Pacific Journal of Mathematics

HIERARCHIES AND COMPATIBILITY ON COURANT ALGEBROIDS

PAULO ANTUNES, CAMILLE LAURENT-GENGOUX
AND JOANA M. NUNES DA COSTA

Volume 261 No. 1 January 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 261, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.261.1

HIERARCHIES AND COMPATIBILITY ON COURANT ALGEBROIDS

PAULO ANTUNES, CAMILLE LAURENT-GENGOUX AND JOANA M. NUNES DA COSTA

We introduce Poisson–Nijenhuis, deforming-Nijenhuis and Nijenhuis pairs that extend to Courant algebroids the notion of a Poisson–Nijenhuis manifold, both the Poisson and the Nijenhuis structures being (1, 1)-tensors on a Courant algebroid. In each case, we construct the natural hierarchies by successive deformation by one of the (1, 1)-tensors.

1. Introduction

The purpose of this article is to explain how (1, 1)-tensors with vanishing Nijenhuis torsion on a Courant algebroid naturally give rise to several types of hierarchies, using as much as possible the supergeometric approach. We first briefly review Courant algebroids, supergeometric approach, Leibniz algebroids, Nijenhuis torsion and hierarchies. We then end this introduction by a more detailed summary of the content of this work.

1A. *On Courant structures, supergeometry, Leibniz algebroids, Nijenhuis torsion and hierarchies.*

Courant structures. It has been noticed by Roytenberg [1999] that the original $\mathbb{R}$-bilinear skew-symmetric bracket introduced by Courant [1990] on the space of sections of $TM \oplus T^*M$, for M a manifold, can be equivalently defined as the skew-symmetrization of the bracket:

$$[(X,\alpha),(Y,\beta)] := ([X,Y], L_X\beta - i_Y \mathrm{d}\alpha), \tag{1}$$

with $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$. This bracket still satisfies the Jacobi identity and, as mentioned in [Ševera and Weinstein 2001], this fact was already noticed by several authors: Kosmann-Schwarzbach, Ševera and Xu (all unpublished). The bracket (1) is a Loday bracket and was used in [Dorfman 1993], hence its name *Dorfman bracket*. The original bracket on $TM \oplus T^*M$ yields to the definition of Courant algebroid given by Liu, Weinstein and Xu [Liu et al. 1997], while the

MSC2010: 53D17.
Keywords: Courant algebroid, Nijenhuis, Poisson–Nijenhuis.

version with non-skew-symmetric bracket (1) yields to the equivalent definition of Courant algebroid by Roytenberg [1999] (see also [Kosmann-Schwarzbach 2005] for a simpler version). Relaxing the Jacobi identity of the Loday bracket, one gets the weaker notion of pre-Courant algebroid (see Definition 2.1 below).

Supergeometric approach. Dealing with Courant bracket can be a difficult task when it comes to computation (see for example [Kosmann-Schwarzbach 1992; Voronov 2002]), due to the many structures that involve it, and to the unnatural aspects of some of the operations that define them. However, in supergeometric formalism, all these structures and conditions are encoded in two objects and one condition, as follows. To every vector bundle equipped with a fiberwise nondegenerate bilinear form is associated a graded commutative algebra, equipped with a Poisson bracket denoted by $\{\cdot\,,\cdot\}$ (which coincides with the big bracket [Kosmann-Schwarzbach 1992] in particular cases) [Roytenberg 2002]. Pre-Courant structures are in one-to-one correspondence with elements of degree 3 in this graded algebra and Courant structures are those elements that satisfy

$$\{\Theta, \Theta\} = 0$$

(see [Roytenberg 2002; Antunes 2010]).

Leibniz algebroids. Courant structures on vector bundles can be viewed as special cases of Leibniz algebroids [Ibáñez et al. 1999]. These are vector bundles $E \to M$ equipped with a $\mathbb{R}$-bilinear bracket on the space of sections and a vector bundle morphism $\rho : E \to TM$ satisfying the Leibniz rule:

$$[X, fY] = f[X, Y] + (\rho(X).f)Y$$

and the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

for all $X, Y, Z \in \Gamma(E)$ and $f \in C^\infty(M)$. Relaxing the Jacobi identity, one gets the weaker notion of pre-Leibniz algebroid. When the base manifold reduces to a point, a Leibniz algebroid is just a Leibniz algebra (also called Loday algebra), while a pre-Leibniz algebroid is simply an algebra, i.e., a space equipped with a bilinear product. Pre-Courant algebroids are pre-Leibniz algebroids; see [Kosmann-Schwarzbach 2005]. But it is important to stress that the supergeometric approach, referred above for pre-Courant and Courant structures, does not extend to the more general pre-Leibniz and Leibniz algebroid framework.

Nijenhuis torsion. The Nijenhuis torsion of a $(1, 1)$-tensor on M, that is, a fiberwise linear endomorphism of TM, is the $(1, 2)$-tensor given by

$$X, Y \mapsto [NX, NY] - N[X, Y]_N, \text{ where } [X, Y]_N := [NX, Y] + [X, NY] - N[X, Y].$$

We call Nijenhuis tensors $(1, 1)$-tensors whose Nijenhuis torsion vanishes. The previous definition can be extended from TM to arbitrary Lie algebroids [Kosmann-Schwarzbach and Magri 1990; Grabowski and Urbański 1997], then from Lie algebroids to Courant algebroids [Cariñena et al. 2004; Kosmann-Schwarzbach 2011] and Leibniz algebroids [Cariñena et al. 2004].

By means of Nijenhuis $(1, 1)$-tensors, a Lie algebroid bracket $[\cdot, \cdot]$ can be deformed into the bracket $[\cdot, \cdot]_N$ above, which can be shown to be a Lie algebroid bracket again. Also, Poisson structures can be deformed into Poisson structures.

Hierarchies. There is no mathematical definition of what a *hierarchy* is, but, within the context of integrable systems, the name has been commonly given either to families (indexed by $\mathbb{N}$ or $\mathbb{Z}$) of Hamiltonian functions that commute for a fixed Poisson structure, or of Poisson structures/Lie algebroid structures which commute pairwise — and sometimes families of both Poisson structures and Hamiltonian functions such that two functions in that family commute with respect to any Poisson structure. We use that name in the same spirit: that is, for us a hierarchy is either a family of commuting Courant structures or a family of Nijenhuis tensors that commute pairwise with respect to some Courant structure.

To obtain a hierarchy, the idea is to start from a structure and a Nijenhuis tensor by means of which we deform the initial structure into a sequence of structures of the same nature [Kosmann-Schwarzbach and Magri 1990; Magri and Morosi 1984].

1B. ***Purpose and content of the present article.*** Our goal is, as we already stated, to construct hierarchies. More precisely, we wish to construct

(i) hierarchies of Courant structures, given a Nijenhuis tensor on a Courant algebroid,

(ii) hierarchies of Poisson structures, given a Nijenhuis tensor compatible with a given Poisson structure on a Courant algebroid, and

(iii) hierarchies of Courant structures and pairs of tensors that we call deforming-Nijenhuis pairs or Nijenhuis pairs.

Indeed, for the two last points, pre-Courant structures are enough. The idea behind item (i) is simply that what is true for manifolds and Lie algebroids should be true for Courant structures as well, and that, in particular, deforming a Courant structure by a Nijenhuis tensor should give a hierarchy of compatible Courant structures. The idea behind items (ii) and (iii) is more involved. We invite the reader to have in mind the case of Poisson–Nijenhuis structures to obtain some intuitive picture [Magri and Morosi 1984; Kosmann-Schwarzbach and Magri 1990; Grabowski and Urbański 1997]. In terms of Courant algebroids, a Poisson–Nijenhuis structure can be seen as a pair (J_π, I_N) of skew-symmetric $(1, 1)$-tensors on $TM \oplus T^*M$

(see Examples 2.6 and 2.9). The pair (π, N) is Poisson–Nijenhuis when π and N are compatible, which means that J_π and I_N anticommute and their concomitant with respect to the Courant structure vanishes; see Example 4.14. These conditions yield our Definition 4.12 of Poisson–Nijenhuis pair on a (pre-)Courant algebroid, Poisson–Nijenhuis pairs for which we generalize the hierarchies of [Magri and Morosi 1984]. Poisson–Nijenhuis pairs being slightly too restrictive, we indeed do it in the more general context of deforming-Nijenhuis pairs and Nijenhuis pairs.

The statements of most results in this article are written in the pre-Courant algebroid framework and are proved using the supergeometric approach. However, for some of them, the proofs only use the pre-Leibniz structure induced by the pre-Courant structure, so that these results hold not only for pre-Courant algebroids, but also for the more general setting of pre-Leibniz algebroids. This happens, for example, with most results in Sections 3A and 3B and the whole Section 5. The lack of convincing examples prevented us from going to such an unnecessary level of generality.

Let us give a more precise content of the article. In Section 2, we make a brief introduction of the supergeometric setting for (pre-)Courant structures and we recall the notions of deforming and Nijenhuis tensors.

In Section 3, we show that a Courant structure Θ can be deformed k times by a Nijenhuis tensor I, and that the henceforth obtained objects $(\Theta_k)_{k\in\mathbb{N}}$ are compatible (Theorem 3.6). Then, we show that the property of being compatible is, for a given compatible pair (I, J), also preserved when deforming n times J by I, provided that I is Nijenhuis (or at least satisfies a weaker condition involving the vanishing of torsion of I on the image of J), and that this result still holds true with respect to pre-Courant structures Θ_k obtained when deforming Θ by I (Theorem 3.16). An even more general case is obtained when considering the tensor I^{2s+1}, $s \in \mathbb{N}$, which is the deformation of I by itself an odd number of times, and, if J is also Nijenhuis, J is replaced by $I^n \circ J^{2m+1}$, $n, m \in \mathbb{N}$ (Theorem 3.20).

In Section 4, we turn our attention to deforming-Nijenhuis pairs, that is, compatible pairs (J, I) where J is a deforming tensor and I is Nijenhuis for Θ. We show that if (J, I) is a deforming-Nijenhuis pair for Θ, then (J, I^{2n+1}) is a deforming-Nijenhuis pair for Θ_k for all $k, n \in \mathbb{N}$ (Theorem 4.7). Then, we consider Poisson–Nijenhuis pairs (J, I), that is, deforming-Nijenhuis pairs where the deforming tensor J is supposed to be Poisson for Θ, and we state one of the main results of the article, which is the construction of a hierarchy of Poisson–Nijenhuis pairs for Θ_k, for all $k \in \mathbb{N}$, that includes pairs of compatible Poisson tensors (Theorem 4.19).

Last, in Section 5, we conclude with the case of Nijenhuis pairs, that is, pairs (I, J) of Nijenhuis tensors compatible with respect to Θ. More precisely, we show that if (I, J) is a Nijenhuis pair for Θ, then for all $m, n, t \in \mathbb{N}$, $(I^{2m+1} \circ J^n, J^{2t+1})$

is a Nijenhuis pair for Θ, and, more generally, for all the Courant structures obtained by deforming Θ several times, either by I or by J (Theorem 5.11).

2. Skew-symmetric tensors on Courant algebroids

2A. ***Courant algebroids in supergeometric terms.*** We introduce the supergeometric setting following the approach in [Roytenberg 1999; 2002; Vaintrob 1997]. Given a vector bundle $A \to M$, we denote by $A[n]$ the graded manifold obtained by shifting the degree of coordinates on the fiber by n. The graded manifold $T^*[2]A[1]$[1] is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is called the *big bracket*; see [Kosmann-Schwarzbach 1992; 2005].

In local coordinates x^i, p_i, ξ^a, θ_a, $i \in \{1, \dots, n\}$, $a \in \{1, \dots, d\}$, in $T^*[2]A[1]$, where x^i, ξ^a are local coordinates on $A[1]$ and p_i, θ_a are the conjugate coordinates, the Poisson bracket is given by

$$\{p_i, x^i\} = \{\theta_a, \xi^a\} = 1, \quad i = 1, \dots, n, a = 1, \dots, d,$$

while the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with an $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree locally as follows: the coordinates on the base manifold M, x^i, $i \in \{1, \dots, n\}$, have bidegree $(0, 0)$, while the coordinates on the fibers, ξ^a, $a \in \{1, \dots, d\}$, have bidegree $(0, 1)$ and their associated moment coordinates, p_i and θ_a, have bidegrees $(1, 1)$ and $(1, 0)$, respectively.[2] We denote by $\mathcal{F}^{k,l}$ the space of functions of bidegree (k, l). The *total degree* of a function $f \in \mathcal{F}^{k,l}$ is equal to $k + l$ and the subset of functions of total degree t is denoted by $\mathcal{F}^t$. We can verify that the big bracket has bidegree $(-1, -1)$, that is,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1,l_1+l_2-1}.$$

This construction is a particular case of a more general one [Roytenberg 2002] in which we consider a vector bundle E equipped with a fiberwise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, \cdot \rangle)$. We denote

[1] This graded manifold is in fact an N-manifold because the parity of a homogeneous function on $T^*[2]A[1]$ is compatible with its degree. For more details on these notions see [Voronov 2002] and for this particular N-manifold (of degree 2) see [Roytenberg 2002]. We should observe that a similar work to the present one could be done, with more complicated computations, on graded manifolds. However, since we want to restrict to the Courant algebroid setting, the N-manifold $T^*[2]A[1]$ is the appropriate one.

