

*Pacific
Journal of
Mathematics*

GEOMETRIZATION OF CONTINUOUS CHARACTERS OF \mathbb{Z}_p^\times

CLIFTON CUNNINGHAM AND MASOUD KAMGARPOUR

GEOMETRIZATION OF CONTINUOUS CHARACTERS OF \mathbb{Z}_p^\times

CLIFTON CUNNINGHAM AND MASOUD KAMGARPOUR

We define the p -adic trace of certain rank-one local systems on the multiplicative group over p -adic numbers, using Sekiguchi and Suwa's unification of Kummer and Artin–Schreier–Witt theories. Our main observation is that, for every nonnegative integer n , the p -adic trace defines an isomorphism of abelian groups between local systems whose order divides $(p - 1)p^n$ and ℓ -adic characters of the multiplicative group of p -adic integers of depth less than or equal to n .

Introduction. Let p and ℓ be distinct primes and let q be a power of p . Let G be a connected commutative algebraic group over \mathbb{F}_q ; that is, a smooth commutative group scheme of finite type over a field. To geometrize a character $\psi : G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ one pushes forward the Lang central extension

$$0 \rightarrow G(\mathbb{F}_q) \rightarrow G \xrightarrow{\text{Lang}} G \rightarrow 0, \quad \text{Lang}(x) = \text{Fr}(x) - x,$$

by ψ^{-1} and obtains a local system \mathcal{L}_ψ on G . The trace of Frobenius of \mathcal{L}_ψ equals ψ ; which is to say that \mathcal{L}_ψ and ψ correspond under the functions–sheaves dictionary. Thus, we think of \mathcal{L}_ψ as *the geometrization of ψ* . Let $C(G)$ be the abelian group (under tensor product) consisting of \mathcal{L}_ψ as ψ ranges over $\text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times)$; in other words, $C(G)$ is the group of irreducible summands of $\text{Lang}_! \overline{\mathbb{Q}}_\ell$. Trace of Frobenius defines an isomorphism of abelian groups

$$(1) \quad t_{\text{Fr}} : C(G) \xrightarrow{\cong} \text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times);$$

see [Deligne 1977, Sommes Trig.] and [Laumon 1987, Example 1.1.3].

Here we obtain an analogue of this isomorphism for \mathbb{G}_m over p -adic numbers.

Theorem. *The work of Sekiguchi and Suwa, on unification of Kummer with Artin–Schreier theories, provides an isomorphism between the abelian group of rank-one*

Cunningham was supported by NSERC. Kamgarpour acknowledges the hospitality and support of the University of Calgary.

MSC2010: primary 20C15; secondary 14G20, 14G15.

Keywords: geometrization, character sheaves, continuous multiplicative characters of p -adic fields, p -adic trace function, continuous characters of \mathbb{Z}_p^\times .

local systems on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and the abelian group of characters of \mathbb{Z}_p^\times of depth less than or equal to n , for every nonnegative integer n .

Motivation and relation to character sheaves. Before proving the theorem, we take a moment to explain our motivation. Deligne used the local systems \mathcal{L}_ψ , appearing above, to prove bounds on the trigonometric sums over finite fields. A key fact used by Deligne in his computation is Grothendieck trace formula. An analogue of this trace formula is missing over p -adic fields. This is the main hurdle for pursuing an analogue of Deligne's results. We hope that the local systems we study here will be of use in obtaining bounds for corresponding sums over p -adic fields.

