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**LOWER ESTIMATE OF MILNOR NUMBER AND CHARACTERIZATION OF ISOLATED HOMOGENEOUS HYPERSURFACE SINGULARITIES**


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# LOWER ESTIMATE OF MILNOR NUMBER AND CHARACTERIZATION OF ISOLATED HOMOGENEOUS HYPERSURFACE SINGULARITIES

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**Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{ z \in \mathbb{C}^n : f(z) = 0 \}$. A beautiful theorem of Saito [1971] gives a necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial. It is a natural and important question to characterize (up to a biholomorphic change of coordinates) a homogeneous polynomial with an isolated critical point at the origin. For a two-dimensional isolated hypersurface singularity $V$, Xu and Yau [1992; 1993] found a coordinate-free characterization for $V$ to be defined by a homogeneous polynomial. Lin and Yau [2004] and Chen, Lin, Yau, and Zuo [2001] gave necessary and sufficient conditions for 3- and 4-dimensional isolated hypersurface singularities with $p_g \geq 0$ and $p_g > 0$, respectively. However, it is quite difficult to generalize their methods to give characterization of homogeneous polynomials. In 2005, Yau formulated the Yau Conjecture 1.1: (1) Let $\mu$ and $\nu$ be the Milnor number and multiplicity of $(V, 0)$, respectively. Then $\mu \geq (\nu - 1)^n$, and the equality holds if and only if $f$ is a semihomogeneous function. (2) If $f$ is a quasihomogeneous function, then $\mu = (\nu - 1)^n$ if and only if $f$ is a homogeneous polynomial after change of coordinates. In this paper we solve part (1) of Yau Conjecture 1.1 for general $n$. We introduce a new method, which allows us to solve the part (2) of Yau Conjecture 1.1 for $n = 5$ and 6. As a result we have shown that for $n = 5$ or 6, $f$ is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^n$. As a by-product we have also proved Yau Conjecture 1.2 in some special cases.**

## 1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{ z \in \mathbb{C}^n : f(z) = 0 \}$. It is a natural question to ask when $V$ is defined by a weighted homogeneous polynomial or a homogeneous

Dedicated to Professor Banghe Li on the occasion of his 70th birthday.

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polynomial up to biholomorphic change of coordinates. Recall that the multiplicity of the singularity $V$ is defined to be the order of the lowest nonvanishing term in the power series Taylor expansion of $f$ at 0, and the Milnor number $\mu$ and the Tjurina number $\tau$ of the singularity $(V, 0)$ are defined respectively by

$$\mu = \dim \mathbb{C}\{z_1, z_2, \dots, z_n\}/(f_{z_1}, \dots, f_{z_n}),$$
$$\tau = \dim \mathbb{C}\{z_1, z_2, \dots, z_n\}/(f, f_{z_1}, \dots, f_{z_n}).$$

The following theorem gives a necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial:

**Theorem 1.1** [Saito 1971]. *The function $f$ is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$.*

Let $\pi : (M, A) \to (V, 0)$ be a resolution of singularity with exceptional set $A = \pi^{-1}(0)$. The geometric genus $p_g$ of the singularity $(V, 0)$ is the dimension of $H^{n-2}(M, \mathbb{O})$ and is independent of the resolution $M$. Xu and Yau [1993] gave necessary and sufficient conditions for a 2-dimensional $V$ to be defined by a homogeneous polynomial.

**Theorem 1.2** [Xu and Yau 1993]. *Let $(V, 0)$ be a 2-dimensional isolated hypersurface singularity defined by a holomorphic function $f(z_1, z_2, z_3) = 0$. Let $\mu$ be the Milnor number, $\tau$ the Tjurina number, $p_g$ the geometric genus, and $\nu$ the multiplicity of the singularity. Then $f$ is a homogeneous polynomial after a biholomorphic change of variables if and only if $\mu = \tau$ and $\mu - \nu + 1 = 6p_g$.*

Based on above theorem, a conjecture was made by Yau in 2005 as follows:

**Yau Conjecture 1.1** [Lin et al. 2006b]. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{ z : f(z) = 0 \}$ at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of $(V, 0)$, respectively. Then*

$$\mu \geq (\nu - 1)^n, \tag{1-1}$$

*and equality holds if and only if $f$ is a semihomogeneous function (i.e., $f = f_\nu + g$, where $f_\nu$ is a nondegenerate homogeneous polynomial of degree $\nu$ and $g$ consists of terms of degree at least $\nu + 1$) after a biholomorphic change of coordinates. Furthermore, if $f$ is a quasihomogeneous function, i.e., $f \in (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$, then the equality in (1-1) holds if and only if $f$ is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Yau Conjecture 1.2** [Chen et al. 2011]. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let $\mu$, $p_g$, and $\nu$ be the Milnor number, geometric genus, and multiplicity of the singularity $V = \{ z : f(z) = 0 \}$. Then*

$$\mu - p(\nu) \geq n!\, p_g,$$

*where* $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \ldots (\nu - n + 1)$, *and equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

These conjectures are sharp estimates and have some important applications in geometry. The Yau conjectures were proved only for very low dimensional singularities. For Yau Conjecture 1.1, Lin, Wu, Yau, and Luk proved the following two theorems:

**Theorem 1.3** [Lin et al. 2006b]. *Let* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *be a germ of a holomorphic function defining an isolated plane curve singularity* $V = \{ z \in \mathbb{C}^2 : f(z) = 0 \}$ *at the origin. Let* $\mu$ *and* $\nu$ *be the Milnor number and multiplicity of* $(V, 0)$, *respectively. Then*

$$\mu \geq (\nu - 1)^2.$$

*Furthermore, if* $V$ *has at most two irreducible branches at the origin, or if* $f$ *is a quasihomogeneous function, then equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Theorem 1.4** [Lin et al. 2006b]. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a germ of a holomorphic function defining an isolated hypersurface singularity* $V = \{ z \in \mathbb{C}^n : f(z) = 0 \}$ *at the origin. Let* $\mu$, $\nu$, *and* $\tau = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f, \partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ *be the Milnor number, multiplicity, and Tjurina number of* $(V, 0)$, *respectively. Suppose* $\mu = \tau$ *and* $n$ *is either* 3 *or* 4. *Then*

$$\mu \geq (\nu - 1)^n,$$

*and equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

For Yau Conjecture 1.2, Lin, Tu, and Yau have the following theorem:

**Theorem 1.5** [Lin and Yau 2004; Lin et al. 2006a]. *Let* $(V, 0)$ *be a* 3*-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial* $f(x, y, z, w) = 0$. *Let* $\mu$ *be the Milnor number,* $p_g$ *the geometric genus, and* $\nu$ *the multiplicity of the singularity. Then*

$$\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geq 4!\, p_g,$$

*and equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

*Remark.* The above theorem is proved in [Lin and Yau 2004] with $p_g > 0$. For $p_g = 0$, the theorem is proved in [Lin et al. 2006a].

**Corollary 1.1.** *Let* $(V, 0)$ *be a* 3*-dimensional isolated hypersurface singularity defined by a polynomial* $f(x, y, z, w) = 0$. *Let* $\mu$, $p_g$, $\nu$, *and* $\tau$ *be the Milnor number, geometric genus, multiplicity, and Tjurina number of the singularity,* *respectively. Then $f$ is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$ and $\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 4!\, p_g$.*

Recently, Chen, Lin, Yau, and Zuo [Chen et al. 2011] generalized the above theorem to any 4-dimensional isolated hypersurface singularity with an additional assumption $p_g > 0$.

