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HIERARCHIES AND COMPATIBILITY ON COURANT ALGEBROIDS

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We introduce Poisson–Nijenhuis, deforming-Nijenhuis and Nijenhuis pairs that extend to Courant algebroids the notion of a Poisson–Nijenhuis manifold, both the Poisson and the Nijenhuis structures being (1, 1)-tensors on a Courant algebroid. In each case, we construct the natural hierarchies by successive deformation by one of the (1, 1)-tensors.

1. Introduction

The purpose of this article is to explain how (1, 1)-tensors with vanishing Nijenhuis torsion on a Courant algebroid naturally give rise to several types of hierarchies, using as much as possible the supergeometric approach. We first briefly review Courant algebroids, supergeometric approach, Leibniz algebroids, Nijenhuis torsion and hierarchies. We then end this introduction by a more detailed summary of the content of this work.

1A. *On Courant structures, supergeometry, Leibniz algebroids, Nijenhuis torsion and hierarchies.*

Courant structures. It has been noticed by Roytenberg [1999] that the original $\mathbb{R}$-bilinear skew-symmetric bracket introduced by Courant [1990] on the space of sections of $TM \oplus T^*M$, for M a manifold, can be equivalently defined as the skew-symmetrization of the bracket:

$$[(X, \alpha), (Y, \beta)] := ([X, Y], L_X \beta - i_Y \mathrm{d}\alpha), \tag{1}$$

with $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$. This bracket still satisfies the Jacobi identity and, as mentioned in [Ševera and Weinstein 2001], this fact was already noticed by several authors: Kosmann-Schwarzbach, Ševera and Xu (all unpublished). The bracket (1) is a Loday bracket and was used in [Dorfman 1993], hence its name *Dorfman bracket*. The original bracket on $TM \oplus T^*M$ yields to the definition of Courant algebroid given by Liu, Weinstein and Xu [Liu et al. 1997], while the

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version with non-skew-symmetric bracket (1) yields to the equivalent definition of Courant algebroid by Roytenberg [1999] (see also [Kosmann-Schwarzbach 2005] for a simpler version). Relaxing the Jacobi identity of the Loday bracket, one gets the weaker notion of pre-Courant algebroid (see Definition 2.1 below).

Supergeometric approach. Dealing with Courant bracket can be a difficult task when it comes to computation (see for example [Kosmann-Schwarzbach 1992; Voronov 2002]), due to the many structures that involve it, and to the unnatural aspects of some of the operations that define them. However, in supergeometric formalism, all these structures and conditions are encoded in two objects and one condition, as follows. To every vector bundle equipped with a fiberwise nondegenerate bilinear form is associated a graded commutative algebra, equipped with a Poisson bracket denoted by $\{\cdot\,,\cdot\}$ (which coincides with the big bracket [Kosmann-Schwarzbach 1992] in particular cases) [Roytenberg 2002]. Pre-Courant structures are in one-to-one correspondence with elements of degree 3 in this graded algebra and Courant structures are those elements that satisfy

$$\{\Theta, \Theta\} = 0$$

(see [Roytenberg 2002; Antunes 2010]).

Leibniz algebroids. Courant structures on vector bundles can be viewed as special cases of Leibniz algebroids [Ibáñez et al. 1999]. These are vector bundles $E \to M$ equipped with a $\mathbb{R}$-bilinear bracket on the space of sections and a vector bundle morphism $\rho : E \to TM$ satisfying the Leibniz rule:

$$[X, fY] = f[X, Y] + (\rho(X).f)Y$$

and the Jacobi identity:

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

for all $X, Y, Z \in \Gamma(E)$ and $f \in C^\infty(M)$. Relaxing the Jacobi identity, one gets the weaker notion of pre-Leibniz algebroid. When the base manifold reduces to a point, a Leibniz algebroid is just a Leibniz algebra (also called Loday algebra), while a pre-Leibniz algebroid is simply an algebra, i.e., a space equipped with a bilinear product. Pre-Courant algebroids are pre-Leibniz algebroids; see [Kosmann-Schwarzbach 2005]. But it is important to stress that the supergeometric approach, referred above for pre-Courant and Courant structures, does not extend to the more general pre-Leibniz and Leibniz algebroid framework.

Nijenhuis torsion. The Nijenhuis torsion of a $(1, 1)$-tensor on M, that is, a fiberwise linear endomorphism of TM, is the $(1, 2)$-tensor given by

$$X, Y \mapsto [NX, NY] - N[X, Y]_N, \text{ where } [X, Y]_N := [NX, Y] + [X, NY] - N[X, Y].$$

We call Nijenhuis tensors $(1, 1)$-tensors whose Nijenhuis torsion vanishes. The previous definition can be extended from TM to arbitrary Lie algebroids [Kosmann-Schwarzbach and Magri 1990; Grabowski and Urbański 1997], then from Lie algebroids to Courant algebroids [Cariñena et al. 2004; Kosmann-Schwarzbach 2011] and Leibniz algebroids [Cariñena et al. 2004].

By means of Nijenhuis $(1, 1)$-tensors, a Lie algebroid bracket $[\cdot, \cdot]$ can be deformed into the bracket $[\cdot, \cdot]_N$ above, which can be shown to be a Lie algebroid bracket again. Also, Poisson structures can be deformed into Poisson structures.

Hierarchies. There is no mathematical definition of what a *hierarchy* is, but, within the context of integrable systems, the name has been commonly given either to families (indexed by $\mathbb{N}$ or $\mathbb{Z}$) of Hamiltonian functions that commute for a fixed Poisson structure, or of Poisson structures/Lie algebroid structures which commute pairwise — and sometimes families of both Poisson structures and Hamiltonian functions such that two functions in that family commute with respect to any Poisson structure. We use that name in the same spirit: that is, for us a hierarchy is either a family of commuting Courant structures or a family of Nijenhuis tensors that commute pairwise with respect to some Courant structure.

To obtain a hierarchy, the idea is to start from a structure and a Nijenhuis tensor by means of which we deform the initial structure into a sequence of structures of the same nature [Kosmann-Schwarzbach and Magri 1990; Magri and Morosi 1984].

1B. ***Purpose and content of the present article.*** Our goal is, as we already stated, to construct hierarchies. More precisely, we wish to construct

(i) hierarchies of Courant structures, given a Nijenhuis tensor on a Courant algebroid,

(ii) hierarchies of Poisson structures, given a Nijenhuis tensor compatible with a given Poisson structure on a Courant algebroid, and

(iii) hierarchies of Courant structures and pairs of tensors that we call deforming-Nijenhuis pairs or Nijenhuis pairs.

Indeed, for the two last points, pre-Courant structures are enough. The idea behind item (i) is simply that what is true for manifolds and Lie algebroids should be true for Courant structures as well, and that, in particular, deforming a Courant structure by a Nijenhuis tensor should give a hierarchy of compatible Courant structures. The idea behind items (ii) and (iii) is more involved. We invite the reader to have in mind the case of Poisson–Nijenhuis structures to obtain some intuitive picture [Magri and Morosi 1984; Kosmann-Schwarzbach and Magri 1990; Grabowski and Urbański 1997]. In terms of Courant algebroids, a Poisson–Nijenhuis structure can be seen as a pair (J_π, I_N) of skew-symmetric $(1, 1)$-tensors on $TM \oplus T^*M$

(see Examples 2.6 and 2.9). The pair (π, N) is Poisson–Nijenhuis when π and N are compatible, which means that J_π and I_N anticommute and their concomitant with respect to the Courant structure vanishes; see Example 4.14. These conditions yield our Definition 4.12 of Poisson–Nijenhuis pair on a (pre-)Courant algebroid, Poisson–Nijenhuis pairs for which we generalize the hierarchies of [Magri and Morosi 1984]. Poisson–Nijenhuis pairs being slightly too restrictive, we indeed do it in the more general context of deforming-Nijenhuis pairs and Nijenhuis pairs.

The statements of most results in this article are written in the pre-Courant algebroid framework and are proved using the supergeometric approach. However, for some of them, the proofs only use the pre-Leibniz structure induced by the pre-Courant structure, so that these results hold not only for pre-Courant algebroids, but also for the more general setting of pre-Leibniz algebroids. This happens, for example, with most results in Sections 3A and 3B and the whole Section 5. The lack of convincing examples prevented us from going to such an unnecessary level of generality.

Let us give a more precise content of the article. In Section 2, we make a brief introduction of the supergeometric setting for (pre-)Courant structures and we recall the notions of deforming and Nijenhuis tensors.

In Section 3, we show that a Courant structure Θ can be deformed k times by a Nijenhuis tensor I, and that the henceforth obtained objects $(\Theta_k)_{k\in\mathbb{N}}$ are compatible (Theorem 3.6). Then, we show that the property of being compatible is, for a given compatible pair (I, J), also preserved when deforming n times J by I, provided that I is Nijenhuis (or at least satisfies a weaker condition involving the vanishing of torsion of I on the image of J), and that this result still holds true with respect to pre-Courant structures Θ_k obtained when deforming Θ by I (Theorem 3.16). An even more general case is obtained when considering the tensor I^{2s+1}, $s \in \mathbb{N}$, which is the deformation of I by itself an odd number of times, and, if J is also Nijenhuis, J is replaced by $I^n \circ J^{2m+1}$, $n, m \in \mathbb{N}$ (Theorem 3.20).

In Section 4, we turn our attention to deforming-Nijenhuis pairs, that is, compatible pairs (J, I) where J is a deforming tensor and I is Nijenhuis for Θ. We show that if (J, I) is a deforming-Nijenhuis pair for Θ, then (J, I^{2n+1}) is a deforming-Nijenhuis pair for Θ_k for all $k, n \in \mathbb{N}$ (Theorem 4.7). Then, we consider Poisson–Nijenhuis pairs (J, I), that is, deforming-Nijenhuis pairs where the deforming tensor J is supposed to be Poisson for Θ, and we state one of the main results of the article, which is the construction of a hierarchy of Poisson–Nijenhuis pairs for Θ_k, for all $k \in \mathbb{N}$, that includes pairs of compatible Poisson tensors (Theorem 4.19).

Last, in Section 5, we conclude with the case of Nijenhuis pairs, that is, pairs (I, J) of Nijenhuis tensors compatible with respect to Θ. More precisely, we show that if (I, J) is a Nijenhuis pair for Θ, then for all $m, n, t \in \mathbb{N}$, $(I^{2m+1} \circ J^n, J^{2t+1})$

is a Nijenhuis pair for Θ, and, more generally, for all the Courant structures obtained by deforming Θ several times, either by I or by J (Theorem 5.11).

2. Skew-symmetric tensors on Courant algebroids

2A. ***Courant algebroids in supergeometric terms.*** We introduce the supergeometric setting following the approach in [Roytenberg 1999; 2002; Vaintrob 1997]. Given a vector bundle $A \to M$, we denote by $A[n]$ the graded manifold obtained by shifting the degree of coordinates on the fiber by n. The graded manifold $T^*[2]A[1]$[1] is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is called the *big bracket*; see [Kosmann-Schwarzbach 1992; 2005].

In local coordinates $x^i, p_i, \xi^a, \theta_a, i \in \{1, \dots, n\}, a \in \{1, \dots, d\}$, in $T^*[2]A[1]$, where x^i, ξ^a are local coordinates on $A[1]$ and p_i, θ_a are the conjugate coordinates, the Poisson bracket is given by

$$\{p_i, x^i\} = \{\theta_a, \xi^a\} = 1, \quad i = 1, \dots, n, a = 1, \dots, d,$$

while the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with an $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree locally as follows: the coordinates on the base manifold M, x^i, $i \in \{1, \dots, n\}$, have bidegree $(0, 0)$, while the coordinates on the fibers, ξ^a, $a \in \{1, \dots, d\}$, have bidegree $(0, 1)$ and their associated moment coordinates, p_i and θ_a, have bidegrees $(1, 1)$ and $(1, 0)$, respectively.[2] We denote by $\mathcal{F}^{k,l}$ the space of functions of bidegree (k, l). The *total degree* of a function $f \in \mathcal{F}^{k,l}$ is equal to $k + l$ and the subset of functions of total degree t is denoted by $\mathcal{F}^t$. We can verify that the big bracket has bidegree $(-1, -1)$, that is,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1,l_1+l_2-1}.$$

This construction is a particular case of a more general one [Roytenberg 2002] in which we consider a vector bundle E equipped with a fiberwise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, \cdot \rangle)$. We denote

[1] This graded manifold is in fact an N-manifold because the parity of a homogeneous function on $T^*[2]A[1]$ is compatible with its degree. For more details on these notions see [Voronov 2002] and for this particular N-manifold (of degree 2) see [Roytenberg 2002]. We should observe that a similar work to the present one could be done, with more complicated computations, on graded manifolds. However, since we want to restrict to the Courant algebroid setting, the N-manifold $T^*[2]A[1]$ is the appropriate one.

[2] Notice that this bidegree can be defined globally using the double vector bundle structure of $T^*[2]A[1]$; see [Roytenberg 1999; Voronov 2002].

by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, that is, $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{\cdot,\cdot\}$, which has degree -2. Notice that $\mathcal{F}_E^0 = C^\infty(M)$ and $\mathcal{F}_E^1 = \Gamma(E)$. Under these identifications, the Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \quad \{f, X\} = 0 \quad \text{and} \quad \{X, Y\} = \langle X, Y\rangle,$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

When $E := A \oplus A^*$ (with A a vector bundle over M) and when $\langle \cdot,\cdot\rangle$ is the usual symmetric bilinear form

$$\langle X+\alpha, Y+\beta\rangle = \alpha(Y) + \beta(X), \quad \text{for all } X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*), \tag{2}$$

the algebras $\mathcal{F} = C^\infty(T^*[2]A[1])$ and $\mathcal{F}_{A\oplus A^*}$ are isomorphic Poisson algebras [Roytenberg 2002].

Definition 2.1. A *pre-Courant* structure on $(E, \langle\cdot,\cdot\rangle)$ is a pair $(\rho, [\cdot,\cdot])$, where ρ is a bundle map from E to TM, called the *anchor*, and $[\cdot,\cdot]$ is a $\mathbb{R}$-bilinear (not necessarily skew-symmetric) assignment on $\Gamma(E)\times\Gamma(E) \to \Gamma(E)$, called the *Dorfman bracket*, satisfying the relations

$$\rho(X)\cdot\langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle, \tag{3}$$

$$\rho(X)\cdot\langle Y, Z\rangle = \langle X, [Y, Z] + [Z, Y]\rangle, \tag{4}$$

for all $X, Y, Z \in \Gamma(E)$.[3]

If the Jacobi identity, $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$, is satisfied for all $X, Y, Z \in \Gamma(E)$, then the pair $(\rho, [\cdot,\cdot])$ is called a *Courant* structure on $(E, \langle\cdot,\cdot\rangle)$.

The Dorfman bracket is a Leibniz bracket when the pair $(\rho, [\cdot,\cdot])$ is a Courant structure. There is a one-to-one correspondence between pre-Courant structures on $(E, \langle\cdot,\cdot\rangle)$ and elements in $\mathcal{F}_E^3$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}_E^3$ are defined for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by

$$\rho(X)\cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X, Y] = \{\{X, \Theta\}, Y\}.$$

The following theorem addresses how the Jacobi identity is expressed in this supergeometric setting.

Theorem 2.2 [Roytenberg 2002]. *There is a one-to-one correspondence between Courant structures on $(E, \langle\cdot,\cdot\rangle)$ and functions $\Theta \in \mathcal{F}_E^3$ such that $\{\Theta, \Theta\} = 0$.*

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[3] From (3) and (4), we obtain $[X, fY] = f[X, Y] + (\rho(X).f)Y$ for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$ [Kosmann-Schwarzbach 2005]. Thus, as we already mentioned in the Introduction, a pre-Courant algebroid is always a pre-Leibniz algebroid.

If Θ is a (pre-)Courant structure on $(E, \langle\cdot,\cdot\rangle)$, then the triple $(E, \langle\cdot,\cdot\rangle, \Theta)$ is called a *(pre-)Courant algebroid*. For the sake of simplicity, we will often denote a (pre-)Courant algebroid by the pair (E, Θ) instead of the triple $(E, \langle\cdot,\cdot\rangle, \Theta)$.

When $E = A \oplus A^*$ and $\langle\cdot,\cdot\rangle$ is the usual symmetric bilinear form (2), a pre-Courant structure $\Theta \in \mathcal{F}^3_E$ can be decomposed as a sum of homogeneous terms with respect to its bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}^{1,2}_{A\oplus A^*}$, $\gamma \in \mathcal{F}^{2,1}_{A\oplus A^*}$, $\phi \in \mathcal{F}^{0,3}_{A\oplus A^*} = \Gamma(\bigwedge^3 A^*)$ and $\psi \in \mathcal{F}^{3,0}_{A\oplus A^*} = \Gamma(\bigwedge^3 A)$.

We recall from [Roytenberg 1999] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ if and only if (A, μ) is a Lie algebroid. Also, when $\phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle\cdot,\cdot\rangle)$ if and only if $((A, \mu), (A^*, \gamma))$ is a Lie bialgebroid [Liu et al. 1997].

2B. ***Deformation of Courant structures by skew-symmetric tensors.*** Suppose that $(E, \langle\cdot,\cdot\rangle, \Theta)$ is a pre-Courant algebroid and $J : E \to E$ is a vector bundle endomorphism of E. The *deformation* of the Dorfman bracket $[\cdot,\cdot]$ by J is the bracket $[\cdot,\cdot]_J$ defined for all sections X, Y of E, by

$$[X, Y]_J = [JX, Y] + [X, JY] - J[X, Y].$$

The $(1, 1)$-tensors on E will be seen as vector bundle endomorphisms of E. A $(1, 1)$-tensor $J : E \to E$ is said to be *skew-symmetric* if

$$\langle Ju, v\rangle + \langle u, Jv\rangle = 0,$$

for all $u, v \in E$. If we consider the endomorphism J^* defined by $\langle u, J^*v\rangle = \langle Ju, v\rangle$, then J is skew-symmetric if and only if $J + J^* = 0$. If J is skew-symmetric, then $[\cdot,\cdot]_J$ satisfies (3) and (4), so that $(\rho\circ J, [\cdot,\cdot]_J)$ is a pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$.[4]

When the $(1, 1)$-tensor $J : E \to E$ is skew-symmetric, the deformed pre-Courant structure $(\rho\circ J, [\cdot,\cdot]_J)$ is associated to the element $\Theta_J := \{J, \Theta\} \in \mathcal{F}^3_E$. The deformation of Θ_J by the skew-symmetric $(1, 1)$-tensor I is denoted by $\Theta_{J,I}$, that is, $\Theta_{J,I} = \{I, \{J, \Theta\}\}$, while the deformed Dorfman bracket $([\cdot,\cdot]_J)_I$ is denoted by $[\cdot,\cdot]_{J,I}$. Although the equality $\Theta_J = \{J, \Theta\}$ only makes sense when J is skew-symmetric, aiming to simplify the notation, we shall denote by Θ_J the pre-Courant structure $(\rho\circ J, [\cdot,\cdot]_J)$, even in the case where J is not skew-symmetric.

