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(DN)-(Ω)-TYPE CONDITIONS FOR FRÉCHET OPERATOR SPACES

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We introduce (DN)-(Ω)-type conditions for Fréchet operator spaces. We investigate which quantizations carry over the above conditions from the underlying Fréchet space onto the operator space structure. This holds in particular for the minimal and maximal quantizations in case of a Fréchet space and — additionally — for the row, column and Pisier quantizations in case of a Fréchet–Hilbert space. We also reformulate these conditions in the language of matrix polars.

1. Introduction

The aim of this paper is to continue building a satisfactory theory for Fréchet operator spaces. The first motivation comes from the work of Effros and Webster [1997] and Effros and Winkler [1997] who started to build such a theory. The setting in both of these articles is very general — they define the operator analogues of arbitrary locally convex spaces. Another paper dealing with local analogues of operator spaces is [Beien and Dierolf 2001]. Motivated by the preface to the book of Effros and Ruan [2000] we restrict ourselves to the class of Fréchet spaces. Moreover the structure theory of Fréchet spaces is highly developed. One of the aspects of this structure theory are the so-called (DN)-(Ω) type conditions which play a very important role in several problems. They appear in the splitting theory of short exact sequences; see [Meise and Vogt 1997, Chapter 30; Poppenberg and Vogt 1995]. They play a role in characterizing when $L(X, Y) = LB(X, Y)$ that is, when every linear and continuous operator between Fréchet spaces is bounded in the sense that it maps some zero neighborhood into a bounded set; see [Meise and Vogt 1997, Chapter 29; Vogt 1983]. These conditions appear also in the lately defined concept of tameness; see [Dubinsky and Vogt 1989; Piszczek 2009]. Both boundedness and tameness are strongly connected with the longstanding open problem of Pełczyński of whether every complemented subspace of a nuclear Fréchet space with a basis

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has a basis itself. So far all known tame Fréchet spaces bring a positive answer to Pełczyński's question. In this paper we will try to build a theory that will enable us to follow the above described point of view.

Section 2 recalls the basic and necessary definitions of the objects we deal with together with the definitions of our conditions. In Section 3 we investigate which quantizations satisfy the operator (DN)-(Ω)-type conditions whenever the underlying Fréchet space possesses any of these properties. The main result is contained in Theorem 14. Recall that many natural Fréchet spaces are nuclear and by [Effros and Webster 1997, Theorem 7.4] such a space has only one quantization (up to a complete isomorphism). Therefore if some quantization of a nuclear space carries over our conditions then any other does, and so it does not seem to be interesting to consider various quantizations for such spaces. However there do exist Fréchet spaces that are not nuclear but seem to be important (see [Taskinen 1991]). Therefore we believe the content of Section 3 is useful. Section 4 shows our conditions from another point of view. We are able to rewrite (oDN) and $(o\Omega)$ in the language of matrix polars.

For unexplained details we refer the reader to [Meise and Vogt 1997] in case of the structure theory of Fréchet spaces and to [Effros and Ruan 2000] and [Pisier 2003] in case of the operator space theory.

2. Preliminaries

Recall that a Fréchet space X is a locally convex space that is metrizable and complete. The topology of such a space can always be given by a nondecreasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms and in this case $X = \operatorname{proj}_k X_k$, where $(X_k, \|\cdot\|_k)$ are local Banach spaces and $\iota_k^{k+1}\colon X_{k+1} \to X_k$ are the linking maps. The closed unit ball in the k-th seminorm in the space X will be usually denoted by U_k and its polar by U_k°, i.e., $U_k^\circ = \{x' \in X'\colon |x'x| \leqslant 1\ \forall\, x \in U_k\}$. The closed unit ball in the k-th local Banach space X_k will be denoted by B_{X_k}. Following [Effros and Webster 1997] we define a *Fréchet operator space* to be the projective limit of a sequence of operator spaces with the linking maps being completely bounded. To indicate this we will sometimes write $X = \text{m-}\operatorname{proj}_k X_k$. Usually it will be clear from the context what kind of projective limit we deal with, therefore we will omit the symbol m-. This means that the Fréchet space $M_n(X)$ of $n \times n$ matrices with entries in X is given by $M_n(X) = \operatorname{proj}_k M_n(X_k)$ and the linking maps are just

$$\big(\iota_k^{k+1}\big)_n\colon\ M_n(X_{k+1}) \to M_n(X_k), \qquad \big(\iota_k^{k+1}\big)_n\big((x_{ij})_{i,j=1}^n\big) := \big(\iota_k^{k+1} x_{ij}\big)_{i,j=1}^n.$$

By $M_n(X')$ we mean the linear space of all completely bounded maps $\phi\colon X \to M_n$. Using [Effros and Ruan 2000, Lemma 4.1.1] we see that $M_n(X') = T_n(X)'$ linearly and this isomorphism allows us to endow $M_n(X')$ with the (DF)-topology (recall that here $T_n(X) = \mathrm{proj}_k T_n(X_k)$ is a Fréchet space therefore its dual is a (DF)-space).

We can also quantize Fréchet spaces. If $X = \mathrm{proj}_k X_k$ is a Fréchet space and $Q\colon \mathfrak{B} \to \mathfrak{O}$ is a strict quantization from the category of Banach spaces into the category of operator spaces then by definition

$$Q(X) := \text{m-proj}_k \mathfrak{Q}(X_k).$$

For convenience we will write

$$\min X = \mathrm{proj}_k \min X_k, \quad \max X = \mathrm{proj}_k \max X_k,$$

and in case of Fréchet–Hilbert spaces

$$H_c = \mathrm{proj}_k (H_k)_c, \quad H_r = \mathrm{proj}_k (H_k)_r, \quad OH = \mathrm{proj}_k OH_k.$$

Let us recall that by [Effros and Webster 1997, Theorem 7.4], all the quantizations for nuclear Fréchet spaces are equal (up to a complete isomorphism).

Examples. 1. The space $C(\mathbb{R}) = \mathrm{proj}_k C([-k, k])$ of continuous functions on the real line is a Fréchet space that carries an operator space structure. In $M_n(C(\mathbb{R}))$ we define seminorms

$$\|(f_{ij})\|_k := \sup\{\|(f_{ij}(x))\|_{M_n} : x \in [-k, k]\}.$$

In a similar fashion we can introduce an operator space structure on the spaces $C^\infty(K)$, $C^\infty(\Omega)$ for arbitrary subsets K compact and Ω open of $\mathbb{R}^d$.

