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A NOTE ON LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN SUBMANIFOLDS OF $\mathbb{R}^{2n+1}$

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We study the relation of an embedded Lagrangian cobordism between two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. More precisely, we investigate the behavior of the Thurston–Bennequin number and (linearized) Legendrian contact homology under this relation. The result about the Thurston–Bennequin number can be considered as a generalization of the result of Chantraine which holds when $n = 1$. In addition, we provide a few constructions of Lagrangian cobordisms and prove that there are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$.

1. Introduction

Basic definitions. A *contact manifold* (M, ξ) is a $(2n+1)$-dimensional manifold M equipped with a smooth maximally nonintegrable hyperplane field $\xi \subset TM$, that is, locally $\xi = \ker \alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. ξ is a *contact structure* and α is a *contact* 1*-form* which locally defines ξ. The *Reeb vector field* R_α of a contact form α is uniquely defined by the conditions $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$. The most basic contact manifold is $(\mathbb{R}^{2n+1}, \xi)$, where $\mathbb{R}^{2n+1}$ has coordinates $(x_1, y_1, \dots, x_n, y_n, z)$, and ξ is given by $\alpha = dz - \sum_{i=1}^n y_i \, dx_i$. Note that $R_\alpha = \partial_z$. From now on, for ease of notation, we write $\mathbb{R}^{2n+1}$ instead of $(\mathbb{R}^{2n+1}, \xi)$.

A *Legendrian submanifold* of $\mathbb{R}^{2n+1}$ is an n-dimensional submanifold Λ which is everywhere tangent to ξ, that is, $T_x\Lambda \subset \xi_x$ for every $x \in \Lambda$. The *Lagrangian projection* is a map $\Pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ defined by

$$\Pi(x_1, y_1, \dots, x_n, y_n, z) = (x_1, y_1, \dots, x_n, y_n).$$

Moreover, for Λ in an open dense subset of all Legendrian submanifolds with C^∞ topology, the self-intersection of $\Pi(\Lambda)$ consists of a finite number of transverse double points. Legendrian submanifolds which satisfy this property are called

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chord generic. A *Reeb chord* of Λ is a path along the flow of the Reeb vector field which begins and ends on Λ. Since $R_\alpha = \partial_z$, there is a one-to-one correspondence between Reeb chords of Λ and double points of $\Pi(\Lambda)$. From now on we assume that all Legendrian submanifolds of $\mathbb{R}^{2n+1}$ are connected and chord-generic.

The *symplectization* of $\mathbb{R}^{2n+1}$ is the symplectic manifold $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha))$, where t is a coordinate on $\mathbb{R}$.

Definition 1.1. Let Λ_- and Λ_+ be two Legendrian submanifolds of $\mathbb{R}^{2n+1}$. We say that Λ_- is cobordant to Λ_+ if there exists a smooth cobordism $(L; \Lambda_-, \Lambda_+)$, and an embedding from L to $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha))$ such that

$$L|_{(-\infty,-T_L]\times\mathbb{R}^{2n+1}} = (-\infty, -T_L] \times \Lambda_-,$$
$$L|_{[T_L,\infty)\times\mathbb{R}^{2n+1}} = [T_L, \infty) \times \Lambda_+$$

for some $T_L \gg 0$ and $L^c := L|_{[-T_L-1,T_L+1]\times\mathbb{R}^{2n+1}}$ is compact. In the case of a Lagrangian (exact Lagrangian) embedding, we say that Λ_- is Lagrangian (exact Lagrangian) cobordant to Λ_+. We will in general not distinguish between L and L^c and call both L.

From now on we assume that all embedded cobordisms in the symplectization of $\mathbb{R}^{2n+1}$ are orientable.

We next define some notations. If L is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism from Λ_- to Λ_+, we write

$$\Lambda_- \prec_L \Lambda_+, \quad \Lambda_- \prec_L^{\text{lag}} \Lambda_+, \quad \text{or } \Lambda_- \prec_L^{\text{ex}} \Lambda_+,$$

respectively. If L_Λ is a filling, Lagrangian filling, or exact Lagrangian filling of Λ in the symplectization of $\mathbb{R}^{2n+1}$, that is, L_Λ is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism with empty $-\infty$-boundary and $+\infty$-boundary Λ, then we write $\varnothing \prec_{L_\Lambda} \Lambda$, $\varnothing \prec_{L_\Lambda}^{\text{lag}} \Lambda$ or $\varnothing \prec_{L_\Lambda}^{\text{ex}} \Lambda$, respectively.

For the discussion about Lagrangian cobordisms between Legendrian knots, we refer to [Chantraine 2010; Ekholm et al. $\geq$ 2013], and for the obstructions to the existence of Lagrangian cobordisms defined using the theory of generating families, we refer to [Sabloff and Traynor 2010; Sabloff and Traynor 2011].

Legendrian contact homology. Legendrian contact homology was independently introduced by Eliashberg, Givental, and Hofer [Eliashberg et al. 2000] and, for Legendrian knots in $\mathbb{R}^3$, by Chekanov [2002]. We now briefly remind the reader of the definition of the linearized Legendrian contact homology complex of a closed, orientable, chord-generic Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$; for more details see [Ekholm et al. 2005a].