By definition, a vector bundle endomorphism $I : E \to E$ is a *Nijenhuis* tensor on the Courant algebroid (E, Θ) if its torsion vanishes, where the torsion $\mathcal{T}_\Theta I$ is

[4]In fact, it suffices that J satisfies the condition $J + J^* = \lambda\,\mathrm{id}_E$, for some $\lambda \in \mathbb{R}$, to guarantee that $(\rho\circ J, [\cdot,\cdot]_J)$ is a pre-Courant structure on $(E, \langle\cdot,\cdot\rangle)$; see [Cariñena et al. 2004].

given, for all $X, Y \in \Gamma(E)$, by

$$\mathcal{T}_\Theta I(X,Y) = [IX, IY] - I[X,Y]_I.$$

A short computation shows that

$$\mathcal{T}_\Theta I(X,Y) = \tfrac{1}{2}([X,Y]_{I,I} - [X,Y]_{I^2}), \tag{5}$$

where $I^2 = I \circ I$. When I is skew-symmetric and $I^2 = \alpha\,\mathrm{id}_E$ for some $\alpha \in \mathbb{R}$, then $\mathcal{T}_\Theta I$ is an element of degree 3 in the supergeometric setting [Kosmann-Schwarzbach 2011], and (5) is given by [Grabowski 2006]:

$$\mathcal{T}_\Theta I = \tfrac{1}{2}(\Theta_{I,I} - \alpha\Theta). \tag{6}$$

In the case of pre-Courant algebroids, the definition of Nijenhuis tensors is the same as in the case of a Courant algebroids.

Example 2.3. Let $\mathcal{G}$ be a Lie algebra. A linear operator $I : \mathcal{G} \to \mathcal{G}$ that takes values in the center and such that, in addition, the kernel of I^2 contains the derived algebra $[\mathcal{G}, \mathcal{G}]$ is a Nijenhuis operator.

The notion of *deforming* tensor for a Courant structure Θ on E was introduced in [Kosmann-Schwarzbach 2011]. The definition holds in the case of a pre-Courant algebroid and it will play an important role in this article.

Definition 2.4. Let (E, Θ) be a pre-Courant algebroid. A skew-symmetric $(1,1)$-tensor J on (E, Θ) is said to be *deforming for* Θ if $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$.

Remark 2.5. If I is Nijenhuis for Θ and satisfies $I^2 = \alpha\,\mathrm{id}_E$ for some $\alpha \in \mathbb{R}$, then, it follows from (6) that I is also deforming for Θ. This was noticed in [Kosmann-Schwarzbach 2011].

When $E = A \oplus A^*$ and $\langle \cdot, \cdot \rangle$ is the usual symmetric bilinear form, a skew-symmetric $(1,1)$-tensor $J : A \oplus A^* \to A \oplus A^*$ is of the type

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}, \tag{7}$$

with $N : A \to A$, $\pi \in \Gamma(\bigwedge^2 A)$ and $\omega \in \Gamma(\bigwedge^2 A^*)$. In the supergeometric framework, J corresponds to the function $N + \pi + \omega$, which we also denote by J. Therefore, we have $\Theta_J = \{N + \pi + \omega, \Theta\}$.

We shall now present examples of skew-symmetric deforming or/and Nijenhuis tensors in the case where $(E = A \oplus A^*, \Theta)$ is the Courant algebroid associated to a Lie algebroid, that is $\Theta = \mu$, with μ a Lie algebroid structure on A.

Example 2.6. Let π be a bivector on A and $J_\pi = \left(\begin{smallmatrix} 0 & \pi^\# \\ 0 & 0 \end{smallmatrix}\right)$. Then, J_π is deforming for $\Theta = \mu$ if and only if π is a Poisson bivector on the Lie algebroid (A, μ).

If π is a Poisson bivector on (A, μ) then, denoting by $[\cdot,\cdot]_\mu$ the Gerstenhaber bracket on $\Gamma(\bigwedge^\bullet A)$, we have $0 = [\pi,\pi]_\mu = \{\pi, \{\pi, \mu\}\} = \mu_{J_\pi, J_\pi}$, so that J_π is deforming for μ. If J_π is deforming for μ, then $\mu_{J_\pi, J_\pi} = \eta\,\mu$, with $\eta \in \mathbb{R}$. Since μ and μ_{J_π, J_π} do not have the same bidegree, we obtain

$$\mu_{J_\pi, J_\pi} = \eta\,\mu \Leftrightarrow (\eta = 0 \text{ and } \{\pi, \{\pi, \mu\}\} = 0).$$

Thus, π is a Poisson bivector on the Lie algebroid (A, μ).

Example 2.7. Let J_π be as in Example 2.6. The $(1, 1)$-tensor J_π is a Nijenhuis tensor for $\Theta = \mu$ if and only if π is a Poisson bivector on the Lie algebroid (A, μ).

We remark that $J_\pi \circ J_\pi = 0$ so that, using (6) with $\alpha = 0$, we deduce that the torsion of J_π is given by $\mathcal{T}_\mu J_\pi = \frac{1}{2}\{\pi, \{\pi, \mu\}\}$. Therefore, the torsion of J_π with respect to $\Theta = \mu$ vanishes if and only if $[\pi, \pi]_\mu = 0$.

Example 2.8. Let ω be a 2-form on A. Then, $J_\omega = \left(\begin{smallmatrix} 0 & 0 \\ \omega^\flat & 0 \end{smallmatrix}\right)$ is a deforming and a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$.

This is an immediate consequence of $J_\omega \circ J_\omega = 0$ and $\mu_{J_\omega, J_\omega} = \{\omega, \{\omega, \mu\}\} = 0$.

Example 2.9. Let $N : A \to A$ be a $(1, 1)$-tensor on A, such that $N^2 = \alpha\,\mathrm{id}_A$ for some $\alpha \in \mathbb{R}$. Then, $I_N = \left(\begin{smallmatrix} N & 0 \\ 0 & -N^* \end{smallmatrix}\right)$ is a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$ if and only if N is Nijenhuis tensor for the Lie algebroid (A, μ) [Kosmann-Schwarzbach 2011].

Example 2.10. Let π be a bivector on A and $N : A \to A$ a $(1, 1)$-tensor on A. Then, $J = \left(\begin{smallmatrix} N & \pi^\# \\ 0 & -N^* \end{smallmatrix}\right)$ is deforming for $\Theta = \mu$ if and only if N is a deforming tensor on (A, μ),[5] π is a Poisson bivector on (A, μ) and $\mu_{N,\pi} + \mu_{\pi,N} = 0$.

We have

$$\begin{aligned} \mu_{J,J} &= \{N + \pi, \{N + \pi, \mu\}\} \\ &= \{N, \{N, \mu\}\} + \{\pi, \{N, \mu\}\} + \{N, \{\pi, \mu\}\} + \{\pi, \{\pi, \mu\}\} \\ &= \mu_{N,N} + \mu_{N,\pi} + \mu_{\pi,N} + \mu_{\pi,\pi} \end{aligned}$$

and, by counting the bidegrees, we deduce that $\mu_{J,J} = \eta\,\mu$ if and only if

$$\mu_{N,N} = \eta\,\mu, \quad \mu_{N,\pi} + \mu_{\pi,N} = 0, \quad [\pi, \pi]_\mu = 0.$$

Let us consider the Courant algebroid $(A \oplus A^*, \mu + \gamma)$, which is the double of a Lie bialgebroid $((A, \mu), (A^*, \gamma))$ and the skew-symmetric $(1, 1)$-tensor $J : A \oplus A^* \to A \oplus A^*$:

$$J = \begin{pmatrix} \frac{1}{2}\,\mathrm{id}_A & \pi^\# \\ 0 & -\frac{1}{2}\,\mathrm{id}_{A^*} \end{pmatrix}, \tag{8}$$

where π is a bivector on A.

[5] An $(1, 1)$-tensor N on a Lie algebroid (A, μ) is a *deforming* tensor if $\mu_{N,N} = \eta\mu$, for some $\eta \in \mathbb{N}$.

Proposition 2.11. *Let* $((A,\mu),(A^*,\gamma))$ *be a Lie bialgebroid. Then, the* $(1,1)$-*tensor* J *given by* (8) *is a deforming tensor for the Courant structure* $\mu+\gamma$ *if and only if* π *is a solution of the Maurer–Cartan equation*

$$\mathrm{d}_\gamma \pi = \tfrac{1}{2}[\pi,\pi]_\mu.$$

Proof. The $(1,1)$-tensor $J=\frac{1}{2}\,\mathrm{id}_A+\pi$ is a deforming tensor for $\mu+\gamma$ if there exists $\eta\in\mathbb{R}$ such that

$$\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\mu+\gamma\right\}\right\}=\eta(\mu+\gamma).$$

We have, using the fact that $\{\mathrm{id}_A,u\}=(q-p)u$ for all u of bidegree (p,q),

$$\begin{aligned}\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\left\{\tfrac{1}{2}\,\mathrm{id}_A+\pi,\mu+\gamma\right\}\right\}&\\ &=\tfrac{1}{4}\left\{\mathrm{id}_A,\{\mathrm{id}_A,\mu\}+\{\mathrm{id}_A,\gamma\}\right\}+\tfrac{1}{2}\left\{\mathrm{id}_A,\{\pi,\mu\}+\{\pi,\gamma\}\right\}\\ &\qquad+\tfrac{1}{2}\left\{\pi,\{\mathrm{id}_A,\mu\}+\{\mathrm{id}_A,\gamma\}\right\}+\left\{\pi,\{\pi,\mu\}+\{\pi,\gamma\}\right\}\\ &=\tfrac{1}{4}(\mu+\gamma)-2\{\pi,\gamma\}-\left\{\{\pi,\mu\},\pi\right\},\end{aligned}$$

since $\{\pi,\{\pi,\gamma\}\}=0$ for reasons of bidegree. Therefore, J is a deforming $(1,1)$-tensor if and only if

$$\eta=\tfrac{1}{4}\quad\text{and}\quad \mathrm{d}_\gamma\pi=\tfrac{1}{2}[\pi,\pi]_\mu.\qquad\square$$

3. Hierarchies of compatible tensors and structures

We construct a hierarchy of compatible Courant structures on $(E,\langle\cdot,\cdot\rangle)$ that are obtained deforming an initial Courant structure by a Nijenhuis tensor. Then, we consider hierarchies of pairs of tensors which are compatible, in a certain sense, with respect to some deformed pre-Courant structures.

We introduce the following notation, where $I,J,\dots,T$ are skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ):

- $\Theta_{I,J,\dots,T}=(((\Theta_I)_J)_{\dots})_T$,
- $\Theta_k=(((\Theta_I)_I)\overset{k}{\dots})_I=\Theta_{I,\overset{k}{\dots},I}$, $k\in\mathbb{N}$, $\Theta_0=\Theta$.

3A. *Hierarchies of compatible Courant structures.* In this section we construct a hierarchy of compatible Courant structures on $(E,\langle\cdot,\cdot\rangle)$.

The next proposition generalizes a result in [Kosmann-Schwarzbach and Magri 1990].

Proposition 3.1. *Let* I *be a* $(1,1)$-*tensor on a pre-Courant algebroid* (E,Θ). *For all sections* X,Y *of* E *and* $k\geq 1$,

$$\mathcal{T}_{\Theta_k}I(X,Y)=\mathcal{T}_{\Theta_{k-1}}I(IX,Y)+\mathcal{T}_{\Theta_{k-1}}I(X,IY)-I(\mathcal{T}_{\Theta_{k-1}}I(X,Y)).\tag{9}$$

Proof. Let us denote by $[\,\cdot\,,\cdot\,]_k$ the Dorfman bracket associated to Θ_k. It is obvious that

$$[X, Y]_k = [IX, Y]_{k-1} + [X, IY]_{k-1} - I[X, Y]_{k-1},$$

and therefore we have

$$\begin{aligned}
\mathcal{T}_{\Theta_k} I(X, Y) &= [IX, IY]_k - I[IX, Y]_k - I[X, IY]_k + I^2[X, Y]_k \\
&= [I^2X, IY]_{k-1} - I[I^2X, Y]_{k-1} - I[IX, IY]_{k-1} + I^2[IX, Y]_{k-1} \\
&\quad + [IX, I^2Y]_{k-1} - I[IX, IY]_{k-1} - I[X, I^2Y]_{k-1} + I^2[X, IY]_{k-1} \\
&\quad - I([IX, IY]_{k-1} - I[IX, Y]_{k-1} - I[X, IY]_{k-1} + I^2[X, Y]_{k-1}) \\
&= \mathcal{T}_{\Theta_{k-1}} I(IX, Y) + \mathcal{T}_{\Theta_{k-1}} I(X, IY) - I(\mathcal{T}_{\Theta_{k-1}} I(X, Y)). \qquad \square
\end{aligned}$$

Corollary 3.2. *If I is Nijenhuis for Θ, then I is Nijenhuis for Θ_k, $\forall k \in \mathbb{N}$.*

It is well known [Grabowski 2006] that for every skew-symmetric $(1, 1)$-tensor I on a Courant algebroid (E, Θ), the deformation of Θ by I, Θ_I, is a Courant structure on $(E, \langle\,\cdot\,,\cdot\,\rangle)$ provided that I is Nijenhuis. Applying (9) we get, by recursion:

Proposition 3.3. *Let (E, Θ) be a Courant algebroid and I a skew-symmetric Nijenhuis tensor for Θ. Then, (E, Θ_k) is a Courant algebroid for all $k \in \mathbb{N}$.*

We introduce the notation $I^n = I \circ \overset{n}{\ldots} \circ I$, for $n \geq 1$ and $I^0 = \mathrm{id}_E$.

Let us compute the torsion $\mathcal{T}_\Theta I^n$, for all $n \in \mathbb{N}$.

Proposition 3.4. *Let I be a $(1, 1)$-tensor on a pre-Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$\begin{aligned}
(10)\quad \mathcal{T}_\Theta I^n(X, Y) &= \mathcal{T}_\Theta I(I^{n-1}X, I^{n-1}Y) + I(\mathcal{T}_\Theta I^{n-1}(IX, Y) \\
&\quad + \mathcal{T}_\Theta I^{n-1}(X, IY)) - I^2(\mathcal{T}_\Theta I^{n-2}(IX, IY)) + I^{2n-2}(\mathcal{T}_\Theta I(X, Y)),
\end{aligned}$$

for $n \geq 2$.

Proof. It suffices to use the definition of Nijenhuis torsion to compute each term on the right hand side of (10). $\square$

As an immediate consequence of the previous proposition and Corollary 3.2, we have:

Proposition 3.5. *Let (E, Θ) be a pre-Courant algebroid and I a $(1, 1)$-tensor on E. If I is a Nijenhuis tensor for Θ, then I^n is Nijenhuis for Θ_k, for all $n, k \in \mathbb{N}$.*

Recall that a pair of Courant structures Θ_1 and Θ_2 on a vector bundle $(E, \langle\,\cdot\,,\cdot\,\rangle)$ are said to be *compatible* if their sum $\Theta_1 + \Theta_2$ is a Courant structure on $(E, \langle\,\cdot\,,\cdot\,\rangle)$. As an immediate consequence, we have that Θ_1 and Θ_2 are compatible if and only if

$$\{\Theta_1, \Theta_2\} = 0.$$

Theorem 3.6. *Let I be a skew-symmetric $(1,1)$-tensor on a Courant algebroid (E,Θ). If I is Nijenhuis for Θ, then the Courant structures Θ_k and Θ_m on $(E,\langle\cdot,\cdot\rangle)$ are compatible for all $k,m\in\mathbb{N}$.*

Proof. We first remark that if $m=k$, then we have $\{\Theta_m,\Theta_m\}=0$ by Proposition 3.3. Also, for any Courant structure Θ and any skew-symmetric $(1,1)$-tensor I, the relation $\{\Theta,\Theta_I\}=0$ follows from the Jacobi identity and the graded symmetry of the Poisson bracket. We use induction on $m+k$ to complete the proof. Assume first that $m+k=2$; then either $m=k=1$ and it is clear that $\{\Theta_I,\Theta_I\}=0$, or $m=2$ and $k=0$ and it is clear that $\{\Theta_{I,I},\Theta\}=\{I,\{\Theta,\Theta_I\}\}-\{\Theta_I,\Theta_I\}=0$.

Now, suppose that $\{\Theta_m,\Theta_k\}=0$ holds for $m+k=s-1$ and take m and k such that $m+k=s$.

i) If $m=k$, we already noticed that $\{\Theta_m,\Theta_m\}=0$.

ii) If $m\neq k$, suppose that $m>k$. Then,

$$\begin{aligned}\{\Theta_m,\Theta_k\}&=\{\{I,\Theta_{m-1}\},\Theta_k\}=\{I,\{\Theta_k,\Theta_{m-1}\}\}-\{\Theta_{k+1},\Theta_{m-1}\}\\&=-\{\Theta_{m-1},\Theta_{k+1}\}\\&=-\{I,\{\Theta_{m-2},\Theta_{k+1}\}\}+\{\Theta_{m-2},\Theta_{k+2}\}=\{\Theta_{m-2},\Theta_{k+2}\}.\end{aligned}$$

Applying the Jacobi identity several times, we get

$$\begin{aligned}\{\Theta_m,\Theta_k\}&=\begin{cases}(-1)^{m-l}\{\Theta_l,\Theta_l\} & \text{if } m+k=2l,\\ (-1)^{m-(l+1)}\{\Theta_{l+1},\Theta_l\} & \text{if } m+k=2l+1.\end{cases}\\&=\begin{cases}0 & \text{if } m+k=2l,\\ (-1)^{m-(l+1)}\frac{1}{2}\{I,\{\Theta_l,\Theta_l\}\}=0 & \text{if } m+k=2l+1.\end{cases}\end{aligned}$$

□

Remark 3.7. The statement of Theorem 3.6 still holds if we replace the assumption of I being Nijenhuis for Θ by I deforming for Θ. In fact, if $\Theta_{I,I}=\eta\Theta$ for some $\eta\in\mathbb{R}$, then a straightforward computation yields

$$\Theta_{2k}=\eta^k\Theta,\quad \Theta_{2k+1}=\eta^k\Theta_I\quad\text{for all } k\in\mathbb{N}.$$

We have investigated so far the pre-Courant structure Θ_n, obtained by deforming n times the original pre-Courant structure Θ by a Nijenhuis tensor I. It is logical to ask what happens when one deforms Θ by I^n. We shall show that we obtain precisely the same pre-Courant structure Θ_n.

Proposition 3.8. *Let $(\rho,[\cdot,\cdot])$ be a pre-Courant structure on $(E,\langle\cdot,\cdot\rangle)$ and I a $(1,1)$-tensor on E. Let X and Y be sections of E and let $n\in\mathbb{N}^*$. Then:*

a) $[X,Y]_{I^{2n+1}}=[X,Y]_{I^{2n},I}-\sum_{\substack{0\leq i,j\leq 2n-1\\ i+j=2n-1}} I^j(\mathcal{T}_\Theta I(I^iX,Y)+\mathcal{T}_\Theta I(X,I^iY))$.

b) *If I is Nijenhuis for $(\rho,[\cdot,\cdot])$, then $[X,Y]_{I^n}=[X,Y]_{I,\overset{n}{\dots},I}$ for all $n\in\mathbb{N}$.*

c) *if I is Nijenhuis for $(\rho, [\cdot, \cdot])$, then $[X, Y]_{I^m, I^n} = [X, Y]_{I^{m+n}}$ for all $m, n \in \mathbb{N}$.*

Proof. Statement a) is an easy but cumbersome computation.

For b), first, observe that if a pair of skew-symmetric $(1, 1)$-tensors I and J commute, then $[X, Y]_{I,J} = [X, Y]_{J,I}$ for all sections X and Y of E. In particular, we have, for all $m, n \in \mathbb{N}$,

$$[X, Y]_{I^m, I^n} = [X, Y]_{I^n, I^m}. \tag{11}$$

We now prove the result by recursion on $n \geq 1$. If $n = 2k + 1$, we use a):

$$[X, Y]_{I^n} = [X, Y]_{I^{2k+1}} = [X, Y]_{I^{2k}, I}$$

and we use the recursion hypothesis. If $n = 2k$, since I^k is Nijenhuis, using (5) we may write

$$[X, Y]_{I^n} = [X, Y]_{I^k \circ I^k} = [X, Y]_{I^k, I^k},$$

and we use, again, the recursion hypothesis.