[2] Notice that this bidegree can be defined globally using the double vector bundle structure of $T^*[2]A[1]$; see [Roytenberg 1999; Voronov 2002].

by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, that is, $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{\cdot\,,\cdot\}$, which has degree -2. Notice that $\mathcal{F}_E^0 = C^\infty(M)$ and $\mathcal{F}_E^1 = \Gamma(E)$. Under these identifications, the Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \quad \{f, X\} = 0 \quad \text{and} \quad \{X, Y\} = \langle X, Y\rangle,$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

When $E := A \oplus A^*$ (with A a vector bundle over M) and when $\langle\cdot\,,\cdot\rangle$ is the usual symmetric bilinear form

$$\langle X+\alpha, Y+\beta\rangle = \alpha(Y) + \beta(X), \quad \text{for all } X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*), \tag{2}$$

the algebras $\mathcal{F} = C^\infty(T^*[2]A[1])$ and $\mathcal{F}_{A\oplus A^*}$ are isomorphic Poisson algebras [Roytenberg 2002].

Definition 2.1. A *pre-Courant* structure on $(E, \langle\cdot\,,\cdot\rangle)$ is a pair $(\rho, [\cdot\,,\cdot])$, where ρ is a bundle map from E to TM, called the *anchor*, and $[\cdot\,,\cdot]$ is a $\mathbb{R}$-bilinear (not necessarily skew-symmetric) assignment on $\Gamma(E)\times\Gamma(E) \to \Gamma(E)$, called the *Dorfman bracket*, satisfying the relations

$$\rho(X)\cdot\langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle, \tag{3}$$
$$\rho(X)\cdot\langle Y, Z\rangle = \langle X, [Y, Z] + [Z, Y]\rangle, \tag{4}$$

for all $X, Y, Z \in \Gamma(E)$.[3]

If the Jacobi identity, $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$, is satisfied for all $X, Y, Z \in \Gamma(E)$, then the pair $(\rho, [\cdot\,,\cdot])$ is called a *Courant* structure on $(E, \langle\cdot\,,\cdot\rangle)$.

The Dorfman bracket is a Leibniz bracket when the pair $(\rho, [\cdot\,,\cdot])$ is a Courant structure. There is a one-to-one correspondence between pre-Courant structures on $(E, \langle\cdot\,,\cdot\rangle)$ and elements in $\mathcal{F}_E^3$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}_E^3$ are defined for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by

$$\rho(X)\cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X, Y] = \{\{X, \Theta\}, Y\}.$$

The following theorem addresses how the Jacobi identity is expressed in this supergeometric setting.

Theorem 2.2 [Roytenberg 2002]. *There is a one-to-one correspondence between Courant structures on $(E, \langle\cdot\,,\cdot\rangle)$ and functions $\Theta \in \mathcal{F}_E^3$ such that $\{\Theta, \Theta\} = 0$.*

[3]From (3) and (4), we obtain $[X, fY] = f[X, Y] + (\rho(X).f)Y$ for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$ [Kosmann-Schwarzbach 2005]. Thus, as we already mentioned in the Introduction, a pre-Courant algebroid is always a pre-Leibniz algebroid.

If Θ is a (pre-)Courant structure on $(E, \langle\cdot,\cdot\rangle)$, then the triple $(E, \langle\cdot,\cdot\rangle, \Theta)$ is called a *(pre-)Courant algebroid*. For the sake of simplicity, we will often denote a (pre-)Courant algebroid by the pair (E, Θ) instead of the triple $(E, \langle\cdot,\cdot\rangle, \Theta)$.

When $E = A \oplus A^*$ and $\langle\cdot,\cdot\rangle$ is the usual symmetric bilinear form (2), a pre-Courant structure $\Theta \in \mathcal{F}^3_E$ can be decomposed as a sum of homogeneous terms with respect to its bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}^{1,2}_{A\oplus A^*}$, $\gamma \in \mathcal{F}^{2,1}_{A\oplus A^*}$, $\phi \in \mathcal{F}^{0,3}_{A\oplus A^*} = \Gamma(\bigwedge^3 A^*)$ and $\psi \in \mathcal{F}^{3,0}_{A\oplus A^*} = \Gamma(\bigwedge^3 A)$.

We recall from [Roytenberg 1999] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ if and only if (A, μ) is a Lie algebroid. Also, when $\phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ if and only if $((A, \mu), (A^*, \gamma))$ is a Lie bialgebroid [Liu et al. 1997].

2B. ***Deformation of Courant structures by skew-symmetric tensors.*** Suppose that $(E, \langle\cdot,\cdot\rangle, \Theta)$ is a pre-Courant algebroid and $J : E \to E$ is a vector bundle endomorphism of E. The *deformation* of the Dorfman bracket $[\cdot,\cdot]$ by J is the bracket $[\cdot,\cdot]_J$ defined for all sections X, Y of E, by

$$[X, Y]_J = [JX, Y] + [X, JY] - J[X, Y].$$

The $(1, 1)$-tensors on E will be seen as vector bundle endomorphisms of E. A $(1, 1)$-tensor $J : E \to E$ is said to be *skew-symmetric* if

$$\langle Ju, v\rangle + \langle u, Jv\rangle = 0,$$

for all $u, v \in E$. If we consider the endomorphism J^* defined by $\langle u, J^*v\rangle = \langle Ju, v\rangle$, then J is skew-symmetric if and only if $J + J^* = 0$. If J is skew-symmetric, then $[\cdot,\cdot]_J$ satisfies (3) and (4), so that $(\rho \circ J, [\cdot,\cdot]_J)$ is a pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$.[4]

When the $(1, 1)$-tensor $J : E \to E$ is skew-symmetric, the deformed pre-Courant structure $(\rho \circ J, [\cdot,\cdot]_J)$ is associated to the element $\Theta_J := \{J, \Theta\} \in \mathcal{F}^3_E$. The deformation of Θ_J by the skew-symmetric $(1, 1)$-tensor I is denoted by $\Theta_{J,I}$, that is, $\Theta_{J,I} = \{I, \{J, \Theta\}\}$, while the deformed Dorfman bracket $([\cdot,\cdot]_J)_I$ is denoted by $[\cdot,\cdot]_{J,I}$. Although the equality $\Theta_J = \{J, \Theta\}$ only makes sense when J is skew-symmetric, aiming to simplify the notation, we shall denote by Θ_J the pre-Courant structure $(\rho \circ J, [\cdot,\cdot]_J)$, even in the case where J is not skew-symmetric.

By definition, a vector bundle endomorphism $I : E \to E$ is a *Nijenhuis* tensor on the Courant algebroid (E, Θ) if its torsion vanishes, where the torsion $\mathcal{T}_\Theta I$ is

[4]In fact, it suffices that J satisfies the condition $J + J^* = \lambda\, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}$, to guarantee that $(\rho \circ J, [\cdot,\cdot]_J)$ is a pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$; see [Cariñena et al. 2004].

given, for all $X, Y \in \Gamma(E)$, by

$$\mathcal{T}_\Theta I(X, Y) = [IX, IY] - I[X, Y]_I.$$

A short computation shows that

$$\mathcal{T}_\Theta I(X, Y) = \tfrac{1}{2}([X, Y]_{I,I} - [X, Y]_{I^2}), \tag{5}$$

where $I^2 = I \circ I$. When I is skew-symmetric and $I^2 = \alpha\, \mathrm{id}_E$ for some $\alpha \in \mathbb{R}$, then $\mathcal{T}_\Theta I$ is an element of degree 3 in the supergeometric setting [Kosmann-Schwarzbach 2011], and (5) is given by [Grabowski 2006]:

$$\mathcal{T}_\Theta I = \tfrac{1}{2}(\Theta_{I,I} - \alpha\Theta). \tag{6}$$

In the case of pre-Courant algebroids, the definition of Nijenhuis tensors is the same as in the case of a Courant algebroids.

Example 2.3. Let $\mathcal{G}$ be a Lie algebra. A linear operator $I : \mathcal{G} \to \mathcal{G}$ that takes values in the center and such that, in addition, the kernel of I^2 contains the derived algebra $[\mathcal{G}, \mathcal{G}]$ is a Nijenhuis operator.

The notion of *deforming* tensor for a Courant structure Θ on E was introduced in [Kosmann-Schwarzbach 2011]. The definition holds in the case of a pre-Courant algebroid and it will play an important role in this article.

Definition 2.4. Let (E, Θ) be a pre-Courant algebroid. A skew-symmetric $(1, 1)$-tensor J on (E, Θ) is said to be *deforming for* Θ if $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$.

Remark 2.5. If I is Nijenhuis for Θ and satisfies $I^2 = \alpha\, \mathrm{id}_E$ for some $\alpha \in \mathbb{R}$, then, it follows from (6) that I is also deforming for Θ. This was noticed in [Kosmann-Schwarzbach 2011].

When $E = A \oplus A^*$ and $\langle \cdot, \cdot \rangle$ is the usual symmetric bilinear form, a skew-symmetric $(1, 1)$-tensor $J : A \oplus A^* \to A \oplus A^*$ is of the type

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}, \tag{7}$$

with $N : A \to A$, $\pi \in \Gamma(\bigwedge^2 A)$ and $\omega \in \Gamma(\bigwedge^2 A^*)$. In the supergeometric framework, J corresponds to the function $N + \pi + \omega$, which we also denote by J. Therefore, we have $\Theta_J = \{N + \pi + \omega, \Theta\}$.

We shall now present examples of skew-symmetric deforming or/and Nijenhuis tensors in the case where $(E = A \oplus A^*, \Theta)$ is the Courant algebroid associated to a Lie algebroid, that is $\Theta = \mu$, with μ a Lie algebroid structure on A.

Example 2.6. Let π be a bivector on A and $J_\pi = \left(\begin{smallmatrix} 0 & \pi^\# \\ 0 & 0 \end{smallmatrix}\right)$. Then, J_π is deforming for $\Theta = \mu$ if and only if π is a Poisson bivector on the Lie algebroid (A, μ).

If π is a Poisson bivector on (A, μ) then, denoting by $[\cdot, \cdot]_\mu$ the Gerstenhaber bracket on $\Gamma(\bigwedge^\bullet A)$, we have $0 = [\pi, \pi]_\mu = \{\pi, \{\pi, \mu\}\} = \mu_{J_\pi, J_\pi}$, so that J_π is deforming for μ. If J_π is deforming for μ, then $\mu_{J_\pi, J_\pi} = \eta\, \mu$, with $\eta \in \mathbb{R}$. Since μ and μ_{J_π, J_π} do not have the same bidegree, we obtain

$$\mu_{J_\pi, J_\pi} = \eta\, \mu \Leftrightarrow (\eta = 0 \text{ and } \{\pi, \{\pi, \mu\}\} = 0).$$

Thus, π is a Poisson bivector on the Lie algebroid (A, μ).

Example 2.7. Let J_π be as in Example 2.6. The $(1, 1)$-tensor J_π is a Nijenhuis tensor for $\Theta = \mu$ if and only if π is a Poisson bivector on the Lie algebroid (A, μ).

We remark that $J_\pi \circ J_\pi = 0$ so that, using (6) with $\alpha = 0$, we deduce that the torsion of J_π is given by $\mathcal{T}_\mu J_\pi = \frac{1}{2}\{\pi, \{\pi, \mu\}\}$. Therefore, the torsion of J_π with respect to $\Theta = \mu$ vanishes if and only if $[\pi, \pi]_\mu = 0$.

Example 2.8. Let ω be a 2-form on A. Then, $J_\omega = \left(\begin{smallmatrix} 0 & 0 \\ \omega^\flat & 0 \end{smallmatrix}\right)$ is a deforming and a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$.

This is an immediate consequence of $J_\omega \circ J_\omega = 0$ and $\mu_{J_\omega, J_\omega} = \{\omega, \{\omega, \mu\}\} = 0$.