Let $\mathscr{C}$ be the set of Reeb chords of Λ. Since Λ is generic, $\mathscr{C}$ is a finite set. Let A_Λ be the vector space over $\mathbb{Z}_2$ generated by the elements of $\mathscr{C}$ and $\mathscr{A}_\Lambda$ the unital tensor algebra over A_Λ, that is,

$$\mathcal{A}_\Lambda = \bigotimes_{k=0}^{\infty} A_\Lambda^{\otimes k}.$$

$\mathcal{A}_\Lambda$ is a differential graded algebra whose grading is denoted by $|\cdot|$ and whose differential is denoted by ∂_Λ. $\mathcal{A}_\Lambda$ is called a Legendrian contact homology differential graded algebra of Λ. For the definitions of $|\cdot|$ and ∂_Λ we refer to Section 2 of [Ekholm et al. 2005b].

Note that it is difficult to use Legendrian contact homology in practical applications, as it is the homology of an infinite dimensional noncommutative algebra with a nonlinear differential. One of the ways to extract useful information from the Legendrian contact homology differential graded algebra is to follow Chekanov's [2002] linearization method, which uses an augmentation $\varepsilon : \mathcal{A}_\Lambda \to \mathbb{Z}_2$ to produce a finite-dimensional chain complex $\mathrm{LC}^\varepsilon(\Lambda)$ whose homology is denoted by $\mathrm{LCH}^\varepsilon(\Lambda)$. More precisely, ε is a graded algebra map $\varepsilon : \mathcal{A}_\Lambda \to \mathbb{Z}_2$ that satisfy the following two conditions:

(1) $\varepsilon(1) = 1$;

(2) $\varepsilon \circ \partial_\Lambda = 0$.

Consider the graded isomorphism $\varphi^\varepsilon : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ defined by $\varphi^\varepsilon(c) = c + \varepsilon(c)$. This map defines a new differential $\partial^\varepsilon(c) := \varphi^\varepsilon \circ \partial_\Lambda \circ (\varphi^\varepsilon)^{-1}(c)$ and $\mathrm{LC}^\varepsilon(\Lambda) := (A_\Lambda, \partial_1^\varepsilon)$, where $\partial_1^\varepsilon : A_\Lambda \to A_\Lambda$ is a 1-component of ∂^ε. We let $\mathrm{LCH}_\varepsilon(\Lambda)$ be the homology of the dual complex $\mathrm{LC}_\varepsilon(\Lambda) := \mathrm{Hom}(\mathrm{LC}^\varepsilon(\Lambda), \mathbb{Z}_2)$.

Following Ekholm [2008], we observe that exact Lagrangian cobordism between two Legendrian submanifolds can be used to define a map between the Legendrian contact homology algebras.

In this paper, we establish the following two long exact sequences.

Theorem 1.2. *Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$ such that $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$. Then from the condition $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$ it follows that there is an exact sequence*

$$(1\text{-}1)\quad \to H_i(\Lambda_-) \to H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to .$$

In addition, $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$ implies that there is an exact sequence

$$(1\text{-}2)\quad \to \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \to H_i(L, \Lambda_-) \to \mathrm{LCH}^{n-i+3}_{\varepsilon_-}(\Lambda_-) \to .$$

Here $\mathrm{LCH}^i_{\varepsilon_\pm}(\Lambda_\pm)$ *is the linearized Legendrian contact cohomology of* $\Lambda_\pm$ *over* $\mathbb{Z}_2$, *linearized with respect to the augmentation* $\varepsilon_\pm$. ε_- *is the augmentation induced by* L_{Λ_-}, *and* ε_+ *is the augmentation induced by* L *and* ε_-.

We thank Joshua Sabloff and Lisa Traynor for pointing out how to get the second long exact sequence in Theorem 1.2.

The Thurston–Bennequin invariant. The Thurston–Bennequin invariant (number) of a closed, orientable, connected Legendrian submanifold Λ of $\mathbb{R}^{2n+1}$ was independently defined for $n = 1$ by Bennequin [1983] and by Thurston, and was generalized to the case when $n \geq 1$ by Tabachnikov [1988].

Pick an orientation on $\Lambda \subset \mathbb{R}^{2n+1}$. Push Λ slightly off of itself along $R_\alpha = \partial_z$ to get another oriented submanifold Λ' disjoint from Λ. The Thurston–Bennequin invariant of Λ is the linking number

$$\mathrm{tb}(\Lambda) = \mathrm{lk}(\Lambda, \Lambda').$$

Note that $\mathrm{tb}(\Lambda)$ is independent of the choice of orientation on Λ, since changing it also changes the orientation of Λ'.

Our goal is to prove the following theorem.

Theorem 1.3. *Let* Λ_- *and* Λ_+ *be two closed*, *orientable Legendrian submanifolds of* $\mathbb{R}^{2n+1}$.