For c), we use b) and (11):

$$[X, Y]_{I^n, I^m} = [X, Y]_{I, \overset{n}{\dots}, I, I^m} = [X, Y]_{I^m, I, \overset{n}{\dots}, I} = [X, Y]_{I, \overset{m+n}{\dots}, I} = [X, Y]_{I^{m+n}}. \qquad \square$$

If I is a Nijenhuis tensor on a pre-Courant algebroid (E, Θ), then, from parts b) and c) of Proposition 3.8, we have

$$\Theta_{I^{k_1}, \dots, I^{k_n}} = \Theta_{\underbrace{I, \dots, I}_{k_1 + \dots + k_n}} = \Theta_{I^{k_1 + \dots + k_n}}, \tag{12}$$

for all $k_1, \dots, k_n \in \mathbb{N}$, $n \in \mathbb{N}$.

3B. ***Hierarchy of compatible tensors with respect to Θ.*** In this section, we introduce the notion of compatible pair of $(1, 1)$-tensors with respect to a pre-Courant algebroid (E, Θ) and construct a hierarchy of compatible pairs of tensors.

The Magri–Morosi concomitant of a bivector and a $(1, 1)$-tensor on a manifold was introduced in [Magri and Morosi 1984] and then extended to Lie algebroids in [Kosmann-Schwarzbach and Magri 1990]. For a pre-Courant algebroid (E, Θ), we introduce a concomitant of two skew-symmetric $(1, 1)$-tensors I and J by setting

$$C_\Theta(I, J) = \{J, \{I, \Theta\}\} + \{I, \{J, \Theta\}\} = \Theta_{I,J} + \Theta_{J,I}. \tag{13}$$

If $(\rho, [\cdot, \cdot])$ is the pre-Courant structure on E corresponding to Θ, (13) reads as follows:

$$\begin{aligned} \{\{X, C_\Theta(I, J)\}, Y\} &= [X, Y]_{I,J} + [X, Y]_{J,I}, \\ \{\{X, C_\Theta(I, J)\}, f\} &= (\rho \circ (I \circ J + J \circ I))(X).f, \end{aligned} \tag{14}$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

In the sequel, we denote the left-hand side of (14) by $C_\Theta(I, J)(X, Y)$. When I and J anticommute, we have $\{\{X, C_\Theta(I, J)\}, f\} = 0$ for all $X \in \Gamma(E)$ and $f \in C^\infty(M)$. Therefore, in this case,

$$C_\Theta(I, J) = 0 \quad \Longleftrightarrow \quad C_\Theta(I, J)(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(E). \tag{15}$$

Remark 3.9. Let (A, μ) be a Lie algebroid. Recall that the Magri–Morosi concomitant of a bivector π and a $(1, 1)$-tensor N on A is given by [Kosmann-Schwarzbach and Magri 1990]:

$$C_\mu(\pi, N) = \{N, \{\pi, \mu\}\} + \{\pi, \{N, \mu\}\}. \tag{16}$$

If we consider the Courant algebroid $(A \oplus A^*, \mu)$ and the $(1, 1)$-tensors J_π and I_N as in Examples 2.6 and 2.9, respectively, we have that the concomitant of J_π and I_N given by (13) and the concomitant of π and N given by (16) coincide.

For the various classes of pairs of skew-symmetric $(1, 1)$-tensors that will be introduced in the sequel, we shall require that the skew-symmetric $(1, 1)$-tensors are compatible in the following sense:

Definition 3.10. A pair (I, J) of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) is said to be a *compatible pair with respect to* Θ if I and J anticommute and $C_\Theta(I, J) = 0$.

Let I and J be two $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Recall that the *Nijenhuis concomitant* of I and J is the map $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ (in general not a tensor) defined for all sections X and Y of E as follows [Kobayashi and Nomizu 1963]:

$$\begin{aligned}\mathcal{N}_\Theta(I, J)(X, Y) = [IX, JY] - I[X, JY] - J[IX, Y] + IJ[X, Y] \\ +[JX, IY] - J[X, IY] - I[JX, Y] + JI[X, Y].\end{aligned} \tag{17}$$

Notice that $\mathcal{N}_\Theta(I, I) = 2\mathcal{T}_\Theta I$, while if I and J anticommute, then

$$\mathcal{N}_\Theta(I, J) = \tfrac{1}{2} C_\Theta(I, J).$$

Lemma 3.11. *Let* (I, J) *be a pair of skew-symmetric* $(1, 1)$*-tensors on a pre-Courant algebroid* (E, Θ)*. Then,* $\mathcal{T}_\Theta(I + J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J + \mathcal{N}_\Theta(I, J)$.

Proof. Using the definition of the Nijenhuis torsion we get, for all $X, Y \in \Gamma(E)$,

$$\begin{aligned}\mathcal{T}_\Theta(I+J)(X,Y) &= \mathcal{T}_\Theta I(X,Y) + \mathcal{T}_\Theta J(X,Y) + [IX,JY] + [JX,IY] - I[X,JY] \\ &\quad - J[X,IY] - I[JX,Y] - J[IX,Y] + IJ[X,Y] + JI[X,Y] \\ &= \mathcal{T}_\Theta I(X,Y) + \mathcal{T}_\Theta J(X,Y) + \mathcal{N}_\Theta(I,J)(X,Y).\end{aligned}$$ ☐

The next proposition gives a characterization of compatible pairs.

Proposition 3.12. *Let (I, J) be a pair of anticommuting skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then, (I, J) is a compatible pair with respect to Θ if and only if $\mathcal{T}_\Theta(I+J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J$.*

Proposition 3.13. *Let (I, J) be a pair of anticommuting skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then for all sections X, Y of E and $n \geq 1$,*

$$C_\Theta(I, I^n \circ J)(X, Y) = I(C_\Theta(I, I^{n-1} \circ J)(X, Y)) \\ +2\,\mathcal{T}_\Theta I((I^{n-1} \circ J)X, Y) + 2\,\mathcal{T}_\Theta I(X, (I^{n-1} \circ J)Y). \tag{18}$$

Proof. A simple computation gives

$$\begin{aligned} C_\Theta(I, I^n \circ J)(X, Y) &= [X, Y]_{I, I^n \circ J} + [X, Y]_{I^n \circ J, I} \\ &= 2([I^n(JX), IX] - I[I^n(JX), Y] + [IX, I^n(JY)] \\ &\quad - I[X, I^n(JY)] - I^n \circ J[IX, Y] - I^n \circ J[X, IY]), \end{aligned}$$

for all sections X, Y of E and $n \geq 1$. Thus, we have

$$\begin{aligned} &I(C_\Theta(I, I^{n-1} \circ J)(X, Y)) \\ &\quad = 2(I[I^{n-1}(JX), IY] - I^2[I^{n-1}(JX), Y] + I[IX, I^{n-1}(JY)] \\ &\qquad - I^2[X, I^{n-1}(JY)] - I^n \circ J[IX, Y] - I^n \circ J[X, IY]). \end{aligned}$$

Since

$$\begin{aligned} &\mathcal{T}_\Theta I(I^{n-1}(JX), Y) \\ &\quad = [I^n(JX), IY] - I([I^n(JX), Y] + [I^{n-1}(JX), IY] - I[I^{n-1}(JX), Y]) \end{aligned}$$

and

$$\begin{aligned} &\mathcal{T}_\Theta I(X, I^{n-1}(JY)) \\ &\quad = [IX, I^n(JY)] - I([IX, I^{n-1}(JY)] + [X, I^n(JY)] - I[X, I^{n-1}(JY)]), \end{aligned}$$

the result follows. $\square$

Theorem 3.14. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then*

$$C_\Theta(I, I^n \circ J) = 0, \tag{19}$$

and $(I, I^n \circ J)$ is a compatible pair with respect to Θ for all $n \in \mathbb{N}$.

Proof. For $n = 0$, (19) reduces to $C_\Theta(I, J) = 0$ and (I, J) is a compatible pair with respect to Θ, which is one of the assumptions. From (18), we get

$$C_\Theta(I, I^n \circ J)(X, Y) = I(C_\Theta(I, I^{n-1} \circ J)(X, Y)), \quad n \geq 1,$$

for all sections X, Y of E, where we used $I^{n-1} \circ J = (-1)^{n-1} J \circ I^{n-1}$ to obtain

$$\mathcal{T}_\Theta I((I^{n-1}(JX), Y) = (-1)^{n-1} \mathcal{T}_\Theta I(J(I^{n-1}X), Y) = 0$$

and analogously

$$\mathcal{T}_\Theta I(X, I^{n-1}(JY)) = 0.$$

Therefore, using (15), it is obvious that if $C_\Theta(I, I^{n-1} \circ J) = 0$, then $C_\Theta(I, I^n \circ J) = 0$ and (19) follows by recursion. Since I anticommutes with $I^n \circ J$, the proof is complete. $\square$

3C. ***Compatible tensors with respect to Θ_k, $k \in \mathbb{N}$.*** In this section, we address the general case of hierarchies of tensors that are compatible with respect to each term of a family $(\Theta_k)_{k \in \mathbb{N}}$ of pre-Courant structures on E.

Proposition 3.15. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$C_{\Theta_I}(I, J) = C_\Theta(I, \{J, I\}) + \{I, C_\Theta(I, J)\}.$$

In particular, if I and J anticommute, then,

$$C_{\Theta_I}(I, J) = 2C_\Theta(I, I \circ J) + \{I, C_\Theta(I, J)\}. \tag{20}$$

Proof. Applying the Jacobi identity of the bracket $\{\cdot, \cdot\}$ twice, we get

$$\Theta_{I,I,J} = \Theta_{I,\{J,I\}} + \Theta_{I,J,I} = \Theta_{I,\{J,I\}} + \Theta_{\{J,I\},I} + \Theta_{J,I,I},$$

which can be written as

$$C_\Theta(I, \{J, I\}) = \Theta_{I,I,J} - \Theta_{J,I,I}.$$

From the definition of $C_\Theta(I, J)$, we have $\Theta_{J,I,I} = \{I, C_\Theta(I, J)\} - \Theta_{I,J,I}$. Substituting this result in the last equality, we get

$$C_\Theta(I, \{J, I\}) = \Theta_{I,I,J} - \{I, C_\Theta(I, J)\} + \Theta_{I,J,I} = C_{\Theta_I}(I, J) - \{I, C_\Theta(I, J)\},$$

proving the first statement. If I and J anticommute, then $\{J, I\} = 2I \circ J$ and the second statement follows. $\square$

The next theorem extends the result of Theorem 3.14.

Theorem 3.16. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then $C_{\Theta_k}(I, I^n \circ J) = 0$ and $(I, I^n \circ J)$ is a compatible pair with respect to Θ_k for all $k, n \in \mathbb{N}$.*

Proof. Suppose that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. We will prove, by induction on k, that

$$C_{\Theta_k}(I, I^n \circ J) = 0, \quad \text{for all } k, n \in \mathbb{N}.$$

For $k = 0$, this is the content of Theorem 3.14.

Suppose now that, for some $k \in \mathbb{N}$, $C_{\Theta_k}(I, I^n \circ J) = 0$ for all $n \in \mathbb{N}$. Then, from (20) we have, for all $n \in \mathbb{N}$,

$$C_{\Theta_{k+1}}(I, I^n \circ J) = 2C_{\Theta_k}(I, I^{n+1} \circ J) + \{I, C_{\Theta_k}(I, I^n \circ J)\} = 0,$$

where we used the induction hypothesis in the last equality. Since the skew-symmetric tensor $I^n \circ J$ anticommutes with I for all $n \in \mathbb{N}$, $(I, I^n \circ J)$ is a compatible pair with respect to Θ_k, for all $k, n \in \mathbb{N}$. □

In order to establish the main results of this section, we need the following lemmas.

Lemma 3.17. *Let (I, J) be a pair of anticommuting skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$C_\Theta(I, J) = 2(\Theta_{I,J} - \Theta_{I \circ J}).$$

Proof. Since I and J anticommute, $\{I, J\} = -2\, I \circ J$. Using the Jacobi identity of the bracket $\{\cdot\,, \cdot\}$, we have $\Theta_{J,I} = -2\Theta_{I \circ J} + \Theta_{I,J}$. Therefore,

$$C_\Theta(I, J) = \Theta_{I,J} + \Theta_{J,I} = 2(\Theta_{I,J} - \Theta_{I \circ J}). \qquad \square$$

Lemma 3.18. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that I is Nijenhuis for Θ. If (I, J) is a compatible pair with respect to Θ, then, for all sections X and Y of E,*

$$[X, Y]_{I, \overset{n}{\dots}, I, J} = [X, Y]_{I^n \circ J}.$$

Proof. Theorem 3.16 ensures that, for all $n \in \mathbb{N}$, $C_{\Theta_n}(I, J) = 0$ and, applying Lemma 3.17 for the pre-Courant structure Θ_{n-1}, we get

$$\begin{aligned} [X, Y]_{I, \overset{n}{\dots}, I, J} &= [X, Y]_{I, \overset{n-1}{\dots}, I, I, J} = \{\{X, (\Theta_{n-1})_{I,J}\}, Y\} \\ &= \{\{X, (\Theta_{n-1})_{I \circ J}\}, Y\} = [X, Y]_{I, \overset{n-1}{\dots}, I, I \circ J}. \end{aligned}$$

Since, for every $k \in \mathbb{N}$, I anticommutes with $I^k \circ J$, we may repeat $n-1$ times this procedure to yield

$$[X, Y]_{I, \overset{n}{\dots}, I, J} = [X, Y]_{I^n \circ J}. \qquad \square$$

Remark 3.19. In Lemma 3.18, we may replace the assumption that I is Nijenhuis for Θ by the weaker assumption $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E.

Theorem 3.20. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that I is Nijenhuis and (I, J) is a compatible pair with respect to Θ. Then,*

$$C_{\Theta_k}(I^{2s+1}, I^n \circ J) = 0 \tag{21}$$

and $(I^{2s+1}, I^n \circ J)$ is a compatible pair with respect to Θ_k for all $k, n, s \in \mathbb{N}$. Moreover, if J is Nijenhuis tensor, then

$$C_{\Theta_k}(I^{2s+1}, I^n \circ J^{2m+1}) = 0 \tag{22}$$

and $(I^{2s+1}, I^n \circ J^{2m+1})$ is a compatible pair with respect to Θ_k for all $k, m, n, s \in \mathbb{N}$.

Proof. Let I and J be two skew-symmetric $(1,1)$-tensors which are compatible with respect to Θ and such that $\mathcal{T}_\Theta I = 0$. Firstly, we prove that

$$C_\Theta(I^{2s+1}, I^n \circ J) = 0, \quad \text{for all } s, n \in \mathbb{N}.$$

Since I^{2s+1} anticommutes with $I^n \circ J$, we may apply Lemma 3.17 to obtain

$$C_\Theta(I^{2s+1}, I^n \circ J)(X, Y) = 2([X, Y]_{I^{2s+1}, I^n \circ J} - [X, Y]_{I^{2s+1} \circ (I^n \circ J)}).$$

From Theorem 3.14, $(I, I^n \circ J)$ is a compatible pair with respect to Θ and, applying Lemma 3.18, we get

$$\begin{aligned} C_\Theta(I^{2s+1}, I^n \circ J)(X, Y) &= 2([X, Y]_{I^{2s+1}, I^n \circ J} - [X, Y]_{I, \overset{2s+1}{\ldots}, I, I^n \circ J}) \\ &= 2([X, Y]_{I^{2s+1}} - [X, Y]_{I^{2s+1}})_{I^n \circ J} = 0, \end{aligned}$$

where we have used Proposition 3.8b) in the second equality. From (15), we obtain $C_\Theta(I^{2s+1}, I^n \circ J) = 0$.

In order to prove the result for a general Θ_k, notice that, due to Corollary 3.2 and Theorem 3.16, the assumptions originally satisfied for Θ are also satisfied for any of the pre-Courant structures Θ_k, $k \in \mathbb{N}$. Therefore, in the above arguments, we can replace Θ by any Θ_k, $k \in \mathbb{N}$.

Now, suppose that I and J are both Nijenhuis for Θ. Since they play symmetric roles, we may exchange them in (21) and, taking $k = 0$, $n = 0$ and $s = m$, we obtain $C_\Theta(I, J^{2m+1}) = 0$. Because I and J^{2m+1} anticommute, we conclude that (I, J^{2m+1}) is a compatible pair with respect to Θ. Thus, we may apply (21) again, replacing J by J^{2m+1}, to obtain $C_{\Theta_k}(I^{2s+1}, I^n \circ J^{2m+1}) = 0$ and, because I^{2s+1} anticommutes with $I^n \circ J^{2m+1}$, the pair $(I^{2s+1}, I^n \circ J^{2m+1})$ is a compatible pair with respect to Θ_k, for all $k, m, n, s \in \mathbb{N}$. $\square$

4. Hierarchies of deforming-Nijenhuis pairs

We introduce the notion of deforming-Nijenhuis pair as well as the definitions of Poisson tensor and Poisson–Nijenhuis pair on a pre-Courant algebroid. We construct several hierarchies of deforming-Nijenhuis and Poisson–Nijenhuis pairs.

4A. ***Hierarchy of deforming-Nijenhuis pairs for Θ_k, $k \in \mathbb{N}$.*** Starting with a deforming-Nijenhuis pair (J, I) for Θ, we prove, in a first step, that it is also a deforming-Nijenhuis pair for Θ_k for all $k \in \mathbb{N}$. Then, we construct a hierarchy $(J, I^{2n+1})_{n\in\mathbb{N}}$ of deforming-Nijenhuis pairs for Θ_k for all $k \in \mathbb{N}$.

Definition 4.1. Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *deforming-Nijenhuis pair* for Θ if

- (J, I) is a compatible pair with respect to Θ,
- J is deforming for Θ,
- I is Nijenhuis for Θ.

We need the following lemmas.

Lemma 4.2. *Let (I, J) be a pair of anticommuting skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then, for all $k \in \mathbb{N}$,*

$$\big((\Theta_r)_{\{J,\{I,J\}\}}\big)_{I,\overset{s}{\dots},I} = (\Theta_{\{J,\{I,J\}\}})_{I,\overset{k}{\dots},I} \tag{23}$$

for all $r, s \in \mathbb{N}$ such that $r + s = k$.

