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REMARKS ON SOME ISOPERIMETRIC PROPERTIES OF THE $k - 1$ FLOW

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We consider the evolution of a convex closed plane curve γ_0 along its inward normal direction with speed $k - 1$, where k is the curvature. This flow has the feature that it is the gradient flow of the (length − area) functional and has been previously studied by Chou and Zhu, and Yagisita. We revisit the flow and point out some interesting isoperimetric properties not discussed before.

We first prove that if the curve γ_t converges to the unit circle S^1 (without rescaling), its length $L(t)$ and area $A(t)$ must satisfy certain monotonicity properties and inequalities.

On the other hand, if the curve γ_t (assume γ_0 is not a circle) expands to infinity as $t \to \infty$ and we interpret Yagisita's result in the right way, the isoperimetric difference $L^2(t) - 4\pi A(t)$ of γ_t will decrease to a *positive* constant as $t \to \infty$. Hence, without rescaling, the expanding curve γ_t will not become circular. It is asymptotically close to some expanding curve C_t, where C_0 is not a circle and each C_t is *parallel* to C_0. The asymptotic speed of C_t is given by the constant 1.

1. Introduction

Let γ_0 be a smooth embedded convex closed curve in $\mathbb{R}^2$ (with positive curvature everywhere) parametrized by $X_0 := X_0(\varphi) : S^1 \to \mathbb{R}^2$, where S^1 is the unit circle. We study the geometric behavior of γ_0 driven by the equation

$$\frac{\partial X}{\partial t}(\varphi, t) = (k(\varphi, t) - 1)N_{\text{in}}(\varphi, t), \quad X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1 \tag{1}$$

where $k(\varphi, t)$ is the curvature of the curve γ_t (parametrized by $X(\varphi, t)$), $N_{\text{in}}(\varphi, t)$ is the unit inward normal vector of γ_t.

Without the constant term, (1) is the well-known *curve shortening flow*. See [Gage and Hamilton 1986] for the case when γ_0 is convex and [Grayson 1987] for the case when γ_0 is a simple closed curve. Also see [Andrews 1998] for more

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general types of curvature flows. Unlike the situation in [Grayson 1987], a simple closed curve γ_0 may develop self-intersections under the flow (1) due to the constant term -1. This will make the problem hard to manage. Thus we focus only on the case when γ_0 is convex.

According to [Gage and Hamilton 1986], (1) is a parabolic flow and there exists a unique smooth solution $X(\varphi, t) : S^1 \times [0, T) \to \mathbb{R}^2$ to the flow for a short time $T > 0$, $T < \infty$. We want to study its long-time convergence behavior. The flow (1) has the interesting property that it is the *gradient flow* of the functional

$$E(\gamma) = \text{ length} - \text{area} = \int_\gamma ds - \tfrac{1}{2} \int_\gamma (x\, dy - y\, dx) \tag{2}$$

with respect to the L^2 inner product $\langle u, v\rangle = \int_\gamma uv\, ds$ on the space of all normal variations of γ. One can also view it as a competition between the curve shortening flow $\partial X/\partial t = k\boldsymbol{N}_{\text{in}}$ (contraction) and the unit-speed outward normal flow $\partial X/\partial t = -\boldsymbol{N}_{\text{in}} = \boldsymbol{N}_{out}$ (expansion). See [Gage and Hamilton 1986; Green and Osher 1999].

It is expected intuitively that, depending on the convex initial curve, the flow (1) will either converge to a point, converge to a round circle S^1, or expand to infinity, with each γ_t remaining smooth and convex. This is indeed true due to Theorem 3.12 (see also Remark 3.14) of [Chou and Zhu 2001]. Moreover, for given initial data $X_0(\varphi) : S^1 \to \mathbb{R}^2$, if we consider its homothetic class

$$\mathscr{H}(X_0) = \{\lambda X_0(\varphi) : \lambda > 0,\ \varphi \in S^1\}, \tag{3}$$

there exists a unique number $\Lambda > 0$ (for convenience we call it the *critical number* of X_0) such that under the flow (1) with initial data $\lambda X_0(\varphi)$, $\lambda = \Lambda$, γ_t will converge to the unit circle S^1 (without rescaling) smoothly as $t \to \infty$. For $0 < \lambda < \Lambda$, the flow exists on a maximal finite time interval $[0, T_{\max})$, $T_{\max} < \infty$, and γ_t converges to a point $p \in \mathbb{R}^2$ as $t \to T_{\max}$; and for $\lambda > \Lambda$, the flow expands to infinity as $t \to \infty$. Thus the generic behavior of the $k - 1$ flow is either converging to a point or expanding to infinity.

The asymptotic behavior of γ_t as $t \to T_{\max}$ (or $t \to \infty$) in the above three cases are known due to [Chou and Zhu 2001, Theorem 3.12; Gage 1984; Gage and Hamilton 1986; Chow and Tsai 1996]. Also see [Yagisita 2005] for a more refined estimate in the expanding case.

Our purpose is to give an estimate of the number Λ and to point out some monotonicity properties of length $L(t)$ and area $A(t)$ not observed before. See Theorem 2.1. We also reinterpret Yagisita's estimate in terms of the asymptotic behavior of the isoperimetric difference $D(t) := L^2(t) - 4\pi A(t)$. See Lemmas 3.7 and 3.9.

Remark 1.1. This is to explain how to convert Chou and Zhu's results for $k - \lambda$ flow into results for $k - 1$ flow. Chou and Zhu [2001, Theorem 3.12] considered a general flow which includes the following as a special case:

$$\frac{\partial X}{\partial t}(\varphi, t) = (k(\varphi, t) - \lambda)\boldsymbol{N}_{\text{in}}(\varphi, t), \quad X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1. \tag{4}$$

Here $\lambda \in \mathbb{R}$ is a number serving as a parameter. For a given initial curve $\gamma_0 := X_0(\varphi)$, there exists a unique number Λ such that the flow (4) with $\lambda = \Lambda$ will evolve γ_0 smoothly into a circle with radius $1/\Lambda$ as $t \to \infty$, $t \in [0, \infty)$. We assert that if we replace γ_0 by $\tilde{\gamma}_0 := \Lambda\gamma_0$ and rescale time (denote the new time as τ), then, under the $k-1$ flow (1), $\tilde{\gamma}_0$ will converge to the unit circle S^1 as $\tau \to \infty$. More precisely, let $X(\varphi, t)$ be the solution to the $k - \Lambda$ flow with initial condition $X_0(\varphi)$ and set

$$\widetilde{X}(\varphi, \tau) = \Lambda X(\varphi, t), \quad \tau = \Lambda^2 t \in [0, \infty).$$