According to [Lusztig 1985, Section 2], character sheaves on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ are perverse sheaves on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ (cohomologically) concentrated in degree 1 where they are rank-one Kummer local systems. We restrict our attention to those character sheaves on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and find that these are precisely those that admit a $\mathbb{Q}_p(\mu_{p^n})$ -rational structure; that is, they can be defined on $\mathbb{G}_{m, \mathbb{Q}_p(\mu_{p^n})}$. In this language, the above theorem states the following: *The p -adic trace (defined below) of every $\mathbb{Q}_p(\mu_{p^\infty})$ -rational character sheaf on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ is a continuous character $\mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and, moreover, every continuous ℓ -adic character of \mathbb{Z}_p^\times is obtained in this manner, each one from a unique character sheaf of $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$.*

Our idea for defining a function from a $\mathbb{Q}_p(\mu_{p^n})$ -rational character sheaf \mathcal{H} on $\mathbb{G}_{m, \overline{\mathbb{Q}}_p}$ is to consider $\mathbb{Z}_p[\mu_{p^n}]$ -models for $\mathbb{G}_{m, \mathbb{Q}_p(\mu_{p^n})}$ such that \mathcal{H} extends to a local system on the model; then, after restriction to the special fibre of the model, we recover a local system to which we may apply the trace of Frobenius function, as above. Using the work of Sekiguchi and Suwa we find that this idea can be realized if one additional step is introduced: we must consider $\mathbb{Z}_p[\mu_{p^n}]$ -models for $\mathbb{G}_{m, \mathbb{Q}_p(\mu_{p^n})}^{n+1}$, rather than $\mathbb{G}_{m, \mathbb{Q}_p(\mu_{p^n})}$. We believe that this strategy for passing from character sheaves on p -adic groups with rational structure to smooth characters by judicious use of integral models may be of wider applicability in establishing a relationship between character sheaves on p -adic groups and admissible characters. This note is meant to illustrate a case of this strategy.

Unification of Kummer with Artin–Schreier–Witt. Henceforth, we assume that p is an odd prime. Fix a nonnegative integer n and a primitive p^n -th root of unity $\zeta \in \overline{\mathbb{Q}}_p$. Set $R = \mathbb{Z}_p[\zeta]$, $K = \mathbb{Q}_p(\zeta)$. The main theorem of Sekiguchi and Suwa on the unification of Kummer and Artin–Schreier–Witt theories provides us with

- an exact sequence

$$0 \rightarrow \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathfrak{y} \xrightarrow{f} \mathfrak{X} \rightarrow 0$$

of commutative group schemes over R ,

- isomorphisms $\mathfrak{Y}_K := \mathfrak{Y} \otimes_R K \xrightarrow{\cong} \mathbb{G}_{m,K}^{n+1}$ and $\mathfrak{X}_K \rightarrow \mathbb{G}_{m,K}^{n+1}$,
- isomorphisms $\mathfrak{Y}_{\mathbb{F}_p} \xrightarrow{\cong} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$ and $\mathfrak{X}_{\mathbb{F}_p} \xrightarrow{\cong} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$,

where $\mathbb{W}_{n,\mathbb{F}_p}$ is the Witt ring scheme of dimension n over \mathbb{F}_p , such that the following diagram commutes:

$$\begin{array}{ccccccccc}
 \mathbb{G}_{m,K} & \xleftarrow{m} & \mathbb{G}_{m,K}^{n+1} & \longleftarrow & \mathfrak{Y}_K & \longrightarrow & \mathfrak{Y} & \longleftarrow & \mathfrak{Y}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \\
 \downarrow \theta & & \downarrow \gamma & & \downarrow f_K & & \downarrow f & & \downarrow f_{\mathbb{F}_p} & & \downarrow \text{Lang} \\
 \mathbb{G}_{m,K} & \xleftarrow{\alpha} & \mathbb{G}_{m,K}^{n+1} & \longleftarrow & \mathfrak{X}_K & \longrightarrow & \mathfrak{X} & \longleftarrow & \mathfrak{X}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}.
 \end{array}$$