In particular,

i) *if $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$, for some $\lambda_0 \in \mathbb{R}$, then*

$$(\Theta_k)_{\{J,\{I,J\}\}} = \lambda_0 (\Theta_{J,J})_{I,\overset{k+1}{\dots},I} \quad \textit{for all } k \in \mathbb{N},$$

ii) *if $\{J, \{I, J\}\}$ is a Θ-cocycle, then it is a Θ_k-cocycle for all $k \in \mathbb{N}$.*

Proof. Since I and J anticommute, we have

$$I \circ (I \circ J^2) = (I \circ J^2) \circ I \;\Leftrightarrow\; \{I, I \circ J^2\} = 0 \;\Leftrightarrow\; \{I, \{J, \{J, I\}\}\} = 0. \tag{24}$$

Using the Jacobi identity of the bracket $\{\cdot, \cdot\}$, it follows from (24) that

$$\Theta_{I,\{J,\{J,I\}\}} = \Theta_{\{J,\{J,I\}\},I}. \tag{25}$$

Since (25) holds for any pre-Courant structure on E, we may write

$$(\Theta_{I,\overset{r+s}{\dots},I})_{\{J,\{I,J\}\}} = (\Theta_{I,\overset{r+s-1}{\dots},I})_{\{J,\{I,J\}\},I}.$$

Repeating the procedure $(s - 1)$ times, we obtain (23). The particular cases follow immediately. $\square$

Lemma 4.3. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). Then,*

$$\Theta_{J,I,J} = \tfrac{1}{3}\big(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} + \{J, C_\Theta(I, J)\}\big), \tag{26}$$

$$\Theta_{I,J,J} = -\tfrac{1}{3}\big(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} - 2\{J, C_\Theta(I, J)\}\big). \tag{27}$$

Proof. The formulae are obtained by application of the Jacobi identity. □

As a particular case of the previous lemma, we have the following:

Corollary 4.4. *If $C_\Theta(I, J) = 0$ and $\Theta_{\{J,\{I,J\}\}} = \lambda_0\Theta_{J,J,I}$, $\lambda_0 \in \mathbb{R}$, then*

$$\Theta_{I,J,J} = \alpha\Theta_{J,J,I} \quad \text{with } \alpha = -\frac{\lambda_0+1}{3}. \tag{28}$$

Moreover, if J is deforming for Θ, that is, $\Theta_{J,J} = \eta\Theta$ with $\eta \in \mathbb{R}$, then J is deforming for Θ_I. More precisely, $\Theta_{I,J,J} = \eta\,\alpha\,\Theta_I$.

Lemma 4.5. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that (I, J) is a compatible pair with respect to Θ and $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. Suppose that $\Theta_{\{J,\{I,J\}\}} = \lambda_0\Theta_{J,J,I}$ for some $\lambda_0 \in \mathbb{R}\backslash\{4/((-3)^m - 1), m \in \mathbb{N}\}$. Then, for all $k \in \mathbb{N}$:*

(a) *$(\Theta_k)_{\{J,\{I,J\}\}} = \lambda_k(\Theta_k)_{J,J,I}$, where λ_k is defined by recursion*[6] *as follows: $\lambda_k = -3\lambda_{k-1}/(1 + \lambda_{k-1})$, $k \geq 1$.*

(b) *$\lambda_k(\Theta_k)_{J,J,I} = \lambda_0\Theta_{J,J,I,\overset{k+1}{\dots},I}$.*

(c) *If, in particular, $\lambda_0 = 0$, then $(\Theta_k)_{J,J} = \big(-\tfrac{1}{3}\big)^k\Theta_{J,J,I,\overset{k}{\dots},I}$ for all $k \in \mathbb{N}$.*

Proof.

(a) We will prove this statement by induction. Suppose that, for some $k \geq 1$, $(\Theta_{k-1})_{\{J,\{I,J\}\}} = \lambda_{k-1}(\Theta_{k-1})_{J,J,I}$. Using Lemma 4.2 and the induction hypothesis, we have

$$(\Theta_k)_{\{J,\{I,J\}\}} = (\Theta_{k-1})_{\{J,\{I,J\}\},I} = \lambda_{k-1}(\Theta_{k-1})_{J,J,I,I}.$$

Applying formula (28) for Θ_{k-1}, we obtain

$$(\Theta_k)_{\{J,\{I,J\}\}} = \frac{-3\lambda_{k-1}}{1+\lambda_{k-1}}(\Theta_{k-1})_{I,J,J,I} = \lambda_k(\Theta_k)_{J,J,I}, \quad \text{with } \lambda_k = \frac{-3\lambda_{k-1}}{1+\lambda_{k-1}}.$$

(b) Starting from the previous statement, then using the Lemma 4.2 and the hypothesis, we have

$$\lambda_k(\Theta_k)_{J,J,I} = (\Theta_k)_{\{J,\{I,J\}\}} = \Theta_{\{J,\{I,J\}\},I,\overset{k}{\dots},I} = \lambda_0\Theta_{J,J,I,\overset{k+1}{\dots},I}.$$

[6] Explicitly, $\lambda_k = \dfrac{(-3)^k\lambda_0}{1 + \frac{1-(-3)^k}{4}\lambda_0}$ for all $k \in \mathbb{N}$.

(c) From Lemma 4.2i), we get

$$(\Theta_k)_{\{J,\{I,J\}\}} = 0 \quad \text{for all } k \in \mathbb{N},$$

while Theorem 3.16 gives

$$C_{\Theta_k}(I,J) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, applying the formula (27) several times yields

$$(\Theta_k)_{J,J} = -\tfrac{1}{3}(\Theta_{k-1})_{J,J,I} = \cdots = \left(-\tfrac{1}{3}\right)^k \Theta_{J,J,I,\overset{k}{\dots},I}\,. \qquad \square$$

Proposition 4.6. *Let* (I,J) *be a pair of skew-symmetric* $(1,1)$*-tensors on a pre-Courant algebroid* (E,Θ) *such that* (I,J) *is a compatible pair with respect to* Θ *and* $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ *for some* $\lambda_0 \in \mathbb{R}\backslash\{4/((-3)^m-1), m \in \mathbb{N}\}$. *Assume moreover that* $\mathcal{T}_\Theta I(JX,Y) = \mathcal{T}_\Theta I(X,JY) = 0$ *for all sections* X *and* Y *of* E. *If* J *is a deforming tensor for* Θ, *then* J *is also a deforming tensor for* Θ_k *for all* $k \in \mathbb{N}$.

Proof. We consider two cases, depending on the value of λ_0.

i) Case $\lambda_0 \neq 0$. From Theorem 3.16, we have that $C_{\Theta_k}(I,J) = 0$, for all $k \in \mathbb{N}$. We compute,[7] using Lemma 4.3 and both statements of Lemma 4.5,

$$\begin{aligned}(\Theta_k)_{J,J} = (\Theta_{k-1})_{I,J,J} &= -\tfrac{1}{3}((\Theta_{k-1})_{J,J,I} + (\Theta_{k-1})_{\{J,\{I,J\}\}})\\ &= -\tfrac{1}{3}((\Theta_{k-1})_{J,J,I} + \lambda_{k-1}(\Theta_{k-1})_{J,J,I})\\ &= -\frac{1+\lambda_{k-1}}{3}(\Theta_{k-1})_{J,J,I}\\ &= -\frac{(1+\lambda_{k-1})\lambda_0}{3\lambda_{k-1}}\,\Theta_{J,J,I,\overset{k}{\dots},I}\\ &= \frac{\lambda_0}{\lambda_k}\,\Theta_{J,J,I,\overset{k}{\dots},I}.\end{aligned}$$

The tensor J being deforming for Θ, we have $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$, and the last equality becomes

$$(\Theta_k)_{J,J} = \frac{\lambda_0}{\lambda_k}\,\eta\Theta_k,$$

which means that J is a deforming tensor for Θ_k.

(ii) Case $\lambda_0 = 0$. If J is deforming for Θ, that is, $\Theta_{J,J} = \eta\Theta$ with $\eta \in \mathbb{R}$, then, from Lemma 4.5c) we immediately get

$$(\Theta_k)_{J,J} = \left(-\tfrac{1}{3}\right)^k \eta\Theta_k \quad \text{for all } k \in \mathbb{N},$$

which means that J is deforming for Θ_k. $\square$

Now, we establish the main result of this section.

[7] Notice that if $\lambda_0 \neq 0$ then $\lambda_k \neq 0$ for all $k \in \mathbb{N}$.

Theorem 4.7. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant (respectively, Courant) algebroid (E, Θ) such that $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ for some $\lambda_0 \in \mathbb{R}\backslash\{4/((-3)^m - 1), m \in \mathbb{N}\}$. If (J, I) is a deforming-Nijenhuis pair for Θ, then (J, I^{2n+1}) is a deforming-Nijenhuis pair for the pre-Courant (respectively, Courant) structures Θ_k for all $k, n \in \mathbb{N}$.*

Proof. Let (J, I) be a deforming-Nijenhuis pair for Θ. Combining Corollary 3.2, Theorem 3.16 and Proposition 4.6, we have that (J, I) is a deforming-Nijenhuis pair for Θ_k for all $k \in \mathbb{N}$. From Proposition 3.5 we obtain that I^{2n+1} is Nijenhuis for Θ_k for all $k, n \in \mathbb{N}$. Since I and J anticommute, the tensors I^{2n+1} and J also anticommute and, from Theorem 3.20, we have that $C_{\Theta_k}(I^{2n+1}, J) = 0$, for all $k, n \in \mathbb{N}$. Thus, (J, I^{2n+1}) is a deforming-Nijenhuis pair for Θ_k for all $k, n \in \mathbb{N}$. □

4B. *Hierarchy of Poisson–Nijenhuis pairs for Θ_k, $k \in \mathbb{N}$.* We introduce the notions of Poisson tensor, Poisson–Nijenhuis pair and compatible Poisson tensors for a pre-Courant algebroid (E, Θ) and construct a hierarchy of Poisson–Nijenhuis pairs.

We start by introducing the notion of Poisson tensor.

Definition 4.8. A skew-symmetric $(1, 1)$-tensor J on a pre-Courant algebroid (E, Θ) satisfying $\Theta_{J,J} = 0$ is said to be a *Poisson* tensor for Θ.

In the next example, we show that the previous definition extends the usual definition of a Poisson bivector on a Lie algebroid.

Example 4.9. Let (A, μ) be a Lie algebroid. Consider the Courant algebroid $(A \oplus A^*, \Theta = \mu)$ and the $(1, 1)$-tensor J_π of Example 2.6. Then, J_π is a Poisson tensor for $\Theta = \mu$ if and only if π is a Poisson tensor on (A, μ).

Example 4.10. The tensors introduced in Example 2.3 are Poisson tensors on Lie algebras.

The next theorem follows directly from Lemma 4.5c).

Theorem 4.11. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) such that (I, J) is a compatible pair with respect to Θ, $\Theta_{\{J,\{I,J\}\}} = 0$ and $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If J is Poisson for Θ, then J is Poisson for Θ_k for all $k \in \mathbb{N}$.*

Requiring $\Theta_{\{J,\{I,J\}\}} = 0$ might seem somewhat arbitrary, but it is not. In fact, in the case where I and J anticommute, this condition may be interpreted as $I \circ J^2$ being a Θ-cocycle. When $E = A \oplus A^*$, a $(1, 1)$-tensor J_π of the type considered in Example 2.6 trivially satisfies this condition because $J_\pi^2 = 0$.

Now, we introduce the main notion of this section.

Definition 4.12. Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *Poisson–Nijenhuis pair* for Θ if

- (J, I) is a compatible pair with respect to Θ,
- J is Poisson for Θ,
- I is Nijenhuis for Θ.

Remark 4.13. If (J, I) is a Poisson–Nijenhuis pair for Θ, then it is a deforming-Nijenhuis pair for Θ.

Recall that a Poisson–Nijenhuis structure on a Lie algebroid (A, μ) is a pair (π, N), where π is a Poisson bivector and $N : A \to A$ is a Nijenhuis tensor such that $N\pi^\# = \pi^\# N^*$ and $C_\mu(\pi, N) = 0$.

The next example shows the relation between Definition 4.12 and the notion of Poisson–Nijenhuis structure on a Lie algebroid.

Example 4.14. Let (π, N) be a Poisson–Nijenhuis structure on a Lie algebroid (A, μ) with $N^2 = \alpha\, \mathrm{id}_A$, $\alpha \in \mathbb{R}$. Consider the Courant algebroid (E, Θ), with $E = A \oplus A^*$ and $\Theta = \mu$, J_π and I_N as in Examples 2.6 and 2.9, respectively. Then, (J_π, I_N) is a Poisson–Nijenhuis pair for Θ. In fact,

$$N\pi^\# = \pi^\# N^* \Leftrightarrow I_N \circ J_\pi = -J_\pi \circ I_N$$

and $C_\mu(\pi, N) = C_\mu(J_\pi, I_N) = 0$, so that (J_π, I_N) is a compatible pair with respect to μ. Moreover, π is a Poisson bivector on (A, μ) if and only if J_π is Poisson for $\Theta = \mu$ (see Example 4.9) and I_N is Nijenhuis for $\Theta = \mu$ (see Example 2.9). The above arguments show that conversely, if (J_π, I_N) is a Poisson–Nijenhuis pair for $\Theta = \mu$ with $N^2 = \alpha\, \mathrm{id}_A$, then (π, N) is a Poisson–Nijenhuis structure on (A, μ).

Definition 4.15. Let J and J' be two Poisson tensors for the pre-Courant structure Θ on the vector bundle $(E, \langle \cdot, \cdot \rangle)$. The tensors J and J' are said to be *compatible* Poisson tensors for Θ if $J + J'$ is a Poisson tensor for Θ, i.e., $\Theta_{J+J', J+J'} = 0$.

An immediate consequence of this definition is the following:

Lemma 4.16. *Let J and J' be two Poisson tensors for Θ. Then, J and J' are compatible Poisson tensors for Θ if and only if $\Theta_{J,J'} + \Theta_{J',J} = 0$. In other words, J and J' are compatible Poisson tensors for Θ if and only if $C_\Theta(J, J') = 0$.*

Example 4.17. Let (A, μ) be a Lie algebroid, consider the Courant algebroid $(A \oplus A^*, \Theta = \mu)$ and take two Poisson tensors for $\Theta = \mu$, J_π and $J_{\pi'}$, of the type considered in Example 2.6. Then,

$$\Theta_{J_\pi, J_{\pi'}} + \Theta_{J_{\pi'}, J_\pi} = \{\pi', \{\pi, \mu\}\} + \{\pi, \{\pi', \mu\}\} = 2\{\pi', \{\pi, \mu\}\} = -2[\pi, \pi']_\mu,$$

so that J_π and $J_{\pi'}$ are compatible Poisson tensors on $(A \oplus A^*, \mu)$ if and only if π and π' are compatible Poisson tensors on the Lie algebroid (A, μ).

In order to construct a hierarchy of Poisson–Nijenhuis pairs, we need the next proposition.

Proposition 4.18. *Let* (I, J) *be a pair of anticommuting skew-symmetric* $(1,1)$*-tensors on a pre-Courant algebroid* (E, Θ)*. Then, for all sections* X *and* Y *of* E,

$$\mathcal{T}_{\Theta_I} J(X,Y) = -J(C_\Theta(I,J)(X,Y)) - \mathcal{T}_\Theta J(IX,Y) - \mathcal{T}_\Theta J(X,IY) - I(\mathcal{T}_\Theta J(X,Y)) \tag{29}$$

and

$$\mathcal{T}_{\Theta_J} I(X,Y) = -I(C_\Theta(I,J)(X,Y)) - \mathcal{T}_\Theta I(JX,Y) - \mathcal{T}_\Theta I(X,JY) - J(\mathcal{T}_\Theta I(X,Y)). \tag{30}$$

Proof. Since the roles of I and J can be exchanged, we only prove (29). We compute $\mathcal{T}_{\Theta_I} J$ and $C_\Theta(I,J)$. For any sections X, Y of E, we have

$$\begin{aligned}\mathcal{T}_{\Theta_I} J(X,Y) &= [JX,JY]_I - J[JX,Y]_I - J[X,JY]_I + J^2[X,Y]_I \\ &= [IJX,JY] + [JX,IJY] - I[JX,JY] - J[IJX,Y] \\ &\quad - J[JX,IY] + JI[JX,Y] - J[IX,JY] - J[X,IJY] \\ &\quad + JI[X,JY] + J^2[IX,Y] + J^2[X,IY] - J^2I[X,Y]\end{aligned}$$

and

$$\begin{aligned}&C_\Theta(I,J)(X,Y) \\ &\quad = 2([JX,IY] + [IX,JY] - I([JX,Y]+[X,JY]) - J([IX,Y]+[X,IY])).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathcal{T}_{\Theta_I} J(X,Y) + J(C_\Theta(I,J)(X,Y)) \\ &\quad = -[JIX,JY] - [JX,JIY] - I[JX,JY] + J[JIX,Y] + J[JX,IY] + IJ[JX,Y] \\ &\qquad + J[IX,JY] + J[X,JIY] + IJ[X,JY] - J^2[IX,Y] - J^2[X,IY] - IJ^2[X,Y] \\ &\quad = -\mathcal{T}_\Theta J(IX,Y) - \mathcal{T}_\Theta J(X,IY) - I(\mathcal{T}_\Theta J(X,Y)).\end{aligned}$$

□

The next theorem defines a hierarchy of Poisson–Nijenhuis pairs.

Theorem 4.19. *Let* (J, I) *be a pair of skew-symmetric* $(1,1)$*-tensors on a pre-Courant algebroid* (E, Θ) *such that* (J, I) *is a Poisson–Nijenhuis pair for* Θ *and* $\Theta_{\{J,\{I,J\}\}} = 0$*. Then*:

1. $I^n \circ J$ *is a Poisson tensor for* Θ_k *for all* $n, k \in \mathbb{N}$.
2. $(I^n \circ J)_{n\in\mathbb{N}}$ *is a hierarchy of pairwise compatible Poisson tensors for* Θ_k*, for all* $k \in \mathbb{N}$.
3. $(I^n \circ J, I^{2m+1})$ *is a Poisson–Nijenhuis pair for* Θ_k*, for all* $m, n, k \in \mathbb{N}$.

The proof of this theorem needs two auxiliary lemmas.

Lemma 4.20. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that (I, J) is a compatible pair with respect to Θ. If I is Nijenhuis for Θ, then I is Nijenhuis for $(\Theta_k)_J$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. From Corollary 3.2, I is Nijenhuis for Θ_k. Also, applying Theorem 3.16, we obtain $C_{\Theta_k}(I, J) = 0$. Finally, using (30) for the pre-Courant structure Θ_k, we conclude that I is Nijenhuis for $(\Theta_k)_J$. □

Lemma 4.21. *Let (I, J) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that J is Poisson for Θ and $\Theta_{\{J,\{I,J\}\}} = 0$. Assume, moreover, that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$ for all sections X and Y of E. If (I, J) is a compatible pair with respect to Θ, then (I, J) is a compatible pair with respect to $(\Theta_k)_J$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. By definition, $C_{(\Theta_k)_J}(I, J) = (\Theta_k)_{J,I,J} + (\Theta_k)_{J,J,I}$. In order to compute $(\Theta_k)_{J,I,J}$, recall formula (26) for the pre-Courant structure Θ_k:

$$(\Theta_k)_{J,I,J} = \tfrac{1}{3}\big((\Theta_k)_{J,J,I} + (\Theta_k)_{\{J,\{I,J\}\}} + \{J, C_{\Theta_k}(I, J)\}\big).$$

Since (I, J) is a compatible pair with respect to Θ, applying Theorem 3.16, we obtain $C_{\Theta_k}(I, J) = 0$. Furthermore, the relation $(\Theta_k)_{\{J,\{I,J\}\}} = 0$ follows from Lemma 4.2(ii). Then, the above formula yields $(\Theta_k)_{J,I,J} = \frac{1}{3}(\Theta_k)_{J,J,I}$, so that $C_{(\Theta_k)_J}(I, J) = \frac{4}{3}(\Theta_k)_{J,J,I}$.

Now, using Theorem 4.11, we obtain $(\Theta_k)_{J,J,I} = 0$. Therefore, (I, J) is a compatible pair with respect to $(\Theta_k)_J$. □

We now the prove the above theorem.