Then by $\tilde{k}(\varphi, \tau) = (1/\Lambda)k(\varphi, t)$, $\widetilde{\boldsymbol{N}}_{\text{in}}(\varphi, \tau) = \boldsymbol{N}_{\text{in}}(\varphi, t)$, we have

$$\widetilde{X}(\varphi, 0) = \Lambda X_0(\varphi) = \tilde{\gamma}_0,$$

$$\frac{\partial \widetilde{X}}{\partial \tau}(\varphi, \tau) = \Lambda \frac{dt}{d\tau}\frac{\partial X}{\partial t}(\varphi, t) = \frac{1}{\Lambda}(k(\varphi, t) - \Lambda)\boldsymbol{N}_{\text{in}}(\varphi, t) = (\tilde{k}(\varphi, \tau) - 1)\widetilde{\boldsymbol{N}}_{\text{in}}(\varphi, \tau)$$

for all $(\varphi, \tau) \in S^1 \times [0, \infty)$. That is, $\widetilde{X}(\varphi, \tau)$ satisfies the $k-1$ flow (with initial condition $\tilde{\gamma}_0 = \Lambda\gamma_0$) and converges to the unit circle S^1 as $\tau \to \infty$.

2. Estimate of the critical number Λ

According to [Chou and Zhu 2001], the critical number Λ is obtained via a contradiction argument and for a given curve $X_0(\varphi)$ we do not know what it is. However, we can use the following theorem to give an estimate of Λ (see Corollary 2.7).

Theorem 2.1. *Let γ_0 be a convex closed curve* (*which is not a unit circle*) *and consider* (1) *with initial data γ_0. If the flow is defined on time interval* $[0, \infty)$ *and γ_t converges* (*without rescaling*) *to the unit circle S^1 as $t \to \infty$, its length $L(t)$ and enclosed area $A(t)$ must satisfy the estimate*

$$L(t) > 2\pi, \quad A(t) < \pi, \quad L(t) - 2\pi > \pi - A(t) \tag{5}$$

for all $t \in [0, \infty)$. Moreover, $L(t)$ is strictly decreasing, $A(t)$ is strictly increasing, and $(L(t) - 2\pi) - (\pi - A(t))$ is strictly decreasing on $[0, \infty)$.

The proof consists of several simple lemmas. Recall that for a family of time-dependent simple closed curves $\gamma_t = \gamma(\cdot, t)$ in the plane its length $L(t)$ and enclosed area $A(t)$ satisfy the equations

$$\frac{dL}{dt}(t) = -\int_{\gamma(\cdot,t)} \left\langle \frac{\partial \gamma}{\partial t}, k\boldsymbol{N}_{\text{in}} \right\rangle ds, \quad \frac{dA}{dt}(t) = -\int_{\gamma(\cdot,t)} \left\langle \frac{\partial \gamma}{\partial t}, \boldsymbol{N}_{\text{in}} \right\rangle ds, \tag{6}$$

where $\boldsymbol{N}_{\text{in}}$ is the unit inward normal of γ_t and k is its signed curvature with respect to $\boldsymbol{N}_{\text{in}}$. Therefore

$$\frac{d}{dt}(L(t)-A(t)) = -\int_{\gamma(\cdot,t)}\left\langle \frac{\partial\gamma}{\partial t}, (k-1)\boldsymbol{N}_{\text{in}}\right\rangle ds, \tag{7}$$

which explains why (1) is the gradient flow of the functional $E(\gamma)$ in (2). In particular, under the flow (1), we have

$$\frac{dL}{dt}(t) = -\int_{\gamma(\cdot,t)} k^2 ds + 2\pi, \quad \frac{dA}{dt}(t) = L(t) - 2\pi. \tag{8}$$

As γ_0 is strictly convex, γ_t will remain so for a short time (we may assume γ_t is convex on $[0, T)$ for some $T > 0$). Thus one can use the outward normal angle $\theta \in S^1$ of γ_t as a parametrization variable. In terms of $(\theta, t) \in S^1 \times [0, T)$ we have

$$\frac{\partial k}{\partial t}(\theta,t) = k^2(\theta,t)[k_{\theta\theta}(\theta,t)+k(\theta,t)] - k^2(\theta,t) \tag{9}$$

and

$$\frac{\partial u}{\partial t}(\theta,t) = 1 - k(\theta,t) = 1 - \frac{1}{u_{\theta\theta}(\theta,t)+u(\theta,t)}, \tag{10}$$

where $u(\theta, t)$ is the support function of γ_t. We also have

$$L(t) = \int_0^{2\pi} u(\theta,t)\,d\theta, \quad A(t) = \tfrac{1}{2}\int_0^{2\pi} [u^2(\theta,t) - u_\theta^2(\theta,t)]\,d\theta. \tag{11}$$

Let $w(\theta, t) = k(\theta, t)e^{t/4}$ and compute

$$\frac{\partial w}{\partial t}(\theta,t) = k^2(\theta,t)w_{\theta\theta}(\theta,t) + \big(k(\theta,t) - \tfrac{1}{2}\big)^2 w(\theta,t).$$

By the maximum principle we can obtain a lower bound of the curvature:

$$k(\theta,t) \geq k_{\min}(0)e^{-t/4} > 0 \tag{12}$$

for all $(\theta, t) \in S^1 \times [0, T)$, where $k_{\min}(0) = \min_{\theta\in[0,2\pi]} k(\theta, 0)$. By Theorem 3.12 of [Chou and Zhu 2001], the flow γ_t (each γ_t remains smooth and convex) is either defined on a finite maximal time interval $[0, T_{\max})$ with $\lim_{t\to T_{\max}} k_{\max}(t) = \infty$ or on an infinite time interval $[0, \infty)$ with $\lim_{t\to\infty} k(\theta, t) = 1$ or $\lim_{t\to\infty} k(\theta, t) = 0$ uniformly on S^1.