Here, $\theta(x) = x^{(p-1)p^n}$, m denotes the multiplication map, γ and α are defined by

$$\begin{aligned}
 \gamma(x_0, \dots, x_n) &= \left(x_0^{p-1}, \frac{x_1^p}{x_2}, \frac{x_2^p}{x_3}, \dots, \frac{x_n^p}{x_{n-1}} \right), \\
 \alpha(x_0, x_1, \dots, x_n) &= \frac{(x_0 x_1 x_2 x_3 \cdots x_n)^{p^n}}{x_1 x_2^p x_3^{p^2} \cdots x_n^{p^{n-1}}},
 \end{aligned}$$

and f_K and $f_{\mathbb{F}_p}$ are the restrictions of f to the generic and special fibre, respectively. The theorem in question was announced in [Suwa and Sekiguchi 1995] and a proof appeared in the preprint [Sekiguchi and Suwa 1999]. According to Sekiguchi, the main tools of this preprint have been published in [Sekiguchi and Suwa 2003]. For a general overview see [Tsuchiya 2003].

The p -adic trace function. Let $\mathcal{K}(\mathbb{G}_{m,K})$ denote the group (under tensor product) of local systems that are irreducible summands of $\theta_! \overline{\mathcal{Q}}_\ell$. One can easily check that all the squares in the above diagram are Cartesian; moreover, it is clear that all the vertical arrows are Galois covers of order $(p-1)p^n$. It follows that the diagram above determines a canonical isomorphism of groups

$$(2) \quad \mathcal{S} : \mathcal{K}(\mathbb{G}_{m,K}) \xrightarrow{\cong} \mathcal{C}(\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}).$$

We define the p -adic trace function by

$$(3) \quad \begin{aligned} \mathfrak{T}\tau_n : \mathcal{K}(\mathbb{G}_{m,K}) &\longrightarrow \text{Hom}(\mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p), \overline{\mathcal{Q}}_\ell^\times) \\ \mathcal{H} &\longmapsto t_{\text{Fr}}(\mathcal{S}(\mathcal{H})). \end{aligned}$$

It follows at once from (1) and (2) that $\mathfrak{T}\tau_n$ is a canonical isomorphism.

Relationship to continuous characters of \mathbb{Z}_p^\times . Since p is odd, the exponential map defines an isomorphism of algebraic \mathbb{F}_p -groups

$$(4) \quad \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \xrightarrow{\cong} \mathbb{W}_{n+1,\mathbb{F}_p}^*$$

where $\mathbb{W}_{n+1, \mathbb{F}_p}^*$ refers to the group scheme of units in the Witt ring scheme $\mathbb{W}_{n+1, \mathbb{F}_p}$ (see [Greenberg 1962]) and therefore an isomorphism

$$(5) \quad \mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p) = \mathbb{Z}/(p-1) \times \mathbb{Z}/p^n \xrightarrow{\sim} \mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p).$$

Accordingly, we can think of the p -adic trace as a character of $\mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)$. Composing with the quotient $\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)$, we see that the p -adic trace can be interpreted as a continuous ℓ -adic character of \mathbb{Z}_p^\times .

Conversely, for every continuous character $\chi : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$, there is a nonnegative integer n such that $\chi(\mathbb{Z}_p^\times / (1 + p^{n+1}\mathbb{Z}_p)) = \{1\}$. The smallest such n is known as the depth of χ . We propose to think of $\mathcal{K}_\chi := \mathfrak{T}\mathfrak{r}_n^{-1}(\chi)$ as *the geometrization of χ* , when $\chi : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is a continuous character of depth n . We do not discuss how to vary n in the present text.

We note that choosing an isomorphism of the form (5) is unappetizing. We hope, in time, to give a construction which does not depend on this choice.

Acknowledgement

We would like to thank T. Sekiguchi for sending us a copy of his unpublished manuscript (joint with Suwa) and for answering our questions. We thank A.-A. Aubert, J. Noel, R. Pries, T. Schedler, P. Scholze and J. Weinstein for helpful discussions and comments. Finally we would like to thank P. Deligne for carefully reading an earlier draft and providing insightful comments.