**Theorem 1.6** [Chen et al. 2011]. *Let $(V, 0)$ be a 4-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial $f(x, y, z, w, t) = 0$. Let $\mu$ be the Milnor number, $p_g$ the geometric genus, and $\nu$ the multiplicity of the singularity. If $p_g > 0$, then*

$$\mu - [(\nu-1)^5 + \nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)] \geq 5!\, p_g,$$

*and equality holds if and only if $f$ is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Corollary 1.2.** *Let $(V, 0)$ be a 4-dimensional isolated hypersurface singularity defined by a polynomial $f(x, y, z, w, t) = 0$. Let $\mu$, $p_g$, $\nu$, and $\tau$ be the Milnor number, geometric genus, multiplicity, and Tjurina number of the singularity, respectively. Moreover, if $p_g > 0$, then $f$ is a homogeneous polynomial after a biholomorphic change of coordinate if and only if $\mu = \tau$ and*

$$\mu - [(\nu-1)^5 + \nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)] = 5!\, p_g.$$

The purpose of this paper is to prove the following results:

**Proposition A.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z : f(z) = 0\}$ at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of $(V, 0)$, respectively. Then*

$$\mu \geq (\nu-1)^n,$$

*and equality holds if and only if $f$ is a semihomogeneous function (i.e., $f = f_\nu + g$, where $f$ is a nondegenerate homogeneous polynomial of degree $\nu$ and $g$ consists of terms of degree at least $\nu + 1$) after a biholomorphic change of coordinates.*

**Theorem B.** *Let $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$, where $k$ is either 5 or 6, be a weighted homogeneous polynomial with an isolated singularity at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of the singularity $V = \{z : f(z) = 0\}$, respectively. Then*

$$\mu \geq (\nu-1)^k,$$

*and equality holds if and only if $f$ is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Theorem C.** *Let* $f : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let* $\mu$, $p_g$, *and* $\nu$ *be the Milnor number, geometric genus, and multiplicity of the singularity* $V = \{ z : f(z) = 0 \}$. *Then*

$$\mu - p(\nu) \geq 5!\, p_g,$$

*where* $p(\nu) = (\nu-1)^5 - \nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)$, *and equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Theorem D.** *Let* $f : (\mathbb{C}^6, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let* $\mu$, $p_g$, *and* $\nu$ *be the Milnor number, geometric genus, and multiplicity of the singularity* $V = \{ z : f(z) = 0 \}$. *If* $p_g = 0$, *then*

$$\mu - p(\nu) \geq 6!\, p_g,$$

*where* $p(\nu) = (\nu-1)^6 - \nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)$ (*which equals* 0), *and equality holds if and only if* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates.*

**Corollary E.** *Let* $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$, *where* $k$ *is either* 5 *or* 6, *be a polynomial with an isolated singularity at the origin. Let* $\mu$, $\tau$, *and* $\nu$ *be the Milnor number, Tjurina number, and multiplicity of the singularity* $V = \{ z : f(z) = 0 \}$, *respectively. Then* $f$ *is a homogeneous polynomial after a biholomorphic change of coordinates if and only if* $\mu = \tau = (\nu - 1)^k$.

In Section 2, we recall the necessary materials needed to prove the main theorems. In Section 3, we prove the main theorems.

## 2. Preliminary

In this section, we recall some known results that are needed to prove the main theorems. Let $f(z_1, \dots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose $f$ has an isolated critical point at the origin. It can be developed in a convergent Taylor series $f(z_1, \dots, z_n) = \sum a_\lambda z^\lambda$, where $z^\lambda = z_1^{\lambda_1} \dots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of subsets $\{\lambda + \mathbb{R}^n_+\}$ for $\lambda$ such that $a_\lambda \neq 0$. Let $\Gamma_-(f)$, the Newton polyhedron of $f$, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face $\Delta$ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that $f$ is nondegenerate if $f_\Delta$ has no critical point in $(\mathbb{C}^*)^n$ for any $\Delta \in \Gamma(f)$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. We say that a point $p$ of the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ is positive if all coordinates of $p$ are positive. The following beautiful theorem holds:

**Theorem 2.1** [Merle and Teissier 1980]. *Let* $(V, 0)$ *be an isolated hypersurface singularity defined by a nondegenerate holomorphic function* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. *Then the geometric genus* $p_g = \#\{ p \in \mathbb{Z}^n \cap \Gamma_-(f) : p \text{ is positive} \}$.