Note that for any simple closed curve γ in the plane, we have $\int_\gamma k\,ds = 2\pi$ and

$$\int_\gamma k^2\,ds \geq \frac{4\pi^2}{L} \quad \text{(Hölder inequality)}, \tag{13}$$

and, by Gage's isoperimetric inequality [1983], we have

$$\int_\gamma k^2\,ds \geq \frac{\pi L}{A} \tag{14}$$

for any convex closed curve γ in $\mathbb{R}^2$. We also need the fact that the equality holds in (13) or (14) if and only if γ is a circle.

As a consequence of (14), the isoperimetric difference and ratio of γ_t, under the $k-1$ flow, are both decreasing (strictly decreasing if γ_0 is not a circle) in time due to

$$\frac{d}{dt}(L^2 - 4\pi A) \leq -\frac{2\pi}{A}(L^2 - 4\pi A) \leq 0 \tag{15}$$

and

$$\frac{d}{dt}\Big(\frac{L^2}{4\pi A} - 1\Big) \leq \frac{-L}{A}\Big(\frac{L^2}{4\pi A} - 1\Big) \leq 0. \tag{16}$$

Thus, in any case of convergence, γ_t is getting more and more circular.

Let $L(0)$ and $A(0)$ be the length and area of γ_0 (γ_0 is not a unit circle).

Lemma 2.2. *If $L(0) \leq 2\pi$, the flow* (1) *strictly decreases $L(t)$ and $A(t)$, and γ_t converges to a point $p \in \mathbb{R}^2$ in finite time $T_{\max}$.*

Remark 2.3. If $L(0) \leq 2\pi$, γ_0 may not be enclosed by a circle with radius less than 1. Otherwise the result is trivial due to the maximum principle.

Proof. Since γ_0 is not a unit circle, if $L(0) \leq 2\pi$, we must have $A(0) < \pi$ due to $L^2(0) > 4\pi A(0)$. We also have strict inequality in (13). By (8), we have

$$\frac{dL}{dt} = -\int_{\gamma(\cdot,t)} k^2\,ds + 2\pi \leq \frac{2\pi}{L(t)}(L(t) - 2\pi),$$

and at $t = 0$ we have $(dL/dt)(0) < 0$. Thus the flow (1) strictly decreases $L(t)$. By $dA/dt = L(t) - 2\pi$, it also strictly decreases $A(t)$. As a consequence of Theorem 3.12 of [Chou and Zhu 2001], γ_t will converge to a point $p \in \mathbb{R}^2$ in finite time $T_{\max}$. □

Lemma 2.4. *If $A(0) \geq \pi$, the flow* (1) *strictly increases $A(t)$, and γ_t expands to infinity as $t \to \infty$.*

Remark 2.5. If $A(0) \geq \pi$, γ_0 may not enclose a circle with radius larger than 1. Otherwise the result is trivial due to the maximum principle.

Proof. We now have $L(0) > 2\pi$ and $(dA/dt)(0) = L(0) - 2\pi > 0$. By continuity $L(t)$ remains $L(t) > 2\pi$ for a short time $[0, T)$ and $A(t)$ is strictly increasing with $A(t) > \pi$ on $(0, T)$. As time proceeds, the inequality $L^2 \geq 4\pi A$ forces $L(t)$ to remain $L(t) > 2\pi$ and $A(t)$ keeps strictly increasing. Again by Theorem 3.12 of

[Chou and Zhu 2001], the flow is defined on $[0, \infty)$ and γ_t expands to infinity as $t \to \infty$. □

Lemma 2.6. *If $L(0) > 2\pi$, $A(0) < \pi$, and $L(0) - 2\pi \leq \pi - A(0)$, γ_t converges to a point $p \in \mathbb{R}^2$ in finite time $T_{\max}$.*

Proof. By continuity $L(t) > 2\pi$ and $A(t) < \pi$ for a short time $[0, T)$, and during this time interval we have

$$\text{(17)} \qquad \frac{d}{dt}(L(t) - 2\pi) = -\int_{\gamma(\cdot,t)} k^2\, ds + 2\pi < -\frac{\pi L(t)}{A(t)} + 2\pi < -(L(t) - 2\pi) = \frac{d}{dt}(\pi - A(t)) < 0.$$

Thus $L(t)$ is strictly decreasing and $A(t)$ is strictly increasing.

Equation (17) says that $L(t) - 2\pi$ decreases more rapidly than $\pi - A(t)$ (as long as $L(t) > 2\pi$ and $A(t) < \pi$). Since $L(0) - 2\pi$ is closer to 0 than $\pi - A(0)$, $L(t)$ must touch 2π earlier than $A(t)$ touches π. More precisely, let $t_* > 0$ be the first time at which $L(t) > 2\pi$ and $A(t) < \pi$ on $[0, t_*)$ and $L(t_*) = 2\pi$. Such a t_* must exist and is finite. Otherwise we would have $L(t) > 2\pi$ and $A(t) < \pi$ on $[0, \infty)$, and, by [Chou and Zhu 2001, Theorem 3.12], the flow would have to converge to the unit circle S^1 (without rescaling), which is impossible due to the inequality

$$(L(t) - 2\pi) - (\pi - A(t)) < (L(t_1) - 2\pi) - (\pi - A(t_1)) < [(L(0) - 2\pi) - (\pi - A(0))] \leq 0$$

for all $t > t_1 > 0$ in $[0, \infty)$. (Note that now we have $L(t) \to 2\pi$ and $A(t) \to \pi$ as $t \to \infty$.) Therefore $t_* > 0$ is finite and $L(t_*) = 2\pi$. By Lemma 2.2, γ_t must converge to a point $p \in \mathbb{R}^2$ in finite time $T_{\max}$. □

Proof of Theorem 2.1. Combining these three lemmas, the proof of Theorem 2.1 is now clear. Since we assume that γ_t converges to the unit circle S^1 (without rescaling) as $t \to \infty$, if at some time $t_0 \in [0, \infty)$ we have $L(t_0) \leq 2\pi$ or $A(t_0) \geq \pi$, the curve will either converge to a point or expand to infinity. Hence we must have $L(t) > 2\pi$ and $A(t) < \pi$ for all time. By Lemma 2.6 we also have $L(t) - 2\pi > \pi - A(t)$ for all time. The monotonicity of $L(t)$, $A(t)$, and $(L(t) - 2\pi) - (\pi - A(t))$ can all be seen from (17). □

As a consequence of Theorem 2.1, we can give an estimate of the number Λ in Theorem 3.12 of [Chou and Zhu 2001].