(1) *If* n *is even and* $\Lambda_- \prec_L \Lambda_+$,

$$\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L).$$

(2) *If* n *is odd*, $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$, *and* $\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+$,

$$\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = (-1)^{((n-2)(n-1))/2+1}\chi(L).$$

Constructions and examples. Chantraine [2010] described the way to construct Lagrangian cobordisms from Legendrian isotopies of Legendrian knots. We show that the construction of Chantraine works in high dimensions. More precisely, we prove the following:

Proposition 1.4. *Let* Λ_-, Λ_+ *be two closed*, *orientable Legendrian submanifolds of* $\mathbb{R}^{2n+1}$ *that are Legendrian isotopic. Then there exists an exact Lagrangian cobordism* L *such that*

$$\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+.$$

Front spinning is a procedure invented by Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b] to construct a closed, orientable Legendrian submanifold $\Sigma\Lambda \subset \mathbb{R}^{2n+3}$ from a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$. We will provide a detailed description of this procedure in Section 4, and prove the following property of it.

Proposition 1.5. *Let Λ_-, Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. If $\Lambda_- \prec_L^{\text{lag}} \Lambda_+$, there exists a Lagrangian cobordism ΣL such that*

$$\Sigma\Lambda_- \prec_{\Sigma L}^{\text{lag}} \Sigma\Lambda_+.$$

In addition, if $\Lambda_- \prec_L^{\text{ex}} \Lambda_+$, there exists an exact Lagrangian cobordism ΣL such that $\Sigma\Lambda_- \prec_{\Sigma L}^{\text{ex}} \Sigma\Lambda_+$.

Finally, we apply Proposition 1.5 to the exact Lagrangian cobordisms from [Ekholm et al. $\geq$ 2013] and construct exact Lagrangian cobordisms between the nonisotopic Legendrian tori described in [Ekholm et al. 2005b].

Proposition 1.6. *There are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$.*

2. Proof of Theorem 1.2

Proof. In this section, we prove the existence of the long exact sequences described in Theorem 1.2. We first construct an exact Lagrangian filling of Λ_+.

Since Λ_- is connected, and L, L_{Λ_-} are exact Lagrangian cobordisms in the symplectization of $\mathbb{R}^{2n+1}$ such that the $(-\infty)$-boundary of L, which is Λ_-, agrees with the $(+\infty)$-boundary of L_{Λ_-}, L and L_{Λ_-} can be joined to the exact Lagrangian cobordism L_{Λ_+} in the symplectization of $\mathbb{R}^{2n+1}$, where L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L. Since the $-\infty$-boundary of L_{Λ_-} is empty, the $-\infty$-boundary of L_{Λ_+} is also empty.

We now use the Mayer–Vietoris long exact sequence for $L_{\Lambda_-}, L \subset L_{\Lambda_+}$. We extend L_{Λ_-} and L in such a way that $L_{\Lambda_-} \cap L$ is diffeomorphic to $\mathbb{R} \times \Lambda_-$. Hence the Mayer–Vietoris long exact sequence can be written as

$$\to H_i(\mathbb{R} \times \Lambda_-) \to H_i(L) \oplus H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_{i-1}(\mathbb{R} \times \Lambda_-) \to .$$

Now we note that $H_i(\mathbb{R} \times \Lambda_-) \simeq H_i(\Lambda_-)$ for all i. Hence we can rewrite the Mayer–Vietoris long exact sequence as

$$\to H_i(\Lambda_-) \to H_i(L) \oplus H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_{i-1}(\Lambda_-) \to . \tag{2-1}$$

We now remind the reader of the following fact, which comes from certain observations of Seidel in wrapped Floer homology [Abouzaid and Seidel 2010; Fukaya et al. 2009].

Fact 2.1 [Ekholm 2012]. Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$ and $\varnothing \prec_{L_\Lambda}^{\text{ex}} \Lambda$. Then

$$H_{n-i+2}(L_\Lambda) \simeq \text{LCH}_\varepsilon^i(\Lambda). \tag{2-2}$$

Here ε is the augmentation induced by L_Λ.

For the definition of the augmentation induced by a filling, we refer to Section 3 of [Ekholm 2008]. Also, [Ekholm 2012] provides a fairly complete sketch of a proof of Fact 2.1.

We change the indices in (2-2) and write it as

$$H_i(L_{\Lambda_\pm}) \simeq \mathrm{LCH}^{n-i+2}_{\varepsilon_\pm}(\Lambda_\pm). \tag{2-3}$$

Using (2-3), we rewrite the Mayer–Vietoris long exact sequence (2-1) as

$$\begin{aligned} \to H_i(\Lambda_-) \to H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \\ \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to . \end{aligned} \tag{2-4}$$

We now write the long exact sequence for the pair $(L_{\Lambda_-}, L_{\Lambda_+})$

$$\to H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_i(L_{\Lambda_+}, L_{\Lambda_-}) \to H_{i-1}(L_{\Lambda_-}) \to . \tag{2-5}$$