A polynomial $f(z_1, \dots, z_n)$ is weighted homogeneous of type $(w_1, \dots, w_n)$, where $w_1, \dots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_1^{i_1} \dots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. As a consequence of the theorem of Merle–Teissier, for isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in the tetrahedron defined by $x_1/w_1 + \cdots + x_n/w_n \le 1$ and $x_1 \ge 0, \dots, x_n \ge 0$. We also need the following result:

**Theorem 2.2** [Milnor and Orlik 1970]. *Let $f(z_1, \dots, z_n)$ be a weighted homogeneous polynomial of type $(w_1, \dots, w_n)$ with isolated singularity at the origin. Then the Milnor number is $\mu = (w_1 - 1) \dots (w_n - 1)$.*

The following theorem is about the relation of weight and multiplicity:

**Theorem 2.3** [Sękalski 2008]. *If $f$ is a quasihomogeneous isolated singularity of type $(\omega_1, \dots, \omega_n)$, then* $\mathrm{mult}(f) = \min\{ m \in \mathbb{N} : m \ge \min\{ \omega_i : i = 1, \dots, n \} \}$.

There is a lower bound for the $p_g$ of a hypersurface singularity:

**Theorem 2.4** [Yau 1977]. *Let*

$$f(z_1, \dots, z_{n-1}, z_n) = z_n^m + a_1(z_1, \dots, z_{n-1}) z_n^{m-1} + \cdots + a_m(z_1, \dots, z_{n-1})$$

*be holomorphic near $(0, \dots, 0)$. Let $d_i$ be the order of the zero of $a_i(z_1, \dots, z_{n-1})$ at $(0, \dots, 0)$ with $d_i \ge i$. Let $d = \min_{1 \le i \le m}(d_i/i)$. Suppose that*

$$V = \{ (z_1, \dots, z_n) : f(z_1, \dots, z_n) = 0 \},$$

*defined in a suitably small polydisc, has $p = (0, \dots, 0)$ as its only singularity. Let $\pi : M \to V$ be resolution of $V$. Then* $\dim H^{n-2}(M, \mathbb{O}) > (m-1)d - (n-1)$.

In the following theorem, it is convenient for us to use another definition of weight type. Let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ define an isolated singularity at the origin. Let $w = (w_1, \dots, w_n)$ be a weight on the coordinates $(z_1, \dots, z_n)$ by positive integer numbers $w_i$ for $i = 1, \dots, n$. We have the weighted Taylor expansion $f = f_\rho + f_{\rho+1} + \cdots$ with respect to $w$ and $f_\rho \ne 0$, where $f_k$ is a weighted homogeneous of type $(w_1, \dots, w_n; k)$ for $k \ge \rho$, i.e., $f_k$ is linear combination of monomials $z_1^{i_1} \dots z_n^{i_n}$ for which $i_1 w_1 + \cdots + i_n w_n = k$. We only use this definition of weight for the following theorem as well as in the proof of Proposition A. For any other place we use the previous definition before Theorem 2.2 for weight type.

**Theorem 2.5** [Furuya and Tomari 2004]. *Let the situation be as above, and let $f \in \mathbb{C}\{z_1, \dots, z_n\}$ define an isolated singularity at the origin. Then*

$$\mu(f) \ge \left(\frac{\rho}{w_1} - 1\right) \dots \left(\frac{\rho}{w_n} - 1\right), \tag{2-1}$$

*and equality holds if and only if $f_\rho$ defines an isolated singularity at the origin.*

Here we recall that $f$ is called a semiquasihomogeneous function if the initial term $f_\rho$ defines an isolated singularity at the origin.

**Definition 2.1.** Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions defining the respective isolated hypersurface singularities $V_f = \{ z : f(z) = 0 \}$ and $V_g = \{ z : g(z) = 0 \}$. Let $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ of biholomorphic map.

(1) If $\phi(V_f) = V_g$, then $f$ is *contact equivalent* to $g$.

(2) If $g = f \circ \phi$, then $f$ is *right equivalent* to $g$.

The Milnor number is an invariant of contact equivalence [Teissier 1975].