Corollary 2.7. *Let γ_0 be a convex closed curve (which is not a unit circle) with length $L(0)$ and area $A(0)$. Then its critical number Λ satisfies*

$$\text{(18)} \qquad \Lambda L(0) > 2\pi, \quad \Lambda^2 A(0) < \pi, \quad \Lambda L(0) - 2\pi > \pi - \Lambda^2 A(0),$$

which implies

$$\max\left\{\frac{2\pi}{L(0)}, \frac{-L(0)+\sqrt{L^2(0)+12\pi A(0)}}{2A(0)}\right\} < \Lambda < \sqrt{\frac{\pi}{A(0)}}. \tag{19}$$

Remark 2.8. Let $k_0(\theta)$ be the curvature of γ_0. As $\Lambda\gamma_0$ converges to the unit circle S^1 as $t \to \infty$, its curvature $(1/\Lambda)k_0(\theta)$ must satisfy $(1/\Lambda)\max_{\theta\in S^1} k_0(\theta) > 1$ and $(1/\Lambda)\min_{\theta\in S^1} k_0(\theta) < 1$. This gives a rough estimate of Λ in terms of the curvature of γ_0, that is,

$$\min_{\theta\in S^1} k_0(\theta) < \Lambda < \max_{\theta\in S^1} k_0(\theta). \tag{20}$$

We explain that (19) is better than (20). To see this, by the identity

$$L(0) = \int_0^{2\pi} \frac{1}{k_0(\theta)}\, d\theta$$

and Gage's inequality (14), we have

$$\min_{\theta\in S^1} k_0(\theta) < \frac{2\pi}{L(0)} \quad \text{and} \quad \frac{L(0)}{2A(0)} < \max_{\theta\in S^1} k_0(\theta).$$

Combined with the classical isoperimetric inequality

$$\frac{2\pi}{L(0)} < \sqrt{\frac{\pi}{A(0)}} < \frac{L(0)}{2A(0)},$$

we conclude

$$\min_{\theta\in S^1} k_0(\theta) < \frac{2\pi}{L(0)} < \sqrt{\frac{\pi}{A(0)}} < \frac{L(0)}{2A(0)} < \max_{\theta\in S^1} k_0(\theta). \tag{21}$$

Hence (19) is better than the curvature estimate (20).

Remark 2.9. Under the assumption of Theorem 2.1, the curvature $k(\theta, t) \to 1$ uniformly as $t \to \infty$. One can follow a similar proof to that of [Gage and Hamilton 1986, Theorem 5.7.1] to conclude the following curvature estimate: for any $m \in \mathbb{N}$ and any $\alpha \in (0, 1)$, there exists a constant C depending only on m and γ_0 such that

$$\left\| \frac{\partial^m k}{\partial\theta^m}(\theta, t) \right\|_{L^\infty(S^1)} \le C(m)e^{-2\alpha t} \tag{22}$$

for time t large enough.

Given initial curve γ_0, if we replace the $k-1$ flow by $ck-d$ flow, the critical number $\widetilde{\Lambda}$ for the $ck-d$ flow and the critical number Λ for the $k-1$ flow are related by the following.

Corollary 2.10. *Let c, d be two positive constants and let γ_0 be a convex closed curve parametrized by $X_0(\varphi)$, $\varphi \in S^1$. Then the critical number Λ in the $k-1$ flow and the critical number $\widetilde{\Lambda}$ in the $ck-d$ flow are related by*

$$\widetilde{\Lambda} = \frac{c}{d}\Lambda. \tag{23}$$

Proof. This is a consequence of scaling. By definition, the solution $\widetilde{X}(\varphi, t)$ to the initial value problem

$$\frac{\partial \widetilde{X}}{\partial t}(\varphi, t) = (c\tilde{k}(\varphi, t) - d)\widetilde{\boldsymbol{N}}_{\text{in}}(\varphi, t), \quad \widetilde{X}(\varphi, 0) = \widetilde{\Lambda} X_0(\varphi)$$

will converge to the circle with radius $R = c/d$ as $t \to \infty$. Let

$$Y(\varphi, t) = \frac{d}{c}\widetilde{X}\Big(\varphi, \frac{c}{d^2}t\Big).$$

Then it satisfies

$$\frac{\partial Y}{\partial t}(\varphi, t) = (k^{(Y)}(\varphi, t) - 1)\boldsymbol{N}_{\text{in}}^{(Y)}(\varphi, t), \quad Y(\varphi, 0) = \frac{d}{c}\widetilde{\Lambda} X_0(\varphi),$$

where $k^{(Y)}(\varphi, t)$ and $\boldsymbol{N}_{\text{in}}^{(Y)}(\varphi, t)$ are the curvature and normal at $Y(\varphi, t)$, respectively. Since $Y(\varphi, t)$ will converge to the unit circle, we have $(d/c)\widetilde{\Lambda} = \Lambda$. □

3. The asymptotic behavior of $L^2 - 4\pi A$ in the expanding case

There is another interesting property of the flow (1) not discussed before when, given an initial curve γ_0, it expands to infinity. It is about the value of $D(t) := L^2(t) - 4\pi A(t)$ as $t \to \infty$. From (8) it is easy to see that $L(t)$ has scale t and $A(t)$ has scale t^2 as $t \to \infty$ (since $k(\theta, t) \to 0$ uniformly on S^1). If we integrate (16) with respect to time, we get

$$0 \le \frac{L^2(t)}{4\pi A(t)} - 1 \le \Big(\frac{L^2(0)}{4\pi A(0)} - 1\Big)e^{-\int_0^t (L(z)/A(z))\,dz}, \quad \lim_{t\to\infty}\int_0^t \frac{L(z)}{A(z)}\,dz = \infty.$$

This implies $L^2(t)/(4\pi A(t)) \to 1$ exponentially as $t \to \infty$. But if we integrate (15), we only get

$$0 \le D(t) \le D(0)e^{-\int_0^t (2\pi/A(z))\,dz}, \quad \lim_{t\to\infty}\int_0^t \frac{2\pi}{A(z)}dz < \infty,$$

which implies that $L^2(t) - 4\pi A(t)$ decreases to some number bounded above by

$$D(0)e^{-\int_0^\infty (2\pi/A(z))\,dz}.$$

We shall see that, unless γ_0 is a circle, $D(t)$ will not decrease to zero as $t \to \infty$.