2. In order to give an example of a Fréchet operator space arising in quantum physics, let

$$s = \left\{ x = (x_j)_{j \in \mathbb{N}} : \|x\|_k^2 := \sum_{j=1}^{+\infty} |x_j|^2 j^{2k} < +\infty \ \forall\, k \in \mathbb{N} \right\},$$

be the (nuclear) Fréchet space of rapidly decreasing sequences, with the topology given by the sequence of norms $(\|\cdot\|_k)_{k\in\mathbb{N}}$: in short,

$$s = \mathrm{proj}_k \ell_2((j^k)_j).$$

Following [Dubin and Hennings 1990] we call $s \tilde{\otimes}_\pi s$ the space of physical states and we endow it with the Fréchet operator space structure

$$\mathcal{T}_{op} = s_r \hat{\otimes}_{op} s_c,$$

where $\hat{\otimes}_{op}$ stands for the operator projective tensor product.

3. A moment's reflection shows that the above space $\mathcal{T}_{op}$ is in fact $L(s', s)$ with a suitable operator space structure. We can generalize this by introducing such a

structure on $L(X', Y)$ for arbitrary Fréchet spaces X and Y. Recall that $L(X', Y)$ is a Fréchet space with a sequence $(|||\cdot|||_k)_k$ of seminorms defined by

$$|||T|||_k := \sup\{\|Tf\|_k : f \in U_k^\circ\},$$

where $(U_k)_k$ is a zero neighborhood basis in X and $(\|\cdot\|_k)_k$ defines the topology of Y. Then the linear isomorphism $M_n(L(X', Y)) = L(\ell_p^n(X)', \ell_p^n(Y))$ for arbitrary $1 \leqslant p \leqslant +\infty$ provides $L(X', Y)$ with an operator space structure.

Let us now define the operator analogues of the conditions (DN) and (Ω).

Definition 1. (i) We will say that a Fréchet operator space X satisfies the property (oDN) if there exists a seminorm p such that for any other seminorm q and arbitrary number $\tau \in (0, 1)$ there exist another seminorm r and a constant $C > 0$ such that the inequality

$$\|(x_{ij})\|_q \leqslant C\big(\|(x_{ij})\|_p\big)^{1-\tau}\big(\|(x_{ij})\|_r\big)^{\tau} \tag{2-1}$$

holds for every matrix $(x_{ij}) \in M_n(X)$ of arbitrary size $n \in \mathbb{N}$.

(ii) We will say that a Fréchet operator space X satisfies the property $(o\Omega)$ if for every seminorm p there exists another seminorm q such that for any other seminorm r there exist a number $\theta \in (0, 1)$ and a constant $C > 0$ such that the inequality

$$\|(\phi_{ij})\|_q^* \leqslant C\big(\|(\phi_{ij})\|_p^*\big)^{\theta}\big(\|(\phi_{ij})\|_r^*\big)^{1-\theta} \tag{2-2}$$

holds for every matrix $(\phi_{ij}) \in M_n(X')$ of arbitrary size $n \in \mathbb{N}$.

Remarks. 1. If one of the above conditions holds for a Fréchet operator space X then we write (respectively) $X \in (oDN)$, $X \in (o\Omega)$.

2. If $X \in (oDN)$ then the seminorm p is in fact a norm and so all the seminorms become norms.

3. In the above definition the symbol $\|(\phi_{ij})\|_k^*$ stands for the cb-norm of a map $(\phi_{ij})\colon X \to M_n$. We stress that — in general — it is finite for all but finitely many k.

4. If we restrict the above definitions to $n = 1$ then we get the classical (DN) and (Ω) conditions of Vogt; see [Meise and Vogt 1997, page 367].

5. There are other versions of these conditions: if we change the quantifiers in (1) to "$\ldots\exists\, r \in \mathbb{N}, \tau \in (0, 1)\ldots$" then we get the condition $(o\underline{DN})$. If we change in (2) the quantifiers to "$\forall\, p \in \mathbb{N}, \theta \in (0, 1)\ldots$" then we get the condition $(o\overline{\Omega})$ and the change to "$\ldots\forall\, r \in \mathbb{N}, \theta \in (0, 1)\ldots$" leads to the condition $(o\overline{\overline{\Omega}})$. We have obvious implications

$$(o\overline{\overline{\Omega}}) \Rightarrow (o\overline{\Omega}) \Rightarrow (o\Omega), \quad (oDN) \Rightarrow (o\underline{DN}).$$

6. It is not difficult to show (see [Meise and Vogt 1997, Lemma 29.10]) that (oDN) is satisfied whenever (2-1) holds with $\tau = \frac{1}{2}$.

7. Recall that by [Tomiyama 1983, Lemma 1.1] (compare also [Paulsen 2002, page 41]) we have for all operator spaces

$$\|(a_{ij})\| \leqslant \left(\sum_{i,j=1}^{n} \|a_{ij}\|^2\right)^{1/2} \leqslant n\|(a_{ij})\|.$$

Therefore if X has (DN) as a Fréchet space then all the Fréchet spaces $M_n(X)$ satisfy this property with some constant $C_n = C_{p,q,r}(n)$. The point is that these constants be uniformly bounded (with respect to the matrix size n). The same can be observed for (Ω).

8. Both conditions are invariants in the category of Fréchet operator spaces.

Proposition 2. *The Fréchet operator space $\mathcal{T}_{op} = s_r \hat{\otimes}_{op} s_c$ of physical states satisfies both properties (DN) and (Ω).*

Proof. By [Effros and Webster 1997, Theorem 7.5] and the commutativity of $\hat{\otimes}_{op}$ we have the complete isomorphism

$$\mathcal{T}_{op} = s_c \check{\otimes}_{op} s_r = \operatorname{proj}_k\big(\ell_2(j^k)_c \check{\otimes}_{op} \ell_2(j^k)_r\big),$$

where $\check{\otimes}_{op}$ stands for the operator injective tensor product. Applying [Effros and Ruan 2000, 9.3.1 and 9.3.4] we get the complete isometry

$$\ell_2(j^k)_c \check{\otimes}_{op} \ell_2(j^k)_r \cong \mathcal{K}\big(\ell_2(j^k)', \ell_2(j^k)\big),$$

where $\mathcal{K}$ stands for the compact operators. Therefore

$$\begin{aligned} M_n(s_c \check{\otimes}_{op} s_r) &= \operatorname{proj}_k \mathcal{K}\big(\ell_2^n(\ell_2(j^k))', \ell_2^n(\ell_2(j^k))\big) \\ &= \operatorname{proj}_k\big(\ell_2^n(\ell_2(j^k)) \widetilde{\otimes}_\varepsilon \ell_2^n(\ell_2(j^k))\big) = \ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s). \end{aligned}$$

By [Meise and Vogt 1997, Lemma 29.2] the space s satisfies (DN) and it is easy to see that $\ell_2^n(s)$ satisfies this condition with exactly the same constant C in (2-1). Applying [Piszczek 2010, Theorem 4] we observe that $\ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s)$ satisfies (DN) with constant C independent of n therefore $\mathcal{T}_{op} \in (oDN)$. In order to show the other property let us recall that by [Meise and Vogt 1997, Lemma 29.11] $s \in (\Omega)$ therefore $\ell_2^n(s) \in (\Omega)$ (with unchanged constants). By [Piszczek 2010, Theorem 5] $\ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s)$ satisfies (Ω) with constants C independent of the matrix size. This shows that $\mathcal{T}_{op} \in (o\Omega)$. □

The (DN)-(Ω)-type conditions have equivalent forms which are often used in proofs. Since we will be using these equivalent forms extensively in the sequel we state them below for convenience of the reader.