## 3. Proof of the main theorems

*Proof of Proposition A.* Let $f(z_1, \dots, z_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of the singularity $V = \{ z : f(z) = 0 \}$. By an analytic change of coordinates, one can assume that the $z_n$-axis is not contained in the tangent cones of $V$ so that $f(0, \dots, 0, z_n) \neq 0$. By the Weierstrass preparation theorem, near 0, the germ $f$ can be represented as a product

$$f(z_1, \dots, z_n) = u(z_1, \dots, z_n)\, g(z_1, \dots, z_n),$$

where $u(0, \dots, 0) \neq 0$, and

$$g(z_1, \dots, z_{n-1}, z_n) = z_n^\nu + a_1(z_1, \dots, z_{n-1}) z_n^{\nu-1} + \dots + a_\nu(z_1, \dots, z_{n-1}),$$

where $\nu$ is the multiplicity of $f(z_1, \dots, z_n)$ and $a_i \in (x_1, \dots, x_{n-1})^i$ for $i = 1, \dots, \nu$. Therefore, $f(z_1, \dots, z_n)$ is contact equivalent to $g(z_1, \dots, z_n)$.

Let $d_i$ be the order of the zero of $a_i(z_1, \dots, z_{n-1})$ at $(0, \dots, 0)$, $d_i \geq i$. Let $d = \min_{1 \leq i \leq \nu} [d_i / i]$, so $d \geq 1$. We define a weight $w$ on the new coordinate systems by $w(z_n) = d$ with $w(z_i) = 1$ for $1 \leq i \leq n-1$. Here the definition of weight type is the same as in Theorem 2.5. With respect to the new weights, $z_n^\nu$ has degree $d\nu$, and $a_i(z_1, \dots, z_{n-1}) z_n^{\nu-i}$ has degree at least $d(\nu - i) + d_i \geq d\nu - di + di = d\nu$. Thus, the initial term of $f(z_1, \dots, z_n)$ has the degree $\rho = d\nu$. Because the Milnor number is an invariant under contact equivalence, by (2-1) we have $\mu = \mu(g) \geq (d\nu/d - 1)(d\nu/1 - 1) \dots (d\nu/1 - 1) = (\nu - 1)(d\nu - 1)^{n-1} \geq (\nu - 1)^n$.

Suppose $f$ is a semihomogeneous polynomial. Since the Milnor number of $f$ is the same as its initial part (see [Arnold 1974]), $\mu = (\nu - 1)^n$ is obvious.

If $\mu = (\nu - 1)^n$, then by $\mu \geq (\nu - 1)(d\nu - 1)^{n-1} \geq (\nu - 1)^n$, we have $d = 1$, and by the last part of Theorem 2.5, $g_{d\nu}(z_1, \dots, z_n) = g_\nu(z_1, \dots, z_n)$ is a homogeneous polynomial of degree $\nu$ defining an isolated singularity. Hence, $f(z_1, \dots, z_n)$ is contact equivalent to a semihomogeneous singularity. □

We prove a lemma that is useful in the proof of Theorem B.

**Lemma 3.1.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial of weight type $(w_1, \dots, w_n)$ with an isolated singularity at the origin. If $w_i$ is not an integer, $z_i^{a_i} z_{j_i} \in \operatorname{supp}(f)$, where $a_i$ is a positive integer, $j_i \neq i$, and $[w_i] < \nu$, then $\nu = a_i + 1$ and $w_i/w_{j_i} \neq 1$.*

*Proof.* Since $z_i^{a_i} z_{j_i} \in \operatorname{supp}(f)$, $a_i/w_i + 1/w_{j_i} = 1$. It follows from the fact that $w_i$ is not an integer that $w_i/w_{j_i} \neq 1$, and $a_i/w_i + 1/w_{j_i} = 1$ implies that $w_i > a_i$. Since $[w_i] < \nu$, by Theorem 2.3, we have $\nu = [w_i] + 1 \geq a_i + 1$. By the definition of multiplicity, we also have $\nu \leq a_i + 1$. Therefore, $\nu = a_i + 1$. □

*Proof of Theorem B.* We shall give a detailed proof for $k = 5$.