Proof of Theorem 4.19. Let (I, J) be a Poisson–Nijenhuis pair for Θ such that $\Theta_{\{J,\{I,J\}\}} = 0$. We start by proving that

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = 0 \tag{31}$$

for all $m, n, k \in \mathbb{N}$. From the above auxiliary lemmas, (I, J) is a compatible pair with respect to $(\Theta_{k+m})_J$ and I is Nijenhuis for $(\Theta_{k+m})_J$. Then, using Lemma 3.18 for the pre-Courant structure $(\Theta_{k+m})_J$, we obtain

$$\begin{aligned}(\Theta_k)_{I^m \circ J, I^n \circ J} &= ((\Theta_{k+m})_J)_{I^n \circ J} = ((\Theta_{k+m})_J)_{I, \overset{n}{\dots}, I, J} \\ &= \Theta_{I, \overset{k+m}{\dots}, I, J, I, \overset{n}{\dots}, I, J} = (-1)^n\, \Theta_{I, \overset{k+m+n}{\dots}, I, J, J},\end{aligned}$$

where in the last equality we used n times that $C_{\Theta_s}(I, J) = 0$ for all $s \in \mathbb{N}$ (see Theorem 3.16). Using Theorem 4.11, we obtain (31), from which statements (1) and (2) follow. From Theorem 3.20, $(I^n \circ J, I^{2m+1})$ is a compatible pair with respect to Θ_k and, from Proposition 3.5, I^{2m+1} is Nijenhuis for Θ_k. Combining this with statement (1), we obtain statement (3). □

Using the Poisson–Nijenhuis pair arising from a Poisson–Nijenhuis structure as in Example 4.14, we recover most of the hierarchy already studied in [Kosmann-Schwarzbach and Magri 1990], up to a minor difference. In this general setting it is not possible to consider I^{2n} since it is not a skew-symmetric $(1,1)$-tensor.

We conclude this section with a particular case of deforming-Nijenhuis pairs.

Proposition 4.22. *Let (J, I) be a pair of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that $I^2 = \alpha\,\mathrm{id}_E$ and $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ for some $\alpha, \lambda_0 \in \mathbb{R}$. If (J, I) is a deforming-Nijenhuis pair for Θ, then $(I^n \circ J, I)$ is a deforming-Nijenhuis pair for Θ, for all $n \in \mathbb{N}$.*

Proof. Let (J, I) be a deforming-Nijenhuis pair for Θ. First, we prove that $I^n \circ J$ is deforming for Θ. Since $I^2 = \alpha\,\mathrm{id}_E$, $I^n \circ J$ is proportional either to J or to $I \circ J$. So, we only need to prove that $I \circ J$ is deforming for Θ. Using Lemma 3.17 and the fact that I and J anticommute, we have

$$\Theta_{I\circ J, I\circ J} = \Theta_{I,J,I\circ J} = \tfrac{1}{2}\Theta_{I,J,\{J,I\}} = \tfrac{1}{2}\big(\Theta_{I,J,I,J} - \Theta_{I,J,J,I}\big),$$

where in the last equality we used the Jacobi identity of the bracket $\{\cdot,\cdot\}$. Using (26) for Θ_I and Lemma 4.2, we get

$$2\Theta_{I\circ J, I\circ J} = \tfrac{1}{3}\big(\Theta_{I,J,J,I} + \Theta_{I,\{J,\{I,J\}\}}\big) - \Theta_{I,J,J,I} = -\tfrac{2}{3}\Theta_{I,J,J,I} + \tfrac{1}{3}\Theta_{\{J,\{I,J\}\},I}.$$

Now, from the equality (27), we obtain

$$2\Theta_{I\circ J, I\circ J} = \tfrac{2}{9}\Theta_{J,J,I,I} + \tfrac{5}{9}\Theta_{\{J,\{I,J\}\},I}.$$

Since $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$ and $\Theta_{J,J} = \eta\Theta$ for some $\eta \in \mathbb{R}$, we get

$$\Theta_{I\circ J, I\circ J} = \frac{2+5\lambda_0}{18}\eta\Theta_{I,I} = \frac{2+5\lambda_0}{18}\eta\Theta_{I^2} = \frac{2+5\lambda_0}{18}\eta\alpha\Theta,$$

where, in the last equalities, we used the fact that I is Nijenhuis and satisfies $I^2 = \alpha\,\mathrm{id}_E$. Therefore, $I \circ J$ is deforming for Θ.

The tensors I and $I^n \circ J$ anticommute and, from Theorem 3.14, $C_\Theta(I, I^n \circ J) = 0$. Thus, $(I^n \circ J, I)$ is a deforming-Nijenhuis pair for Θ. □

Notice that $(I^n \circ J, I)_{n\in\mathbb{N}}$ is a very poor hierarchy of deforming-Nijenhuis pairs since, as we already mentioned, all the pairs are of type either (J, I) or $(I \circ J, I)$. In fact we have, for all $n \in \mathbb{N}$,

$$I^{2n} \circ J = \alpha^n J, \qquad I^{2n+1} \circ J = \alpha^n I \circ J.$$

5. Hierarchies of Nijenhuis pairs

The last part of this article is devoted to the study of pairs of Nijenhuis tensors on pre-Courant algebroids.

5A. ***Nijenhuis pair for a hierarchy of pre-Courant structures.*** We introduce the notion of Nijenhuis pair for a pre-Courant algebroid (E, Θ) and prove that a Nijenhuis pair (I, J) for Θ is still a Nijenhuis pair for any deformation of Θ, either by I or J.

We first introduce the notion of Nijenhuis pair for a pre-Courant algebroid.

Definition 5.1. Let I and J be two skew-symmetric tensors on a pre-Courant algebroid (E, Θ). The pair (I, J) is called a *Nijenhuis pair* for Θ, if it is a compatible pair with respect to Θ and I and J are both Nijenhuis for Θ.

Example 5.2. Let J be a deforming tensor on (E, Θ), that is $\Theta_{J,J} = \eta\Theta$, for some $\eta \in \mathbb{R}$. If (J, I) is a deforming-Nijenhuis pair, with $J^2 = \eta\, \mathrm{id}_E$, then (J, I) is a Nijenhuis pair. In particular, if (J, I) is Poisson–Nijenhuis pair, and $J^2 = 0$, then (J, I) is a Nijenhuis pair. Notice that this happens when $J = J_\pi$ as in Example 2.6.

In the next proposition we compute the torsion of the composition $I \circ J$.

Proposition 5.3. *Let (I, J) be a pair of anticommuting tensors on a pre-Courant algebroid (E, Θ). Then, for all sections X and Y of E,*

$$2\mathcal{T}_\Theta(I \circ J)(X, Y) = \big(\mathcal{T}_\Theta I(JX, JY) - J\big(\mathcal{T}_\Theta I(JX, Y) + \mathcal{T}_\Theta I(X, JY)\big) - J^2(\mathcal{T}_\Theta I(X, Y))\big) + \underset{I,J}{\circlearrowleft}, \tag{32}$$

where $\underset{I,J}{\circlearrowleft}$ stands for permutation of I and J.

Proof. Let us compute the first four terms of the right hand side of (32):

$$\begin{aligned}
\mathcal{T}_\Theta I(JX, JY) &= [IJX, IJY] - I[IJX, JY] - I[JX, IJY] + I^2[JX, JY] \\
-J(\mathcal{T}_\Theta I(JX, Y)) &= -J[IJX, IY] + JI[IJX, Y] + JI[JX, IY] - JI^2[JX, Y] \\
-J(\mathcal{T}_\Theta I(X, JY)) &= -J[IX, IJY] + JI[IX, JY] + JI[X, IJY] - JI^2[X, JY] \\
-J^2(\mathcal{T}_\Theta I(X, Y)) &= -J^2[IX, IY] + J^2 I[IX, Y] + J^2 I[X, IY] - J^2 I^2[X, Y].
\end{aligned}$$

The terms appearing on the right hand sides of the above equalities can be written in a matrix form:

$$M(I, J)(X, Y) = \begin{bmatrix} [IJX, IJY] & -I[IJX, JY] & -I[JX, IJY] & I^2[JX, JY] \\ -J[IJX, IY] & JI[IJX, Y] & JI[JX, IY] & -JI^2[JX, Y] \\ -J[IX, IJY] & JI[IX, JY] & JI[X, IJY] & -JI^2[X, JY] \\ -J^2[IX, IY] & J^2 I[IX, Y] & J^2 I[X, IY] & -J^2 I^2[X, Y] \end{bmatrix}.$$

Because I and J anticommute, exchanging the tensors I and J, we obtain the matrix $M(J, I)$ with entries given by

$$M(J, I)_{m,n} = \begin{cases} -M(I, J)_{n,m} & \text{if } m \neq n, \\ M(I, J)_{m,n} & \text{if } m = n, \end{cases}$$

for all $m, n = 1, \dots, 4$.

Note that the right hand side of (32) is the sum of all the entries of both matrices $M(I, J)(X, Y)$ and $M(J, I)(X, Y)$. Thus,

$$\begin{aligned} \mathcal{T}_\Theta I(JX, JY) - J\big(\mathcal{T}_\Theta I(JX, Y) + \mathcal{T}_\Theta I(X, JY)\big) - J^2(\mathcal{T}_\Theta I(X, Y)) + \underset{I,J}{\circlearrowleft} \\ = 2\big([IJX, IJY] + JI[IJX, Y] + JI[X, IJY] - J^2 I^2[X, Y]\big) \\ = 2\mathcal{T}_\Theta(I \circ J)(X, Y), \end{aligned}$$

and the proof is complete. ☐

Proposition 5.4. *Let (I, J) be a pair of skew-symmetric tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then $(I, I \circ J)$ and $(J, I \circ J)$ are also Nijenhuis pairs for Θ.*

Proof. It is obvious that I and $I \circ J$ anticommute, as well as J and $I \circ J$. From (32) we conclude that $I \circ J$ is a Nijenhuis tensor and from (19), with $n = 1$, we obtain $C_\Theta(I, I \circ J) = 0$ and $C_\Theta(J, I \circ J) = 0$. ☐

Using Proposition 5.4, we may establish a relationship between Nijenhuis pairs and hypercomplex triples.

A triple (I, J, K) of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ) is called a *hypercomplex triple* if $I^2 = J^2 = K^2 = I \circ J \circ K = -\operatorname{id}_E$ and all the six Nijenhuis concomitants $\mathcal{N}_\Theta(I, I)$, $\mathcal{N}_\Theta(J, J)$, $\mathcal{N}_\Theta(K, K)$, $\mathcal{N}_\Theta(I, J)$, $\mathcal{N}_\Theta(J, K)$ and $\mathcal{N}_\Theta(I, K)$ vanish [Stiénon 2009]. (See (17) for the definition of $\mathcal{N}_\Theta$).

Example 5.5. Given a Nijenhuis pair (I, J) such that $I^2 = J^2 = -\operatorname{id}_E$, the triple $(I, J, I \circ J)$ is a hypercomplex structure. Conversely, for every hypercomplex structure (I, J, K), the pairs (I, J), (J, K) and (K, I) are Nijenhuis pairs.

The main result of this section is the following.

Theorem 5.6. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then (I, J) is a Nijenhuis pair for $\Theta_{T_1, T_2, \dots, T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J, for every $i = 1, \dots, s$.*

Proof. Combining formulae (18) and (20) we get, for all $X, Y \in \Gamma(E)$,

$$\begin{aligned} C_{\Theta_I}(I, J)(X, Y) &= 2I(C_\Theta(I, J)(X, Y)) + 4\mathcal{T}_\Theta I(JX, Y) + 4\mathcal{T}_\Theta I(X, JY) \\ &= 0. \end{aligned} \tag{33}$$

Now, from Corollary 3.2, (29) and (33), we conclude that (I, J) is a Nijenhuis pair for Θ_I. Since we may exchange the roles of I and J, we also conclude that (I, J) is a Nijenhuis pair for Θ_J.

Since Corollary 3.2 and the formulae (29) and (33) hold for any anticommuting tensors I and J and for any pre-Courant structure Θ on E, we can repeat the previous argument iteratively to conclude that (I, J) is a Nijenhuis pair for $\Theta_{T_1,T_2,\dots,T_s}$ for all $s \in \mathbb{N}$, where T_i stands either for I or for J for every $i = 1, \dots, s$. □

As a consequence of the above theorem, we deduce:

Corollary 5.7. *Let (I, J) be a pair of $(1, 1)$-tensors on a Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $s \in \mathbb{N}$, $\Theta_{T_1,T_2,\dots,T_s}$ is a Courant structure on E, where T_i stands either for I or for J, for every $i = 1, \dots, s$.*

5B. *Hierarchies of Nijenhuis pairs.* Starting with a Nijenhuis pair (I, J) for a pre-Courant algebroid (E, Θ), we construct several hierarchies of Nijenhuis pairs for any deformation of Θ, either by I or J.

We start with the construction of a hierarchy $(I^{2m+1}, J)_{m\in\mathbb{N}}$ of Nijenhuis pairs where one of the Nijenhuis tensors remains unchanged.

Proposition 5.8. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair for $\Theta_{T_1,T_2,\dots,T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J for every $i = 1, \dots, s$.*

Proof. The proof follows from Proposition 3.5, Theorem 3.20 and Theorem 5.6. □

Now we consider the hierarchy $(I^{2m+1}, J^{2n+1})_{m,n\in\mathbb{N}}$. This case follows from the previous one: for every $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair. Applying Proposition 5.8 to each of these pairs, we get that $(I^{2m+1}, J^{2n+1})_{m,n\in\mathbb{N}}$ is a hierarchy of Nijenhuis pairs and we obtain the following.

Theorem 5.9. *Let (I, J) be a pair of $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m, n \in \mathbb{N}$, (I^{2m+1}, J^{2n+1}) is a Nijenhuis pair for $\Theta_{T_1,T_2,\dots,T_s}$, for all $s \in \mathbb{N}$, where T_i stands either for I or for J, for every $i = 1, \dots, s$.*

Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If I and J are Nijenhuis tensors, we know (see Proposition 3.5) that, for any $m, n \in \mathbb{N}$, I^m and J^n are also Nijenhuis tensors for Θ. The next lemma gives a condition ensuring that $I^m \circ J^n$ is also Nijenhuis.

Lemma 5.10. *Let (I, J) be a pair of skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). If I and J are anticommuting Nijenhuis tensors, then $I^m \circ J^n$ is a Nijenhuis tensor provided that at least one of the integers m, n is odd.*

Proof. As the roles of the tensors I and J are symmetric, we can suppose that m is odd (and n is even or odd). If n is also odd then I^m and J^n anticommute and the result follows from Proposition 5.3. Suppose now that m is odd and n is even. By the previous case, $I^m \circ J^{n-1}$ is Nijenhuis and anticommutes with J:

$$(I^m \circ J^{n-1}) \circ J = I^m \circ J^n = -J \circ (I^m \circ J^{n-1}).$$

Then, using again Proposition 5.3, we conclude that $I^m \circ J^n$ is a Nijenhuis tensor. □

The main result of this section is the following theorem.

Theorem 5.11. *Let* (I, J) *be a pair of skew-symmetric* $(1, 1)$*-tensors on a pre-Courant algebroid* (E, Θ). *If* (I, J) *is a Nijenhuis pair for* Θ, *then for all* $m, n, t \in \mathbb{N}$, $(I^{2m+1} \circ J^n, J^{2t+1})$ *is a Nijenhuis pair for* $\Theta_{T_1,T_2,\ldots,T_s}$, *for all* $s \in \mathbb{N}$, *where* T_i *stands either for* I *or for* J *for every* $i = 1, \ldots, s$.

Proof. First, we prove that $(I^{2m+1} \circ J^n, J^{2t+1})$ is a Nijenhuis pair for Θ for all $m, n, t \in \mathbb{N}$. We already know that $I^{2m+1} \circ J^n$ is Nijenhuis (see Lemma 5.10) and that J^{2t+1} is Nijenhuis (see Proposition 3.5). Moreover, $I^{2m+1} \circ J^n$ anticommutes with J^{2t+1} and, applying (22), we obtain $C_\Theta(I^{2m+1} \circ J^n, J^{2t+1}) = 0$.

Using Theorem 5.6, this result can be extended to all pre-Courant structures $\Theta_{T_1,T_2,\ldots,T_s}$, where T_i stands either for I or for J for every $i = 1, \ldots, s$. □

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A NEW CHARACTERIZATION OF COMPLETE LINEAR WEINGARTEN HYPERSURFACES IN REAL SPACE FORMS

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We apply the Hopf's strong maximum principle in order to obtain a suitable characterization of the complete linear Weingarten hypersurfaces immersed in a real space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c. Under the assumption that the mean curvature attains its maximum and supposing an appropriated restriction on the norm of the traceless part of the second fundamental form, we prove that such a hypersurface must be either totally umbilical or isometric to a Clifford torus, if $c = 1$, a circular cylinder, if $c = 0$, or a hyperbolic cylinder, if $c = -1$.

1. Introduction and statement of the main result

Many authors have approached the problem of characterizing hypersurfaces immersed with constant mean curvature or with constant scalar curvature in a real space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c. In this setting, Cheng and Yau [1977] introduced a new self-adjoint differential operator $\square$ acting on smooth functions defined on Riemannian manifolds. As a byproduct of this approach they were able to classify closed hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in $\mathbb{Q}_c^{n+1}$. Later on, Li [1996] extended the results of Cheng and Yau in terms of the squared norm of the second fundamental form of the hypersurface M^n. Shu [2007] applied the generalized Omori–Yau maximum principle [Omori 1967; Yau 1975] to prove that a complete hypersurface M^n in the hyperbolic space $\mathbb{H}^{n+1}$ with constant normalized scalar curvature and nonnegative sectional curvature must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$.

Li [1997] studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed in a unit sphere with scalar curvature proportional to mean curvature. Next, Li et al. [2009] extended the result of [Cheng and Yau 1977; Li 1997] by considering *linear Weingarten* hypersurfaces immersed in the

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unit sphere $\mathbb{S}^{n+1}$, that is, hypersurfaces of $\mathbb{S}^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy $R = aH + b$, for some $a, b \in \mathbb{R}$. In this setting, they showed that if M^n is a compact linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$, such that $R = aH + b$ with $(n-1)a^2 + 4n(b-1) \geq 0$, then M^n is either totally umbilical or isometric to a Clifford torus $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r)$, where $1 \leq k \leq n-1$. Thereafter, Shu [2010] obtained some rigidity theorems concerning to linear Weingarten hypersurfaces with two distinct principal curvatures immersed in $\mathbb{Q}_c^{n+1}$.

In [Brasil et al. 2010], Brasil Jr., Colares and Palmas used the generalized maximum principle of Omori–Yau to characterize complete hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$. By applying a weak Omori–Yau maximum principle due to Pigola, Rigoli and Setti [Pigola et al. 2005], Alías and García-Martínez [2010] studied the behavior of the scalar curvature R of a complete hypersurface immersed with constant mean curvature into a real space form $\mathbb{Q}_c^{n+1}$, deriving a sharp estimate for the infimum of R. More recently, Alías, García-Martínez and Rigoli [Alías et al. 2012] obtained another suitable weak maximum principle for complete hypersurfaces with constant scalar curvature in $\mathbb{Q}_c^{n+1}$, and gave some applications of it in order to estimate the norm of the traceless part of its second fundamental form. In particular, they extended the main theorem of [Brasil et al. 2010] for the context of $\mathbb{Q}_c^{n+1}$.

Here, our purpose is to establish a new characterization theorem concerning the complete linear Weingarten hypersurfaces immersed in a real space form $\mathbb{Q}_c^{n+1}$. Under the assumption that the mean curvature H attains its maximum along the hypersurface M^n and supposing an appropriated restriction on the norm of the traceless part Φ of the second fundamental form of M^n, we get the following theorem.