Theorem 3 [Vogt 1977; Vogt and Wagner 1980]. *Let X be a Fréchet space.*

(1) *X satisfies the property (DN) if and only if there exists a seminorm p such that for any other seminorm q there exist another seminorm r and a constant $C > 0$ such that the inclusion*

$$U_q^\circ \subset sU_p^\circ + \frac{C}{s} U_r^\circ$$

is satisfied for all numbers $s > 0$.

(2) *X satisfies the property (Ω) if and only if for every seminorm p there exists another seminorm q such that for any other seminorm r there exist a number $\gamma > 0$ and a constant $C > 0$ such that the inclusion*

$$U_q \subset sU_p + \frac{C}{s^\gamma} U_r$$

is satisfied for all numbers $s > 0$.

3. Hereditary properties of quantizations

Now we will try to answer the following question: Suppose X has one of the properties (DN) or (Ω). Which quantizations of X automatically satisfy the operator analogues of these conditions? Let us indicate that the proofs are exactly the same for all versions of (DN)-type conditions as well as those of (Ω) type. Therefore we will always give proofs precisely for (DN) and (Ω). Moreover the results are formulated in such a way that only sufficiency will require an argument. First we focus on the condition (DN). We start this a little bit technical section with the following result.

Proposition 4. *Let X be a Fréchet space. Then X satisfies (DN) if and only if* $\min X$ *satisfies (oDN).*

Proof. Recall that for arbitrary $n \in \mathbb{N}$ we have in $M_n(\min X)$ the seminorms

$$\|(x_{ij})\|_k = \sup\left\{\|\xi(x_{ij})\|_{M_n} : \xi \in U_k^\circ\right\},$$

where U_k° is the polar of the zero neighborhood U_k. Choosing all the parameters according to (2-1) and assuming $X \in (DN)$ we obtain by Theorem 3 a chain of inequalities

$$\begin{aligned}\|(x_{ij})\|_q &\leqslant \sup\left\{\left\|\left(s\xi(x_{ij}) + Cs^{-1}\eta(x_{ij})\right)\right\|_{M_n} : \xi \in U_p^\circ, \eta \in U_r^\circ\right\}\\ &\leqslant s\|(x_{ij})\|_p + Cs^{-1}\|(x_{ij})\|_r.\end{aligned}$$

Taking the infimum over positive s we get

$$\|(x_{ij})\|_q^2 \leqslant 4C\|(x_{ij})\|_p\|(x_{ij})\|_r,$$

and since the constant C is independent of the matrix size (that is, C does not depend on n) we obtain the condition (oDN). □

In order to prove the analogous result for the max quantization we will need two lemmata.

Lemma 5. *Let X, E be locally convex spaces. Suppose $U, V \subset X$ and $B \subset E$ are absolutely convex subsets. Then $(U \cap V) \otimes B = U \otimes B \cap V \otimes B$.*

Proof. Since the inclusion $\subset$ is obvious we take an element ϕ with representations $\phi = x \otimes b \in U \otimes B$ and $\phi = y \otimes c \in V \otimes B$. If $g(b) = 0$ for all functionals $g \in E'$ then $b = 0$ and so $\phi = 0 \in (U \cap V) \otimes B$. Therefore we may suppose $f(b) \neq 0$ for some functional $f \in E'$. If $f(c) = 0$ then for every functional $x' \in X'$ we have

$$(x' \otimes f)(x \otimes b) = (x' \otimes f)(y \otimes c),$$

which means $x'x = 0$ for every $x' \in X'$. This gives $x = 0$ and $\phi = 0 \in (U \cap V) \otimes B$. So let us suppose $f(c) \neq 0$. We may also assume $|f(b)/f(c)| \leqslant 1$ (otherwise we take the inverse). For every $x' \in X'$ we get $x'(x)f(b) = x'(y)f(c)$ which leads to $y = (f(b)/f(c))x$. But we deal with absolutely convex sets, therefore $y \in U$ and so $\phi \in (U \cap V) \otimes B$, which shows the other inclusion. □

Lemma 6. *Let X be a Fréchet space and E a Banach space. If X has the property (DN) or (Ω) then their projective tensor product as well as the injective one satisfy the same condition too.*

Proof. We start with the projective tensor product. By [Köthe 1979, Chapter VIII, §41, 2(4)] one basis of zero neighborhoods in $X \tilde{\otimes}_\pi E$ has the form

$$\left(\overline{\Gamma(U_k \otimes B_E)}\right)_{k \in \mathbb{N}},$$

where $(U_k)_k$ is a basis of zero neighborhoods in X and B_E is the closed unit ball in E. Let us now assume $X \in (DN)$. By Theorem 3(1) we have

$$\frac{1}{2s} U_p \cap \frac{s}{2C} U_r \subset U_q.$$

Tensoring by B_E and taking polars we obtain, with the help of Lemma 5,

$$\left(U_q \otimes B_E\right)^\circ \subset \left(\frac{1}{2s}\left(U_p \otimes B_E\right) \cap \frac{s}{2C}\left(U_r \otimes B_E\right)\right)^\circ.$$

If now U and V are arbitrary zero neighborhoods in a locally convex space Y then by the Bipolar theorem we have

$$(U \cap V)^\circ \subset \overline{\Gamma(U^\circ + V^\circ)}^{\sigma(Y',Y)} = U^\circ + V^\circ,$$

the last equality being a consequence of absolute convexity and weak* compactness of U° and V°. Adapting the above inclusion to $Y = X \tilde{\otimes}_\pi E$ and the considered zero neighborhoods we get

$$\big(U_q \otimes B_E\big)^\circ \subset 2s\big(U_p \otimes B_E\big)^\circ + \frac{2C}{s}\big(U_r \otimes B_E\big)^\circ.$$