Let $f : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of the singularity $V = \{ z : f(z) = 0 \}$, respectively. We want to show $\mu \geq (\nu - 1)^5$ and that the equality holds if and only if $f$ is a homogeneous polynomial. By Proposition A, it suffices to show that equality holds if and only if $f$ is a homogeneous polynomial. Set $w(z_i) = w_i$ for $1 \leq i \leq 5$. We assume that $2 \leq w_1 \leq \min\{w_2, \dots, w_5\}$, where $w_i$ for $i = 1, \dots, 5$ are positive rational numbers, without loss of generality.

If $\nu = 2$, then the theorem is trivial by the Milnor–Orlik formula (Theorem 2.2). In the following, we only consider $\nu \geq 3$ or, equivalently, $w_1 > 2$.

If $w_1$ is an integer, then by Theorem 2.3, $\nu = w_1$. Since $\mu = (w_1 - 1) \dots (w_5 - 1)$, $\mu = (\nu - 1)^5$ if and only if $w_1 = w_2 = \cdots = w_5$, i.e., $f$ is an homogeneous polynomial.

If $w_1$ is not an integer, by Theorem 2.3, $\nu = [w_1] + 1$, where $[w_1]$ denotes the integer part of $w_1$. We want to show that $\mu > (\nu - 1)^5$. Since $f$ is an isolated singularity, for every $i \in \{1, \dots, 5\}$, either $z_i^{a_i}$ or $z_i^{a_i} z_j$ is in the support of $f$, where $j \neq i$ and $a_i$ is a positive integer. By assumption, $w_1$ is not an integer, so $z_1^{a_1} z_{j_1} \in \operatorname{supp}(f)$. By Lemma 3.1, we have $\nu = a_1 + 1$. We shall show that $(\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1)$. Since $a_1/w_1 + 1/w_{j_1} = 1$, $a_1 = w_1 - w_1/w_{j_1}$ and $\nu = w_1 - w_1/w_{j_1} + 1$. Therefore, the fraction part of $w_1$ is $w_1/w_{j_1}$. In order to make the notation simple, we set $x = [w_1]$, where $x \geq 2$, and $y = w_1/w_{j_1}$, where $0 < y < 1$, and then $x = \nu - 1$, $w_1 = x + y$, and $w_{j_1} = (x + y)/y$. By a simple calculation, $(\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1)$ is the same as $x^2 < (x + y - 1)((x + y)/y - 1)$, which is true for $x \geq 2$.

We consider $\{w_1, \dots, w_5\} \setminus \{w_1, w_{j_1}\}$, the set of three rational numbers obtained from $\{w_1, w_2, w_3, w_4, w_5\}$ by removing $w_1$ and $w_{j_1}$. Without loss of generality, we assume that $w_2 \in \{w_1, \dots, w_5\} \setminus \{w_1, w_{j_1}\}$ (which is not the empty set) is the minimal weight in this set.

If $w_2$ is a positive integer, then $\nu \leq w_2$; hence, $\nu - 1 \leq w_2 - 1$. Since $w_2$ is the minimal weight in the set $\{w_1, \dots, w_5\} \setminus \{w_1, w_{j_1}\}$, we have $\mu > (\nu - 1)^5$.

If $w_2$ is not a positive integer and $[w_2] > [w_1]$, then we have $\nu - 1 < w_2 - 1$. The same reason as before gives $\mu > (\nu - 1)^5$.

If $w_2$ is not a positive integer and $[w_2]=[w_1]$, then our goal is to prove that $(\nu-1)^2<(w_2-1)(w_{j_2}-1)$, where $w_{j_2}$ depends on $w_2$. Since $w_2$ is not an integer, there exists $a_2$, a positive integer number such that $z_2^{a_2}z_{j_2}\in\operatorname{supp} f$, where $j_2\neq 2$. There are three cases to consider:

Case 1. If $j_2=1$, then $z_2^{a_2}z_1\in\operatorname{supp} f$. Then $a_2/w_2+1/w_1=1$, and $w_2\geq w_1$ implies $a_2+1/w_2\leq 1$ and $w_2\geq(a_2+1)\geq\nu$, which contradicts $\nu=[w_1]+1=[w_2]+1$. This case cannot happen.