Remark 3.1. Note that if we have a family of curves γ_t, $t \in [0, T)$, $T \le \infty$, so that $A(t)$ has uniform positive upper and lower bounds, then $\lim_{t\to T} D(t) = 0$ is equivalent to $\lim_{t\to T} L^2(t)/(4\pi A(t)) = 1$. But if $\lim_{t\to T} A(t) = 0$ or ∞, then they may not be equivalent.

Recall that if we evolve a convex closed curve γ_0 by the *unit-speed* (that is, the constant 1) *outward normal flow*, we get a family of *parallel curves* γ_t expanding to infinity. We get a similar result if we replace the speed constant 1 by a positive time function $a(t)$, where $\lim_{t\to\infty} a(t) = \infty$. Moreover, any two parallel convex closed curves (or simple closed curves) γ_{t_1} and γ_{t_2} have the *same* isoperimetric difference.

The intuitive observation is that when γ_0 expands to infinity under the $k-1$ flow, its asymptotic behavior is given by the unit-speed outward normal flow. As the unit-speed outward normal flow preserves the isoperimetric difference, we expect that $D(t)$ will not decrease to zero as $t \to \infty$. This is indeed the case based on results in [Yagisita 2005], which are explained below.

From now on we use $\theta \in S^1$ to denote the outward normal angle of γ_t (γ_t is convex) and use $\sigma \in S^1$ to denote the *polar angle* of γ_t (we may assume that γ_t encloses the origin of $\mathbb{R}^2$). In Theorem 2 of [Yagisita 2005], he looked at the radial function $r(\sigma, t)$ of γ_t and proved that there exists a smooth function $\ell(\sigma)$ defined on S^1 such that

$$\lim_{t\to\infty} \|r(\sigma, t) - (R(t) + \ell(\sigma))\|_{C^k(S^1)} = 0 \tag{24}$$

for any $k \in \mathbb{N}$, where $R(t)$ is the solution to the ODE

$$\frac{dR}{dt}(t) = 1 - \frac{1}{R(t)}, \qquad R(0) = 2. \tag{25}$$

Note that $R(t)$ is strictly increasing on $[0, \infty)$ with $\lim_{t\to\infty} R'(t) = 1$.

To see that $D(t)$ decreases to a *positive* constant asymptotically, we need to see what (24) implies in terms of the support function $u(\theta, t)$ of γ_t. By (11), it suffices to look at the asymptotic behavior of $u(\theta, t)$ and $u_\theta(\theta, t)$.

Remark 3.2. Yagisita [2005] used the radial function $r(\sigma, t)$ to study the flow (1) instead of the support function $u(\theta, t)$. The advantage is that one can get a *quasilinear* uniformly parabolic equation for the difference $A(\sigma, \tau) := r(\sigma, t) - R(t)$ (see pages 227–230 of [Yagisita 2005]) if we also rescale time. More precisely, let

$$\tau(t) = \log\Big(1 - \frac{1}{R(t)}\Big) : [0, \infty) \to [-\log 2, 0), \qquad \frac{d\tau}{dt} = \frac{1}{R^2(t)}.$$

Then we have

$$\begin{aligned} \frac{\partial A}{\partial \tau}(\sigma, \tau) &\\ &= \frac{R^2(t(\tau))}{[A(\sigma, \tau) + R(t(\tau))]^2 + A_\sigma^2(\sigma, \tau)} A_{\sigma\sigma}(\sigma, \tau) + \text{lower order terms} \end{aligned} \tag{26}$$

for all $(\sigma, \tau) \in S^1 \times [-\log 2, 0)$, where $R(t(\tau)) = 1/(1-e^\tau)$. On the other hand, the evolution equation for $B(\theta, \tau) := u(\theta, t) - R(t)$ is also uniformly parabolic but *fully nonlinear*, that is,

$$\frac{\partial B}{\partial \tau}(\theta, \tau) = \frac{R(t(\tau))}{B_{\theta\theta}(\theta, \tau)+B(\theta, \tau)+R(t(\tau))}(B_{\theta\theta}(\theta, \tau) + B(\theta, \tau)) \tag{27}$$

for all $(\theta, \tau) \in S^1 \times [-\log 2, 0)$. Equation (26) is easier to handle than (27). However, the disadvantage is that it is very awkward to use $r(\sigma, t)$ to study the isoperimetric difference $D(t)$.

Lemma 3.3. *Let γ_0 be a convex closed curve and consider the flow* (1) *with initial data γ_0. Then* (24) *implies*

$$\lim_{t\to\infty} \|u(\theta, t) - (R(t) + \ell(\theta))\|_{C^1(S^1)} = 0. \tag{28}$$

In particular, we have

$$\lim_{t\to\infty} D(t) = \left(\int_0^{2\pi} \ell(\theta)\, d\theta\right)^2 - 2\pi \int_0^{2\pi} \left(\ell^2(\theta) - (\ell'(\theta))^2\right) d\theta \geq 0. \tag{29}$$

Remark 3.4. We may get higher order convergence of $u(\theta, t)$. But (28) is sufficient.

Proof. For a point $p \in \gamma_t$ with position vector P, its support function $u(\theta, t)$ and radial function $r(\sigma, t)$ are related by

$$P = u(\theta, t)(\cos\theta, \sin\theta) + u_\theta(\theta, t)(-\sin\theta, \cos\theta) = r(\sigma, t)(\cos\sigma, \sin\sigma). \tag{30}$$

From this we get

$$u(\theta, t) = r(\sigma, t)\cos(\sigma - \theta), \quad u_\theta(\theta, t) = r(\sigma, t)\sin(\sigma - \theta), \tag{31}$$

and

$$\sigma = \sigma(\theta, t) = \tan^{-1}\left(\frac{u(\theta, t)\sin\theta + u_\theta(\theta, t)\cos\theta}{u(\theta, t)\cos\theta - u_\theta(\theta, t)\sin\theta}\right), \quad \theta \in S^1. \tag{32}$$

In particular, at any point p where $\theta = \sigma$, we have $u(\theta, t) = r(\sigma, t)$ and $u_\theta(\theta, t) = 0$.