Example 2.9. Let $N : A \to A$ be a $(1, 1)$-tensor on A, such that $N^2 = \alpha\, \mathrm{id}_A$ for some $\alpha \in \mathbb{R}$. Then, $I_N = \left(\begin{smallmatrix} N & 0 \\ 0 & -N^* \end{smallmatrix}\right)$ is a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$ if and only if N is Nijenhuis tensor for the Lie algebroid (A, μ) [Kosmann-Schwarzbach 2011].

Example 2.10. Let π be a bivector on A and $N : A \to A$ a $(1, 1)$-tensor on A. Then, $J = \left(\begin{smallmatrix} N & \pi^\# \\ 0 & -N^* \end{smallmatrix}\right)$ is deforming for $\Theta = \mu$ if and only if N is a deforming tensor on (A, μ),[5] π is a Poisson bivector on (A, μ) and $\mu_{N,\pi} + \mu_{\pi,N} = 0$.

We have

$$\begin{aligned} \mu_{J,J} &= \{N + \pi, \{N + \pi, \mu\}\} \\ &= \{N, \{N, \mu\}\} + \{\pi, \{N, \mu\}\} + \{N, \{\pi, \mu\}\} + \{\pi, \{\pi, \mu\}\} \\ &= \mu_{N,N} + \mu_{N,\pi} + \mu_{\pi,N} + \mu_{\pi,\pi} \end{aligned}$$

and, by counting the bidegrees, we deduce that $\mu_{J,J} = \eta\, \mu$ if and only if

$$\mu_{N,N} = \eta\, \mu, \quad \mu_{N,\pi} + \mu_{\pi,N} = 0, \quad [\pi, \pi]_\mu = 0.$$

Let us consider the Courant algebroid $(A \oplus A^*, \mu + \gamma)$, which is the double of a Lie bialgebroid $((A, \mu), (A^*, \gamma))$ and the skew-symmetric $(1, 1)$-tensor $J : A \oplus A^* \to A \oplus A^*$:

$$J = \begin{pmatrix} \frac{1}{2}\, \mathrm{id}_A & \pi^\# \\ 0 & -\frac{1}{2}\, \mathrm{id}_{A^*} \end{pmatrix}, \tag{8}$$

where π is a bivector on A.

[5] An $(1, 1)$-tensor N on a Lie algebroid (A, μ) is a *deforming* tensor if $\mu_{N,N} = \eta\mu$, for some $\eta \in \mathbb{N}$.

Proposition 2.11. *Let* $((A,\mu),(A^*,\gamma))$ *be a Lie bialgebroid. Then, the* $(1,1)$*-tensor* J *given by* (8) *is a deforming tensor for the Courant structure* $\mu+\gamma$ *if and only if* π *is a solution of the Maurer–Cartan equation*

$$\mathrm{d}_\gamma \pi = \tfrac{1}{2}[\pi,\pi]_\mu.$$

Proof. The $(1,1)$-tensor $J=\frac{1}{2}\,\mathrm{id}_A+\pi$ is a deforming tensor for $\mu+\gamma$ if there exists $\eta\in\mathbb{R}$ such that

$$\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\mu+\gamma\right\}\right\}=\eta(\mu+\gamma).$$

We have, using the fact that $\{\mathrm{id}_A,u\}=(q-p)u$ for all u of bidegree (p,q),

$$\begin{aligned}
&\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\mu+\gamma\right\}\right\}\\
&\qquad=\tfrac{1}{4}\left\{\mathrm{id}_A,\{\mathrm{id}_A,\mu\}+\{\mathrm{id}_A,\gamma\}\right\}+\tfrac{1}{2}\left\{\mathrm{id}_A,\{\pi,\mu\}+\{\pi,\gamma\}\right\}\\
&\qquad\qquad+\tfrac{1}{2}\left\{\pi,\{\mathrm{id}_A,\mu\}+\{\mathrm{id}_A,\gamma\}\right\}+\left\{\pi,\{\pi,\mu\}+\{\pi,\gamma\}\right\}\\
&\qquad=\tfrac{1}{4}(\mu+\gamma)-2\{\pi,\gamma\}-\left\{\{\pi,\mu\},\pi\right\},
\end{aligned}$$

since $\{\pi,\{\pi,\gamma\}\}=0$ for reasons of bidegree. Therefore, J is a deforming $(1,1)$-tensor if and only if

$$\eta=\tfrac{1}{4}\quad\text{and}\quad \mathrm{d}_\gamma\pi=\tfrac{1}{2}[\pi,\pi]_\mu. \qquad\square$$

3. Hierarchies of compatible tensors and structures

We construct a hierarchy of compatible Courant structures on $(E,\langle\cdot,\cdot\rangle)$ that are obtained deforming an initial Courant structure by a Nijenhuis tensor. Then, we consider hierarchies of pairs of tensors which are compatible, in a certain sense, with respect to some deformed pre-Courant structures.

We introduce the following notation, where $I,J,\dots,T$ are skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ):

- $\Theta_{I,J,\dots,T}=(((\Theta_I)_J)_{\dots})_T$,
- $\Theta_k=(((\Theta_I)_I)\overset{k}{\cdots})_I=\Theta_{I,\overset{k}{\dots},I}$, $k\in\mathbb{N}$, $\Theta_0=\Theta$.

3A. ***Hierarchies of compatible Courant structures.*** In this section we construct a hierarchy of compatible Courant structures on $(E,\langle\cdot,\cdot\rangle)$.

The next proposition generalizes a result in [Kosmann-Schwarzbach and Magri 1990].

Proposition 3.1. *Let* I *be a* $(1,1)$*-tensor on a pre-Courant algebroid* (E,Θ)*. For all sections* X,Y *of* E *and* $k\geq 1$,

$$\mathcal{T}_{\Theta_k}I(X,Y)=\mathcal{T}_{\Theta_{k-1}}I(IX,Y)+\mathcal{T}_{\Theta_{k-1}}I(X,IY)-I(\mathcal{T}_{\Theta_{k-1}}I(X,Y)). \tag{9}$$

Proof. Let us denote by $[\,\cdot\,,\cdot\,]_k$ the Dorfman bracket associated to Θ_k. It is obvious that

$$[X, Y]_k = [IX, Y]_{k-1} + [X, IY]_{k-1} - I[X, Y]_{k-1},$$

and therefore we have

$$\begin{aligned}\mathcal{T}_{\Theta_k} I(X, Y) &= [IX, IY]_k - I[IX, Y]_k - I[X, IY]_k + I^2[X, Y]_k \\ &= [I^2X, IY]_{k-1} - I[I^2X, Y]_{k-1} - I[IX, IY]_{k-1} + I^2[IX, Y]_{k-1} \\ &\quad + [IX, I^2Y]_{k-1} - I[IX, IY]_{k-1} - I[X, I^2Y]_{k-1} + I^2[X, IY]_{k-1} \\ &\quad - I([IX, IY]_{k-1} - I[IX, Y]_{k-1} - I[X, IY]_{k-1} + I^2[X, Y]_{k-1}) \\ &= \mathcal{T}_{\Theta_{k-1}} I(IX, Y) + \mathcal{T}_{\Theta_{k-1}} I(X, IY) - I(\mathcal{T}_{\Theta_{k-1}} I(X, Y)).\end{aligned}$$ □

Corollary 3.2. *If I is Nijenhuis for Θ, then I is Nijenhuis for Θ_k, $\forall k \in \mathbb{N}$.*

It is well known [Grabowski 2006] that for every skew-symmetric $(1, 1)$-tensor I on a Courant algebroid (E, Θ), the deformation of Θ by I, Θ_I, is a Courant structure on $(E, \langle\,\cdot\,,\cdot\,\rangle)$ provided that I is Nijenhuis. Applying (9) we get, by recursion:

Proposition 3.3. *Let (E, Θ) be a Courant algebroid and I a skew-symmetric Nijenhuis tensor for Θ. Then, (E, Θ_k) is a Courant algebroid for all $k \in \mathbb{N}$.*

We introduce the notation $I^n = I \circ \overset{n}{\ldots} \circ I$, for $n \geq 1$ and $I^0 = \mathrm{id}_E$.

Let us compute the torsion $\mathcal{T}_\Theta I^n$, for all $n \in \mathbb{N}$.

Proposition 3.4. *Let I be a $(1, 1)$-tensor on a pre-Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$\begin{aligned}\mathcal{T}_\Theta I^n(X, Y) &= \mathcal{T}_\Theta I(I^{n-1}X, I^{n-1}Y) + I(\mathcal{T}_\Theta I^{n-1}(IX, Y) \\ &\quad + \mathcal{T}_\Theta I^{n-1}(X, IY)) - I^2(\mathcal{T}_\Theta I^{n-2}(IX, IY)) + I^{2n-2}(\mathcal{T}_\Theta I(X, Y)),\end{aligned} \tag{10}$$

for $n \geq 2$.

Proof. It suffices to use the definition of Nijenhuis torsion to compute each term on the right hand side of (10). □

As an immediate consequence of the previous proposition and Corollary 3.2, we have:

Proposition 3.5. *Let (E, Θ) be a pre-Courant algebroid and I a $(1, 1)$-tensor on E. If I is a Nijenhuis tensor for Θ, then I^n is Nijenhuis for Θ_k, for all $n, k \in \mathbb{N}$.*

Recall that a pair of Courant structures Θ_1 and Θ_2 on a vector bundle $(E, \langle\,\cdot\,,\cdot\,\rangle)$ are said to be *compatible* if their sum $\Theta_1 + \Theta_2$ is a Courant structure on $(E, \langle\,\cdot\,,\cdot\,\rangle)$. As an immediate consequence, we have that Θ_1 and Θ_2 are compatible if and only if

$$\{\Theta_1, \Theta_2\} = 0.$$

Theorem 3.6. *Let I be a skew-symmetric $(1,1)$-tensor on a Courant algebroid (E,Θ). If I is Nijenhuis for Θ, then the Courant structures Θ_k and Θ_m on $(E,\langle\cdot,\cdot\rangle)$ are compatible for all $k,m\in\mathbb{N}$.*

Proof. We first remark that if $m=k$, then we have $\{\Theta_m,\Theta_m\}=0$ by Proposition 3.3. Also, for any Courant structure Θ and any skew-symmetric $(1,1)$-tensor I, the relation $\{\Theta,\Theta_I\}=0$ follows from the Jacobi identity and the graded symmetry of the Poisson bracket. We use induction on $m+k$ to complete the proof. Assume first that $m+k=2$; then either $m=k=1$ and it is clear that $\{\Theta_I,\Theta_I\}=0$, or $m=2$ and $k=0$ and it is clear that $\{\Theta_{I,I},\Theta\}=\{I,\{\Theta,\Theta_I\}\}-\{\Theta_I,\Theta_I\}=0$.

Now, suppose that $\{\Theta_m,\Theta_k\}=0$ holds for $m+k=s-1$ and take m and k such that $m+k=s$.

i) If $m=k$, we already noticed that $\{\Theta_m,\Theta_m\}=0$.

ii) If $m\neq k$, suppose that $m>k$. Then,

$$\begin{aligned}\{\Theta_m,\Theta_k\}&=\{\{I,\Theta_{m-1}\},\Theta_k\}=\{I,\{\Theta_k,\Theta_{m-1}\}\}-\{\Theta_{k+1},\Theta_{m-1}\}\\&=-\{\Theta_{m-1},\Theta_{k+1}\}\\&=-\{I,\{\Theta_{m-2},\Theta_{k+1}\}\}+\{\Theta_{m-2},\Theta_{k+2}\}=\{\Theta_{m-2},\Theta_{k+2}\}.\end{aligned}$$

Applying the Jacobi identity several times, we get

$$\begin{aligned}\{\Theta_m,\Theta_k\}&=\begin{cases}(-1)^{m-l}\{\Theta_l,\Theta_l\} & \text{if } m+k=2l,\\ (-1)^{m-(l+1)}\{\Theta_{l+1},\Theta_l\} & \text{if } m+k=2l+1.\end{cases}\\&=\begin{cases}0 & \text{if } m+k=2l,\\ (-1)^{m-(l+1)}\frac{1}{2}\{I,\{\Theta_l,\Theta_l\}\}=0 & \text{if } m+k=2l+1.\end{cases}\end{aligned}$$

□

Remark 3.7. The statement of Theorem 3.6 still holds if we replace the assumption of I being Nijenhuis for Θ by I deforming for Θ. In fact, if $\Theta_{I,I}=\eta\Theta$ for some $\eta\in\mathbb{R}$, then a straightforward computation yields

$$\Theta_{2k}=\eta^k\Theta,\quad \Theta_{2k+1}=\eta^k\Theta_I\quad \text{for all } k\in\mathbb{N}.$$

We have investigated so far the pre-Courant structure Θ_n, obtained by deforming n times the original pre-Courant structure Θ by a Nijenhuis tensor I. It is logical to ask what happens when one deforms Θ by I^n. We shall show that we obtain precisely the same pre-Courant structure Θ_n.