Taking $t = 2s$, $D = 4C$ and recalling that $\overline{\Gamma(A)}^\circ = A^\circ$ for every set A we arrive at

$$\overline{\Gamma(U_q \otimes B_E)}^\circ \subset t\overline{\Gamma(U_p \otimes B_E)}^\circ + \frac{D}{t}\overline{\Gamma(U_r \otimes B_E)}^\circ.$$

Again by Theorem 3(1) we obtain the property (DN) for the space $X\tilde{\otimes}_\pi E$. Moreover the crucial constant D in the above inclusion does not depend on the Banach space E but only on the Fréchet space X. The case of the condition (Ω) is even simpler since by Theorem 3(2) we have

$$\begin{aligned}\overline{\Gamma(U_q \otimes B_E)} &\subset \overline{s\Gamma(U_p \otimes B_E) + Cs^{-\gamma}\Gamma(U_q \otimes B_E)}\\ &\subset s\overline{\Gamma(U_p \otimes B_E)} + (C+1)s^{-\gamma}\overline{\Gamma(U_r \otimes B_E)}.\end{aligned}$$

Again the constant $C + 1$ depends only on the Fréchet space X.

In the case of the injective tensor product we recall that by [Köthe 1979, Chapter VIII, §44, 2(3)] one basis of zero neighborhoods in $X\tilde{\otimes}_\varepsilon E$ is of the form

$$\big((U_k^\circ \otimes B_E^\circ)^\circ\big)_{k\in\mathbb{N}}.$$

Suppose now that $X \in (\Omega)$. By Theorem 3(2) this gives

$$U_q \subset sU_p + Cs^{-\gamma}U_r,$$

whence the meaning of all the parameters follows. Using a technique similar to the one in the beginning of the proof we obtain

$$(U_q^\circ \otimes B_E^\circ)^\circ \subset \Big(\frac{1}{2s}(U_p^\circ \otimes B_E^\circ) \cap \frac{s^\gamma}{2C}(U_r^\circ \otimes B_E^\circ)\Big)^\circ.$$

By [Köthe 1969, Chapter IV, §20, 8(10)] we get

$$\begin{aligned}(U_q^\circ \otimes B_E^\circ)^\circ &\subset \overline{\Gamma\big\{2s\big(U_p^\circ \otimes B_E^\circ\big)^\circ \cup 2Cs^{-\gamma}\big(U_r^\circ \otimes B_E^\circ\big)^\circ\big\}}\\ &\subset \overline{2s\big(U_p^\circ \otimes B_E^\circ\big)^\circ + 2Cs^{-\gamma}\big(U_r^\circ \otimes B_E^\circ\big)^\circ}.\end{aligned}$$

But the sets under consideration are zero neighborhoods therefore we may drop the closure by increasing one of them which leads to

$$(U_q^\circ \otimes B_E^\circ)^\circ \subset 2s(U_p^\circ \otimes B_E^\circ)^\circ + 3Cs^{-\gamma}(U_r^\circ \otimes B_E^\circ)^\circ.$$

Now taking $t = 2s$, $D = 3 \cdot 2^\gamma C$ and applying Theorem 3(2) we arrive at the property (Ω) in the injective tensor product. To show that the property (DN) also passes onto $X\tilde{\otimes}_\varepsilon E$ we start with $u = \sum_{j=1}^{m} x_j \otimes a_j \in X \otimes_\varepsilon E$. By [Köthe 1979, Chapter VIII, §44, 2(5)] its seminorms are calculated as

$$\|u\|_k = \sup\left\{\sum_{j=1}^{m} |f(x_j)g(a_j)| : f \in U_k^\circ,\ g \in B_{E'}\right\}.$$

If X satisfies (*DN*) then for $f \in U_q^\circ$ we obtain by Theorem 3(1) functionals $f_1 \in U_p^\circ$ and $f_2 \in U_r^\circ$ with $f = sf_1 + Cs^{-1} f_2$. Consequently,

$$\sum_{j=1}^{m} |f(x_j)g(a_j)| \leqslant s \sum_{j=1}^{m} |f_1(x_j)g(a_j)| + Cs^{-1} \sum_{j=1}^{m} |f_2(x_j)g(a_j)|.$$

Taking the supremum over all such f, f_1, f_2 we get

$$\|u\|_q \leqslant s\|u\|_p + Cs^{-1}\|u\|_r,$$

and taking the infimum over all $s > 0$ we arrive at our condition. Finally it is easy to observe that the above property passes onto the completion, therefore $X\tilde{\otimes}_\varepsilon E$ satisfies (*DN*) and the constant C does not depend on the Banach space E. □

Proposition 7. *Let X be a Fréchet space. Then X satisfies* (*DN*) *if and only if* $\max X$ *satisfies* (*oDN*).

Proof. Recall that by [Effros and Ruan 2000, 3.3] for $(x_{ij}) \in M_n(\max(X))$ we have

$$\|(x_{ij})\|_k = \sup \left\| \big(f_{uv}(x_{ij})\big) \right\|_{M_{nm}},$$

where the supremum runs over all $(f_{uv}) \in L(X, M_m)$ with $\|(f_{uv})\|_{L(X_k,M_m)} \leqslant 1$ and all $m \in \mathbb{N}$. We have $L(X, M_m) = (X \otimes_\pi M_m)'$ by [Köthe 1979, Chapter VIII, §41, 3(3)] (since M_m is finite-dimensional we may drop the tensor product completion). Moreover, by [Meise and Vogt 1997, Remark 24.5(b)], $B_{L(X_k,M_m)} = (U_k \otimes B_{M_m})^\circ$. If X satisfies the property (*DN*) then by Lemma 6 we get

$$B_{L(X_q,M_m)} \subset sB_{L(X_p,M_m)} + Cs^{-1}B_{L(X_r,M_m)},$$

where all the parameters are chosen according to Theorem 3(1). Choosing (f_{uv}) in $L(X, M_m)$ with $\|(f_{uv})\|_{L(X_q,M_m)} \leqslant 1$ we obtain

$$\left\| \big(f_{uv}(x_{ij})\big) \right\|_{M_{nm}} = \left\| \big((sg_{uv} + Cs^{-1}h_{uv})(x_{ij})\big) \right\|_{M_{nm}}$$

for some $(g_{uv}) \in B_{L(X_p,M_m)}$, $(h_{uv}) \in B_{L(X_r,M_m)}$. Now taking the supremum over all such (f_{uv}), (g_{uv}), (h_{uv}) and all natural m we obtain

$$\|(x_{ij})\|_q \leqslant s\|(x_{ij})\|_p + Cs^{-1}\|(x_{ij})\|_r.$$

Finally, taking the infimum over all $s > 0$ we arrive at

$$\|(x_{ij})\|_q^2 \leqslant 4C\|(x_{ij})\|_p\|(x_{ij})\|_r,$$

and since the constant C is independent of the matrix size we obtain the condition (oDN). □

We now move to row and column quantizations of Fréchet–Hilbert spaces.