Using (2-3) and the excision theorem for $L_{\Lambda_+}, L \subset L_{\Lambda_+}$, we write the long exact sequence (2-5) as

$$\begin{aligned} \to \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \\ \to H_i(L, \Lambda_-) \to \mathrm{LCH}^{n-i+3}_{\varepsilon_-}(\Lambda_-) \to . \end{aligned} \tag{2-6}$$

□

Remark 2.2. Under the conditions of Theorem 1.2, if $H_i(\Lambda_-) = H_{i-1}(\Lambda_-) = 0$ for some i, say when $\Lambda_- = S^n$ and $i, i-1 \neq 0, n$, then long exact sequence (2-4) implies that

$$\mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \simeq H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-).$$

Hence, for such i, we get

$$H_i(L) \simeq \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+)/\,\mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-).$$

Remark 2.3. We can rewrite the long exact sequences (2-4) and (2-6) using the relative symplectic field theory of $((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha)), L_{\Lambda_\pm})$, since

$$E^i_1\big((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha)), L_{\Lambda_\pm}\big) \simeq \mathrm{LCH}^i_{\varepsilon_\pm}(\Lambda_\pm) \tag{2-7}$$

over $\mathbb{Z}_2$. For the definition of the relative symplectic field theory, we refer to [Ekholm 2008], and for the details about the isomorphism described in (2-7), we refer to [Ekholm 2012]. (We observe that since $L_{\Lambda_\pm}$ are connected, the associated spectral sequences have only one level.)

3. Proof of Theorem 1.3

Let n be even. We recall the following result:

Proposition 3.1 [Eliashberg 1990]. *Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$, where n is even. Then*

$$\mathrm{tb}(\Lambda) = (-1)^{n/2+1}\tfrac{1}{2}\chi(\Lambda).$$

We now note that

$$\chi(\partial L) = 2\chi(L), \tag{3-1}$$

since the Euler characteristic of an even-dimensional boundary is twice the Euler characteristic of its bounded manifold; see Chapter 21 of [May 1999]. We now observe that $\partial L = \Lambda_+ \sqcup \Lambda_-$ and hence, from (3-1), we get that

$$2\chi(L) = \chi(\partial L) = \chi(\Lambda_+) + \chi(\Lambda_-). \tag{3-2}$$

Then we use Proposition 3.1 and rewrite (3-2) as

$$2\chi(L) = \chi(\Lambda_+) + \chi(\Lambda_-) = 2(-1)^{-n/2-1}(\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-)). \tag{3-3}$$

From (3-3) it follows that

$$\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L). \tag{3-4}$$

This finishes the proof of Theorem 1.3 in the case when n is even.

We now prove case (2) of the theorem. First we provide an alternate definition of the Thurston–Bennequin number, found in [Ekholm et al. 2005a].

Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$ and let c be a Reeb chord of Λ with end points a and b such that $z(a) > z(b)$. We define $V_a := d\Pi(T_a\Lambda)$ and $V_b := d\Pi(T_b\Lambda)$. Given an orientation on Λ, V_a and V_b are oriented n-dimensional transverse subspaces of $\mathbb{R}^{2n}$. If the orientation of $V_a \oplus V_b$ agrees with that of $\mathbb{R}^{2n}$, we say that the sign of c, denoted by $\mathrm{sign}(c)$, is $+1$, otherwise we say that it is -1. Then

$$\mathrm{tb}(\Lambda) = \sum_c \mathrm{sign}(c), \tag{3-5}$$

where the sum is taken over all Reeb chords c of Λ.

The following proposition was proven using (3-5):

Proposition 3.2 [Ekholm et al. 2005b]. *If $\Lambda \subset \mathbb{R}^{2n+1}$ is a closed, orientable, connected, chord generic Legendrian submanifold,*

$$\mathrm{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2}\sum_{c\in\mathscr{C}}(-1)^{|c|}.$$

We now construct an exact Lagrangian filling of Λ_+. We do it the same way as in the proof of Theorem 1.2, namely L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L in the symplectization of $\mathbb{R}^{2n+1}$.

By using Proposition 3.2 and taking Euler characteristics of the long exact sequence (1-2), we get

$$\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = (-1)^{((n-2)(n-1))/2+1}\chi(L). \tag{3-6}$$

This finishes the proof of Theorem 1.3 when n is odd. □

Remark 3.3. When $n = 1$ we can write (3-6) as

$$\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = -\chi(L),$$

which coincides with the formula from Theorem 1.2 of [Chantraine 2010].

Remark 3.4. Observe that the condition of Theorem 1.3 in the case when n is odd is much stronger than the condition of Theorem 1.3 in the case when n is even. If n is even, $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$ and $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$, then, taking Euler characteristics of the long exact sequence (1-2) and using Proposition 3.2, we get that

$$\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L).$$

The proof of Theorem 1.3 can be easily modified to become a proof of the following remark.

Remark 3.5. Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$.