Proposition 3.8. *Let $(\rho,[\cdot,\cdot])$ be a pre-Courant structure on $(E,\langle\cdot,\cdot\rangle)$ and I a $(1,1)$-tensor on E. Let X and Y be sections of E and let $n\in\mathbb{N}^*$. Then:*

a) $[X,Y]_{I^{2n+1}}=[X,Y]_{I^{2n},I}-\displaystyle\sum_{\substack{0\leq i,j\leq 2n-1\\ i+j=2n-1}} I^j(\mathcal{T}_\Theta I(I^iX,Y)+\mathcal{T}_\Theta I(X,I^iY)).$

b) *If I is Nijenhuis for $(\rho,[\cdot,\cdot])$, then $[X,Y]_{I^n}=[X,Y]_{I,\overset{n}{\dots},I}$ for all $n\in\mathbb{N}$.*

c) *if I is Nijenhuis for $(\rho, [\cdot,\cdot])$, then* $[X,Y]_{I^m,I^n} = [X,Y]_{I^{m+n}}$ *for all* $m, n \in \mathbb{N}$.

Proof. Statement a) is an easy but cumbersome computation.

For b), first, observe that if a pair of skew-symmetric $(1,1)$-tensors I and J commute, then $[X,Y]_{I,J} = [X,Y]_{J,I}$ for all sections X and Y of E. In particular, we have, for all $m, n \in \mathbb{N}$,

$$[X,Y]_{I^m,I^n} = [X,Y]_{I^n,I^m}. \tag{11}$$

We now prove the result by recursion on $n \geq 1$. If $n = 2k+1$, we use a):

$$[X,Y]_{I^n} = [X,Y]_{I^{2k+1}} = [X,Y]_{I^{2k},I}$$

and we use the recursion hypothesis. If $n = 2k$, since I^k is Nijenhuis, using (5) we may write

$$[X,Y]_{I^n} = [X,Y]_{I^k \circ I^k} = [X,Y]_{I^k,I^k},$$

and we use, again, the recursion hypothesis.

For c), we use b) and (11):

$$[X,Y]_{I^n,I^m} = [X,Y]_{I,\overset{n}{\dots},I,I^m} = [X,Y]_{I^m,I,\overset{n}{\dots},I} = [X,Y]_{I,\overset{m+n}{\dots},I} = [X,Y]_{I^{m+n}}. \qquad \square$$

If I is a Nijenhuis tensor on a pre-Courant algebroid (E,Θ), then, from parts b) and c) of Proposition 3.8, we have

$$\Theta_{I^{k_1},\dots,I^{k_n}} = \Theta_{\underbrace{I,\dots,I}_{k_1+\dots+k_n}} = \Theta_{I^{k_1+\dots+k_n}}, \tag{12}$$

for all $k_1, \dots, k_n \in \mathbb{N}$, $n \in \mathbb{N}$.

3B. ***Hierarchy of compatible tensors with respect to Θ.*** In this section, we introduce the notion of compatible pair of $(1,1)$-tensors with respect to a pre-Courant algebroid (E,Θ) and construct a hierarchy of compatible pairs of tensors.

The Magri–Morosi concomitant of a bivector and a $(1,1)$-tensor on a manifold was introduced in [Magri and Morosi 1984] and then extended to Lie algebroids in [Kosmann-Schwarzbach and Magri 1990]. For a pre-Courant algebroid (E,Θ), we introduce a concomitant of two skew-symmetric $(1,1)$-tensors I and J by setting

$$C_\Theta(I,J) = \{J,\{I,\Theta\}\} + \{I,\{J,\Theta\}\} = \Theta_{I,J} + \Theta_{J,I}. \tag{13}$$

If $(\rho, [\cdot,\cdot])$ is the pre-Courant structure on E corresponding to Θ, (13) reads as follows:

$$\begin{aligned} \{\{X, C_\Theta(I,J)\}, Y\} &= [X,Y]_{I,J} + [X,Y]_{J,I}, \\ \{\{X, C_\Theta(I,J)\}, f\} &= (\rho \circ (I \circ J + J \circ I))(X).f, \end{aligned} \tag{14}$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

In the sequel, we denote the left-hand side of (14) by $C_\Theta(I, J)(X, Y)$. When I and J anticommute, we have $\{\{X, C_\Theta(I, J)\}, f\} = 0$ for all $X \in \Gamma(E)$ and $f \in C^\infty(M)$. Therefore, in this case,

$$C_\Theta(I, J) = 0 \quad \Longleftrightarrow \quad C_\Theta(I, J)(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(E). \tag{15}$$

Remark 3.9. Let (A, μ) be a Lie algebroid. Recall that the Magri–Morosi concomitant of a bivector π and a $(1, 1)$-tensor N on A is given by [Kosmann-Schwarzbach and Magri 1990]:

$$C_\mu(\pi, N) = \{N, \{\pi, \mu\}\} + \{\pi, \{N, \mu\}\}. \tag{16}$$

If we consider the Courant algebroid $(A \oplus A^*, \mu)$ and the $(1, 1)$-tensors J_π and I_N as in Examples 2.6 and 2.9, respectively, we have that the concomitant of J_π and I_N given by (13) and the concomitant of π and N given by (16) coincide.

For the various classes of pairs of skew-symmetric $(1, 1)$-tensors that will be introduced in the sequel, we shall require that the skew-symmetric $(1, 1)$-tensors are compatible in the following sense:

Definition 3.10. A pair (I, J) of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) is said to be a *compatible pair with respect to* Θ if I and J anticommute and $C_\Theta(I, J) = 0$.

Let I and J be two $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Recall that the *Nijenhuis concomitant* of I and J is the map $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ (in general not a tensor) defined for all sections X and Y of E as follows [Kobayashi and Nomizu 1963]:

$$\begin{aligned} \mathcal{N}_\Theta(I, J)(X, Y) = [IX, JY] - I[X, JY] - J[IX, Y] + IJ[X, Y] \\ +[JX, IY] - J[X, IY] - I[JX, Y] + JI[X, Y]. \end{aligned} \tag{17}$$

Notice that $\mathcal{N}_\Theta(I, I) = 2\mathcal{T}_\Theta I$, while if I and J anticommute, then

$$\mathcal{N}_\Theta(I, J) = \tfrac{1}{2} C_\Theta(I, J).$$

Lemma 3.11. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then, $\mathcal{T}_\Theta(I + J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J + \mathcal{N}_\Theta(I, J)$.*

Proof. Using the definition of the Nijenhuis torsion we get, for all $X, Y \in \Gamma(E)$,

$$\begin{aligned} \mathcal{T}_\Theta(I+J)(X, Y) &= \mathcal{T}_\Theta I(X, Y) + \mathcal{T}_\Theta J(X, Y) + [IX, JY] + [JX, IY] - I[X, JY] \\ &\quad - J[X, IY] - I[JX, Y] - J[IX, Y] + IJ[X, Y] + JI[X, Y] \\ &= \mathcal{T}_\Theta I(X, Y) + \mathcal{T}_\Theta J(X, Y) + \mathcal{N}_\Theta(I, J)(X, Y). \end{aligned}$$

□

The next proposition gives a characterization of compatible pairs.

Proposition 3.12. *Let (I, J) be a pair of anticommuting skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then, (I, J) is a compatible pair with respect to Θ if and only if $\mathcal{T}_\Theta(I + J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J$.*

Proposition 3.13. *Let (I, J) be a pair of anticommuting skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then for all sections X, Y of E and $n \geq 1$,*

$$C_\Theta(I, I^n \circ J)(X, Y) = I(C_\Theta(I, I^{n-1} \circ J)(X, Y)) \\ +2\,\mathcal{T}_\Theta I((I^{n-1} \circ J)X, Y) + 2\,\mathcal{T}_\Theta I(X, (I^{n-1} \circ J)Y). \tag{18}$$

Proof. A simple computation gives

$$\begin{aligned} C_\Theta(I, I^n \circ J)(X, Y) &= [X, Y]_{I, I^n \circ J} + [X, Y]_{I^n \circ J, I} \\ &= 2([I^n(JX), IX] - I[I^n(JX), Y] + [IX, I^n(JY)] \\ &\quad - I[X, I^n(JY)] - I^n \circ J[IX, Y] - I^n \circ J[X, IY]), \end{aligned}$$

for all sections X, Y of E and $n \geq 1$. Thus, we have

$$\begin{aligned} &I(C_\Theta(I, I^{n-1} \circ J)(X, Y)) \\ &\quad = 2(I[I^{n-1}(JX), IY] - I^2[I^{n-1}(JX), Y] + I[IX, I^{n-1}(JY)] \\ &\qquad - I^2[X, I^{n-1}(JY)] - I^n \circ J[IX, Y] - I^n \circ J[X, IY]). \end{aligned}$$

Since

$$\begin{aligned} &\mathcal{T}_\Theta I(I^{n-1}(JX), Y) \\ &\quad = [I^n(JX), IY] - I([I^n(JX), Y] + [I^{n-1}(JX), IY] - I[I^{n-1}(JX), Y]) \end{aligned}$$

and

$$\begin{aligned} &\mathcal{T}_\Theta I(X, I^{n-1}(JY)) \\ &\quad = [IX, I^n(JY)] - I([IX, I^{n-1}(JY)] + [X, I^n(JY)] - I[X, I^{n-1}(JY)]), \end{aligned}$$

the result follows. □

Theorem 3.14. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then*

$$C_\Theta(I, I^n \circ J) = 0, \tag{19}$$

and $(I, I^n \circ J)$ is a compatible pair with respect to Θ for all $n \in \mathbb{N}$.

Proof. For $n = 0$, (19) reduces to $C_\Theta(I, J) = 0$ and (I, J) is a compatible pair with respect to Θ, which is one of the assumptions. From (18), we get

$$C_\Theta(I, I^n \circ J)(X, Y) = I(C_\Theta(I, I^{n-1} \circ J)(X, Y)), \quad n \geq 1,$$

for all sections X, Y of E, where we used $I^{n-1} \circ J = (-1)^{n-1} J \circ I^{n-1}$ to obtain

$$\mathcal{T}_\Theta I((I^{n-1}(JX), Y) = (-1)^{n-1} \mathcal{T}_\Theta I(J(I^{n-1}X), Y) = 0$$

and analogously

$$\mathcal{T}_\Theta I(X, I^{n-1}(JY)) = 0.$$

Therefore, using (15), it is obvious that if $C_\Theta(I, I^{n-1} \circ J) = 0$, then $C_\Theta(I, I^n \circ J) = 0$ and (19) follows by recursion. Since I anticommutes with $I^n \circ J$, the proof is complete. $\square$

3C. ***Compatible tensors with respect to Θ_k, $k \in \mathbb{N}$.*** In this section, we address the general case of hierarchies of tensors that are compatible with respect to each term of a family $(\Theta_k)_{k \in \mathbb{N}}$ of pre-Courant structures on E.

Proposition 3.15. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$C_{\Theta_I}(I, J) = C_\Theta(I, \{J, I\}) + \{I, C_\Theta(I, J)\}.$$

In particular, if I and J anticommute, then,

$$C_{\Theta_I}(I, J) = 2C_\Theta(I, I \circ J) + \{I, C_\Theta(I, J)\}. \tag{20}$$

Proof. Applying the Jacobi identity of the bracket $\{\cdot, \cdot\}$ twice, we get

$$\Theta_{I,I,J} = \Theta_{I,\{J,I\}} + \Theta_{I,J,I} = \Theta_{I,\{J,I\}} + \Theta_{\{J,I\},I} + \Theta_{J,I,I},$$

which can be written as

$$C_\Theta(I, \{J, I\}) = \Theta_{I,I,J} - \Theta_{J,I,I}.$$

From the definition of $C_\Theta(I, J)$, we have $\Theta_{J,I,I} = \{I, C_\Theta(I, J)\} - \Theta_{I,J,I}$. Substituting this result in the last equality, we get

$$C_\Theta(I, \{J, I\}) = \Theta_{I,I,J} - \{I, C_\Theta(I, J)\} + \Theta_{I,J,I} = C_{\Theta_I}(I, J) - \{I, C_\Theta(I, J)\},$$

proving the first statement. If I and J anticommute, then $\{J, I\} = 2I \circ J$ and the second statement follows. $\square$

The next theorem extends the result of Theorem 3.14.

