ge 0.$$

We may assume that $\epsilon_p > 1/C$, then we have

$$(e^{-t/C}f_j(t))' + Ce^{-\epsilon_p j}e^{(\epsilon_p - 1/C)t} \ge 0.$$

Integrating this inequality over $[2, L_j]$ gives

$$\begin{aligned} f_j(2) &\le C\left(e^{-L_j/C}f_j(L_j) + e^{-\epsilon_p j}\int_1^{L_j} e^{(\epsilon_p-1/C)t}\,dt\right) \\ &\le C\big(e^{-L_j/C}f_j(L_j) + e^{-\epsilon_p j}e^{(\epsilon_p-1/C)L_j}\big). \end{aligned}$$

Note that $j \ge L_j$, so

$$f_j(2) \le C(e^{-L_j/C}f_j(L_j) + e^{-j/C}).$$

Since the energy identity was proved in the last section, we can take δ small such that the energy on the neck domain is less than ϵ^2, which implies that $f_j(L_j) < \epsilon^2$. So we get

$$f_j(2) \le C\big(e^{-L_j/C}\epsilon^2 + e^{-j/C}\big).$$

Using Lemma 2 on the domain $P_j = D_{2^{1-j}} \setminus D_{2^{-j}}$ when $j < j_n$, we obtain

$$\begin{aligned} \|u_n\|_{Osc(P_j)} &\le C\big(\|\nabla u_n\|_{L^2(P_{j-1}\cup P_j\cup P_{j+1})} + 2^{\frac{2(1-p)j}{p}}\|\tau(u_n)\|_p\big) \\ &\le C\big(f_j(2) + e^{-2\epsilon_p j}\big). \end{aligned}$$

Summing over j from j_0 to j_n yields

$$
\begin{aligned}
\|u_n\|_{Osc(D_\delta\setminus D_{2r_nR})} &\le \sum_{j=j_0}^{j_n} \|u_n\|_{Osc(P_j)} \\
&\le C \sum_{j=j_0}^{j_n} \big(f_j(2) + e^{-2\epsilon_p j}\big) \\
&\le C \sum_{j=j_0}^{j_n} \big(e^{-L_j/C}\epsilon^2 + e^{-j/C} + e^{-2\epsilon_p j}\big) \\
&\le C\bigg(\sum_{i=0}^{\infty} e^{-i/C}\epsilon^2 + \sum_{j=j_0}^{\infty} e^{-j/C}\bigg) \\
&\le C\big(\epsilon^2 + e^{-j_0/C}\big) \le C\big(\epsilon^2 + \delta^{1/C}\big).
\end{aligned}
$$

Here we used the assumption that $\epsilon_p > 1/C$. So we have proved that there is no neck during the blowing up.

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