Proposition 8. *Let H be a Fréchet–Hilbert space.*

(1) *$H \in (DN)$ if and only if $H_r \in (oDN)$.*

(2) *$H \in (DN)$ of and only if $H_c \in (oDN)$.*

Proof. We start with the row quantization. Recall that by [Pisier 2003, page 22] the seminorms in $M_n(H_r)$ are given by the following formula: If $\phi_{ij} \in H$ for $i, j = 1, \dots, n$, then

$$\|(\phi_{ij})\|_k = \sup\left\{\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2\right)^{1/2} : (x_j)_{j=1}^n \in B_{\ell_2^n(H_k)'}\right\}.$$

If H satisfies the property (DN) then there exists p such that for all q we can find r and $C > 0$ with

$$\|h\|_q^2 \leqslant C\|h\|_p\|h\|_r, \quad \forall\, h \in H.$$

Using the Cauchy–Schwarz inequality we get

$$\|(h_j)\|_q^2 \leqslant C\|(h_j)\|_p\|(h_j)\|_r, \quad \forall\,(h_j) \in \ell_2^n(H)$$

with the same constant C. By Theorem 3(1) this gives

$$B_{\ell_2^n(H_q)'} \subset sB_{\ell_2^n(H_p)'} + Cs^{-1}B_{\ell_2^n(H_r)'}$$

for all positive s and a constant C independent of n. For arbitrary $(x_j)_{j=1}^n \in B_{\ell_2^n(H_q)'}$ the above inclusion allows us to find $(y_j)_{j=1}^n \in B_{\ell_2^n(H_p)'}$ and $(z_j)_{j=1}^n \in B_{\ell_2^n(H_r)'}$ with

$$\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2 = \sum_{i=1}^n \left|\sum_{j=1}^n \langle sy_j + Cs^{-1}z_j, \phi_{ij}\rangle\right|^2.$$

Applying once again the Cauchy–Schwarz inequality we arrive at

$$\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}} \leqslant s\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle y_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}} + Cs^{-1}\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle z_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}}.$$

Taking the supremum over all such (x_j), (y_j), (z_j) we obtain

$$\|(\phi_{ij})\|_q \leqslant s\|(\phi_{ij})\|_p + Cs^{-1}\|(\phi_{ij})\|_r.$$

Finally taking the infimum over all $s > 0$ leads to the condition (oDN). Moving to the column quantization we recall that by [Pisier 2003, page 22] the seminorms in

$M_n(H_c)$ are given by the following formula: If $\phi_{ij} \in H$ for $i, j = 1, \dots, n$, then

$$\|(\phi_{ij})\|_k = \sup\left\{\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_k^2\right)^{1/2} : (\xi_j)_{j=1}^n \in B_{\ell_2^n}\right\}.$$

If $H \in (DN)$ then by the Cauchy–Schwarz inequality we have for arbitrary $(\xi_j)_{j=1}^n$ in $B_{\ell_2^n}$ that

$$\begin{aligned}\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_q^2 &\leqslant C \sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_p \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_r \\ &\leqslant C\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_p^2\right)^{1/2} \left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_r^2\right)^{1/2}.\end{aligned}$$

This leads to

$$\|(\phi_{ij})\|_q^2 \leqslant C\|(\phi_{ij})\|_p\|(\phi_{ij})\|_r$$

with the constant C independent of the matrix size, therefore we conclude that the column quantization also carries over the property (DN). □

Proposition 9. *Let H be a Fréchet–Hilbert space.*

(1) $H \in (DN)$ *if and only if* $OH \in (oDN)$.

(2) $H \in (\Omega)$ *if and only if* $OH \in (o\Omega)$.

Proof. Recall that if K is a Hilbert space then by [Effros and Ruan 2000, Proposition 3.5.2] the norm in $M_n(OK)$ is given by

$$\|\phi\| = \|\langle\langle\phi, \phi\rangle\rangle\|^{1/2} := \left\|\left(\left(\langle\phi_{ij}, \phi_{kl}\rangle\right)_{k,l=1}^n\right)_{i,j=1}^n\right\|^{1/2},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in K. With the above notation the scalar matrix $\langle\langle\phi, \phi\rangle\rangle$ need not be positive in M_{n^2}, therefore (for the reasons that will become apparent shortly) we quickly describe how to change it isometrically into a positive one. Suppose $A = (A_{i,j})$ is in $M_n(M_n)$ and each $A_{i,j} = (a_{i,j,k,l}) \in M_n$. We reorder the first row of A in the following way: the first row of $A_{1,1}$ remains untouched, the first row of $A_{1,2}$ exchanges with the second row of $A_{1,1}$ and in general the first row of $A_{1,j}$ exchanges with the j-th row of $A_{1,1}$. Next the second row of $A_{1,2}$ remains untouched and the second row of $A_{1,j}$ $(j \geqslant 3)$ exchanges with the j-th row of $A_{1,2}$. We continue until the first row of A is completely reordered and apply the same procedure to any other row of A. Such a reordering (call it ρ) is an isometry and $\rho(\langle\langle\phi, \phi\rangle\rangle)$ is positive in M_{n^2}. Indeed, if $\xi = (\xi^i)_i \in \ell_2^n(\ell_2^n)$ and each $\xi^i = (\xi^i_j)_j \in \ell_2^n$ then

$$\langle \rho(\langle\!\langle \phi, \phi \rangle\!\rangle)\xi, \xi \rangle = \left\langle \sum_{i,j=1}^{n} \overline{\xi}^i_j \phi_{ij}, \sum_{k,l=1}^{n} \overline{\xi}^k_l \phi_{kl} \right\rangle = \left\| \sum_{i,j=1}^{n} \overline{\xi}^i_j \phi_{ij} \right\|^2, \tag{3-1}$$

and the last quantity is nonnegative. Suppose now H satisfies the condition (DN) and let p, q, r, C have the same meaning as in (2-1). Take n in $\mathbb{N}$ and $x = (x_{ij})$ in $M_n(OH)$. By (3-1) and positivity of $\rho(\langle\!\langle x, x \rangle\!\rangle)$ we get

$$\begin{aligned}
\|x\|_q^2 &= \|\rho(\langle\!\langle x, x \rangle\!\rangle_q)\|^2 \\
&= \sup\left\{ \left\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \right\|_q^2 : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \right\} \\
&\leqslant C \sup\left\{ \left\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \right\|_p : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \right\} \sup\left\{ \left\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \right\|_r : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \right\} \\
&= C\|x\|_p \|x\|_r.
\end{aligned}$$

Since the constant C does not depend on the matrix size n, we get the condition (oDN).