(1) If n is even and $\varnothing \prec_{L_\Lambda} \Lambda$,

$$\mathrm{tb}(\Lambda) = (-1)^{n/2+1}\chi(L_\Lambda).$$

(2) If n is odd and $\varnothing \prec^{\mathrm{ex}}_{L_\Lambda} \Lambda$,

$$\mathrm{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2+1}\chi(L_\Lambda).$$

4. Examples

In this section, we describe a few examples of Lagrangian cobordisms. These examples are based on [Chantraine 2010; Ekholm et al. 2005b] and the work of Ekholm, Honda, and Kálmán [Ekholm et al. ≥ 2013]. For the constructions of Lagrangian cobordisms based on the generating families technique, we refer to [Bourgeois et al. ≥ 2013].

Example 4.1. *Proof of Proposition 1.4.* Let Λ_- and $\Lambda_+ \subset \mathbb{R}^{2n+1}$ be two closed, orientable Legendrian submanifolds which are Legendrian isotopic. Then there is a smooth isotopy of a closed manifold Λ to $\mathbb{R}^{2n+1}$ given by $\varphi : \Lambda \times [0, 1] \to \mathbb{R}^{2n+1}$ such that $\Lambda_\nu := \varphi(\Lambda, \nu)$ is Legendrian for all $\nu \in [0, 1]$, $\Lambda_- = \Lambda_0$ and $\Lambda_+ = \Lambda_1$. We now construct L such that $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$. Observe that in the construction below one can omit the assumption that Λ_-, Λ_+, L are connected. In the case of Legendrian knots in $\mathbb{R}^3$, the construction of L was described in [Chantraine 2010, Theorem 1.1]. In our case, the construction of Chantraine can be described in the following way.

(1) Note that $\mathbb{R}\times\Lambda_-$ is an exact Lagrangian submanifold of $(\mathbb{R}\times\mathbb{R}^{2n+1}, d(e^t\alpha))$.

(2) Theorem 2.6.2 of [Geiges 2008] implies that there is a compactly supported one-parameter family of contactomorphisms f_ν which realizes the isotopy $(\Lambda_\nu)_{\nu\in[0,1]}$.

(3) Proposition 2.2 from [Chantraine 2010] implies that a contactomorphism of $\mathbb{R}^{2n+1}$ lifts to a Hamiltonian diffeomorphism of the symplectization

$$(\mathbb{R}\times\mathbb{R}^{2n+1}, d(e^t\alpha)).$$

(4) Let H be a Hamiltonian on $\mathbb{R}\times\mathbb{R}^{2n+1}$ whose flow realizes the lifts of f_νs. The existence of H follows from (3). Following Chantraine, we construct

$$H' : \mathbb{R}\times\mathbb{R}^{2n+1}\times[0,1]\to\mathbb{R}$$

such that

$$H'(t,x,\nu)=\begin{cases} H(t,x,\nu) & \text{for } t>T;\\ 0 & \text{for } t<-T.\end{cases}$$

Here $T\gg 0$.

(5) Let ϕ^ν be the Hamiltonian flow of H'. We now observe that $\phi^1(\mathbb{R}\times\Lambda_-)$ coincides with $\mathbb{R}\times\Lambda_-$ near $-\infty$ and with $\mathbb{R}\times\Lambda_+$ near ∞.

(6) Since $\mathbb{R}\times\Lambda_-$ is exact and ϕ^1 a Hamiltonian diffeomorphism, $L:=\phi^1(\mathbb{R}\times\Lambda_-)$ is exact. □

Remark 4.2. Eliashberg and Gromov [1998] provided another proof of the fact that Legendrian isotopy implies Lagrangian cobordism.

Example 4.3. *Proof of Proposition 1.5.* The following construction is based on the front spinning method invented in [Ekholm et al. 2005b].

First we recall the notion of the front projection. The *front projection* is a map Π_F from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{n+1}$ defined by

$$\Pi_F(x_1,y_1,\dots,x_n,y_n,z)=(x_1,x_2,\dots,x_n,z).$$

Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$ parametrized by $f_\Lambda:\Lambda\to\mathbb{R}^{2n+1}$. We write

$$f_\Lambda(p)=(x_1(p),y_1(p),\dots,x_n(p),y_n(p),z(p))$$

for $p\in\Lambda$. The front projection of Λ is parametrized by $\Pi_F\circ f_\Lambda$, and we have

$$\Pi_F\circ f_\Lambda(p)=(x_1(p),x_2(p),\dots,x_n(p),z(p)).$$

Without loss of generality we can assume that $x_1(p) > 0$ for all $p \in \Lambda$. We now embed $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+2}$ via

$$(x_1, \ldots, x_n, z) \to (x_0 = 0, x_1, \ldots, x_n, z)$$

and construct the suspension of Λ, denoted by $\Sigma\Lambda$, such that $\Pi_F(\Sigma\Lambda)$ is obtained from $\Pi_F(\Lambda)$ by rotating it around the subspace $x_0 = x_1 = 0$. $\Pi_F(\Sigma\Lambda)$ can be parametrized by $(x_1(p)\sin\theta, x_1(p)\cos\theta, x_2(p), \ldots, x_n(p), z(p))$ with $\theta \in S^1$ and is the front projection of a Legendrian embedding $\Lambda \times S^1 \to \mathbb{R}^{2n+3}$. For the properties of $\Sigma\Lambda$ we refer to Lemma 4.16 of [Ekholm et al. 2005b].

Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$ such that

$$\Lambda_\pm \subset \{(x_1, y_1, \ldots, x_n, y_n, z) \in \mathbb{R}^{2n+1} \mid x_1 > 0\} \tag{4-1}$$

and $\Lambda_- \prec_L^{\mathrm{lag}} \Lambda_+$. Let L be parametrized by $f_L : L \to \mathbb{R}^{2n+2}$

$$f_L(p) = (t(p), x_1(p), y_1(p), \ldots, x_n(p), y_n(p), z(p)).$$

Without loss of generality we assume that $x_1(p) > 0$ for all p. (Formula (4-1) implies that

$$\{f_L(p) \mid x_1(p) \leq 0\}$$

is compact and we can translate L so that $x_1(p) > 0$ for all p.) Then we construct a Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$ that we call ΣL. We define ΣL to be parametrized by

$$f_{\Sigma L} : L \times S^1 \to \mathbb{R} \times \mathbb{R}^{2n+3}$$

with

$$f_{\Sigma L}(p, \theta) = (t(p), x_1(p)\sin\theta, y_1(p)\sin\theta, x_1(p)\cos\theta, y_1(p)\cos\theta, x_2(p), \ldots, z(p)).$$

Here $p \in L$ and $\theta \in S^1$.

We now show that ΣL is really a Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$. Let

$$\Lambda_+^{T_L} := \{(x_0, \ldots, y_n, z) \mid (T_L, x_0, \ldots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{T_L\} \times \mathbb{R}^{2n+3})\},$$
$$\Lambda_-^{T_L} := \{(x_0, \ldots, y_n, z) \mid (-T_L, x_0, \ldots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{-T_L\} \times \mathbb{R}^{2n+3})\}.$$

From the definition of T_L, it follows that

$$f_{\Sigma L}(\Sigma L) \cap ([T_L, \infty) \times \mathbb{R}^{2n+3}) = [T_L, \infty) \times \Lambda_+^{T_L},$$
$$f_{\Sigma L}(\Sigma L) \cap ((-\infty, -T_L] \times \mathbb{R}^{2n+3}) = (-\infty, -T_L] \times \Lambda_-^{T_L}.$$

In addition, we observe that $\Lambda_{\pm}^{T_L} \subset \mathbb{R}^{2n+3}$ can be parametrized by

$$f_{\Lambda_{\pm}^{T_L}} : \Lambda_{\pm} \times S^1 \to \mathbb{R}^{2n+3}$$

such that

$$f_{\Lambda_{\pm}^{T_L}}(p,\theta) = (x_1(p)\sin\theta,\, y_1(p)\sin\theta,\, x_1(p)\cos\theta,\, y_1(p)\cos\theta,\, x_2(p),\dots,z(p)).$$

Here $p \in \Lambda_{\pm} \subset \partial L$ and $\theta \in S^1$. We now prove that $\Lambda_{\pm}^{T_L}$ coincides with $\Sigma\Lambda_{\pm}$. It is clear that $\Pi_F(\Lambda_{\pm}^{T_L}) = \Pi_F(\Sigma\Lambda_{\pm})$. It remains to prove that $\Lambda_{\pm}^{T_L}$ is a Legendrian submanifold of $\mathbb{R}^{2n+3}$.

It is easy to see that

$$\text{(4-2)}\quad f^*_{\Lambda_{\pm}^{T_L}}\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big) = dz(p) - \sum_{i=2}^{n} y_i(p)\, dx_i(p) - y_1(p)(\sin^2\theta + \cos^2\theta)\, dx_1(p) + (y_1(p)x_1(p)\sin\theta\cos\theta - y_1(p)x_1(p)\sin\theta\cos\theta)\, d\theta.$$

Since $\Lambda_{\pm}$ is a Legendrian submanifold of $\mathbb{R}^{2n+1}$ and so $f^*_{\Lambda_{\pm}}(dz - \sum_{i=1}^{n} y_i\, dx_i) = 0$, we have

$$\text{(4-3)}\quad y_1(p)\, dx_1(p) = dz(p) - \sum_{i=2}^{n} y_i(p)\, dx_i(p).$$

Hence (4-2) and (4-3) imply that

$$\text{(4-4)}\quad f^*_{\Lambda_{\pm}^{T_L}}\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big) = 0.$$

Since

$$f_{\Lambda_{\pm}}(p) := (x_1(p),\dots,y_n(p),z(p)),$$

where $p \in \Lambda_{\pm} \subset \partial L$ is a parametrization of an embedded submanifold of dimension n, and $x_1(p) > 0$ for $p \in \Lambda_{\pm} \subset \partial L$, one easily sees that

$$f_{\Lambda_{\pm}^{T_L}}(p) = (x_1(p)\sin\theta,\, y_1(p)\sin\theta,\, x_1(p)\cos\theta,\, y_1(p)\cos\theta,\, x_2(p),\dots,z(p)),$$

where $p \in \Lambda_{\pm}$, $\theta \in S^1$, is a parametrization of an embedded submanifold of dimension $n+1$. Thus, using (4-4), we see that $\Lambda_{\pm}^{T_L}$ is an embedded Legendrian submanifold of $\mathbb{R}^{2n+3}$ whose front projection coincides with $\Pi_F(\Sigma\Lambda_{\pm})$. Thus we get that $\Lambda_{\pm}^{T_L} = \Sigma\Lambda_{\pm}$.