Theorem 1.1. *Let M^n be a complete linear Weingarten hypersurface immersed in a real space form $\mathbb{Q}_c^{n+1}$, $n \geq 3$, such that $R = aH + b$ with $b > c$. Suppose that $R > 0$, when $c = 0$ or $c = -1$, and that $R > (n-2)/n$, when $c = 1$. If H attains its maximum on M^n and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}, \tag{1-1}$$

then either

i. $|\Phi| \equiv 0$ *and M^n is totally umbilical, or*

ii. $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$ *and M^n is isometric to*

  (a) *a Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, when $c = 1$,*
  (b) *a circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, when $c = 0$, or*
  (c) *a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$, when $c = -1$,*

where in each case $r = \sqrt{\dfrac{n-2}{nR}}$.

The proof of Theorem 1.1 is given in Section 3, jointly with a corollary related to the compact case.

2. Preliminaries

In this section we will introduce some basic facts and notation that will appear on the paper. In what follows, we will suppose that all hypersurfaces are orientable and connect.

Let M^n be an n-dimensional hypersurface in a real space form $\mathbb{Q}_c^{n+1}$. We choose a local field of orthonormal frame $\{e_A\}$ in $\mathbb{Q}_c^{n+1}$, with dual coframe $\{\omega_A\}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \le A, B, C, \ldots \le n+1, \qquad 1 \le i, j, k, \ldots \le n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{Q}_c^{n+1}$, we have that the structure equations of $\mathbb{Q}_c^{n+1}$ are given by

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i + \omega_{An+1} \wedge \omega_{n+1}, \qquad \omega_{AB} + \omega_{BA} = 0, \tag{2-1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \tfrac{1}{2} \sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2-2}$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2-3}$$

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and by *Cartan's Lemma* [1938] we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2-4}$$

This gives the second fundamental form of M^n, $B = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2-5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \tfrac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \tag{2-6}$$

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2-7}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = (n-1)c\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2-8}$$

$$R = \frac{1}{n(n-1)} \sum_i R_{ii}. \tag{2-9}$$

From (2-8) and (2-9) we obtain

$$|B|^2 = n^2H^2 - n(n-1)(R-c), \tag{2-10}$$

where $|B|^2 = \sum_{i,j} h_{ij}^2$ is the square of the length of the second fundamental form B of M^n.

Set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i\omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless and, from (2-10), we get

$$|\Phi|^2 = |B|^2 - nH^2 = n(n-1)H^2 - n(n-1)(R-c). \tag{2-11}$$

The components h_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{jk}\omega_{ki}. \tag{2-12}$$

The *Codazzi equation* and the *Ricci identity* are, respectively, given by

$$h_{ijk} = h_{ikj}, \tag{2-13}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2-14}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From (2-13) and (2-14), we obtain

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,l} h_{kl}R_{lijk} + \sum_{k,l} h_{li}R_{lkjk}. \tag{2-15}$$

Since $\Delta|B|^2 = 2\big(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2\big)$, from (2-15) we get

$$\tfrac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_{i,i,k} h_{ij}h_{kkij} + \sum_{i,j,k,l} h_{ij}h_{lk}R_{lijk} + \sum_{i,j,k,l} h_{ij}h_{il}R_{lkjk}. \tag{2-16}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from (2-16) we obtain the following Simons-type formula

$$\tfrac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_i \lambda_i (nH)_{,ii} + \tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2-17}$$

Let $\phi = \sum_{i,j} \phi_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by $\phi_{ij} = nH\delta_{ij} - h_{ij}$. Following [Cheng and Yau 1977], we introduce a operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{2-18}$$

Since ϕ_{ij} is divergence-free, it follows from the same reference that the operator $\square$ is self-adjoint relative to the L^2 inner product of M^n, that is,

$$\int_M f\,\square g = \int_M g\,\square f,$$

for any smooth functions f and g on M^n.

Now, setting $f = nH$ in (2-18) and taking a local frame field $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from (2-10) we obtain the following:

$$\begin{aligned}\square(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{,ii} \\ &= \tfrac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{n(n-1)}{2}\Delta R + \tfrac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{,ii}.\end{aligned}$$

Hence, taking into account (2-17), we get

$$\square(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2-19}$$

3. Proof of Theorem 1.1 and a corollary

In order to prove our result, to use some auxiliary lemmas are necessary. The first is a classic algebraic lemma due to M. Okumura [1974], and completed with the equality case proved by H. Alencar and M. do Carmo [1994].

Lemma 3.1. *Let $\mu_1, \dots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{3-1}$$

and equality holds if and only if at least $n-1$ of the numbers μ_i are equal.

To obtain the second lemma, we will reason as in the proof of Lemma 2.1 of [Li et al. 2009].

Lemma 3.2. *Let M^n be a linear Weingarten hypersurface in a space form $\mathbb{Q}_c^{n+1}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(b-c) \geq 0. \tag{3-2}$$

Then

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{3-3}$$

Moreover, if the inequality (3-2) is strict and equality holds in (3-3) on M^n, then H is constant on M^n.

Proof. Since we are supposing that $R = aH + b$, from (2-10) we get

$$2\sum_{i,j} h_{ij}h_{ijk} = (2n^2H - n(n-1)a)H_{,k}.$$

Thus,

$$4\sum_k \Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2 = (2n^2H - n(n-1)a)^2 |\nabla H|^2.$$

Consequently, using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} 4|B|^2|\nabla B|^2 &= 4\Big(\sum_{i,j} h_{ij}^2\Big)\Big(\sum_{i,j,k} h_{ijk}^2\Big) \\ &\geq 4\sum_k \Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2 = (2n^2H - n(n-1)a)^2 |\nabla H|^2. \end{aligned} \tag{3-4}$$

On the other hand, since $R = aH + b$, from (2-10) we easily see that

$$(2n^2H - n(n-1)a)^2 = n^2(n-1)((n-1)a^2 + 4n(b-c)) + 4n^2|B|^2.$$

Hence, from (3-4) we have

$$|B|^2|\nabla B|^2 \geq n^2|B|^2|\nabla H|^2.$$

Therefore, we obtain either $|B| = 0$ and $|\nabla B|^2 = n^2|\nabla H|^2$, or $|\nabla B|^2 \geq n^2|\nabla H|^2$. Moreover, if $(n-1)a^2 + 4n(b-c) > 0$, from the previous identity we get that $(2n^2H - n(n-1)a)^2 > 4n^2|B|^2$. Now, let us assume in addition that the equality holds in (3-3) on M^n. In this case, we wish to show that H is constant on M^n. Suppose, by way of contradiction, that it does not occur. Consequently, there exists a point $p \in M^n$ such that $|\nabla H(p)| > 0$. So, one deduces from (3-4) that

$$4\,|B(p)|^2\,|\nabla B(p)|^2 > 4n^2\,|B(p)|^2\,|\nabla H(p)|^2$$

and, since $|\nabla B(p)|^2 = n^2\,|\nabla H(p)|^2 > 0$, we arrive at a contradiction. Hence, in this case, we conclude that H must be constant on M^n. $\square$

In what follows, we will consider the Cheng–Yau modified operator

$$L = \square - \frac{n-1}{2}a\Delta. \tag{3-5}$$

Related to operator, we have the following sufficient criterion for ellipticity.

Lemma 3.3. *Let M^n be a linear Weingarten hypersurface immersed in a space form $\mathbb{Q}_c^{n+1}$, such that $R = aH + b$ with $b > c$. Then, L is elliptic.*

Proof. From (2-10), since $R = aH + b$ with $b > c$, we easily see that H can not vanish on M^n and, by choosing the appropriate Gauss mapping, we may assume that $H > 0$ on M^n.

Let us consider the case that $a = 0$. Since $R = b > c$, from (2-10) if we choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, we have $\sum_{i<j}\lambda_i\lambda_j > 0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j > \lambda_i^2$$

for every $i = 1, \dots, n$ and, hence, we have that $nH - \lambda_i > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose $a \neq 0$. From (2-10) we get that

$$a = -\frac{1}{n(n-1)H}(|B|^2 - n^2H^2 + n(n-1)(b-c)).$$

Hence, for every $i = 1, \dots, n$, a straightforward algebraic computation yields

$$\begin{aligned} nH - \lambda_i - \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH}(|B|^2 - n^2H^2 + n(n-1)(b-c)) \\ &= \frac{1}{2nH}\biggl(\sum_{j\neq i}\lambda_j^2 + \biggl(\sum_{j\neq i}\lambda_j\biggr)^2 + n(n-1)(b-c)\biggr). \end{aligned}$$

Therefore, since $b > c$, we also conclude in this case that L is elliptic. $\square$

Proof of Theorem 1.1. Choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Since $R = aH + b$, from (2-19) and (3-5) we have

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \tfrac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{3-6}$$

Thus, since from (2-7) we have $R_{ijij} = \lambda_i\lambda_j + c$, we get from (3-6)

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nc(|B|^2 - nH^2) - |B|^4 + nH\sum_i \lambda_i^3. \tag{3-7}$$

Moreover, we have $\Phi_{ij} = \mu_i \delta_{ij}$ and, with a straightforward computation, we verify that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{3-8}$$

Thus, using Gauss (2-7) jointly with (3-8) into (3-7), we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \tag{3-9}$$

By applying Lemmas 3.1 and 3.2, from (3-9) we have

$$L(nH) \geq |\Phi|^2\Big(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 + nc\Big). \tag{3-10}$$

On the other hand, from (2-11), we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R-c) \tag{3-11}$$

Thus, from (3-10) and (3-11) we get

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R(|\Phi|), \tag{3-12}$$

where

$$P_R(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(R-c)} + n(n-1)R.$$

Since we are supposing that $R > 0$, $P_R(0) = n(n-1)R > 0$ and the function $P_R(x)$ is strictly decreasing for $x \geq 0$, with $P_R(x^*) = 0$ at

$$x^* = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}} > 0.$$

Thus, the hypothesis (1-1) guarantees that

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R(|\Phi|) \geq 0. \tag{3-13}$$

Consequently, since Lemma 3.3 guarantees that L is elliptic and as we are supposing that H attains its maximum on M^n, from (3-13) we conclude that H is constant on M^n. Thus, taking into account (3-6), we get

$$|\nabla B|^2 = n^2|\nabla H|^2 = 0,$$

and it follows that λ_i is constant for every $i = 1, \dots, n$.

If $|\Phi| < x^*$, then from (3-13) we have that $|\Phi| = 0$ and, hence, M^n is totally umbilical. If $|\Phi| = x^*$, since the equality holds in (3-1) of Lemma 3.1, we conclude that M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Hence, by the classical results on isoparametric hypersurfaces of real space forms [Cartan 1938; Levi-Civita 1937; Segre 1938] and since we are supposing $R > 0$, we conclude that either $|\Phi| = 0$ and M^n is totally umbilical, or

$$|\Phi|^2 = \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$$

and M^n is isometric to

(a) a Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, with $0 < r < 1$, if $c = 1$,

(b) a circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r > 0$, if $c = 0$, or

(c) a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$, with $r > 0$, if $c = -1$.

When $c = 1$, for a given radius $0 < r < 1$, is a standard fact that the product embedding $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \frac{r}{\sqrt{1-r^2}}, \qquad \lambda_2 = \cdots = \lambda_n = -\frac{\sqrt{1-r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2-(n-1)}{nr\sqrt{1-r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1-r^2)}.$$

When $c = 0$, for a given radius $r > 0$, $\mathbb{R} \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = 0, \qquad \lambda_2 = \cdots = \lambda_n = \frac{1}{r}.$$

In this case,

$$H = \frac{n-1}{nr} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2}.$$

Finally, when $c = -1$, for a given radius $r > 0$, $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{H}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \frac{r}{\sqrt{1+r^2}}, \qquad \lambda_2 = \cdots = \lambda_n = \frac{\sqrt{1+r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2+(n-1)}{nr\sqrt{1+r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1+r^2)}.$$

To finish our proof, we use (2-11) and verify with algebraic computations that in all these situations we must have $r = \sqrt{(n-2)/(nR)}$. □

Using the inequality (3-13) and taking into account that the operator L is self-adjoint relative to the L^2 inner product of the hypersurface M^n, we also get the following result:

Corollary 3.4. *Let M^n be a compact linear Weingarten hypersurface immersed in a real space form $\mathbb{Q}_c^{n+1}$, $n \geq 3$, such $R = aH + b$ with $(n-1)a^2 + 4n(b-c) \geq 0$. Suppose that $R > 0$ when $c = 0$ or $c = -1$, and that $R > (n-2)/n$ when $c = 1$. If*

$$\sup_M |\Phi|^2 < \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)},$$

then $|\Phi| \equiv 0$ and M^n is isometric to $\mathbb{S}^n$, up to scaling.

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CALOGERO–MOSER VERSUS KAZHDAN–LUSZTIG CELLS

CÉDRIC BONNAFÉ AND RAPHAËL ROUQUIER

In 1979, Kazhdan and Lusztig developed a combinatorial theory associated with Coxeter groups, defining in particular partitions of the group in left and two-sided cells. In 1983, Lusztig generalized this theory to Hecke algebras of Coxeter groups with unequal parameters. We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero-Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg. We conjecture that these coincide with Kazhdan-Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino, and we provide here a version of left cell representations. The Calogero-Moser cells will be studied in details in a forthcoming paper, providing thus several results supporting our conjecture.

1. Introduction

Kazhdan and Lusztig [1979] developed a combinatorial theory associated with Coxeter groups. They defined in particular partitions of the group in left and two-sided cells. For Weyl groups, these have a representation-theoretic interpretation in terms of primitive ideals, and they play a key role in Lusztig's description [1984] of unipotent characters for finite groups of Lie type. Lusztig [1983; 2003] generalized this theory to Hecke algebras of Coxeter groups with unequal parameters.

We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero–Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg [2002]. We conjecture that these coincide with Kazhdan–Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino [2009], and we provide here a version of left cell representations. The Calogero–Moser cells are studied in detail in [Bonnafé and Rouquier $\geq$ 2013].

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2. Calogero–Moser spaces and cells

Rational Cherednik algebras at $t = 0$. Let us recall some constructions and results from [Etingof and Ginzburg 2002]. Let V be a finite-dimensional complex vector space and W a finite subgroup of $\mathrm{GL}(V)$. Let $\mathcal{S}$ be the set of reflections of W, that is, elements g such that $\ker(g-1)$ is a hyperplane. We assume that W is a reflection group, that is, it is generated by $\mathcal{S}$.

We denote by $\mathcal{S}/{\sim}$ the quotient of $\mathcal{S}$ by the conjugation action of W and we let $\{\underline{c}_s\}_{s\in\mathcal{S}/\sim}$ be a set of indeterminates. We put $A = \mathbb{C}[\mathbb{C}^{\mathcal{S}/\sim}] = \mathbb{C}[\{\underline{c}_s\}_{s\in\mathcal{S}/\sim}]$. Given $s \in \mathcal{S}$, let $v_s \in V$ and $\alpha_s \in V^*$ be eigenvectors for s associated to the nontrivial eigenvalue.

The 0-rational Cherednik algebra $\mathbf{H}$ is the quotient of $A \otimes T(V \oplus V^*) \rtimes W$ by the relations

$$[x, x'] = [\xi, \xi'] = 0,$$
$$[\xi, x] = \sum_{s\in\mathcal{S}} \underline{c}_s \frac{\langle v_s, x\rangle \cdot \langle \xi, \alpha_s\rangle}{\langle v_s, \alpha_s\rangle} s \text{ for } x, x' \in V^* \text{ and } \xi, \xi' \in V.$$

We put $Q = Z(\mathbf{H})$ and $P = A \otimes S(V^*)^W \otimes S(V)^W \subset Q$. The ring Q is normal. It is a free P-module of rank $|W|$.

Galois closure. Let $K = \mathrm{Frac}(P)$ and $L = \mathrm{Frac}(Q)$. Let M be a Galois closure of the extension L/K and R the integral closure of Q in M. Let $G = \mathrm{Gal}(M/K)$ and $H = \mathrm{Gal}(M/L)$. Let $\mathcal{P} = \mathrm{Spec}\, P = \mathbb{A}_{\mathbb{C}}^{\mathcal{S}/\sim} \times V/W \times V^*/W$, $\mathcal{Q} = \mathrm{Spec}\, Q$ the Calogero–Moser space, and $\mathcal{R} = \mathrm{Spec}\, R$.

We denote by $\pi : \mathcal{R} \to \mathcal{Q}$ the quotient by H, and by $\Upsilon : \mathcal{Q} \to \mathcal{P}$ and $\phi : \mathcal{P} \to \mathbb{A}_{\mathbb{C}}^{\mathcal{S}/\sim}$ the canonical maps. We put $p = \Upsilon\pi : \mathcal{R} \to \mathcal{P}$ the quotient by G.

Ramification. Let $\mathfrak{r} \in \mathcal{R}$ be a prime ideal of R. We denote by $D(\mathfrak{r}) \subset G$ its decomposition group and by $I(\mathfrak{r}) \subset D(\mathfrak{r})$ its inertia group.

We have a decomposition into irreducible components

$$\mathcal{R} \times_{\mathcal{P}} \mathcal{Q} = \bigcup_{g\in G/H} \mathcal{O}_g, \text{ where } \mathcal{O}_g = \{(x, \pi(g^{-1}(x))) \mid x \in \mathcal{R}\},$$

inducing a decomposition into irreducible components

$$V(\mathfrak{r}) \times_{\mathcal{P}} \mathcal{Q} = \coprod_{g\in I(\mathfrak{r})\backslash G/H} \mathcal{O}_g(\mathfrak{r}), \text{ where } \mathcal{O}_g(\mathfrak{r}) = \{(x, \pi(g^{-1}g'(x))) \mid x \in V(\mathfrak{r}), g' \in I(\mathfrak{r})\}.$$

Undeformed case. Let $\mathfrak{p}_0 = \phi^{-1}(0) = \sum_{s\in\mathcal{S}/\sim} P\underline{c}_s$. We have

$$P/\mathfrak{p}_0 = \mathbb{C}[V \oplus V^*]^{W\times W}, \qquad Q/\mathfrak{p}_0 Q = \mathbb{C}[V \oplus V^*]^{\Delta W},$$

where $\Delta(W) = \{(w, w) \mid w \in W\} \subset W \times W$. A Galois closure of the extension of $\mathbb{C}(\mathfrak{p}_0 Q) = \mathbb{C}(V \oplus V^*)^{\Delta W}$ over $\mathbb{C}(\mathfrak{p}_0) = \mathbb{C}(V \oplus V^*)^{W\times W}$ is $\mathbb{C}(V \oplus V^*)^{\Delta Z(W)}$.

Let $\mathfrak{r}_0 \in \mathscr{R}$ above $\mathfrak{p}_0$. Since $\mathfrak{p}_0 Q$ is prime, we have $G = D(\mathfrak{r}_0)H = HD(\mathfrak{r}_0)$, $I(\mathfrak{r}_0) = 1$, and $\mathbb{C}(r_0)$ is a Galois closure of the extension $\mathbb{C}(\mathfrak{p}_0 Q)/C(\mathfrak{p}_0)$. Fix an isomorphism $\iota : \mathbb{C}(\mathfrak{r}_0) \xrightarrow{\sim} \mathbb{C}(V \oplus V^*)^{\Delta Z(W)}$ extending the canonical isomorphism of $\mathbb{C}(\mathfrak{p}_0 Q)$ with $\mathbb{C}(V \oplus V^*)^{\Delta W}$.

The application ι induces an isomorphism $D(\mathfrak{r}_0) \xrightarrow{\sim} (W \times W)/\Delta Z(W)$, that restricts to an isomorphism $D(\mathfrak{r}_0) \cap H \xrightarrow{\sim} \Delta W/\Delta Z(W)$. This provides a bijection $G/H \xrightarrow{\sim} (W \times W)/\Delta W$. Composing with the inverse of the bijection

$$W \xrightarrow{\sim} (W \times W)/\Delta W, \quad w \mapsto (1, w),$$

we obtain a bijection $G/H \xrightarrow{\sim} W$.