Theorem 3.16. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then $C_{\Theta_k}(I, I^n \circ J) = 0$ and $(I, I^n \circ J)$ is a compatible pair with respect to Θ_k for all $k, n \in \mathbb{N}$.*

Proof. Suppose that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. We will prove, by induction on k, that

$$C_{\Theta_k}(I, I^n \circ J) = 0, \quad \text{for all } k, n \in \mathbb{N}.$$

For $k = 0$, this is the content of Theorem 3.14.

Suppose now that, for some $k \in \mathbb{N}$, $C_{\Theta_k}(I, I^n \circ J) = 0$ for all $n \in \mathbb{N}$. Then, from (20) we have, for all $n \in \mathbb{N}$,

$$C_{\Theta_{k+1}}(I, I^n \circ J) = 2C_{\Theta_k}(I, I^{n+1} \circ J) + \{I, C_{\Theta_k}(I, I^n \circ J)\} = 0,$$

where we used the induction hypothesis in the last equality. Since the skew-symmetric tensor $I^n \circ J$ anticommutes with I for all $n \in \mathbb{N}$, $(I, I^n \circ J)$ is a compatible pair with respect to Θ_k, for all $k, n \in \mathbb{N}$. □

In order to establish the main results of this section, we need the following lemmas.

Lemma 3.17. *Let (I, J) be a pair of anticommuting skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$C_\Theta(I, J) = 2(\Theta_{I,J} - \Theta_{I \circ J}).$$

Proof. Since I and J anticommute, $\{I, J\} = -2\, I \circ J$. Using the Jacobi identity of the bracket $\{\cdot, \cdot\}$, we have $\Theta_{J,I} = -2\Theta_{I \circ J} + \Theta_{I,J}$. Therefore,

$$C_\Theta(I, J) = \Theta_{I,J} + \Theta_{J,I} = 2(\Theta_{I,J} - \Theta_{I \circ J}). \qquad \square$$

Lemma 3.18. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that I is Nijenhuis for Θ. If (I, J) is a compatible pair with respect to Θ, then, for all sections X and Y of E,*

$$[X, Y]_{I, \overset{n}{\dots}, I, J} = [X, Y]_{I^n \circ J}.$$

Proof. Theorem 3.16 ensures that, for all $n \in \mathbb{N}$, $C_{\Theta_n}(I, J) = 0$ and, applying Lemma 3.17 for the pre-Courant structure Θ_{n-1}, we get

$$\begin{aligned}[X, Y]_{I, \overset{n}{\dots}, I, J} &= [X, Y]_{I, \overset{n-1}{\dots}, I, I, J} = \{\{X, (\Theta_{n-1})_{I,J}\}, Y\} \\ &= \{\{X, (\Theta_{n-1})_{I \circ J}\}, Y\} = [X, Y]_{I, \overset{n-1}{\dots}, I, I \circ J}.\end{aligned}$$

Since, for every $k \in \mathbb{N}$, I anticommutes with $I^k \circ J$, we may repeat $n-1$ times this procedure to yield

$$[X, Y]_{I, \overset{n}{\dots}, I, J} = [X, Y]_{I^n \circ J}. \qquad \square$$

Remark 3.19. In Lemma 3.18, we may replace the assumption that I is Nijenhuis for Θ by the weaker assumption $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E.

Theorem 3.20. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that I is Nijenhuis and (I, J) is a compatible pair with respect to Θ. Then,*

$$C_{\Theta_k}(I^{2s+1}, I^n \circ J) = 0 \tag{21}$$

and $(I^{2s+1}, I^n \circ J)$ is a compatible pair with respect to Θ_k for all $k, n, s \in \mathbb{N}$. Moreover, if J is Nijenhuis tensor, then

$$C_{\Theta_k}(I^{2s+1}, I^n \circ J^{2m+1}) = 0 \tag{22}$$

and $(I^{2s+1}, I^n \circ J^{2m+1})$ is a compatible pair with respect to Θ_k for all $k, m, n, s \in \mathbb{N}$.

Proof. Let I and J be two skew-symmetric $(1,1)$-tensors which are compatible with respect to Θ and such that $\mathcal{T}_\Theta I = 0$. Firstly, we prove that

$$C_\Theta(I^{2s+1}, I^n \circ J) = 0, \quad \text{for all } s, n \in \mathbb{N}.$$

Since I^{2s+1} anticommutes with $I^n \circ J$, we may apply Lemma 3.17 to obtain

$$C_\Theta(I^{2s+1}, I^n \circ J)(X, Y) = 2([X, Y]_{I^{2s+1}, I^n \circ J} - [X, Y]_{I^{2s+1} \circ (I^n \circ J)}).$$

From Theorem 3.14, $(I, I^n \circ J)$ is a compatible pair with respect to Θ and, applying Lemma 3.18, we get

$$\begin{aligned} C_\Theta(I^{2s+1}, I^n \circ J)(X, Y) &= 2([X, Y]_{I^{2s+1}, I^n \circ J} - [X, Y]_{I, \overset{2s+1}{\dots}, I, I^n \circ J}) \\ &= 2([X, Y]_{I^{2s+1}} - [X, Y]_{I^{2s+1}})_{I^n \circ J} = 0, \end{aligned}$$

where we have used Proposition 3.8b) in the second equality. From (15), we obtain $C_\Theta(I^{2s+1}, I^n \circ J) = 0$.

In order to prove the result for a general Θ_k, notice that, due to Corollary 3.2 and Theorem 3.16, the assumptions originally satisfied for Θ are also satisfied for any of the pre-Courant structures Θ_k, $k \in \mathbb{N}$. Therefore, in the above arguments, we can replace Θ by any Θ_k, $k \in \mathbb{N}$.

Now, suppose that I and J are both Nijenhuis for Θ. Since they play symmetric roles, we may exchange them in (21) and, taking $k = 0$, $n = 0$ and $s = m$, we obtain $C_\Theta(I, J^{2m+1}) = 0$. Because I and J^{2m+1} anticommute, we conclude that (I, J^{2m+1}) is a compatible pair with respect to Θ. Thus, we may apply (21) again, replacing J by J^{2m+1}, to obtain $C_{\Theta_k}(I^{2s+1}, I^n \circ J^{2m+1}) = 0$ and, because I^{2s+1} anticommutes with $I^n \circ J^{2m+1}$, the pair $(I^{2s+1}, I^n \circ J^{2m+1})$ is a compatible pair with respect to Θ_k, for all $k, m, n, s \in \mathbb{N}$. $\square$

4. Hierarchies of deforming-Nijenhuis pairs

We introduce the notion of deforming-Nijenhuis pair as well as the definitions of Poisson tensor and Poisson–Nijenhuis pair on a pre-Courant algebroid. We construct several hierarchies of deforming-Nijenhuis and Poisson–Nijenhuis pairs.

4A. ***Hierarchy of deforming-Nijenhuis pairs for Θ_k, $k \in \mathbb{N}$.*** Starting with a deforming-Nijenhuis pair (J, I) for Θ, we prove, in a first step, that it is also a deforming-Nijenhuis pair for Θ_k for all $k \in \mathbb{N}$. Then, we construct a hierarchy $(J, I^{2n+1})_{n\in\mathbb{N}}$ of deforming-Nijenhuis pairs for Θ_k for all $k \in \mathbb{N}$.

Definition 4.1. Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *deforming-Nijenhuis pair* for Θ if

- (J, I) is a compatible pair with respect to Θ,
- J is deforming for Θ,
- I is Nijenhuis for Θ.

We need the following lemmas.

Lemma 4.2. *Let (I, J) be a pair of anticommuting skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then, for all $k \in \mathbb{N}$,*

$$\big((\Theta_r)_{\{J,\{I,J\}\}}\big)_{I,\overset{s}{\dots},I} = (\Theta_{\{J,\{I,J\}\}})_{I,\overset{k}{\dots},I} \tag{23}$$

for all $r, s \in \mathbb{N}$ such that $r + s = k$.

In particular,

i) *if $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$, for some $\lambda_0 \in \mathbb{R}$, then*

$$(\Theta_k)_{\{J,\{I,J\}\}} = \lambda_0 (\Theta_{J,J})_{I,\overset{k+1}{\dots},I} \quad \textit{for all } k \in \mathbb{N},$$

ii) *if $\{J, \{I, J\}\}$ is a Θ-cocycle, then it is a Θ_k-cocycle for all $k \in \mathbb{N}$.*

Proof. Since I and J anticommute, we have

$$I \circ (I \circ J^2) = (I \circ J^2) \circ I \;\Leftrightarrow\; \{I, I \circ J^2\} = 0 \;\Leftrightarrow\; \{I, \{J, \{J, I\}\}\} = 0. \tag{24}$$

Using the Jacobi identity of the bracket $\{\cdot, \cdot\}$, it follows from (24) that

$$\Theta_{I,\{J,\{J,I\}\}} = \Theta_{\{J,\{J,I\}\},I}. \tag{25}$$

Since (25) holds for any pre-Courant structure on E, we may write

$$(\Theta_{I,\overset{r+s}{\dots},I})_{\{J,\{I,J\}\}} = (\Theta_{I,\overset{r+s-1}{\dots},I})_{\{J,\{I,J\}\},I}.$$

Repeating the procedure $(s - 1)$ times, we obtain (23). The particular cases follow immediately. □

Lemma 4.3. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$\Theta_{J,I,J} = \tfrac{1}{3}\big(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} + \{J, C_\Theta(I,J)\}\big), \tag{26}$$

$$\Theta_{I,J,J} = -\tfrac{1}{3}\big(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} - 2\{J, C_\Theta(I,J)\}\big). \tag{27}$$

Proof. The formulae are obtained by application of the Jacobi identity. □

As a particular case of the previous lemma, we have the following:

Corollary 4.4. *If $C_\Theta(I,J) = 0$ and $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$, $\lambda_0 \in \mathbb{R}$, then*

$$\Theta_{I,J,J} = \alpha \Theta_{J,J,I} \quad \textit{with } \alpha = -\frac{\lambda_0 + 1}{3}. \tag{28}$$

Moreover, if J is deforming for Θ, that is, $\Theta_{J,J} = \eta\Theta$ with $\eta \in \mathbb{R}$, then J is deforming for Θ_I. More precisely, $\Theta_{I,J,J} = \eta\,\alpha\,\Theta_I$.

Lemma 4.5. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that (I, J) is a compatible pair with respect to Θ and $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. Suppose that $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ for some $\lambda_0 \in \mathbb{R}\backslash\{4/((-3)^m - 1), m \in \mathbb{N}\}$. Then, for all $k \in \mathbb{N}$:*

(a) *$(\Theta_k)_{\{J,\{I,J\}\}} = \lambda_k (\Theta_k)_{J,J,I}$, where λ_k is defined by recursion*[6] *as follows: $\lambda_k = -3\lambda_{k-1}/(1 + \lambda_{k-1})$, $k \geq 1$.*

(b) $\lambda_k (\Theta_k)_{J,J,I} = \lambda_0 \Theta_{J,J,I,\overset{k+1}{\ldots},I}$.

(c) *If, in particular, $\lambda_0 = 0$, then $(\Theta_k)_{J,J} = \left(-\frac{1}{3}\right)^k \Theta_{J,J,I,\overset{k}{\ldots},I}$ for all $k \in \mathbb{N}$.*

Proof.

(a) We will prove this statement by induction. Suppose that, for some $k \geq 1$, $(\Theta_{k-1})_{\{J,\{I,J\}\}} = \lambda_{k-1}(\Theta_{k-1})_{J,J,I}$. Using Lemma 4.2 and the induction hypothesis, we have

$$(\Theta_k)_{\{J,\{I,J\}\}} = (\Theta_{k-1})_{\{J,\{I,J\}\},I} = \lambda_{k-1}(\Theta_{k-1})_{J,J,I,I}.$$

Applying formula (28) for Θ_{k-1}, we obtain

$$(\Theta_k)_{\{J,\{I,J\}\}} = \frac{-3\lambda_{k-1}}{1+\lambda_{k-1}} (\Theta_{k-1})_{I,J,J,I} = \lambda_k (\Theta_k)_{J,J,I}, \quad \text{with } \lambda_k = \frac{-3\lambda_{k-1}}{1+\lambda_{k-1}}.$$

(b) Starting from the previous statement, then using the Lemma 4.2 and the hypothesis, we have

$$\lambda_k (\Theta_k)_{J,J,I} = (\Theta_k)_{\{J,\{I,J\}\}} = \Theta_{\{J,\{I,J\}\},I,\overset{k}{\ldots},I} = \lambda_0 \Theta_{J,J,I,\overset{k+1}{\ldots},I}.$$

[6] Explicitly, $\lambda_k = \dfrac{(-3)^k \lambda_0}{1 + \frac{1-(-3)^k}{4}\lambda_0}$ for all $k \in \mathbb{N}$.