We now note that

$$f_{\Sigma L}^{*}\left(d\left(e^{t}\left(dz-\sum_{i=0}^{n} y_i dx_i\right)\right)\right)=e^{t}(dt(p)\wedge dz(p)-\sum_{i=2}^{n} dy_i(p)\wedge dx_i(p) \tag{4-5}$$
$$-\sum_{i=2}^{n} y_i(p)dt(p)\wedge dx_i(p)-(y_1(p)(\sin^2\theta+\cos^2\theta)dt(p)\wedge dx_1(p)$$
$$+(\sin^2\theta+\cos^2\theta)dy_1(p)\wedge dx_1(p)+(\sin^2\theta+\cos^2\theta)x_1(p)y_1(p)d\theta\wedge d\theta$$
$$+(y_1(p)x_1(p)\sin\theta\cos\theta-y_1(p)x_1(p)\sin\theta\cos\theta)dt(p)\wedge d\theta$$
$$+(y_1(p)\sin\theta\cos\theta-y_1(p)\sin\theta\cos\theta)d\theta\wedge dx_1(p)$$
$$+(x_1(p)\sin\theta\cos\theta-x_1(p)\sin\theta\cos\theta)dy_1(p)\wedge d\theta)).$$

In addition, observe that

$$e^{t}(dt(p)\wedge dz(p)-\sum_{i=2}^{n} dy_i(p)\wedge dx_i(p)-\sum_{i=2}^{n} y_i(p)dt(p)\wedge dx_i(p)) \tag{4-6}$$
$$=e^{t}(y_1(p)dt(p)\wedge dx_1(p)+dy_1(p)\wedge dx_1(p)).$$

Hence (4-5) and (4-6) imply that

$$f_{\Sigma L}^{*}\left(d\left(e^{t}\left(dz-\sum_{i=0}^{n} y_i dx_i\right)\right)\right)=0. \tag{4-7}$$

Since

$$f_L(p)=(t(p),x_1(p),y_1(p),\dots,x_n(p),y_n(p),z(p)),$$

where $p\in L$, is a parametrization of an embedded cobordism of dimension $n+1$ and $x_1(p)>0$ for $p\in L$, one easily sees that

$$f_{\Sigma L}(p,\theta)$$
$$=(t(p),x_1(p)\sin\theta,y_1(p)\sin\theta,x_1(p)\cos\theta,y_1(p)\cos\theta,x_2(p),\dots,z(p)),$$

where $p\in L$ and $\theta\in S^1$, is a parametrization of an embedded cobordism of dimension $n+2$. Hence we use (4-7) and see that ΣL is really an embedded Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$.

We now assume that $\Lambda_- \prec_L^{\mathrm{ex}} \Lambda_+$. Then there is a function $h_L\in C^\infty(f_L(L),\mathbb{R})$ such that

$$dh_L=e^{t}\left(dz-\sum_{i=1}^{n} y_i dx_i\right).$$

From a calculation similar to (4-2) it follows that

$$f_{\Sigma L}^{*}\left(e^{t}\left(dz-\sum_{i=0}^{n} y_i\,dx_i\right)\right)=e^{t(p)}\left(dz(p)-\sum_{i=1}^{n} y_i(p)\,dx_i(p)\right). \tag{4-8}$$

Since $f_{\Sigma L}$ is an embedding, we can define $h_{\Sigma L} \in C^\infty(f_{\Sigma L}(\Sigma L), \mathbb{R})$ by setting

$$(f_{\Sigma L}^* h_{\Sigma L})(p, \theta) := (f_L^* h_L)(p).$$

Hence we use (4-8) and get

(4-9)
$$d(f_{\Sigma L}^* h_{\Sigma L}) = e^{t(p)}\Big(dz(p) - \sum_{i=1}^{n} y_i(p)\, dx_i(p)\Big) = f_{\Sigma L}^*\Big(e^t\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big)\Big).$$

Therefore, since $f_{\Sigma L}$ is an embedding, (4-9) implies that

$$d(h_{\Sigma L}) = e^t\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big).$$

Hence, ΣL is an exact Lagrangian cobordism. □

Note that the proof of Proposition 1.5 can be easily modified to become a proof of the following remark.

Remark 4.4. Let Λ be a closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. If $\varnothing \prec^{\text{lag}}_{L_\Lambda} \Lambda$, there exists a Lagrangian filling $L_{\Sigma\Lambda}$ such that $\varnothing \prec^{\text{lag}}_{L_{\Sigma\Lambda}} \Sigma\Lambda$. In addition, if $\varnothing \prec^{\text{ex}}_{L_\Lambda} \Lambda$, there exists an exact Lagrangian filling $L_{\Sigma\Lambda}$ such that $\varnothing \prec^{\text{ex}}_{L_{\Sigma\Lambda}} \Sigma\Lambda$.