Since we know that $|u_\theta(\theta, t)|$ and $|u_{\theta\theta}(\theta, t)|$ are both uniformly bounded on $S^1 \times [0, \infty)$ (see [Chow and Tsai 1996]) and $u(\theta, t) \to \infty$ uniformly, we have $\lim_{t\to\infty} \sigma(\theta, t) = \theta$ uniformly on S^1 and

$$\lim_{t\to\infty} \frac{\partial\sigma}{\partial\theta}(\theta, t) = \lim_{t\to\infty} \frac{u(\theta, t)(u_{\theta\theta}(\theta, t) + u(\theta, t))}{u^2(\theta, t) + u_\theta^2(\theta, t)} = 1 \tag{33}$$

uniformly on S^1. Since $u_\theta(\theta, t) = r(\sigma, t)\sin(\sigma - \theta)$, we have

$$\begin{aligned}(34)\quad & R(t)\sin(\sigma-\theta)\\ &= u_\theta(\theta,t)-(r(\sigma,t)-R(t)-\ell(\sigma))\sin(\sigma-\theta)-\ell(\sigma)\sin(\sigma-\theta).\end{aligned}$$

This implies that $|R(t)\sin(\sigma-\theta)|$ is also uniformly bounded on $S^1\times[0,\infty)$. Now, by (24), we conclude

$$\begin{aligned}(35)\quad & \lim_{t\to\infty}(u(\theta,t)-R(t))\\ &= \lim_{t\to\infty}\big((r(\sigma,t)-R(t)-\ell(\sigma))\cos(\sigma-\theta)+(R(t)+\ell(\sigma))\cos(\sigma-\theta)-R(t)\big)\\ &= \lim_{t\to\infty}\left((R(t)\sin(\sigma-\theta))\frac{\sigma-\theta}{\sin(\sigma-\theta)}\frac{\cos(\sigma-\theta)-1}{\sigma-\theta}\right)+\ell(\theta)\\ &= \ell(\theta),\end{aligned}$$

uniformly on S^1. We next claim that $u_\theta(\theta,t)\to\ell'(\theta)$ uniformly on S^1 as $t\to\infty$. Apply $\partial/\partial\theta$ to $u(\theta,t)$ and use the chain rule to get

$$u_\theta(\theta,t)=r_\sigma(\sigma,t)\frac{\partial\sigma}{\partial\theta}\cos(\sigma-\theta)-r(\sigma,t)\Big(\frac{\partial\sigma}{\partial\theta}(\theta,t)-1\Big)\sin(\sigma-\theta),\quad \sigma=\sigma(\theta,t),$$

and hence

$$\lim_{t\to\infty}u_\theta(\theta,t)=\ell'(\theta) \tag{36}$$

uniformly on S^1 due to (33) and (24). Since

$$\begin{aligned}\|u(\theta,t)-r(\sigma,t)\|_{C^1(S^1)} &\le \|u(\theta,t)-(R(t)+\ell(\theta))\|_{C^1(S^1)}+\|\ell(\theta)-\ell(\sigma)\|_{C^1(S^1)}\\ &\quad+\|(\ell(\sigma)+R(t))-r(\sigma,t)\|_{C^1(S^1)}\end{aligned}$$

we have

$$\lim_{t\to\infty}\|u(\theta,t)-r(\sigma,t)\|_{C^1(S^1)}=0,\quad \sigma=\sigma(\theta,t), \tag{37}$$

where the C^1 norm is taken with respect to $\theta\in S^1$. By (36) and (34), we also have

$$\begin{aligned}\lim_{t\to\infty}R(t)(\sigma-\theta) &= \lim_{t\to\infty}R(t)\sin(\sigma-\theta)\\ &= \ell'(\theta)\end{aligned} \tag{38}$$

uniformly on S^1.

As a consequence of (11) we have

$$\begin{aligned}(39)\quad & \lim_{t\to\infty}D(t)\\ &= \lim_{t\to\infty}\left[\left(\int_0^{2\pi}u(\theta,t)\,d\theta\right)^2-2\pi\int_0^{2\pi}u^2(\theta,t)\,d\theta\right]+2\pi\int_0^{2\pi}(\ell'(\theta))^2\,d\theta.\end{aligned}$$

Since we have the identity

$$\begin{aligned}(40)\quad \left(\int_0^{2\pi} u(\theta,t)\,d\theta\right)^2 - 2\pi\int_0^{2\pi} u^2(\theta,t)\,d\theta \\ = \left(\int_0^{2\pi}(u(\theta,t)-R(t))\,d\theta\right)^2 - 2\pi\int_0^{2\pi}(u(\theta,t)-R(t))^2\,d\theta,\end{aligned}$$

which is due to the fact that any two parallel convex closed curves have the same isoperimetric difference, we conclude

$$(41)\quad \lim_{t\to\infty} D(t) = \left(\int_0^{2\pi} \ell(\theta)\,d\theta\right)^2 - 2\pi\int_0^{2\pi}\left(\ell^2(\theta)-(\ell'(\theta))^2\right)d\theta \ge 0. \qquad \square$$

Remark 3.5. By (30) and (28) the position vector $P(\theta,t)$ of γ_t satisfies

$$\lim_{t\to\infty} |P(\theta,t)-Q(\theta,t)| = 0$$

uniformly on S^1, where

$$(42)\quad Q(\theta,t) = R(t)(\cos\theta,\sin\theta)+\ell(\theta)(\cos\theta,\sin\theta)+\ell_\theta(\theta)(-\sin\theta,\cos\theta)$$

and it represents a family of expanding circles centered at $(a,b)\in\mathbb{R}^2$ if and only if $\ell(\theta)$ is given by

$$(43)\quad \ell(\theta) = c + a\cos\theta + b\sin\theta, \quad \theta\in S^1$$

for some constants a, b, c. Also note that by the classical Minkowski inequality the right side of (41) is zero if and only if $\ell(\theta)$ has the form (43). So $\lim_{t\to\infty} D(t)=0$ if and only if $P(\theta,t)$ is asymptotically close to a family of expanding circles centered at some $(a,b)\in\mathbb{R}^2$, which can be evaluated by the integral

$$(44)\quad (a,b) = \lim_{t\to\infty}\frac{1}{2\pi}\int_0^{2\pi} P(\theta,t)\,d\theta = \frac{1}{\pi}\int_0^{2\pi}\ell(\theta)(\cos\theta,\sin\theta)\,d\theta.$$

On the other hand, $\lim_{t\to\infty} D(t) = d > 0$ if and only if $P(\theta,t)$ is asymptotically close to a family of *noncircular* parallel curves (described by $Q(\theta,t)$) expanding to infinity. This family of parallel curves have the same fixed *center* (now "*center*" means "*average position vector*") given by (44). The speed of this family of parallel curves is

$$\frac{dR}{dt}(t) = 1-\frac{1}{R(t)} \to 1$$

as $t\to\infty$. Therefore, asymptotically, it is the *unit-speed outward normal flow*.