(c) From Lemma 4.2i), we get

$$(\Theta_k)_{\{J,\{I,J\}\}} = 0 \quad \text{for all } k \in \mathbb{N},$$

while Theorem 3.16 gives

$$C_{\Theta_k}(I, J) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, applying the formula (27) several times yields

$$(\Theta_k)_{J,J} = -\tfrac{1}{3}(\Theta_{k-1})_{J,J,I} = \cdots = \left(-\tfrac{1}{3}\right)^k \Theta_{J,J,I,\overset{k}{\dots},I}\,. \qquad \square$$

Proposition 4.6. *Let* (I, J) *be a pair of skew-symmetric* $(1,1)$*-tensors on a pre-Courant algebroid* (E, Θ) *such that* (I, J) *is a compatible pair with respect to* Θ *and* $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ *for some* $\lambda_0 \in \mathbb{R} \backslash \{4/((-3)^m - 1), m \in \mathbb{N}\}$*. Assume moreover that* $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ *for all sections* X *and* Y *of* E*. If* J *is a deforming tensor for* Θ*, then* J *is also a deforming tensor for* Θ_k *for all* $k \in \mathbb{N}$*.*

Proof. We consider two cases, depending on the value of λ_0.

i) Case $\lambda_0 \neq 0$. From Theorem 3.16, we have that $C_{\Theta_k}(I, J) = 0$, for all $k \in \mathbb{N}$. We compute,[7] using Lemma 4.3 and both statements of Lemma 4.5,

$$\begin{aligned}(\Theta_k)_{J,J} = (\Theta_{k-1})_{I,J,J} &= -\tfrac{1}{3}((\Theta_{k-1})_{J,J,I} + (\Theta_{k-1})_{\{J,\{I,J\}\}}) \\ &= -\tfrac{1}{3}((\Theta_{k-1})_{J,J,I} + \lambda_{k-1}(\Theta_{k-1})_{J,J,I}) \\ &= -\frac{1+\lambda_{k-1}}{3}(\Theta_{k-1})_{J,J,I} \\ &= -\frac{(1+\lambda_{k-1})\lambda_0}{3\lambda_{k-1}}\,\Theta_{J,J,I,\overset{k}{\dots},I} \\ &= \frac{\lambda_0}{\lambda_k}\,\Theta_{J,J,I,\overset{k}{\dots},I}\,.\end{aligned}$$

The tensor J being deforming for Θ, we have $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$, and the last equality becomes

$$(\Theta_k)_{J,J} = \frac{\lambda_0}{\lambda_k}\,\eta\Theta_k,$$

which means that J is a deforming tensor for Θ_k.

(ii) Case $\lambda_0 = 0$. If J is deforming for Θ, that is, $\Theta_{J,J} = \eta\Theta$ with $\eta \in \mathbb{R}$, then, from Lemma 4.5c) we immediately get

$$(\Theta_k)_{J,J} = \left(-\tfrac{1}{3}\right)^k \eta\Theta_k \quad \text{for all } k \in \mathbb{N},$$

which means that J is deforming for Θ_k. $\square$

Now, we establish the main result of this section.

[7] Notice that if $\lambda_0 \neq 0$ then $\lambda_k \neq 0$ for all $k \in \mathbb{N}$.

Theorem 4.7. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant (respectively, Courant) algebroid (E, Θ) such that $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ for some $\lambda_0 \in \mathbb{R} \backslash \{4/((-3)^m - 1), m \in \mathbb{N}\}$. If (J, I) is a deforming-Nijenhuis pair for Θ, then (J, I^{2n+1}) is a deforming-Nijenhuis pair for the pre-Courant (respectively, Courant) structures Θ_k for all $k, n \in \mathbb{N}$.*

Proof. Let (J, I) be a deforming-Nijenhuis pair for Θ. Combining Corollary 3.2, Theorem 3.16 and Proposition 4.6, we have that (J, I) is a deforming-Nijenhuis pair for Θ_k for all $k \in \mathbb{N}$. From Proposition 3.5 we obtain that I^{2n+1} is Nijenhuis for Θ_k for all $k, n \in \mathbb{N}$. Since I and J anticommute, the tensors I^{2n+1} and J also anticommute and, from Theorem 3.20, we have that $C_{\Theta_k}(I^{2n+1}, J) = 0$, for all $k, n \in \mathbb{N}$. Thus, (J, I^{2n+1}) is a deforming-Nijenhuis pair for Θ_k for all $k, n \in \mathbb{N}$. □

4B. *Hierarchy of Poisson–Nijenhuis pairs for Θ_k, $k \in \mathbb{N}$.* We introduce the notions of Poisson tensor, Poisson–Nijenhuis pair and compatible Poisson tensors for a pre-Courant algebroid (E, Θ) and construct a hierarchy of Poisson–Nijenhuis pairs.

We start by introducing the notion of Poisson tensor.

Definition 4.8. A skew-symmetric $(1,1)$-tensor J on a pre-Courant algebroid (E, Θ) satisfying $\Theta_{J,J} = 0$ is said to be a *Poisson* tensor for Θ.

In the next example, we show that the previous definition extends the usual definition of a Poisson bivector on a Lie algebroid.

Example 4.9. Let (A, μ) be a Lie algebroid. Consider the Courant algebroid $(A \oplus A^*, \Theta = \mu)$ and the $(1,1)$-tensor J_π of Example 2.6. Then, J_π is a Poisson tensor for $\Theta = \mu$ if and only if π is a Poisson tensor on (A, μ).

Example 4.10. The tensors introduced in Example 2.3 are Poisson tensors on Lie algebras.

The next theorem follows directly from Lemma 4.5c).

Theorem 4.11. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that (I, J) is a compatible pair with respect to Θ, $\Theta_{\{J,\{I,J\}\}} = 0$ and $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If J is Poisson for Θ, then J is Poisson for Θ_k for all $k \in \mathbb{N}$.*

Requiring $\Theta_{\{J,\{I,J\}\}} = 0$ might seem somewhat arbitrary, but it is not. In fact, in the case where I and J anticommute, this condition may be interpreted as $I \circ J^2$ being a Θ-cocycle. When $E = A \oplus A^*$, a $(1,1)$-tensor J_π of the type considered in Example 2.6 trivially satisfies this condition because $J_\pi^2 = 0$.

Now, we introduce the main notion of this section.

Definition 4.12. Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *Poisson–Nijenhuis pair* for Θ if

- (J, I) is a compatible pair with respect to Θ,
- J is Poisson for Θ,
- I is Nijenhuis for Θ.

Remark 4.13. If (J, I) is a Poisson–Nijenhuis pair for Θ, then it is a deforming-Nijenhuis pair for Θ.

Recall that a Poisson–Nijenhuis structure on a Lie algebroid (A, μ) is a pair (π, N), where π is a Poisson bivector and $N : A \to A$ is a Nijenhuis tensor such that $N\pi^{\#} = \pi^{\#}N^{*}$ and $C_{\mu}(\pi, N) = 0$.

The next example shows the relation between Definition 4.12 and the notion of Poisson–Nijenhuis structure on a Lie algebroid.

Example 4.14. Let (π, N) be a Poisson–Nijenhuis structure on a Lie algebroid (A, μ) with $N^2 = \alpha\,\mathrm{id}_A$, $\alpha \in \mathbb{R}$. Consider the Courant algebroid (E, Θ), with $E = A \oplus A^{*}$ and $\Theta = \mu$, J_{π} and I_N as in Examples 2.6 and 2.9, respectively. Then, (J_{π}, I_N) is a Poisson–Nijenhuis pair for Θ. In fact,

$$N\pi^{\#} = \pi^{\#}N^{*} \Leftrightarrow I_N \circ J_{\pi} = -J_{\pi} \circ I_N$$

and $C_{\mu}(\pi, N) = C_{\mu}(J_{\pi}, I_N) = 0$, so that (J_{π}, I_N) is a compatible pair with respect to μ. Moreover, π is a Poisson bivector on (A, μ) if and only if J_{π} is Poisson for $\Theta = \mu$ (see Example 4.9) and I_N is Nijenhuis for $\Theta = \mu$ (see Example 2.9). The above arguments show that conversely, if (J_{π}, I_N) is a Poisson–Nijenhuis pair for $\Theta = \mu$ with $N^2 = \alpha\,\mathrm{id}_A$, then (π, N) is a Poisson–Nijenhuis structure on (A, μ).

Definition 4.15. Let J and J' be two Poisson tensors for the pre-Courant structure Θ on the vector bundle $(E, \langle \cdot, \cdot \rangle)$. The tensors J and J' are said to be *compatible* Poisson tensors for Θ if $J + J'$ is a Poisson tensor for Θ, i.e., $\Theta_{J+J', J+J'} = 0$.

An immediate consequence of this definition is the following:

Lemma 4.16. *Let J and J' be two Poisson tensors for Θ. Then, J and J' are compatible Poisson tensors for Θ if and only if $\Theta_{J,J'} + \Theta_{J',J} = 0$. In other words, J and J' are compatible Poisson tensors for Θ if and only if $C_{\Theta}(J, J') = 0$.*

Example 4.17. Let (A, μ) be a Lie algebroid, consider the Courant algebroid $(A \oplus A^{*}, \Theta = \mu)$ and take two Poisson tensors for $\Theta = \mu$, J_{π} and $J_{\pi'}$, of the type considered in Example 2.6. Then,

$$\Theta_{J_{\pi}, J_{\pi'}} + \Theta_{J_{\pi'}, J_{\pi}} = \{\pi', \{\pi, \mu\}\} + \{\pi, \{\pi', \mu\}\} = 2\{\pi', \{\pi, \mu\}\} = -2[\pi, \pi']_{\mu},$$

so that J_{π} and $J_{\pi'}$ are compatible Poisson tensors on $(A \oplus A^{*}, \mu)$ if and only if π and π' are compatible Poisson tensors on the Lie algebroid (A, μ).

In order to construct a hierarchy of Poisson–Nijenhuis pairs, we need the next proposition.

Proposition 4.18. *Let (I, J) be a pair of anticommuting skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$\mathcal{T}_{\Theta_I} J(X,Y) = -J(C_\Theta(I,J)(X,Y)) - \mathcal{T}_\Theta J(IX,Y) - \mathcal{T}_\Theta J(X,IY) - I(\mathcal{T}_\Theta J(X,Y)) \tag{29}$$

and

$$\mathcal{T}_{\Theta_J} I(X,Y) = -I(C_\Theta(I,J)(X,Y)) - \mathcal{T}_\Theta I(JX,Y) - \mathcal{T}_\Theta I(X,JY) - J(\mathcal{T}_\Theta I(X,Y)). \tag{30}$$

Proof. Since the roles of I and J can be exchanged, we only prove (29). We compute $\mathcal{T}_{\Theta_I} J$ and $C_\Theta(I,J)$. For any sections X, Y of E, we have

$$\begin{aligned}\mathcal{T}_{\Theta_I} J(X,Y) &= [JX,JY]_I - J[JX,Y]_I - J[X,JY]_I + J^2[X,Y]_I \\ &= [IJX,JY] + [JX,IJY] - I[JX,JY] - J[IJX,Y] \\ &\quad - J[JX,IY] + JI[JX,Y] - J[IX,JY] - J[X,IJY] \\ &\quad + JI[X,JY] + J^2[IX,Y] + J^2[X,IY] - J^2I[X,Y]\end{aligned}$$

and

$$C_\Theta(I,J)(X,Y) = 2([JX,IY] + [IX,JY] - I([JX,Y] + [X,JY]) - J([IX,Y] + [X,IY])).$$

Thus,

$$\begin{aligned}&\mathcal{T}_{\Theta_I} J(X,Y) + J(C_\Theta(I,J)(X,Y)) \\ &= -[JIX,JY] - [JX,JIY] - I[JX,JY] + J[JIX,Y] + J[JX,IY] + IJ[JX,Y] \\ &\quad + J[IX,JY] + J[X,JIY] + IJ[X,JY] - J^2[IX,Y] - J^2[X,IY] - IJ^2[X,Y] \\ &= -\mathcal{T}_\Theta J(IX,Y) - \mathcal{T}_\Theta J(X,IY) - I(\mathcal{T}_\Theta J(X,Y)).\end{aligned}$$ □

The next theorem defines a hierarchy of Poisson–Nijenhuis pairs.