Before we discuss the next example, we briefly recall a few facts about exact Lagrangian cobordisms between Legendrian knots in $\mathbb{R}^3$.

Theorem 4.5 [Ekholm et al. ≥ 2013; Ekholm et al. 2007]. *There exists an exact Lagrangian cobordism for the following*:

(1) *Legendrian isotopy*,

(2) 0-*resolution at a contractible crossing in the Lagrangian projection*,

(3) *capping off a* $\operatorname{tb} = -1$ *unknot with a disk.*

See Figure 1 for the 0-resolution on the Lagrangian projection.

Following Ekholm, Honda, and Kálmán, we say that a *contractible crossing* of Λ is a crossing so that $z_1 - z_0$ can be shrunk to zero without affecting the other crossings. (Here z_1 is the z-coordinate on the upper strand and z_0 is the z-coordinate on the lower strand.)

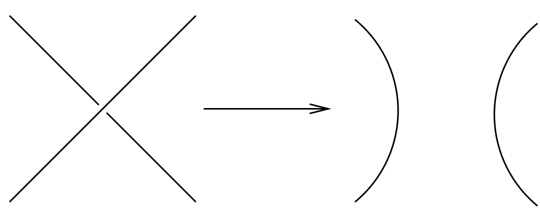

Figure 1. The 0-resolution on the Lagrangian projection.

Remark 4.6. Chantraine [2010] proved the first part of Theorem 4.5.

Remark 4.7. Note that the second part of Theorem 4.5 can be proven using the model from Section 3.3 of [Rizell 2012].

Conjecture 4.8 [Ekholm et al. ≥ 2013; Ekholm et al. 2007]. If $\varnothing \prec^{\mathrm{ex}}_{L_\Lambda} \Lambda$, then L_Λ is obtained by stacking exact Lagrangians cobordisms described in Theorem 4.5.

Example 4.9. *Proof of Proposition 1.6.* We now use Example 4.3 to get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$. We first recall that Theorem 4.5 says that 0-resolution at a contractible crossing in the Lagrangian projection can be realized as an exact Lagrangian cobordism. Let T_{2k+1} be the Legendrian torus knot from Example 4.18 of [Ekholm et al. 2005b]; see Figure 2 for the Lagrangian projection of T_{2k+1}. One observes that all the crossings in the middle part of the Lagrangian projection are contractible (see [Ekholm et al. 2007] for the case of T_3) and hence one can get T_{2k-1} from T_{2k+1} by contracting c_{2k+1} and then c_{2k}. Let L^{2k+1}_{2k} be an exact Lagrangian cobordism which corresponds to the 0-resolution at c_{2k+1} and L^{2k}_{2k-1} an exact Lagrangian cobordism from T_{2k-1} to T_{2k} which corresponds to the resolution of c_{2k}. Then we stack L^{2k+1}_{2k} and L^{2k}_{2k-1} and get an exact Lagrangian cobordism that we call L^{2k+1}_{2k-1} such that

$$T_{2k-1} \prec^{\mathrm{ex}}_{L^{2k+1}_{2k-1}} T_{2k+1}.$$

If we stack L^{2i+1}_{2i-1}s we get an exact Lagrangian cobordism L^{2k+1}_{2j+1} such that

$$T_{2j+1} \prec^{\mathrm{ex}}_{L^{2k+1}_{2j+1}} T_{2k+1}$$

for $k > j$. We use the construction described in Example 4.3 and get

$$\Sigma^n T_{2j+1} \prec^{\mathrm{ex}}_{\Sigma^n L^{2k+1}_{2j+1}} \Sigma^n T_{2k+1}$$

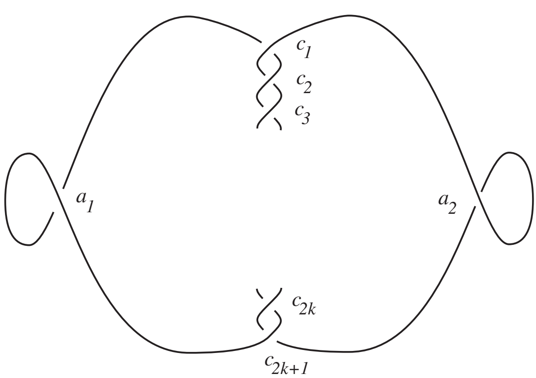

Figure 2. The knot T_{2k+1}; cf. Figure 13 of [Ekholm et al. 2005b] .

for $k > j$. We now recall that Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b, Theorem 4.19] proved that $\Sigma^n T_{2j+1}$ is not Legendrian isotopic to $\Sigma^n T_{2k+1}$ for $k > j+1$ and $j \in \mathbb{N}$.

Hence we get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$. □

Remark 4.10. Given $n \geq 1$, we observe that Theorem 4.19 of [Ekholm et al. 2005b] implies that all the Legendrian n-tori from Proposition 1.6 are not distinguished by the classical invariants.

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