In order to prove the other equivalence recall first that for every functional $\phi \in H'$ we get a sequence of functionals $(\phi_k)_{k \geqslant k_0}$ acting on the local Hilbert steps which satisfy

$$\phi_k \circ \iota_k = \phi \quad (k \geqslant k_0),$$

where ι_ks are the canonical projections. We also have $\|\phi\|_k^* = \|\phi_k\|_{H_k'}$. If now $\phi = (\phi_{ij}) \in M_n((OH)')$ then we may find matrices $\phi_k = (\phi_{k,i,j}) \in M_n((OH_k)')$ of functionals such that

$$\|\phi\|_k^* = \|\phi_k\|_{(OH_k)'}. \tag{3-2}$$

By the selfduality of Pisier's quantization we have $\|\phi_k\|^*_{(OH_k)'} = \|\phi_k\|_{OH_k}$. Suppose now $H \in (\Omega)$ and let p, q, r, θ, C have the same meaning as in (2-2). We take $n \in \mathbb{N}$, $\phi = (\phi_{ij}) \in M_n(H')$ and apply exactly the same reasoning as above to obtain the inequality

$$\|\phi_q\|_{OH_q} \leqslant C \|\phi_p\|_{OH_p}^{\theta} \|\phi_r\|_{OH_r}^{1-\theta},$$

where the constant C does not depend on the matrix size n. Applying (3-2) we get the condition $(o\Omega)$. □

Now we will investigate the minimal and maximal quantizations in view of the condition (Ω). Here the Blecher duality will play an important role.

Proposition 10. *Let X be a Fréchet space. Then X satisfies (Ω) if and only if* $\max X$ *satisfies $(o\Omega)$.*

Proof. Recall that for an arbitrary Banach space X we have by [Blecher 1992, Corollary 2.8] that $(\max X)' = \min X'$ completely isometrically. Therefore if X is a Fréchet space then for $(\phi_{ij}) \in M_n((\max X)')$ we have

$$\|(\phi_{ij})\|_k = \sup\left\{\|x''(\phi_{ij})\|_{M_n} : x'' \in B_{X_k''}\right\}.$$

Taking together the Separation theorem [Köthe 1969, Chapter IV, §20, 7(1)] and the Bipolar theorem [Köthe 1969, Chapter IV, §20, 8(5)] it is enough to take in the above supremum vectors $x \in B_{X_k}$. By the density of U_k in B_{X_k} we may restrict ourselves to vectors $x \in U_k$. If X satisfies (Ω) then by Theorem 3(2) we get for arbitrary p a number q such that for all r there exist positive C and γ with

$$U_q \subset sU_p + \frac{C}{s^\gamma} U_r$$

for all $s > 0$. Repeating the proof of Proposition 4 we obtain the condition $(o\Omega)$. □

In order to prove the analogous result for the minimal quantization we will need a lemma. It seems to be known to specialists but for the sake of convenience we will state and prove it.

Proposition 11. *Let X be a Fréchet space.*

(1) *$X \in (DN)$ if and only if $X'' \in (DN)$.*

(2) *$X \in (\Omega)$ if and only if $X'' \in (\Omega)$.*

Proof. Suppose X satisfies the condition (DN). By Theorem 3(1) we find p such that for all q there exist r and $C > 0$ with

$$U_q^\circ \subset sU_p^\circ + Cs^{-1}U_r^\circ$$

for all $s > 0$. For arbitrary $x'' \in X''$ we have by [Meise and Vogt 1997, Proposition 25.9] that

$$\begin{aligned}\|x''\|_q &= \sup\left\{|x''x'| : x' \in U_q^\circ\right\}\\ &\leqslant\ s\sup\left\{|x''y'| : y' \in U_p^\circ\right\} + Cs^{-1}\sup\left\{|x''z'| : z' \in U_r^\circ\right\}\\ &= s\|x''\|_p + Cs^{-1}\|x''\|_r.\end{aligned}$$

taking the infimum over positive s we get the property (DN) for the bidual. Since by [Meise and Vogt 1997, Corollary 25.10] every Fréchet space is a topological subspace of its bidual, the converse of (1) follows. Suppose now that X satisfies the condition (Ω). By Theorem 3(2) we get

$$U_q \subset sU_p + Cs^{-\gamma}U_r.$$

Taking polars twice (each of which in the consecutive dual) and applying [Köthe 1969, Chapter IV, §20, 8(9)] we obtain

$$U_q^{\circ\circ} \subset (2sU_p + 2Cs^{-\gamma}U_r)^{\circ\circ}.$$

By the Separation theorem [Köthe 1969, Chapter IV, §20, 7(1)] $U_k^{\circ\circ} = \overline{U_k}^{\sigma(X'',X')}$ and these sets constitute a basis of zero neighborhoods in the bidual, therefore

$$\overline{U_q}^{\sigma(X'',X')} \subset \overline{2sU_p + 2Cs^{-\gamma}U_r}^{\sigma(X'',X')} \subset 2s\overline{U_p}^{\sigma(X'',X')} + 2Cs^{-\gamma}\overline{U_r}^{\sigma(X'',X')}.$$

Taking $t = 2s$ and $D = 2^{\gamma+1}C$ we arrive at the (Ω) property in the bidual. The converse of (2) is valid by the Separation theorem which implies that for every functional ϕ acting on X we have $\|\phi\|^*_{k,X'} = \|\phi\|^*_{k,X'''}$. □

Proposition 12. *Let X be a Fréchet space. Then X satisfies (Ω) if and only if* $\min X$ *satisfies $(o\Omega)$.*