Theorem 4.19. *Let (J, I) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that (J, I) is a Poisson–Nijenhuis pair for Θ and $\Theta_{\{J,\{I,J\}\}} = 0$. Then:*

1. *$I^n \circ J$ is a Poisson tensor for Θ_k for all $n, k \in \mathbb{N}$.*
2. *$(I^n \circ J)_{n\in\mathbb{N}}$ is a hierarchy of pairwise compatible Poisson tensors for Θ_k, for all $k \in \mathbb{N}$.*
3. *$(I^n \circ J, I^{2m+1})$ is a Poisson–Nijenhuis pair for Θ_k, for all $m, n, k \in \mathbb{N}$.*

The proof of this theorem needs two auxiliary lemmas.

Lemma 4.20. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that (I, J) is a compatible pair with respect to Θ. If I is Nijenhuis for Θ, then I is Nijenhuis for $(\Theta_k)_J$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. From Corollary 3.2, I is Nijenhuis for Θ_k. Also, applying Theorem 3.16, we obtain $C_{\Theta_k}(I, J) = 0$. Finally, using (30) for the pre-Courant structure Θ_k, we conclude that I is Nijenhuis for $(\Theta_k)_J$. □

Lemma 4.21. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that J is Poisson for Θ and $\Theta_{\{J,\{I,J\}\}} = 0$. Assume, moreover, that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then (I, J) is a compatible pair with respect to $(\Theta_k)_J$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. By definition, $C_{(\Theta_k)_J}(I, J) = (\Theta_k)_{J,I,J} + (\Theta_k)_{J,J,I}$. In order to compute $(\Theta_k)_{J,I,J}$, recall formula (26) for the pre-Courant structure Θ_k:

$$(\Theta_k)_{J,I,J} = \tfrac{1}{3}\big((\Theta_k)_{J,J,I} + (\Theta_k)_{\{J,\{I,J\}\}} + \{J, C_{\Theta_k}(I, J)\}\big).$$

Since (I, J) is a compatible pair with respect to Θ, applying Theorem 3.16, we obtain $C_{\Theta_k}(I, J) = 0$. Furthermore, the relation $(\Theta_k)_{\{J,\{I,J\}\}} = 0$ follows from Lemma 4.2(ii). Then, the above formula yields $(\Theta_k)_{J,I,J} = \frac{1}{3}(\Theta_k)_{J,J,I}$, so that $C_{(\Theta_k)_J}(I, J) = \frac{4}{3}(\Theta_k)_{J,J,I}$.

Now, using Theorem 4.11, we obtain $(\Theta_k)_{J,J,I} = 0$. Therefore, (I, J) is a compatible pair with respect to $(\Theta_k)_J$. □

We now the prove the above theorem.

Proof of Theorem 4.19. Let (I, J) be a Poisson–Nijenhuis pair for Θ such that $\Theta_{\{J,\{I,J\}\}} = 0$. We start by proving that

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = 0 \tag{31}$$

for all $m, n, k \in \mathbb{N}$. From the above auxiliary lemmas, (I, J) is a compatible pair with respect to $(\Theta_{k+m})_J$ and I is Nijenhuis for $(\Theta_{k+m})_J$. Then, using Lemma 3.18 for the pre-Courant structure $(\Theta_{k+m})_J$, we obtain

$$\begin{aligned}(\Theta_k)_{I^m \circ J, I^n \circ J} &= ((\Theta_{k+m})_J)_{I^n \circ J} = ((\Theta_{k+m})_J)_{I, \overset{n}{\dots}, I, J} \\ &= \Theta_{I, \overset{k+m}{\dots}, I, J, I, \overset{n}{\dots}, I, J} = (-1)^n\, \Theta_{I, \overset{k+m+n}{\dots}, I, J, J},\end{aligned}$$

where in the last equality we used n times that $C_{\Theta_s}(I, J) = 0$ for all $s \in \mathbb{N}$ (see Theorem 3.16). Using Theorem 4.11, we obtain (31), from which statements (1) and (2) follow. From Theorem 3.20, $(I^n \circ J, I^{2m+1})$ is a compatible pair with respect to Θ_k and, from Proposition 3.5, I^{2m+1} is Nijenhuis for Θ_k. Combining this with statement (1), we obtain statement (3). □

Using the Poisson–Nijenhuis pair arising from a Poisson–Nijenhuis structure as in Example 4.14, we recover most of the hierarchy already studied in [Kosmann-Schwarzbach and Magri 1990], up to a minor difference. In this general setting it is not possible to consider I^{2n} since it is not a skew-symmetric $(1,1)$-tensor.

We conclude this section with a particular case of deforming-Nijenhuis pairs.

Proposition 4.22. *Let (J, I) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that $I^2 = \alpha\,\mathrm{id}_E$ and $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ for some $\alpha, \lambda_0 \in \mathbb{R}$. If (J, I) is a deforming-Nijenhuis pair for Θ, then $(I^n \circ J, I)$ is a deforming-Nijenhuis pair for Θ, for all $n \in \mathbb{N}$.*

Proof. Let (J, I) be a deforming-Nijenhuis pair for Θ. First, we prove that $I^n \circ J$ is deforming for Θ. Since $I^2 = \alpha\,\mathrm{id}_E$, $I^n \circ J$ is proportional either to J or to $I \circ J$. So, we only need to prove that $I \circ J$ is deforming for Θ. Using Lemma 3.17 and the fact that I and J anticommute, we have

$$\Theta_{I\circ J, I\circ J} = \Theta_{I,J,I\circ J} = \tfrac{1}{2}\Theta_{I,J,\{J,I\}} = \tfrac{1}{2}\big(\Theta_{I,J,I,J} - \Theta_{I,J,J,I}\big),$$

where in the last equality we used the Jacobi identity of the bracket $\{\cdot\,,\cdot\}$. Using (26) for Θ_I and Lemma 4.2, we get

$$2\Theta_{I\circ J, I\circ J} = \tfrac{1}{3}\big(\Theta_{I,J,J,I} + \Theta_{I,\{J,\{I,J\}\}}\big) - \Theta_{I,J,J,I} = -\tfrac{2}{3}\Theta_{I,J,J,I} + \tfrac{1}{3}\Theta_{\{J,\{I,J\}\},I}.$$

Now, from the equality (27), we obtain

$$2\Theta_{I\circ J, I\circ J} = \tfrac{2}{9}\Theta_{J,J,I,I} + \tfrac{5}{9}\Theta_{\{J,\{I,J\}\},I}.$$

Since $\Theta_{\{J,\{I,J\}\}} = \lambda_0\Theta_{J,J,I}$ and $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$, we get

$$\Theta_{I\circ J, I\circ J} = \frac{2+5\lambda_0}{18}\eta\Theta_{I,I} = \frac{2+5\lambda_0}{18}\eta\Theta_{I^2} = \frac{2+5\lambda_0}{18}\eta\alpha\Theta,$$

where, in the last equalities, we used the fact that I is Nijenhuis and satisfies $I^2 = \alpha\,\mathrm{id}_E$. Therefore, $I \circ J$ is deforming for Θ.

The tensors I and $I^n \circ J$ anticommute and, from Theorem 3.14, $C_\Theta(I, I^n \circ J) = 0$. Thus, $(I^n \circ J, I)$ is a deforming-Nijenhuis pair for Θ. □

Notice that $(I^n \circ J, I)_{n\in\mathbb{N}}$ is a very poor hierarchy of deforming-Nijenhuis pairs since, as we already mentioned, all the pairs are of type either (J, I) or $(I \circ J, I)$. In fact we have, for all $n \in \mathbb{N}$,

$$I^{2n} \circ J = \alpha^n J, \qquad I^{2n+1} \circ J = \alpha^n I \circ J.$$

5. Hierarchies of Nijenhuis pairs

The last part of this article is devoted to the study of pairs of Nijenhuis tensors on pre-Courant algebroids.

5A. ***Nijenhuis pair for a hierarchy of pre-Courant structures.*** We introduce the notion of Nijenhuis pair for a pre-Courant algebroid (E, Θ) and prove that a Nijenhuis pair (I, J) for Θ is still a Nijenhuis pair for any deformation of Θ, either by I or J.

We first introduce the notion of Nijenhuis pair for a pre-Courant algebroid.

Definition 5.1. Let I and J be two skew-symmetric tensors on a pre-Courant algebroid (E, Θ). The pair (I, J) is called a *Nijenhuis pair* for Θ, if it is a compatible pair with respect to Θ and I and J are both Nijenhuis for Θ.

Example 5.2. Let J be a deforming tensor on (E, Θ), that is $\Theta_{J,J} = \eta\Theta$, for some $\eta \in \mathbb{R}$. If (J, I) is a deforming-Nijenhuis pair, with $J^2 = \eta\, \mathrm{id}_E$, then (J, I) is a Nijenhuis pair. In particular, if (J, I) is Poisson–Nijenhuis pair, and $J^2 = 0$, then (J, I) is a Nijenhuis pair. Notice that this happens when $J = J_\pi$ as in Example 2.6.

In the next proposition we compute the torsion of the composition $I \circ J$.

Proposition 5.3. *Let (I, J) be a pair of anticommuting tensors on a pre-Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$2\mathcal{T}_\Theta(I \circ J)(X, Y) = \big(\mathcal{T}_\Theta I(JX, JY) - J\big(\mathcal{T}_\Theta I(JX, Y) + \mathcal{T}_\Theta I(X, JY)\big) - J^2(\mathcal{T}_\Theta I(X, Y))\big) + \underset{I,J}{\circlearrowleft}, \tag{32}$$

where $\underset{I,J}{\circlearrowleft}$ stands for permutation of I and J.

Proof. Let us compute the first four terms of the right hand side of (32):

$$\begin{aligned}
\mathcal{T}_\Theta I(JX, JY) &= [IJX, IJY] - I[IJX, JY] - I[JX, IJY] + I^2[JX, JY]\\
-J(\mathcal{T}_\Theta I(JX, Y)) &= -J[IJX, IY] + JI[IJX, Y] + JI[JX, IY] - JI^2[JX, Y]\\
-J(\mathcal{T}_\Theta I(X, JY)) &= -J[IX, IJY] + JI[IX, JY] + JI[X, IJY] - JI^2[X, JY]\\
-J^2(\mathcal{T}_\Theta I(X, Y)) &= -J^2[IX, IY] + J^2 I[IX, Y] + J^2 I[X, IY] - J^2 I^2[X, Y].
\end{aligned}$$

The terms appearing on the right hand sides of the above equalities can be written in a matrix form:

$$M(I, J)(X, Y) = \begin{bmatrix} [IJX, IJY] & -I[IJX, JY] & -I[JX, IJY] & I^2[JX, JY]\\ -J[IJX, IY] & JI[IJX, Y] & JI[JX, IY] & -JI^2[JX, Y]\\ -J[IX, IJY] & JI[IX, JY] & JI[X, IJY] & -JI^2[X, JY]\\ -J^2[IX, IY] & J^2I[IX, Y] & J^2I[X, IY] & -J^2I^2[X, Y] \end{bmatrix}.$$

Because I and J anticommute, exchanging the tensors I and J, we obtain the matrix $M(J, I)$ with entries given by

$$M(J, I)_{m,n} = \begin{cases} -M(I, J)_{n,m} & \text{if } m \neq n, \\ M(I, J)_{m,n} & \text{if } m = n, \end{cases}$$

for all $m, n = 1, \ldots, 4$.

Note that the right hand side of (32) is the sum of all the entries of both matrices $M(I, J)(X, Y)$ and $M(J, I)(X, Y)$. Thus,

$$\begin{aligned} \mathcal{T}_\Theta I(JX, JY) - J\big(\mathcal{T}_\Theta I(JX, Y) + \mathcal{T}_\Theta I(X, JY)\big) - J^2(\mathcal{T}_\Theta I(X, Y)) + \underset{I,J}{\circlearrowleft} \\ = 2\big([IJX, IJY] + JI[IJX, Y] + JI[X, IJY] - J^2I^2[X, Y]\big) \\ = 2\mathcal{T}_\Theta(I \circ J)(X, Y), \end{aligned}$$

and the proof is complete. □

Proposition 5.4. *Let (I, J) be a pair of skew-symmetric tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then $(I, I \circ J)$ and $(J, I \circ J)$ are also Nijenhuis pairs for Θ.*

Proof. It is obvious that I and $I \circ J$ anticommute, as well as J and $I \circ J$. From (32) we conclude that $I \circ J$ is a Nijenhuis tensor and from (19), with $n = 1$, we obtain $C_\Theta(I, I \circ J) = 0$ and $C_\Theta(J, I \circ J) = 0$. □

Using Proposition 5.4, we may establish a relationship between Nijenhuis pairs and hypercomplex triples.