From now on, we identify the sets G/H and W through this bijection. Note that this bijection depends on the choices of $\mathfrak{r}_0$ and of ι. Since M is the Galois closure of L/K, we have $\bigcap_{g \in G} H^g = 1$, hence the left action of G on W induces an injection $G \subset \mathfrak{S}(W)$.

Calogero–Moser cells.

Definition 2.1. Let $\mathfrak{r} \in \mathscr{R}$. The $\mathfrak{r}$-cells of W are the orbits of $I(\mathfrak{r})$ in its action on W.

Let $c \in \mathbb{A}_{\mathbb{C}}^{\mathscr{S}/\sim}$. Choose $\mathfrak{r}_c \in \mathscr{R}$ with $\overline{p(\mathfrak{r}_c)} = \bar{c} \times 0 \times 0$. The $\mathfrak{r}_c$-cells are called the *two-sided Calogero–Moser c-cells* of W. Choose now $\mathfrak{r}_c^{\text{left}} \in \mathscr{R}$ contained in $\mathfrak{r}_c$ with $\overline{p(\mathfrak{r}_c^{\text{left}})} = \bar{c} \times V/W \times 0 \in \mathscr{P}$. The $\mathfrak{r}_c^{\text{left}}$-cells are called the *left Calogero–Moser c-cells* of W. We have $I(\mathfrak{r}_c^{\text{left}}) \subset I(\mathfrak{r}_c)$. Consequently, every left cell is contained in a unique two-sided cell.

The map sending $w \in W$ to $\pi(w^{-1}(\mathfrak{r}_c))$ induces a bijection from the set of two-sided cells to $\Upsilon^{-1}(c \times 0 \times 0)$.

Families and cell multiplicities. Let E be an irreducible representation of $\mathbb{C}[W]$. We extend it to a representation of $S(V) \rtimes W$ by letting V act by 0. Let

$$\Delta(E) = e \cdot \operatorname{Ind}_{S(V)\rtimes W}^{\mathbf{H}}(A \otimes_{\mathbb{C}} E), \quad \text{where } e = \frac{1}{|W|}\sum_{w \in W} w,$$

be the spherical Verma module associated with E. It is a Q-module.

Let $c \in \mathbb{A}_{\mathbb{C}}^{\mathscr{S}/\sim}$ and let $\Delta^{\text{left}}(E) = (R/\mathfrak{r}_c^{\text{left}}) \otimes_P \Delta(E)$.

Definition 2.2. Given a left cell Γ, we define the cell multiplicity $m_\Gamma(E)$ of E as the length of $\Delta^{\text{left}}(E)$ at the component $\mathscr{C}_\Gamma(\mathfrak{r}_c^{\text{left}})$.

Note that $\sum_\Gamma m_\Gamma(E) \cdot [\mathscr{C}_\Gamma(\mathfrak{r}_c^{\text{left}})]$ is the support cycle of $\Delta^{\text{left}}(E)$.

There is a unique two-sided cell Λ containing all left cells Γ such that $m_\Gamma(E) \neq 0$. Its image in $\mathscr{Q}$ is the unique $\mathfrak{q} \in \Upsilon^{-1}(c \times 0 \times 0)$ such that $(Q/\mathfrak{q}) \otimes_Q \Delta(E) \neq 0$. The corresponding map $\operatorname{Irr}(W) \to \Upsilon^{-1}(c \times 0 \times 0)$ is surjective, and its fibers are the *Calogero–Moser families* of $\operatorname{Irr}(W)$, as defined by Gordon [2003].

***Dimension* 1.** Let V be a one-dimensional complex vector space, let $d \geq 2$ and let W be the group of d-th roots of unity acting on V. Let $\zeta = \exp(2i\pi/d)$, let $s = \zeta \in W$ and $\underline{c}_i = \underline{c}_{s^i}$ for $1 \leq i \leq d-1$. We have $A = \mathbb{C}[\underline{c}_1, \ldots, \underline{c}_{d-1}]$ and

$$\mathbf{H} = A\left\langle x, \xi, s \;\middle|\; sxs^{-1} = \zeta^{-1}x,\; s\xi s^{-1} = \zeta\xi \text{ and } [\xi, x] = \sum_{i=1}^{d-1} \underline{c}_i s^i \right\rangle.$$

Let $\mathrm{eu} = \xi x - \sum_{i=1}^{d-1}(1-\zeta^i)^{-1}\underline{c}_i s^i$. We have $P = A[x^d, \xi^d]$ and $Q = A[x^d, \xi^d, \mathrm{eu}]$. Define $\underline{\kappa}_1, \ldots, \underline{\kappa}_d = \underline{\kappa}_0$ by $\underline{\kappa}_1 + \cdots + \underline{\kappa}_d = 0$ and $\sum_{i=1}^{d-1} \underline{c}_i s^i = \sum_{i=0}^{d-1}(\underline{\kappa}_i - \underline{\kappa}_{i+1})\varepsilon_i$, where $\varepsilon_i = \frac{1}{d}\sum_{j=0}^{d-1} \zeta^{ij} s^j$. We have $A = \mathbb{C}[\underline{\kappa}_1, \ldots, \underline{\kappa}_d]/(\underline{\kappa}_1 + \cdots + \underline{\kappa}_d)$.

The normalization of the Galois closure is described as follows. There is an isomorphism of A-algebras

$$A[X, Y, Z]\Big/\Big(XY - \prod_{i=1}^{d}(Z - \underline{\kappa}_i)\Big) \xrightarrow{\sim} Q, \quad X \mapsto x^d, \quad Y \mapsto \xi^d \quad \text{and} \quad Z \mapsto \mathrm{eu}.$$

We have an isomorphism of A-algebras

$$A[X, Y, \lambda_1, \ldots, \lambda_d]\Big/\begin{pmatrix} e_i(\lambda) = e_i(\underline{\kappa}),\ i = 1, \ldots, d-1 \\ e_d(\lambda) = e_d(\underline{\kappa}) + (-1)^{d+1}XY \end{pmatrix} \xrightarrow{\sim} R,$$

where $Z = \lambda_d$ and where e_i denotes the i-th elementary symmetric function. We have $G = \mathfrak{S}_d$, acting by permuting the λ_i, and $H = \mathfrak{S}_{d-1}$.

Let $\mathfrak{p}_0 = (\underline{\kappa}_1, \ldots, \underline{\kappa}_d) \in \operatorname{Spec} P$ and

$$\mathfrak{r}_0 = (\underline{\kappa}_1, \ldots, \underline{\kappa}_d, \lambda_1 - \zeta\lambda_d, \ldots, \lambda_{d-1} - \zeta^{d-1}\lambda_d) \in \operatorname{Spec} R.$$

We have $D(\mathfrak{r}_0) = \langle (1, 2, \ldots, d) \rangle \subset \mathfrak{S}_d$ and

$$\mathbb{C}(\mathfrak{r}_0) = \mathbb{C}(X, Y, \lambda_d = \sqrt[d]{XY}) = \mathbb{C}(X, Y, Z = \sqrt[d]{XY}).$$

The composite bijection $D(\mathfrak{r}_0) \xrightarrow{\sim} G/H \xrightarrow{\sim} W$ is an isomorphism of groups given by $(1, \ldots, d) \mapsto s$.

Fix $c \in \mathbb{C}^{d-1}$ and let $\kappa_1, \ldots, \kappa_d \in \mathbb{C}$ corresponding to c. Consider $\mathfrak{r} = \mathfrak{r}_c$ or $\mathfrak{r}_c^{\text{left}}$ as in Section 2 (see right after Definition 2.1). Then $I(\mathfrak{r})$ is the subgroup of $\mathfrak{S}_d$ stabilizing $(\kappa_1, \ldots, \kappa_d)$. The left c-cells coincide with the two-sided c-cells and two elements s^i and s^j are in the same cell if and only if $\kappa_i = \kappa_j$. Finally, the multiplicity $m_\Gamma(\det^j)$ is 1 if $s^j \in \Gamma$ and 0 otherwise.

3. Coxeter groups

Kazhdan–Lusztig cells. Following [Kazhdan and Lusztig 1979; Lusztig 1983; 2003], let us recall the construction of cells.

We assume here V is the complexification of a real vector space $V_{\mathbb{R}}$ acted on by W. We choose a connected component C of $V_{\mathbb{R}} - \bigcup_{s \in \mathcal{S}} \ker(s-1)$ and we denote by S the set of $s \in \mathcal{S}$ such that $\ker(s-1) \cap \bar{C}$ has codimension 1 in $\bar{C}$. This makes (W, S) into a Coxeter group, and we denote by l the length function.

Let Γ be a totally ordered free abelian group and let $L : W \to \Gamma$ be a weight function, that is, a function such that

$$L(ww') = L(w) + L(w') \quad \text{if } l(ww') = l(w) + l(w').$$

We denote by v^γ the element of the group algebra $\mathbb{Z}[\Gamma]$ corresponding to $\gamma \in \Gamma$.

We denote by H the Hecke algebra of W: this is the $\mathbb{Z}[\Gamma]$-algebra generated by elements T_s with $s \in S$ subject to the relations

$$(T_s - v^{L(s)})(T_s + v^{-L(s)}) = 0 \quad \text{and} \quad \underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}},$$

for $s, t \in S$ with $m_{st} \neq \infty$, where m_{st} is the order of st. Given $w \in W$, we put $T_w = T_{s_1} \cdots T_{s_n}$, where $w = s_1 \cdots s_n$ is a reduced decomposition.

Let i be the ring involution of H given by $i(v^\gamma) = v^{-\gamma}$ for $\gamma \in \Gamma$ and $i(T_s) = T_s^{-1}$. We denote by $\{C_w\}_{w \in W}$ the Kazhdan–Lusztig basis of H. It is uniquely defined by the properties that $i(C_w) = C_w$ and $C_w - T_w \in \bigoplus_{w' \in W} \mathbb{Z}[\Gamma_{<0}] T_{w'}$.

We introduce the partial order $\prec_L$ on W. It is the transitive closure of the relation given by $w' \prec_L w$ if there is $s \in S$ such that the coefficient of $C_{w'}$ in the decomposition of $C_s C_w$ in the Kazhdan–Lusztig basis is nonzero. We define $w \sim_L w'$ to be the corresponding equivalence relation: $w \sim_L w'$ if and only if $w \prec_L w'$ and $w' \prec_L w$. The equivalence classes are the left cells. We define $\prec_{LR}$ as the partial order generated by $w \prec_{LR} w'$ if $w \prec_L w'$ or $w^{-1} \prec_L w'^{-1}$. As above, we define an associated equivalence relation $\sim_{LR}$. Its equivalence classes are the two-sided cells.

When $\Gamma = \mathbb{Z}$, $L = l$, and W is a Weyl group, a definition of left cells based on primitive ideals in enveloping algebras was proposed by Joseph [1980]: let $\mathfrak{g}$ be a complex semisimple Lie algebra with Weyl group W. Let ρ be the half-sum of the positive roots. Given $w \in W$, let I_w be the annihilator in $U(\mathfrak{g})$ of the simple module with highest weight $-w(\rho) - \rho$. Then, w and w' are in the same left cell if and only if $I_w = I_{w'}$.

Representations and families. Let Γ be a left cell. Let $W_{\leq \Gamma}$ and $W_{<\Gamma}$ be the sets of $w \in W$ such that there is $w' \in \Gamma$ with $w \prec_L w'$ and, respectively, $w \prec_L w'$ and $w \notin \Gamma$. The left cell representation of W over $\mathbb{C}$ associated with Γ [Kazhdan and Lusztig 1979; Lusztig 2003] is the unique representation, up to isomorphism, that deforms into the left H-module

$$\Big(\bigoplus_{w \in W_{\leq \Gamma}} \mathbb{Z}[\Gamma] C_w \Big) \Big/ \Big(\bigoplus_{w \in W_{<\Gamma}} \mathbb{Z}[\Gamma] C_w \Big).$$

Lusztig [1982; 2003] has defined the set of constructible characters of W inductively as the smallest set of characters with the following properties: it contains the trivial character, it is stable under tensoring by the sign representation and it is stable under J-induction from a parabolic subgroup. Lusztig's families are the equivalences classes of irreducible characters of W for the relation generated by $\chi \sim \chi'$ if χ and χ' occur in the same constructible character. Lusztig has determined constructible characters and families for all W and all parameters.

Lusztig has shown for equal parameters, and conjectured in general, that the set of left cell characters coincides with the set of constructible characters.

A conjecture. Let $c \in \mathbb{R}^{\mathcal{S}/\sim}$. Let Γ be the subgroup of $\mathbb{R}$ generated by $\mathbb{Z}$ and $\{c_s\}_{s\in\mathcal{S}}$. We endow it with the natural order on $\mathbb{R}$. Let $L : W \to \Gamma$ be the weight function determined by $L(s) = c_s$ if $s \in S$.

The following conjecture is due to Gordon and Martino [2009]. A similar conjecture has been proposed independently by the second author.[1] It is known to hold for types A_n, B_n, D_n and $I_2(n)$ [Gordon 2008; Gordon and Martino 2009; Bellamy 2011; Martino 2010a; 2010b].

Conjecture 3.1. *The Calogero–Moser families of irreducible characters of W coincide with the Lusztig families.*

We propose now a conjecture involving partitions of elements of W, via ramification. The part dealing with left cell characters could be stated in a weaker way, using Q and not R, and thus not needing the choice of prime ideals, by involving constructible characters.

Conjecture 3.2. *There is a choice of $\mathfrak{r}_c^{\text{left}} \subset \mathfrak{r}_c$ such that*

- *the Calogero–Moser two-sided cells and left cells coincide with the Kazhdan–Lusztig two-sided cells and left cells*, *respectively*, *and*
- *the representation $\sum_{E\in\mathrm{Irr}(W)} m_\Gamma(E)E$, where Γ is a Calogero–Moser left cell*, *coincide with the left cell representation of the corresponding Kazhdan–Lusztig cell.*

Various particular cases and general results supporting Conjecture 3.2 are provided in [Bonnafé and Rouquier $\geq$ 2013]. In particular, the conjecture holds for $W = B_2$, for all choices of parameters.

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COARSE MEDIAN SPACES AND GROUPS

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We introduce the notion of a coarse median on a metric space. This satisfies the axioms of a median algebra up to bounded distance. The existence of such a median on a geodesic space is quasi-isometry invariant, and so it applies to finitely generated groups via their Cayley graphs. We show that asymptotic cones of such spaces are topological median algebras. We define a notion of rank for a coarse median and show that this bounds the dimension of a quasi-isometrically embedded euclidean plane in the space. Using the centroid construction of Behrstock and Minsky, we show that the mapping class group has this property, and recover the rank theorem of Behrstock and Minsky and of Hamenstädt. We explore various other properties of such spaces, and develop some of the background material regarding median algebras.

1. Introduction

In this paper we introduce the notion of a "coarse median" on a metric space. The existence of such a structure can be viewed as a kind of coarse nonpositive curvature condition. It can also be applied to finitely generated groups. Many naturally occurring spaces and groups admit such structures. Simple examples include Gromov hyperbolic spaces and CAT(0) cube complexes. It is also preserved under quasi-isometry, relative hyperbolicity and direct products. Moreover (using the construction of [Behrstock and Minsky 2011]), the mapping class group of a surface admits such a structure. One might conjecture that it applies to a much broader class of spaces that are in some sense nonpositively curved, such as CAT(0) spaces. Much of this work is inspired by the results in [Behrstock and Minsky 2008; 2011; Bestvina et al. 2010; Behrstock et al. 2012; 2011; Chatterji et al. 2010]. It seems a natural general setting in which to view some of this work.

A "median algebra" is a set with a ternary operation satisfying certain conditions (see for example [Isbell 1980; Bandelt and Hedlíková 1983; Roller 1998; Chepoi 2000]). As we will see, for many purposes, one can reduce the discussion to a finite subalgebra. Any finite median algebra is canonically the vertex set of a CAT(0)

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cube complex, with the median defined in the usual way. One way to say this is that the median of three points is the unique point which minimises the sum of the distances in the 1-skeleton to these three points. For a fuller discussion, see Sections 2, 4 and 5.

We will also define a "coarse median" as a ternary operation on a metric space. We usually assume this to be a "geodesic space", that is, every pair of points can be connected by a geodesic. The coarse median operation is assumed to satisfy the same conditions as a median algebra up to bounded distance. We can define the "rank" of such a space (which corresponds to the dimension of a CAT(0) complex). We show that the asymptotic cone [van den Dries and Wilkie 1984; Gromov 1993] of such a space is a topological median algebra. It has a "separation dimension" which is at most the rank, when this is finite. We remark that coarse median spaces of rank 1 are the same as Gromov hyperbolic spaces. In such a case, the asymptotic cone is an $\mathbb{R}$-tree.

The existence of a coarse median on a geodesic space is a quasi-isometry invariant, so we can apply this to finitely generated groups via their Cayley graphs. We can thus define a "coarse median group". For example, a hyperbolic group is a coarse median group of rank 1, and a free abelian group is a coarse median group where "rank" agrees with the standard notion. More substantially we show that the mapping class group of a surface has a coarse median structure whose rank equals the maximal rank of a free abelian subgroup. The median we use for this is the centroid constructed in [Behrstock and Minsky 2011]. In particular, the asymptotic cone has at most (in fact precisely) this dimension, thereby giving another proof the rank theorem of [Behrstock and Minsky 2008; Hamenstädt 2005].

Another class of examples arise from relatively hyperbolic groups. We show in [Bowditch 2011b] that a group that is hyperbolic relative to a collection of coarse median groups (of rank at most ν) is itself coarse median (of rank at most ν). Examples of such are geometrically finite kleinian groups (of dimension ν) and Sela's limit groups.

It is natural to ask what other classes of spaces or groups admit such a structure. For example, it is conceivable that every CAT(0) space does, where the rank might be bounded by the dimension. More modestly one could ask this for higher rank symmetric spaces. The only immediately evident constraint is that such a space should satisfy a quadratic isoperimetric inequality.

In [Bowditch 2011a], we show that a metric median algebra of the type that arises as an asymptotic cone of a finite rank coarse median space admits a bilipschitz embedding into a finite product of $\mathbb{R}$-trees. One consequence is that coarse median groups have rapid decay. In fact, their proof of rapid decay of the mapping class groups was the main motivation for introducing centroids in [Behrstock and Minsky 2011].

2. Statement of results

We begin by recalling the notion of a "median algebra". This is a set equipped with a ternary "median" operation satisfying certain axioms. Discussion of these can be found in [Isbell 1980; Bandelt and Hedlíková 1983; Roller 1998; Chepoi 2000]. We will give a more detailed account in Sections 4 to 6. For the moment, we use more intuitive formulations of the definitions. A finite median algebra is essentially an equivalent structure to a finite cube complex. Recall that a (finite) cube complex is a connected metric complex built out of unit euclidean cubes. It is CAT(0) if it is simply connected and the link of every cube is a flag complex. See [Bridson and Haefliger 1999] for a general discussion. Note that a 1-dimensional CAT(0) cube complex is a simplicial tree.

Suppose M is a set, and $\mu : M^3 \to M$ is a ternary operation. Given $a, b \in M$, write $[a, b] = \{e \in M \mid \mu(a, b, e) = e\}$. This is the *interval* from a to b.