Case 2. If $j_2\in\{1,\dots,5\}\setminus\{1,2,j_1\}$, then $a_2/w_2+1/w_{j_2}=1$ since $z_2^{a_2}z_{j_2}\in\operatorname{supp} f$. We want to show that $(\nu-1)^2<(w_2-1)(w_{j_2}-1)$. We have $\nu\leq a_2+1$, and $\nu=a_1+1$ implies $a_2\geq a_1$. Furthermore, $a_2=w_2-w_2/w_{j_2}\geq a_1\geq\nu-1$. Let $x=w_2/w_{j_2}$, so by Lemma 3.1 we have $x\neq 1$. Then $0<x<1$, $w_2\geq\nu-1+x$, and $w_{j_2}\geq(\nu-1+x)/x$. It suffices to show that $(\nu-1)^2<(\nu-1+x-1)((\nu-1+x)/x-1)$, which is true for $\nu>2$ and $0<x<1$.

Case 3. If $j_2=j_1$, then $z_2^{a_2}z_{j_1}\in\operatorname{supp} f$. Since $f$ has isolated singularity and both $z_1^{a_1}z_{j_1}\in\operatorname{supp} f$ and $z_2^{a_2}z_{j_1}\in\operatorname{supp} f$, then either $z_1^{b_1}z_2^{b_2}\in\operatorname{supp} f$, where $b_i>0$ and $i=1,2$, or $z_1^{b_1}z_2^{b_2}z_{j_{12}}\in\operatorname{supp} f$, where $b_i\geq 0$ for $i=1,2$. However, in the latter case $b_1$ and $b_2$ cannot both equal 0 and $j_{12}\in\{1,\dots,5\}\setminus\{1,2,j_1\}$.

Subcase 1. We have $z_1^{b_1}z_2^{b_2}\in\operatorname{supp} f$, where $b_i>0$ for $i=1,2$. In this case, we have $b_1/w_1+b_2/w_2=1$. Then $b_1/w_2+b_2/w_2\leq b_1/w_1+b_2/w_2=1$, which implies $w_2\geq b_1+b_2\geq\nu$, contradicting $\nu-1=[w_1]=[w_2]$. This case cannot happen.

Subcase 2. Now we have $z_1^{b_1}z_2^{b_2}z_{j_{12}}\in\operatorname{supp} f$, where $b_i\geq 0$ for $i=1,2$ and $j_{12}\in\{1,\dots,5\}\setminus\{1,2,j_1\}$. In this case we divide it into three subcases:

(a) If $b_1=0$, then $z_2^{b_2}z_{j_{12}}\in\operatorname{supp} f$. This case is same as the previous Case 2.

(b) If $b_2=0$, then $z_1^{b_1}z_{j_{12}}\in\operatorname{supp} f$. By Lemma 3.1, we have $\nu=b_1+1$. Therefore, $a_1=b_1$. Remember that we also have $a_1/w_1+1/w_{j_1}=1$; thus, $w_{j_1}=w_{j_{12}}$. Since we have proved $(\nu-1)^2<(w_1-1)(w_{j_1}-1)$, then we get $(\nu-1)^2<(w_2-1)(w_{j_{12}}-1)$.

(c) If $b_1\neq 0$ and $b_2\neq 0$, then $b_1/w_1+b_2/w_2+1/w_{j_{12}}=1$, which implies that $(b_1+b_2)/w_2+1/w_{j_{12}}\leq 1$. Since $\nu\leq b_1+b_2+1$ and $\nu=a_1+1$, $a_1\leq b_1+b_2$. Then $a_1/w_2+1/w_{j_{12}}\leq 1$, so $a_1\leq w_2-w_2/w_{j_{12}}$. Since $j_{12}\in\{1,\dots,5\}\setminus\{1,2,j_1\}$, then $w_{j_{12}}\geq w_2$. If $w_{j_{12}}=w_2$, then $w_2\geq a_1+w_2/w_{j_{12}}=a_1+1$, which contradicts $[w_1]=[w_2]=\nu-1$, so $w_{j_{12}}>w_2$. Let $x=w_2/w_{j_{12}}$, so $0<x<1$. Since $w_2\geq a_1+x$, then $w_{j_{12}}\geq(a_1+x)/x$. We want to show that $(\nu-1)^2<(w_2-1)(w_{j_{12}}-1)$. It suffices to show that $a_1^2<(a_1+x-1)((a_1+x)/x-1)$, which follows from $0<(a_1-1)(1-x)$, where $a_1\geq 2$ and $0<x<1$.