Proof. Recall that for an arbitrary Banach space X we have by [Blecher 1992, Corollary 2.8] that $(\min X)' = \max X'$ completely isometrically. Therefore if X is a Fréchet space then for $(\phi_{ij}) \in M_n((\min X)')$ we have

$$\|(\phi_{ij})\|^*_k = \sup \big\| \big(f_{uv}(\phi_{ij})\big) \big\|_{M_{nm}},$$

where the supremum runs over all $(f_{uv}) \in L(X', M_m)$ with $\|(f_{uv})\|_{L(X'_k, M_m)} \leqslant 1$ and all $m \in \mathbb{N}$. We have $L(X', M_m) = X'' \otimes_\varepsilon M_m$ by [Köthe 1979, Chapter VIII, §44, 2(6)] (since M_m is finite-dimensional we may drop the tensor product completion). Moreover $B_{L(X'_k, M_m)} = (V_k^\circ \otimes B_{M'_m})^\circ$, where $V_k = \overline{U_k}^{\sigma(X'',X')}$ and $(U_k)_k$ is a basis of zero neighborhoods in X. By Lemma 6 and Proposition 11(2) we observe that for every p there exists q such that for all r we can find positive C and γ with

$$B_{L(X'_q, M_m)} \subset sB_{L(X'_p, M_m)} + Cs^{-\gamma}B_{L(X'_r, M_m)}.$$

Choosing $\|(f_{uv})\|_{L(X'_q, M_m)} \leqslant 1$ we obtain

$$\big\| \big(f_{uv}(\phi_{ij})\big) \big\|_{M_{nm}} = \big\| \big((sg_{uv} + Cs^{-1}h_{uv})(\phi_{ij})\big) \big\|_{M_{nm}}$$

for some $(g_{uv}) \in B_{L(X'_p, M_m)}$, $(h_{uv}) \in B_{L(X'_r, M_m)}$. Now taking the supremum over all such (f_{uv}), (g_{uv}), (h_{uv}) and all natural m we obtain

$$\|(\phi_{ij})\|^*_q \leqslant s\|(\phi_{ij})\|^*_p + Cs^{-1}\|(\phi_{ij})\|^*_r.$$

Finally, taking the infimum over all $s > 0$ we arrive at

$$\|(\phi_{ij})\|_q \leqslant D\big(\|(\phi_{ij})\|^*_p\big)^{1-\theta}\big(\|(\phi_{ij})\|^*_r\big)^{\theta},$$

where $\theta = 1/(\gamma + 1)$ and $D = (C\gamma)^{1/(\gamma+1)}(1 + \gamma^{-\gamma})$. Since the constant D is independent of the matrix size of (ϕ_{ij}) we obtain the condition $(o\Omega)$. □

By the duality of row and column Hilbert spaces (see [Effros and Ruan 2000, page 59]) we get the following result. Its proof is analogous to that of Proposition 8, therefore we omit it.

Proposition 13. *Let H be a Fréchet–Hilbert space. Then $H \in (\Omega)$ if and only if $H_c \in (o\Omega)$ if and only if $H_r \in (o\Omega)$.*

We now put together all the previously obtained results.

Theorem 14. *Let X be an arbitrary Fréchet space. Then the minimal and maximal quantizations carry over both properties (DN) as well as (Ω) onto their operator space structures. If H is an arbitrary Fréchet–Hilbert space then the above remains valid for the additional row, column and Pisier quantizations.*

4. The conditions of type (oDN)–$(o\Omega)$ in the language of polars

In this section we will prove an analogous version of Theorem 3 for Fréchet operator spaces. In order to do that we will slightly change the notation. So far we have worked with a sequence $(M_n(X))_n$ of spaces where the n-th space denoted $n \times n$ matrices with entries in X. Now we prefer to have one space of infinite matrices. This will enable us to provide an operator space with a suitable notation of weak topologies and polars. Suppose that X is an operator space so that we have a sequence $(M_n(X), \|\cdot\|_n)$ of Banach spaces with $(\|\cdot\|_n)_n$ satisfying Ruan's axioms (see [Effros and Ruan 2000, page 20]). Let us denote by $I(X)$ the linear space of infinite matrices with entries in X and identify $M_n(X)$ with a subspace of $I(X)$ of the form

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \;:\; A \in M_n(X) \right\}.$$

This subspace can be naturally endowed with a norm that makes it isometric to $M_n(X)$. Therefore we will still denote it by $M_n(X)$. The above identification allows us to consider $M_n(X)$ isometrically embedded into $M_{n+1}(X)$. Therefore $\bigcup_n M_n(X)$ has a structure of a normed space and its completion will be denoted by $K(X)$. The norm of $x \in K(X)$ is given by

$$\|x\| = \lim_n \|x^n\|,$$

where x^ns are the truncations of x to $M_n(X)$. Following [Effros and Ruan 2000, Chapter 10] we will also use the notation

$$T(X) := \{w = \alpha x \beta : \alpha, \beta \in HS(\ell_2),\ x \in K(X)\}$$

and endow this space with a norm defined by

$$|||w||| := \inf \|\alpha\|_2 \|x\| \|\beta\|_2,$$

where the infimum runs over all such decompositions. Additionally we write

$$M(X) = \{x \in I(X) : \|x\| < +\infty\}.$$

As a simple example let us note that $K(\mathbb{C}) = \mathcal{K}(\ell_2)$, $T(\mathbb{C}) = \mathcal{N}(\ell_2)$, $M(\mathbb{C}) = \mathcal{B}(\ell_2)$. Moreover by [Effros and Ruan 2000, Theorem 10.1.4], we have isometrically

$$K(X)' = T(X'), \quad T(X)' = M(X'). \tag{4-1}$$

This is in fact a complete isometry but this will be beyond our interests here. The above notation may also be introduced for an arbitrary locally convex operator space. As usual we restrict ourselves to Fréchet operator spaces. If $X = \operatorname{proj}_k X_k$ is such a space then we obtain Fréchet spaces $K(X)$, $T(X)$, $M(X)$. These spaces may be viewed as

$$K(X) = \operatorname{proj}_k K(X_k), \quad T(X) = \operatorname{proj}_k T(X_k), \quad M(X) = \operatorname{proj}_k M(X_k).$$

Equivalently we can easily observe that

$$K(X) = \Big(\bigcup_n M_n(X)\Big)^{\sim}, \quad T(X) = \Big(\bigcup_n T_n(X)\Big)^{\sim},$$
$$M(X) = \{x \in I(X) : \|x\|_k < +\infty \quad \forall k \in \mathbb{N}\},$$

where $\tilde{}$ stands for the completion. We can also define a dual Fréchet operator space to be the linear space

$$K(X') := \bigcup_k K(X_k')$$

equipped with the topology inherited from the space $B(K(X), K(\ell_2))$, as well as the space

$$M(X') := \bigcup_k M(X_k')$$

equipped with the topology inherited from the space $B(K(X), B(\ell_2))$. For the sake of correctness let us point out that if X is a Fréchet operator space then $K(X')$ is no longer a Fréchet space and that the Ruan's axioms are now fulfilled by the dual norms

$$\|\phi\|_k^* = \sup\big\{\|\langle\langle\phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} : x \in K(X),\ \|x\|_k \leqslant 1\big\}.$$

In fact $K(X')$ has the structure of a (DF)-space where the fundamental sequence of bounded sets consists of absolutely matrix convex sets (we recall this definition below). Therefore we may introduce the notion of a (DF)-operator space but we will not go into details since this concept lies beyond our interests. With the above introduced topologies we also obtain for a Fréchet operator space complete isomorphisms (4-1).