References

- [Deligne 1977] P. Deligne, *Cohomologie étale* (Séminaire de Géométrie Algébrique du Bois-Marie 1963-64 = SGA 4 $\frac{1}{2}$), Lecture Notes in Mathematics **569**, Springer, Berlin, 1977. [MR 57 #3132](#) [Zbl 0345.00010](#)
- [Greenberg 1962] M. J. Greenberg, “Unit Witt vectors”, *Proc. Amer. Math. Soc.* **13** (1962), 72–73. [MR 25 #73](#) [Zbl 0104.25406](#)
- [Laumon 1987] G. Laumon, “Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil”, *Inst. Hautes Études Sci. Publ. Math.* **65** (1987), 131–210. [MR 88g:14019](#) [Zbl 0641.14009](#)
- [Lusztig 1985] G. Lusztig, “Character sheaves, I”, *Adv. in Math.* **56**:3 (1985), 193–237. [MR 87b:20055](#) [Zbl 0586.20018](#)
- [Sekiguchi and Suwa 1999] T. Sekiguchi and N. Suwa, “On the unified Kummer–Artin–Schreier–Witt theory”, preprint 11-1, 1999.
- [Sekiguchi and Suwa 2003] T. Sekiguchi and N. Suwa, “A note on extensions of algebraic and formal groups, V”, *Japan. J. Math. (N.S.)* **29**:2 (2003), 221–284. [MR 2004m:14098](#) [Zbl 1075.14045](#)
- [Suwa and Sekiguchi 1995] N. Suwa and T. Sekiguchi, “Théorie de Kummer–Artin–Schreier et applications”, *J. Théor. Nombres Bordeaux* **7**:1 (1995), 177–189. [MR 98d:11139](#) [Zbl 0920.14023](#)
- [Tsuchiya 2003] K. Tsuchiya, “On the descriptions of $\mathbb{Z}/p^2\mathbb{Z}$ -torsors by the Kummer–Artin–Schreier–Witt theory”, *Tokyo J. Math.* **26**:1 (2003), 147–177. [MR 2004h:14050](#)

Received June 14, 2011. Revised November 14, 2012.

CLIFTON CUNNINGHAM
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CALGARY
2500 UNIVERSITY DRIVE NW
CALGARY, AB T2N 1N4
CANADA
cunning@math.ucalgary.ca

MASOUD KAMGARPOUR
SCHOOL OF MATHEMATICS AND PHYSICS
UNIVERSITY OF QUEENSLAND
BRISBANE, QLD 4072
AUSTRALIA
masoud@uq.edu.au

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

V. S. Varadarajan (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Vijayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US \$400/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 1 January 2013

Hierarchies and compatibility on Courant algebroids	1
PAULO ANTUNES, CAMILLE LAURENT-GENGOUX and JOANA M. NUNES DA COSTA	
A new characterization of complete linear Weingarten hypersurfaces in real space forms	33
CÍCERO P. AQUINO, HENRIQUE F. DE LIMA and MARCO A. L. VELÁSQUEZ	
Calogero–Moser versus Kazhdan–Lusztig cells	45
CÉDRIC BONNAFÉ and RAPHAËL ROQUIER	
Coarse median spaces and groups	53
BRIAN H. BOWDITCH	
Geometrization of continuous characters of \mathbb{Z}_p^\times	95
CLIFTON CUNNINGHAM and MASOUD KAMGARPOUR	
A note on Lagrangian cobordisms between Legendrian submanifolds of \mathbb{R}^{2n+1}	101
ROMAN GOLOVKO	
On slope genera of knotted tori in 4-space	117
YI LIU, YI NI, HONGBIN SUN and SHICHENG WANG	
Formal groups of elliptic curves with potential good supersingular reduction	145
ÁLVARO LOZANO-ROBLEDO	
Codimension-one foliations calibrated by nondegenerate closed 2-forms	165
DAVID MARTÍNEZ TORRES	
The trace of Frobenius of elliptic curves and the p -adic gamma function	219
DERMOT MCCARTHY	
(DN) - (Ω) -type conditions for Fréchet operator spaces	237
KRZYSZTOF PISZCZEK	