After the above steps, either we finish the proof, or after reordering the subindex, we have proved $(\nu-1)^4<(w_1-1)(w_2-1)(w_{j_1}-1)(w_{j_2}-1)$, where $z_1$, $z_2$, $z_{j_1}$, and $z_{j_2}$ are different variables. There is only one variable left. Without loss of generality, we use $z_3$ to denote the remaining variable. We know $w_3 \ge w_2 \ge w_1$, and $w_1$ and $w_2$ are not positive integers by the previous arguments.

If $w_3$ is a positive integer, or $w_3$ is not a positive integer and $[w_3] > [w_1]$, then we have $\nu \le w_3$ and $\nu - 1 \le w_3 - 1$. Therefore, $\mu > (\nu - 1)^5$ in this case. The proof ends.

Suppose that $w_3$ is not a positive integer and $[w_3] = [w_1]$. Since $w_3 \ge w_1$, we have $w_3 - 1 \ge w_1 - 1$. We have already proved $(\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1)$ and $(\nu - 1)^2 < (w_2 - 1)(w_{j_2} - 1)$. In order to prove $(\nu - 1)^5 < \mu$, it suffices to show that $(\nu - 1)^3 < (w_1 - 1)^2(w_{j_1} - 1)$. In order to make the notation simple, we set $x = [w_1]$, where $x \ge 2$, and $y = w_1/w_{j_1}$, where $0 < y < 1$. Then $x = \nu - 1$, $w_1 = x + y$, and $w_{j_1} = (x + y)/y$. By simple calculation, $(\nu - 1)^3 < (w_1 - 1)^2(w_{j_1} - 1)$ is equivalent to $x^3 \le (x + y - 1)^2((x + y)/y - 1)$, i.e., $x(x - 2)(1 - y) + (y - 1)^2 > 0$, which follows from $x \ge 2$ and $0 < y < 1$.

In summary, we have proved $(\nu - 1)^5 = \mu$ if and only if $f$ is a homogeneous polynomial.

For $k = 6$, using the same argument as $k = 5$, we obtain $(\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1)$ and $(\nu - 1)^2 < (w_2 - 1)(w_{j_2} - 1)$. Without loss of generality we assume $w_3$ and $w_4$ are the remaining two weights. Then the same argument as above shows that $(\nu - 1)^3 < (w_1 - 1)(w_{j_1} - 1)(w_3 - 1)$ and $(\nu - 1)^3 < (w_2 - 1)(w_{j_2} - 1)(w_4 - 1)$. Thus, we have $(\nu - 1)^6 < (w_1 - 1) \dots (w_6 - 1) = \mu$, which is what we want for proving $(\nu - 1)^6 = \mu$ if and only if the $f$ is homogeneous polynomial. □

*Proof of Theorem C.* If $p_g > 0$, it follows from Theorem 1.6.

If $p_g = 0$, then by Theorem 2.4, $0 > (\nu - 1)d - 4$, where $d = \min_{1 \le i \le \nu}(d_i/i)$, and $d_i$ is the order of the zero of $a_i(x_1, \dots, x_4)$ at $(0, \dots, 0)$ with $d_i \ge i$. Then $\nu < 4/d + 1$. Since $d \ge 1$, $\nu$ is an integer of at least 2 for isolated hypersurface singularities, so $2 \le \nu \le 4$. Therefore, $p(\nu) = (\nu - 1)^5 - \nu(\nu - 1) \dots (\nu - 4) = (\nu - 1)^5$. The theorem is reduced to proving that

$$\mu \ge (\nu - 1)^5,$$

and equality holds if and only if $f$ is a homogeneous polynomial after a biholomorphic change of coordinates. The proof follows from Theorem B. □

*Proof of Theorem D.* It follows from the same argument in the proofs of Theorem C and Theorem B. □

*Proof of Corollary E.* It follows from Theorem B and Theorem 1.1. □

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