To go further, we need the following ODE result.

Lemma 3.6. *For any constant $c \in \mathbb{R}$ there exists a positive solution $s(t)$ to the ODE*

$$\frac{ds}{dt} = 1 - \frac{1}{s}$$

defined on some interval $[T, \infty)$, $T \geq 0$, $s(T) \geq 2$, such that

$$\lim_{t\to\infty} (s(t) - R(t)) = c.$$

Proof. Assume first that $c > 0$. Let $s(t) = R(t+c)$, $t \in [0, \infty)$. It satisfies the same ODE with $s(0) = R(c) > R(0) = 2$. Now

$$\lim_{t\to\infty} (s(t) - R(t)) = \lim_{t\to\infty} \int_t^{t+c} R'(z)\, dz = \lim_{t\to\infty} \int_t^{t+c} \Big(1 - \frac{1}{R(z)}\Big)\, dz = c.$$

For $c < 0$, let $s(t) = R(t+c)$, $t \in [-c, \infty)$. Then $s(t)$ is a positive solution to the ODE on $[-c, \infty)$ with $s(-c) = R(0) = 2$ and

$$\lim_{t\to\infty} (R(t) - s(t)) = \lim_{t\to\infty} \int_{t+c}^{t} R'(z)\, dz = \lim_{t\to\infty} \int_{t+c}^{t} \Big(1 - \frac{1}{R(z)}\Big)\, dz = -c. \qquad \square$$

Our next result is a property about uniqueness.

Lemma 3.7. *If γ_t expands to infinity under the flow* (1), $\lim_{t\to\infty} D(t) = 0$ *if and only if γ_0 is a circle. Therefore if γ_0 is not a circle, $D(t)$ will decrease to a positive constant as $t \to \infty$.*

Proof. Assume that $\lim_{t\to\infty} D(t) = 0$. Then $\ell(\theta) = c + a\cos\theta + b\sin\theta$ in (28), and by Lemma 3.6 there exists a positive solution $s(t)$ to the ODE on some interval $[T, \infty)$, $T \geq 0$, $s(T) \geq 2$, such that

$$\lim_{t\to\infty} \|u(\theta, t) - (s(t) + a\cos\theta + b\sin\theta)\|_{C^1(S^1)} = 0.$$

Now if at time T we consider a circle C_T centered at (a, b) with radius $s(T)$ and evolve it under the flow (1), its support function $U(\theta, t)$ will satisfy

$$U(\theta, t) = s(t) + a\cos\theta + b\sin\theta,$$

$(\theta, t) \in S^1 \times [T, \infty)$ (note that this $U(\theta, t)$ satisfies Equation (10)). By previous discussions, the radial function $r_1(\sigma, t)$ of the evolving curve γ_t (with support function $u(\theta, t)$) and the radial function $r_2(\sigma, t)$ of the evolving circle C_t (with support function $U(\theta, t)$) on the domain $S^1 \times [T, \infty)$ will satisfy the estimate

$$\lim_{t\to\infty} \|r_1(\sigma, t) - r_2(\sigma, t)\|_{C^1(S^1)} = 0, \tag{45}$$

and we can apply Theorem 3 of [Yagisita 2005] to conclude that $\gamma_t \equiv C_t$ for all $t \in [T, \infty)$. In particular γ_T is also a circle. But this is impossible unless γ_0 is a circle. $\square$

Remark 3.8. One can also apply *Andrews' backward uniqueness result* [2002], which follows, to prove the above lemma: Assume $v(\theta, t)$ is a smooth solution to the *uniformly parabolic* equation

$$\frac{\partial v}{\partial t} = a(\theta,t)v_{\theta\theta} + b(\theta,t)v_\theta + c(\theta,t)v, \quad (\theta,t) \in S^1 \times [0,T], \tag{46}$$

where a, b, c are smooth functions on $S^1 \times [0, T]$ with

$$\frac{1}{C} \le a(\theta,t) \le C, \quad (\theta,t) \in S^1 \times [0,T],$$

for some positive constant $C > 0$. If $v(\theta, T) = 0$ for all $\theta \in S^1$, we have $v(\theta, 0) = 0$ for all $\theta \in S^1$ (in particular, $v(\theta, t) \equiv 0$ for all (θ, t)). Now by (24), in terms of the variable $(\sigma, \tau) \in S^1 \times [-\log 2, 0)$ in Remark 3.2, the bounded function $A(\sigma, \tau) = r_1(\sigma, t) - R(t)$ can be smoothly extended to $S^1 \times [-\log 2, 0]$ even though the function $R(t(\tau)) = 1/(1 - e^\tau)$ is undefined at $\tau = 0$. The full equation for (26) is (see the top equation on page 229 of [Yagisita 2005])

$$\begin{aligned}\frac{\partial A}{\partial \tau}(\sigma,\tau) = {} & \frac{1}{C(\sigma,\tau)} A_{\sigma\sigma}(\sigma,\tau) + \frac{1}{D^2(\sigma,\tau) + D(\sigma,\tau)\sqrt{C(\sigma,\tau)}} A_\sigma^2(\sigma,\tau) \\ & + \frac{1}{C(\sigma,\tau)} \Big[D(\sigma,\tau)A(\sigma,\tau) - (1-e^\tau)\Big(\frac{2}{D(\sigma,\tau)} - 1\Big) A_\sigma^2(\sigma,\tau) \Big],\end{aligned} \tag{47}$$

where

$$\begin{aligned} C(\sigma,\tau) &= [A(\sigma,\tau)(1-e^\tau)+1]^2 + A_\sigma^2(\sigma,\tau)(1-e^\tau)^2, \\ D(\sigma,\tau) &= A(\sigma,\tau)(1-e^\tau)+1. \end{aligned}$$