Recall that by the unitary isometry $\ell_2(\ell_2) = \ell_2$ we may always think that $\langle\langle\phi, x\rangle\rangle$ is in $\mathcal{B}(\ell_2)$. Let us now define weak matrix topologies, absolutely matrix convex sets and matrix polars. We follow the notation of [Effros and Webster 1997; Effros and Winkler 1997; Effros and Ruan 2000, Chapter 5.5] with only slight modification coming from the fact that instead of the space $\bigcup_n M_n(X)$ we consider its completion. Suppose X is a Fréchet operator space. We define on $K(X)$ the *weak matrix topology* $m\sigma(K(X), K(X'))$ to be determined by the seminorms

$$\rho_{\xi,\phi,\eta}(x) := |\langle\langle\langle\phi, x\rangle\rangle\eta, \xi\rangle|,$$

where $\phi \in K(X')$, $\xi, \eta \in \ell_2$. Analogously we define on $K(X')$ the *weak* matrix topology* $m\sigma(K(X'), K(X))$ to be determined by the seminorms

$$\rho_{\xi,x,\eta}(\phi) := |\langle\langle\langle\phi, x\rangle\rangle\eta, \xi\rangle|,$$

where $x \in K(X)$, $\xi, \eta \in \ell_2$. It is easy to notice that

$$\begin{aligned} m\sigma(K(X), K(X')) &= \sigma(K(X), K(X)'), \\ m\sigma(K(X'), K(X)) &= \sigma(T(X)', T(X))|_{K(X')}. \end{aligned}$$

A subset $S \subset K(X)$ is called *absolutely matrix convex* if the following two conditions hold:

(1) If $x, y \in S$ then $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in S$.

(2) If $x \in S$ and α, β are contractions on ℓ_2 then $\alpha x \beta \in S$.

Since the intersection of two absolutely matrix convex sets is again absolutely matrix convex we may define amc(S) to be the *absolutely matrix convex hull* of S, i.e., the smallest absolutely matrix convex set containing S. It can be precisely described (see [Effros and Webster 1997, Lemma 3.2]). If $A \subset K(X)$ then its *matrix polar* $A^{\circledcirc} \subset K(X')$ is defined as

$$A^{\circledcirc} := \big\{\phi \in K(X') : \|\langle\langle\phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} \leqslant 1 \quad \text{for all } x \in A\big\}.$$

Similarly for $A \subset K(X')$ we define

$$A^{\circledcirc} := \big\{x \in K(X) : \|\langle\langle\phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} \leqslant 1 \quad \text{for all } \phi \in A\big\}.$$

As in the classical case we have the Bipolar theorem. The original proof is for $\bigcup_n M_n(X)$ while we work with its completion but the argument is analogous.

Theorem 15 [Effros and Webster 1997]. *Let X be a Fréchet operator space.*

(1) *If $A \subset K(X)$ then $A^{\circledcirc\circledcirc} = \overline{\mathrm{amc}(A)}^{m\sigma(K(X), K(X'))}$.*

(2) *If $A \subset K(X')$ then $A^{\circledcirc\circledcirc} = \overline{\mathrm{amc}(A)}^{m\sigma(K(X'), K(X))}$.*

We are now ready to reformulate our (oDN)-$(o\Omega)$ conditions in the spirit of Theorem 3. The proofs are analogous to the ones in [Vogt 1977, Lemma 1.4] and [Vogt and Wagner 1980, Lemma 2.1].

Theorem 16. *Let X be a Fréchet operator space and let $(U_k)_{k\in\mathbb{N}}$ be a basis of zero neighborhoods in $K(X)$.*

(1) *X satisfies the property (oDN) if and only if*

$$\exists\, p\in\mathbb{N}\ \forall\, q\in\mathbb{N}\ \exists\, r\in\mathbb{N}, C>0\ \forall\, s>0\colon\quad U_q^{\circledcirc}\subset sU_p^{\circledcirc}+\frac{C}{s}U_r^{\circledcirc},$$

(2) *X satisfies the property $(o\Omega)$ if and only if*

$$\forall\, p\in\mathbb{N}\ \exists\, q\in\mathbb{N}\ \forall\, r\in\mathbb{N}\ \exists\, \gamma>0, C>0\ \forall\, s>0\colon\quad U_q\subset sU_p+\frac{C}{s^{\gamma}}U_r.$$

We end this section by operator space versions of Lemma 6 and Proposition 11. Let us first note that if X and Y are operator spaces and $U\subset K(X)$, $V\subset K(Y)$ then

$$U\otimes V=\{x\otimes y : x\in U,\ y\in V\}.$$

Recalling the definitions of the operator projective $\hat{\otimes}_{op}$ and injective $\check{\otimes}_{op}$ tensor products and denoting by B_E the unit ball of E we can observe that

$$B_{K(X\hat{\otimes}_{op}Y)}=\overline{\operatorname{amc}(B_{K(X)}\otimes B_{K(Y)})},\quad B_{K(X\check{\otimes}_{op}Y)}=\big(B_{K(X)}^{\circledcirc}\otimes B_{K(Y)}^{\circledcirc}\big)^{\circledcirc}.$$

Therefore repeating the proof of Lemma 6 we obtain the following result.

Theorem 17. *Let X be a Fréchet operator space and E an operator space. If X has the property (oDN) or $(o\Omega)$ then their operator projective tensor product as well as the operator injective one satisfy the same condition too.*

Theorem 18. *Let X be a Fréchet operator space.*

(1) *$X\in(oDN)$ if and only if $X''\in(oDN)$.*

(2) *$X\in(o\Omega)$ if and only if $X''\in(o\Omega)$.*

Proof. (1) Observe that X satisfies (oDN) if and only if $K(X)$ satisfies (DN). By Proposition 11(1) $K(X)''=M(X'')\in(DN)$ and $K(X'')$ is a topological subspace of $M(X'')$, therefore it satisfies the condition (DN) and so $X''\in(oDN)$. Conversely $X''\in(oDN)$ leads to $K(X'')\in(DN)$ and by [Effros and Webster 1997, Corollary 8.2, Proposition 9.1] $K(X)$ is its topological subspace which gives $X\in(oDN)$.

(2) Observe that by [Effros and Ruan 2000, Lemma 4.1.1] $X\in(o\Omega)$ if and only if $T_n(X)\in(\Omega)$ with the constants $C_{p,q,r}(n)$ uniformly bounded with respect to n. By Proposition 11(2) this is equivalent to $T_n(X)''=T_n(X'')\in(\Omega)$ which is then equivalent to $X''\in(o\Omega)$. $\square$

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