A triple (I, J, K) of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) is called a *hypercomplex triple* if $I^2 = J^2 = K^2 = I \circ J \circ K = -\operatorname{id}_E$ and all the six Nijenhuis concomitants $\mathcal{N}_\Theta(I, I)$, $\mathcal{N}_\Theta(J, J)$, $\mathcal{N}_\Theta(K, K)$, $\mathcal{N}_\Theta(I, J)$, $\mathcal{N}_\Theta(J, K)$ and $\mathcal{N}_\Theta(I, K)$ vanish [Stiénon 2009]. (See (17) for the definition of $\mathcal{N}_\Theta$).

Example 5.5. Given a Nijenhuis pair (I, J) such that $I^2 = J^2 = -\operatorname{id}_E$, the triple $(I, J, I \circ J)$ is a hypercomplex structure. Conversely, for every hypercomplex structure (I, J, K), the pairs (I, J), (J, K) and (K, I) are Nijenhuis pairs.

The main result of this section is the following.

Theorem 5.6. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then (I, J) is a Nijenhuis pair for $\Theta_{T_1, T_2, \ldots, T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J, for every $i = 1, \ldots, s$.*

Proof. Combining formulae (18) and (20) we get, for all $X, Y \in \Gamma(E)$,

$$\begin{aligned} C_{\Theta_I}(I, J)(X, Y) &= 2I(C_\Theta(I, J)(X, Y)) + 4\mathcal{T}_\Theta I(JX, Y) + 4\mathcal{T}_\Theta I(X, JY) \\ &= 0. \end{aligned} \tag{33}$$

Now, from Corollary 3.2, (29) and (33), we conclude that (I, J) is a Nijenhuis pair for Θ_I. Since we may exchange the roles of I and J, we also conclude that (I, J) is a Nijenhuis pair for Θ_J.

Since Corollary 3.2 and the formulae (29) and (33) hold for any anticommuting tensors I and J and for any pre-Courant structure Θ on E, we can repeat the previous argument iteratively to conclude that (I, J) is a Nijenhuis pair for $\Theta_{T_1,T_2,\ldots,T_s}$ for all $s \in \mathbb{N}$, where T_i stands either for I or for J for every $i = 1, \ldots, s$. □

As a consequence of the above theorem, we deduce:

Corollary 5.7. *Let (I, J) be a pair of $(1, 1)$-tensors on a Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $s \in \mathbb{N}$, $\Theta_{T_1,T_2,\ldots,T_s}$ is a Courant structure on E, where T_i stands either for I or for J, for every $i = 1, \ldots, s$.*

5B. ***Hierarchies of Nijenhuis pairs.*** Starting with a Nijenhuis pair (I, J) for a pre-Courant algebroid (E, Θ), we construct several hierarchies of Nijenhuis pairs for any deformation of Θ, either by I or J.

We start with the construction of a hierarchy $(I^{2m+1}, J)_{m\in\mathbb{N}}$ of Nijenhuis pairs where one of the Nijenhuis tensors remains unchanged.

Proposition 5.8. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair for $\Theta_{T_1,T_2,\ldots,T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J for every $i = 1, \ldots, s$.*

Proof. The proof follows from Proposition 3.5, Theorem 3.20 and Theorem 5.6. □

Now we consider the hierarchy $(I^{2m+1}, J^{2n+1})_{m,n\in\mathbb{N}}$. This case follows from the previous one: for every $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair. Applying Proposition 5.8 to each of these pairs, we get that $(I^{2m+1}, J^{2n+1})_{m,n\in\mathbb{N}}$ is a hierarchy of Nijenhuis pairs and we obtain the following.

Theorem 5.9. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m, n \in \mathbb{N}$, (I^{2m+1}, J^{2n+1}) is a Nijenhuis pair for $\Theta_{T_1,T_2,\ldots,T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J, for every $i = 1, \ldots, s$.*

Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If I and J are Nijenhuis tensors, we know (see Proposition 3.5) that, for any $m, n \in \mathbb{N}$, I^m and J^n are also Nijenhuis tensors for Θ. The next lemma gives a condition ensuring that $I^m \circ J^n$ is also Nijenhuis.

Lemma 5.10. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If I and J are anticommuting Nijenhuis tensors, then $I^m \circ J^n$ is a Nijenhuis tensor provided that at least one of the integers m, n is odd.*

Proof. As the roles of the tensors I and J are symmetric, we can suppose that m is odd (and n is even or odd). If n is also odd then I^m and J^n anticommute and the result follows from Proposition 5.3. Suppose now that m is odd and n is even. By the previous case, $I^m \circ J^{n-1}$ is Nijenhuis and anticommutes with J:

$$(I^m \circ J^{n-1}) \circ J = I^m \circ J^n = -J \circ (I^m \circ J^{n-1}).$$

Then, using again Proposition 5.3, we conclude that $I^m \circ J^n$ is a Nijenhuis tensor. □

The main result of this section is the following theorem.

Theorem 5.11. *Let* (I, J) *be a pair of skew-symmetric* $(1, 1)$*-tensors on a pre-Courant algebroid* (E, Θ)*. If* (I, J) *is a Nijenhuis pair for* Θ*, then for all* $m, n, t \in \mathbb{N}$*,* $(I^{2m+1} \circ J^n, J^{2t+1})$ *is a Nijenhuis pair for* $\Theta_{T_1,T_2,\ldots,T_s}$*, for all* $s \in \mathbb{N}$*, where* T_i *stands either for* I *or for* J *for every* $i = 1, \ldots, s$*.*

Proof. First, we prove that $(I^{2m+1} \circ J^n, J^{2t+1})$ is a Nijenhuis pair for Θ for all $m, n, t \in \mathbb{N}$. We already know that $I^{2m+1} \circ J^n$ is Nijenhuis (see Lemma 5.10) and that J^{2t+1} is Nijenhuis (see Proposition 3.5). Moreover, $I^{2m+1} \circ J^n$ anticommutes with J^{2t+1} and, applying (22), we obtain $C_\Theta(I^{2m+1} \circ J^n, J^{2t+1}) = 0$.

Using Theorem 5.6, this result can be extended to all pre-Courant structures $\Theta_{T_1,T_2,\ldots,T_s}$, where T_i stands either for I or for J for every $i = 1, \ldots, s$. □

Acknowledgments.

The authors wish to thank Yvette Kosmann-Schwarzbach for many comments and suggestions on a preliminary version of this manuscript. This work was partially supported by CMUC-FCT (Portugal) and FCT grants PEst-C/MAT/UI0324/2011 and PTDC/MAT/099880/2008 through European program COMPETE/FEDER.

References

[Antunes 2010] P. Antunes, *Crochets de Poisson gradués et applications: structures compatibles et généralisations des structures hyperkählériennes*, Ph.D. thesis, École Polytechnique, 2010.

[Cariñena et al. 2004] J. F. Cariñena, J. Grabowski, and G. Marmo, "Courant algebroid and Lie bialgebroid contractions", *J. Phys. A* **37**:19 (2004), 5189–5202. MR 2005h:53139 Zbl 1058.53022

[Courant 1990] T. J. Courant, "Dirac manifolds", *Trans. Amer. Math. Soc.* **319**:2 (1990), 631–661. MR 90m:58065 Zbl 0850.70212

[Dorfman 1993] I. Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, Nonlinear Science: Theory and Applications, John Wiley & Sons Ltd., Chichester, 1993. MR 94j:58081

[Grabowski 2006] J. Grabowski, "Courant–Nijenhuis tensors and generalized geometries", pp. 101–112 in *Groups, geometry and physics*, edited by J. Clemente-Gallardo and E. Martínez, Monogr. Real Acad. Ci. Exact. Fís.-Quím. Nat. Zaragoza **29**, Acad. Cienc. Exact. Fís. Quím. Nat. Zaragoza, Zaragoza, 2006. MR 2008h:53141 Zbl 1137.53021

[Grabowski and Urbański 1997] J. Grabowski and P. Urbański, "Lie algebroids and Poisson–Nijenhuis structures", *Rep. Math. Phys.* **40**:2 (1997), 195–208. MR 99c:58180 Zbl 1005.53061

[Ibáñez et al. 1999] R. Ibáñez, M. de León, J. C. Marrero, and E. Padrón, "Leibniz algebroid associated with a Nambu–Poisson structure", *J. Phys. A* **32**:46 (1999), 8129–8144. MR 2001b:53102 Zbl 0962.53047

[Kobayashi and Nomizu 1963] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I, Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963. MR 27 #2945 Zbl 0119.37502

[Kosmann-Schwarzbach 1992] Y. Kosmann-Schwarzbach, "Jacobian quasi-bialgebras and quasi-Poisson Lie groups", pp. 459–489 in *Mathematical aspects of classical field theory* (Seattle, WA, 1991), edited by M. J. Gotay et al., Contemp. Math. **132**, Amer. Math. Soc., Providence, RI, 1992. MR 94b:17025 Zbl 0847.17020

[Kosmann-Schwarzbach 2005] Y. Kosmann-Schwarzbach, "Quasi, twisted, and all that. . .in Poisson geometry and Lie algebroid theory", pp. 363–389 in *The breadth of symplectic and Poisson geometry*, edited by J. E. Marsden and T. S. Ratiu, Progr. Math. **232**, Birkhäuser, Boston, MA, 2005. MR 2005g:53157 Zbl 1079.53126

[Kosmann-Schwarzbach 2011] Y. Kosmann-Schwarzbach, "Nijenhuis structures on Courant algebroids", *Bull. Braz. Math. Soc.* (*N.S.*) **42**:4 (2011), 625–649. MR 2861782 Zbl 1241.53068

[Kosmann-Schwarzbach and Magri 1990] Y. Kosmann-Schwarzbach and F. Magri, "Poisson–Nijenhuis structures", *Ann. Inst. Henri Poincaré Phys. Théor.* **53**:1 (1990), 35–81. MR 92b:17026 Zbl 0707.58048

[Liu et al. 1997] Z.-J. Liu, A. Weinstein, and P. Xu, "Manin triples for Lie bialgebroids", *J. Differential Geom.* **45**:3 (1997), 547–574. MR 98f:58203 Zbl 0885.58030

[Magri and Morosi 1984] F. Magri and C. Morosi, "A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds", Quaderno S 19, Univ. of Milan, 1984.

[Roytenberg 1999] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, Ph.D. Thesis, University of California, Berkeley, 1999, http://search.proquest.com/docview/304495952.

[Roytenberg 2002] D. Roytenberg, "On the structure of graded symplectic supermanifolds and Courant algebroids", pp. 169–185 in *Quantization, Poisson brackets and beyond* (Manchester, 2001), edited by T. Voronov, Contemp. Math. **315**, Amer. Math. Soc., Providence, RI, 2002. MR 2004i:53116 Zbl 1036.53057

[Ševera and Weinstein 2001] P. Ševera and A. Weinstein, "Poisson geometry with a 3-form background", *Progr. Theoret. Phys. Suppl.* 144 (2001), 145–154. MR 2005e:53132 Zbl 1029.53090

[Stiénon 2009] M. Stiénon, "Hypercomplex structures on Courant algebroids", *C. R. Math. Acad. Sci. Paris* **347**:9-10 (2009), 545–550. MR 2011d:53210 Zbl 1163.53031

[Vaintrob 1997] A. Y. Vaintrob, "Lie algebroids and homological vector fields", *Uspekhi Mat. Nauk* **52**:2(314) (1997), 161–162. In Russian; translated in *Russian Math. Surveys* **52**:2 (1997), 428–429. MR 1480150 Zbl 0955.58017

[Voronov 2002] T. Voronov, "Graded manifolds and Drinfeld doubles for Lie bialgebroids", pp. 131–168 in *Quantization, Poisson brackets and beyond* (Manchester, 2001), edited by T. Voronov, Contemp. Math. **315**, Amer. Math. Soc., Providence, RI, 2002. MR 2004f:53098 Zbl 1042.53056

Received October 4, 2011. Revised September 15, 2012.

PAULO ANTUNES
CMUC
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COIMBRA
APARTADO 3008
3001-454 COIMBRA
PORTUGAL

pantunes@mat.uc.pt

CAMILLE LAURENT-GENGOUX
UMR 7122
UNIVERSITÉ DE METZ
57045 METZ
FRANCE

and

CMUC
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COIMBRA
3001-454 COIMBRA
PORTUGAL

claurent@univ-metz.fr

JOANA M. NUNES DA COSTA
CMUC
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COIMBRA
APARTADO 3008
3001-454 COIMBRA
PORTUGAL

jmcosta@mat.uc.pt

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 1 January 2013