At $\tau = 0$, we have $C(\sigma, \tau) = D(\sigma, \tau) = 1$, which implies that (47) is a *uniformly parabolic* equation with *smooth* coefficients on $S^1 \times [-\delta, 0]$ for some small $\delta > 0$. Moreover, the smooth function

$$w(\sigma,\tau) := (r_1(\sigma,t) - R(t)) - (r_2(\sigma,t) - R(t)), \quad (\sigma,\tau) \in S^1 \times [-\delta, 0],$$

satisfies a uniformly parabolic equation of the form

$$\frac{\partial w}{\partial \tau} = a(\sigma,\tau)w_{\sigma\sigma} + b(\sigma,\tau)w_\sigma + c(\sigma,\tau)w$$

with coefficients smooth on $S^1 \times [-\delta, 0]$, and, by (45), $w(\sigma, 0) = 0$ for all $\sigma \in S^1$. Andrews' result implies $w(\sigma, \tau) \equiv 0$ and the initial curve γ_0 must be a circle.

Lemma 3.9. *For any number $d \ge 0$ and any small $\varepsilon > 0$, one can construct an expanding $k-1$ flow so that its isoperimetric difference $D(t)$ satisfies $|D(t) - d| < \varepsilon$ as long as t is large enough.*

Proof. For any $d \geq 0$ one can find a convex closed curve γ with support function $\ell(\theta)$ such that

$$d = L^2 - 4\pi A = \left(\int_0^{2\pi} \ell(\theta)\, d\theta \right)^2 - 2\pi \int_0^{2\pi} \left(\ell^2(\theta) - (\ell'(\theta))^2 \right) d\theta \geq 0.$$

Following the proof of Theorem 4 of [Yagisita 2005], we can obtain the following: For any smooth function $\ell(\theta)$ defined on S^1 and any small $\delta > 0$, there exists a large time $M > 0$ such that if $T > M$ and γ_T is a convex closed curve with support function $R(T) + \ell(\theta)$, the support function $u(\theta, t)$ of γ_t (γ_t is the evolution of γ_T under (1) on time interval $[T, \infty)$) will satisfy

$$\sup_{t \in [T,\infty)} \| u(\theta, t) - (R(t) + \ell(\theta)) \|_{C^0(S^1)} < \delta. \tag{48}$$

This says that γ_t is close to a parallel curve of γ_T, which is intuitively correct.

For the isoperimetric difference of γ_t we need to be careful, because now the norm in (48) is only in C^0 norm [Yagisita 2005, Proof of Theorem 4], and by (11) we need to know the behavior of $u_\theta(\theta, t)$ in order to control the area. However, there is a result on page 53 of [Schneider 1993], which says that if two compact convex sets K_1, K_2 in $\mathbb{R}^2$ have their support functions $u_1(\theta)$, $u_2(\theta)$ close to each other, their *Hausdorff distance* is also close to each other. In particular, their lengths and areas are also close to each other. But be careful again that the two families of curves γ_t and p_t (p_t is the parallel curve of γ_T with support function $R(t)+\ell(\theta)$ for $t \in [T, \infty)$) are expanding to infinity as $t \to \infty$, so even if their Hausdorff distance is less than δ, $|D(t) - d|$ may not be small as $t \to \infty$ (however, $|D(t) - d| \leq d$ since $D(t) \geq 0$ is decreasing on $[T, \infty)$ with $D(T) = d$). To overcome this we can write (48) as

$$C + \ell(\theta) - \delta \leq u(\theta, t) - R(t) + C \leq C + \ell(\theta) + \delta, \quad (\theta, t) \in S^1 \times [T, \infty),$$

where $C > 0$ is a constant with (recall that $|u_{\theta\theta}(\theta, t)|$ is uniformly bounded on $S^1 \times [T, \infty)$ by [Chow and Tsai 1996])

$$\begin{aligned} &(C + \ell(\theta))_{\theta\theta} + (C + \ell(\theta)) > 0, \\ &(u(\theta, t) - R(t) + C)_{\theta\theta} + (u(\theta, t) - R(t) + C) > 0. \end{aligned} \tag{49}$$

for all $(\theta, t) \in S^1 \times [T, \infty)$. Equation (49) implies the existence of a convex closed curve C_1 with support function $C + \ell(\theta)$ and a convex closed curve $C_2(t)$ with support function $u(\theta, t) - R(t) + C$, where the support function of $C_2(t)$ is close to the support function of C_1 for all $t \in [T, \infty)$. Moreover, the curves C_1 and $C_2(t)$ are both enclosed by two parallel convex curves $C_\pm$ with support functions $C + \ell(\theta) + \delta$ and $C + \ell(\theta) - \delta$, respectively. However, we worry about the situation where, when δ is getting smaller and smaller, the constant C may be getting larger and larger.

We claim that the constant C in (49) can be chosen to be independent of δ. If we let δ tends to zero, the time T in (48) will tend to infinity and the initial value $u(\theta, T) = R(T) + \ell(\theta)$ will also tend to infinity. However, $u_\theta(\theta, T) = \ell'(\theta)$ and $u_{\theta\theta}(\theta, T) = \ell''(\theta)$ are unaffected by T. From the proofs of Proposition 1 and Lemma 4 in [Chow and Tsai 1996], one can see that $|u_\theta(\theta, t)|$ and $|u_{\theta\theta}(\theta, t)|$ are both uniformly bounded on $S^1 \times [T, \infty)$ and the bounds are independent of T. This, together with (48), implies that $C > -u_{\theta\theta}(\theta, t) + R(t) - u(\theta, t)$ is independent of T (and δ).

As $\delta \to 0$, the curve C_1 is unchanged and the Hausdorff distance between $C_2(t)$ and C_1 is getting smaller. We note that the isoperimetric difference of C_1 is given by d and the isoperimetric difference of $C_2(t)$ is the same as $D(t)$ of γ_t. Therefore, for any small $\varepsilon > 0$, by making $\delta > 0$ as small as possible, one can construct an expanding $k - 1$ flow satisfying $|D(t) - d| < \varepsilon$ as long as t is large enough. □

Remark 3.10. In the above proof we use the fact that any smooth function $h(\theta)$ defined on S^1, satisfying $h''(\theta) + h(\theta) > 0$ for all $\theta \in S^1$, is the support function of some convex closed curve.

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