If $M = V(\Pi)$ is the vertex set of a finite cube complex, Π, we can define $[a, b]_\Pi$ to be the set of points of M which lie in some geodesic from a to b in the 1-skeleton of Π. One can show that there is a unique point, $\mu_\Pi(a, b, c)$, lying in $[a, b]_\Pi \cap [b, c]_\Pi \cap [c, a]_\Pi$. (In fact, it is the unique point which minimises the sum of the distances in the 1-skeleton to a, b and c.)

For the purposes of this section, we can define a "finite median algebra" to be a set M with a ternary operation: $\mu : M^3 \to M$ such that M admits a bijection to the vertex set, $V(\Pi)$, of some finite CAT(0) cube complex, Π, such that $\mu = \mu_\Pi$. (This is equivalent to the standard definition.) Given $a, b \in M$, write $[a, b] = \{e \in M \mid \mu(a, b, e) = e\}$. This is the *interval* from a to b. Under the bijection with $V(\Pi)$ it can be seen to agree with $[a, b]_\Pi$. Note that $\mu(a, b, c) = \mu(b, a, c) = \mu(b, c, a)$ and $\mu(a, a, b) = a$ for all $a, b, c \in M$. In fact, the complex Π is determined up to isomorphism by (M, μ), so we can define the "rank" of M to be the dimension of Π. For more details, see Section 4.

In general, we say that a set, M, equipped with a ternary operation, μ, is a "median algebra", if every finite subset $A \subseteq M$ is contained in another finite subset, $B \subseteq M$, which is closed under μ and such that (B, μ) is a finite median algebra. Note that, defining intervals in the same way, we again have $[a,b] \cap [b,c] \cap [c,a] = \{\mu(a,b,c)\}$ for all $a, b, c \in M$. We say that M has "rank at most ν" if every finite subalgebra has rank at most ν. It has "rank ν" if it has rank at most ν but not at most $\nu - 1$.

A median algebra of rank 1 is a treelike structure which has been studied under a variety of different names. They appear in [Sholander 1952] and as "tree algebras" in [Bandelt and Hedlíková 1983]. They have also been called "median pretrees".

We introduce the following notion of a "coarse median space". Let (Λ, ρ) be a metric space and $\mu : \Lambda^3 \to \Lambda$ be a ternary operation. We say that μ is a "coarse median" if it satisfies the following:

(C1) There are constants, k, $h(0)$, such that for all $a, b, c, a', b', c' \in \Lambda$ we have

$$\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).$$

(C2) There is a function, $h : \mathbb{N} \to [0, \infty)$, with the following property. Suppose that $A \subseteq \Lambda$ with $1 \leq |A| \leq p < \infty$, then there is a finite median algebra, (Π, μ_Π) and maps $\pi : A \to \Pi$ and $\lambda : \Pi \to \Lambda$ such that for all $x, y, z \in \Pi$ we have

$$\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)$$

and

$$\rho(a, \lambda\pi a) \leq h(p)$$

for all $a \in A$.

Using (C1) and (C2) we can deduce that, if $a, b, c \in \Lambda$, then $\mu(a, b, c)$, $\mu(b, a, c)$ and $\mu(b, c, a)$ are a bounded distance apart, and that $\rho(\mu(a, a, b), a)$ is bounded. (These facts follow from the corresponding identities in the median algebra (Π, μ_Π); see the discussion in Section 8.) Thus, there is no essential loss in assuming μ to be invariant under permutation of a, b, c and assuming that $\mu(a, a, b) = a$.

If (Λ, ρ) is a geodesic space, then we can replace (C1) by a condition to the effect that if $\rho(c, d)$ is less than some fixed positive constant (for example, 1, for a graph) then $\rho(\mu(a, b, c), \mu(a, b, d))$ is bounded. It then follows for any a, b, c, d that $\rho(\mu(a, b, c), \mu(a, b, d))$ is, in fact, linearly bounded above in terms of $\rho(c, d)$.

Definition. We refer to μ as a *coarse median* on (Λ, ρ) if it satisfies (C1) and (C2) above. We refer to (Λ, ρ, μ) as a *coarse median space*.

If, in the above definition, we can strengthen (C2) to insist that Π has rank most ν (independently of p), then we say that μ is a *coarse median of rank at most* ν, and that (Λ, ρ, μ) is a *coarse median space of rank at most* ν.

We refer to the multiplicative constant k and the additive constants, $h(p)$, featuring in the definitions as the *parameters* of the coarse median space.

Recall that a metric space is a "geodesic space" (or "length space") if every pair of points are connected by a geodesic (that is, a path whose length equals the distance between its endpoints). In this context, coarse median spaces of rank 1 are precisely Gromov hyperbolic spaces (as defined in [Gromov 1987]).

Theorem 2.1. *Let* (Λ, ρ) *be a geodesic space. Then* (Λ, ρ) *is Gromov hyperbolic if and only if it admits a structure as a coarse median space of rank* 1.

In the above one can determine the parameters explicitly in terms of the hyperbolicity constant. The converse we offer here will be nonconstructive and based on the fact that any asymptotic cone is an $\mathbb{R}$-tree. (It is possible to give a constructive argument and explicit constants, but we will not pursue that matter here.)

By a *topological median algebra* we mean a topological space, M, equipped with a median, μ, which is continuous as a map from M^3 to M. Such structures are considered, for example, in [Bandelt and van de Vel 1989]. We will refer to a "metric median algebra" when the topology is induced by some particular metric.

We define a notion of "local convexity" in Section 7. For a finite-rank algebra this is equivalent to saying that an interval connecting two points close together is arbitrarily small. We will also define a notion of "separation dimension" of a topological space. This is analogous to (though weaker than) the standard notion of "inductive dimension". The latter is equivalent to covering dimension [Hurewicz and Wallman 1941; Engelking 1995]. Every locally compact subspace of a space of separation dimension at most ν has covering dimension at most ν. In particular, such a space does not admit any continuous injective map of $\mathbb{R}^{\nu+1}$. We show:

Theorem 2.2. *A locally convex topological median algebra of rank at most ν has separation dimension at most ν.*

This notion of dimension is weaker than the standard notions of topological dimension referred to. For example, there is a totally disconnected space of positive covering dimension [Erdös 1940], but this has separation dimension 0. (I thank Klaas Hart for providing me with this reference.) Nevertheless, we see that every locally compact subspace of such a space has (covering) dimension at most ν. For the mapping class group, this follows from [Behrstock and Minsky 2008].

Topological median algebras arise as ultralimits of coarse median algebras. We will recall the basic definitions in Section 9. Suppose that $((\Lambda_i, \rho_i, \mu_i))_{i\in\mathcal{I}}$ is sequence of coarse median spaces, where the additive constants featuring in (C1) and (C2) tend to zero and where the multiplicative constant, k, featuring in (C1) remains constant. Let $e_i \in \Lambda_i$ be a sequence of basepoints. Given a nonprincipal ultrafilter on $\mathcal{I}$, we can pass to an ultralimit $(\Lambda_\infty, \rho_\infty, \mu_\infty)$, which is a topological median algebra. (In fact, $(\Lambda_\infty, \rho_\infty)$ is a complete metric space.)

Theorem 2.3. *If the $(\Lambda_i, \rho_i, \mu_i)$ all have rank at most ν (with respect to these constants) then $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is a locally convex topological median algebra of rank at most ν.*

Suppose we fix a coarse median space, (Λ, ρ, μ), of rank at most ν. We take any sequence $(t_i)_i$ of positive real numbers tending to 0, rescale the metric $\Lambda_i = \Lambda$, $\rho_i = t_i\rho$ and $\mu_i = \mu$. Fixing a base point $e \in \Lambda$, and an ultrafilter, we then get an "asymptotic" cone, $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ as above. From this, we can deduce:

Corollary 2.4. *If (Λ, ρ) is a geodesic space admitting a coarse median of rank at most ν, then (Λ, ρ) admits no quasi-isometric embedding of $\mathbb{R}^{\nu+1}$ (with the euclidean metric).*

If it did, then an asymptotic cone would contain a bilipschitz copy of $\mathbb{R}^{\nu+1}$. But this contradicts a combination of Theorems 2.2 and 2.3.

The existence, or otherwise, of a coarse median (or rank at most ν) on a geodesic space is easily seen to be quasi-isometry invariant (Lemma 8.1). This justifies the following:

Definition. We say that a finitely generated group Γ is *coarse median* (*of rank at most* ν), if its Cayley graph admits a coarse median (of rank at most ν).

Thus, in view of Theorem 2.1 "coarse median of rank 1" is the same as "hyperbolic". We observed in the Introduction that $\mathbb{Z}^\nu$ is coarse median of rank ν. We also note (Corollary 8.3) that a coarse median group has (at worst) a quadratic Dehn function.

Note that we do not assume that the median is equivariant, though in the examples we describe, it can be assumed to be equivariant up to bounded distance.

One of the main motivations is to study mapping class groups. Let Σ be a compact orientable surface of genus g and with p holes. Let $\mathrm{Map}(\Sigma)$ be its mapping class group. Set $\xi(\Sigma) = 3g - 3 + p$ for the *complexity* of Σ. We assume that $\xi(\Sigma) > 1$, in which case, $\xi(\Sigma)$ is exactly the maximal rank of any free abelian subgroup of $\mathrm{Map}(\Sigma)$. Making use of ideas in [Behrstock and Minsky 2011], we show:

Theorem 2.5. $\mathrm{Map}(\Sigma)$ *is a coarse median group of rank at most* $\xi(\Sigma)$.

We therefore recover the fact that the mapping class group has quadratic Dehn function [Mosher 1995]. Also, applying Corollary 2.4 we recover the result of [Behrstock and Minsky 2008; Hamenstädt 2005]:

Theorem 2.6. *There is no quasi-isometric embedding of* $\mathbb{R}^{\xi(\Sigma)+1}$ *into* $\mathrm{Map}(\Sigma)$.

One can show that some (in fact any) free abelian subgroup of $\mathrm{Map}(\Sigma)$ of rank $\xi(\Sigma)$ is necessarily quasi-isometrically embedded [Farb et al. 2001]. In other words, the rank of $\mathrm{Map}(\Sigma)$ is exactly the maximal rank of a free abelian subgroup.

In Section 12, be briefly discuss a strengthening of rank to the notion of "colourability". We show that the mapping class group has this property.

As mentioned in the Introduction, it is shown in [Bowditch 2011a] that an asymptotic cone that arises in this way admits a bilipschitz embedding into a finite product of $\mathbb{R}$-trees. From this, one can deduce the rapid decay of coarse median groups. For the mapping class group, such an embedding was constructed in [Behrstock et al. 2011] and rapid decay was shown directly using medians in [Behrstock and Minsky 2011].

3. Hyperbolic spaces

In this section, we briefly describe the rank-1 case which corresponds to Gromov hyperbolicity [Gromov 1987]. This case will be used again in Sections 10 and 11.

We suppose throughout this section that (Λ, ρ) is a geodesic space.

Let us suppose first that (Λ, ρ) is K-hyperbolic for some $K \geq 0$. This means that any geodesic triangle (α, β, γ) in Λ has a K-centre, that is, some point d, with $\rho(d, \alpha) \leq K$, $\rho(d, \beta) \leq K$ and $\rho(d, \gamma) \leq K$. If $a, b, c \in \Lambda$ we take a K-centre, d, of any geodesic triangle with vertices at a, b, c, and set $\mu(a, b, c) = d$. (We can assume this to be invariant under permutation of a, b, c.) This is well defined up to bounded distance. We claim:

Lemma 3.1. *(Λ, ρ, μ) is a rank-1 coarse median space whose parameters depend only on K.*

Lemma 3.1 can be viewed as an expression of the "treelike" nature of hyperbolicity. It is a simple consequence of the following standard fact which can be found in Section 6.2 of [Gromov 1987, p. 157]. A more detailed statement and proof is given as Proposition 6.7 of [Bowditch 2006a]. It will be formulated here as Lemma 3.2, and will be used again in Section 10 (see Lemma 10.3).

Before giving the statement, we give a few definitions. Suppose that $\tau \subseteq \Lambda$ is a simplicial tree in Λ (by which we mean a subset homeomorphic to a finite simplicial tree). Given $x, y \in \tau$, we write $[x, y]_\tau$ for the arc in τ with endpoints at x and y. We write $\rho_\tau(x, y)$ for the length of $[x, y]_\tau$, which we will always assume to be finite. (Thus, ρ_τ is the induced path-metric on τ.) Clearly, $\rho(x, y) \leq \rho_\tau(x, y)$.

Definition. Given $t \geq 0$, we say that τ is *t-taut* if $\rho_\tau(x, y) \leq \rho(x, y) + t$ for all $x, y \in \tau$.

Lemma 3.2. *There is some function $h_0 : \mathbb{N} \to [0, \infty)$ such that if (Λ, ρ) is K-hyperbolic and $A \subseteq \Lambda$ with $|A| \leq p$, then there is a $(K h_0(p))$-taut simplicial tree, $\tau \subseteq \Lambda$, with $A \subseteq \tau$.*

Proof. This is essentially due to Gromov. It is a simple consequence of Proposition 6.7 of [Bowditch 2006a]. The conclusion there was stated a little differently, namely that $\rho_\tau(a, b) \leq \rho(a, b) + K h_0(p)$ for all $a, b \in A$. To recover the statement above, first note that we can assume that every extreme (degree-1) vertex of τ is contained in A. (Otherwise, replace τ by the minimal subtree containing A.) Now, given any $x, y \in \tau$, it follows that there exist $a, b \in A$ such that $x, y \in [a, b]_\tau$. The statement that $\rho_\tau(x, y) \leq \rho(x, y) + K h_0(p)$ is now a simple consequence of the same statement for a, b, using the triangle inequalities. $\square$

Note that if τ is a t-taut tree in Λ, and $x, y \in \tau$, then $[x, y]_\tau$ lies a Hausdorff distance at most s from any geodesic in Λ from x to y, where s depends only on t and K. This proven explicitly in [Bowditch 2006a], but is also an immediate consequence of the standard fact that quasigeodesics in a hyperbolic space fellow travel geodesics (where the distance bound depends only on the parameters of the quasigeodesic and the hyperbolicity constant). From this, one can easily deduce that if $x, y, z \in \tau$, then $\mu(x, y, z)$ lies a bounded distance from the τ-median, $\mu_\tau(x, y, z)$, where the bound again depends only on t and K. In the situation described by Lemma 3.2, it therefore depends only on p and K.

We can now deduce Lemma 3.1.

Suppose that Λ is K-hyperbolic and that $A \subseteq \Lambda$ with $|A| \leq p$. Let $\tau \subseteq \Lambda$ be the tree given by Lemma 3.2. Let Π be the vertex set of τ, and let $\pi : A \to \Pi$ and $\lambda : \Pi \to \Lambda$ be the inclusions. Property (C2) is now an immediate consequence of Lemma 3.2 and the subsequent discussion.

Finally, for (C1), it is well known (and also a consequence of Lemma 3.2) that if $a, b, c, d \in \Lambda$, then $\rho(\mu(a, b, c), \mu(a, b, d))$ is linearly bounded above in terms of $\rho(c, d)$. In fact, it is sufficient to note that if we move any one of the points a, b, c a bounded distance, say r, then the median thus defined moves a bounded distance depending only on K and r.

This proves Lemma 3.1, that is, one direction of Theorem 2.1.

For the converse, it is possible to give a constructive argument which gives an explicit constants. However, here we note that it is a consequence of the following statement proven in [Gromov 1993].

Theorem 3.3. *Let (Λ, ρ) be a geodesic space, and suppose that every asymptotic cone of (Λ, ρ) is an $\mathbb{R}$-tree, then (Λ, ρ) is Gromov hyperbolic.*

The notion of an asymptotic cone is due to Van den Dries and Wilkie [1984] and elaborated on in [Gromov 1993] (see Section 9 here). We will see (Theorem 2.2 and Lemma 9.6) that any asymptotic cone of a rank-1 median algebra is an $\mathbb{R}$-tree. From this we deduce the converse to Lemma 3.1. This then proves Theorem 2.1.

4. General median algebras

In this section we discuss some of the general theory regarding median algebras. We will elaborate on particular cases in Sections 5–7. We first describe some general terms, and then, in turn, finite, infinite and topological median algebras. Some of the basic material can be found elsewhere, though the references are somewhat scattered, and often pursued from quite different perspectives. Some general references are [Isbell 1980; Bandelt and Hedlíková 1983; Roller 1998; Chepoi 2000].

We begin with the standard formal definition, which is somewhat unintuitive. In practice, all one needs to know is that every finite subset of a median algebra is contained in a finite subalgebra (Lemma 4.2) which can be identified as the vertex set of a CAT(0) cube complex. (In fact, this could serve as an equivalent definition.)

Let M be a set. A *median* on M is a ternary operation, $\mu : M^3 \to M$, such that, for all $a, b, c, d, e \in M$,

(M1) $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$, (M2) $\mu(a, a, b) = a$,

(M3) $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$.

The axioms are usually given in the above form, though, in fact, (M3) can be replaced by a condition on sets of four points [Kolibiar and Marcisová 1974; Bandelt and Hedlíková 1983].

We refer to (M, μ) as a *median algebra*.

Given $a, b \in M$ the *interval* $[a, b]$ between a an b is defined by $[a, b] = \{c \in M \mid \mu(a, b, c) = c\}$. Clearly $[a, a] = \{a\}$ and $[a, b] = [b, a]$. One can also verify that $[a, b] \cap [b, c] \cap [c, a] = \{\mu(a, b, c)\}$.

Definition. A (*median*) *subalgebra* of M is a subset closed under μ.

Given $A \subseteq M$, we write $\langle A \rangle$ for the subalgebra generated by A, that is, the smallest subalgebra containing A.

Definition. A subset $C \subseteq M$ is *convex* if $[a, b] \subseteq C$ for all $a, b \in C$.

Any convex subset is a subalgebra, but not necessarily conversely. One can check that any interval in M is convex.

Definition. A (*median*) *homomorphism* between median algebras is map which respects medians.

Note that a direct product of median algebras is a median algebra. Also the two-point set, $I = \{-1, 1\}$ has a unique structure as a median algebra. Given any set, X, the direct product, I^X, is naturally a median algebra.

Definition. A *hypercube* is a median algebra isomorphic to I^X for some set X. If $|X| = \nu < \infty$, we refer to it as a ν-*hypercube*. A *square* is a 2-hypercube.

If $Y \subseteq X$, then there is a natural projection epimorphism from I^X to I^Y. If $a \in I^{X \setminus Y}$, then $F = I^Y \times \{a\}$ is a convex hypercube in I^X, which we refer to as a *face* of I^X. There is a natural projection $\phi_F : I^X \to F$.

Let M be a median algebra.

Definition. A *directed wall*, W, is a pair, $(H^-(W), H^+(W))$, where $H^-(W)$ and $H^+(W)$ form a partition of M into two nonempty convex subsets. We refer to the unordered pair, $\{H^-(W), H^+(W)\}$, as an *undirected wall*, or simply a *wall*.

We write $\mathcal{W} = \mathcal{W}(M)$ for the set of all (undirected) walls in M.

Note that a directed wall, W, is equivalent to an epimorphism $\phi : M \to I$, where $H^\pm(W) = \phi^{-1}(\pm 1)$. We say that W, *separates* two subsets, $A, B \subseteq M$, if $A \subseteq H^-(W)$ and $B \subseteq H^+(W)$, or vice versa. We write $A|_W B$ to mean that A, B are separated by the wall W. We write $(A|B)$ or $(A|B)_M$ to mean that there is some $W \in \mathcal{W}$ such that $A|_W B$.

The following gets the whole subject going:

Lemma 4.1. *Any two distinct points of M are separated by a wall.*

A proof can be found in [Bandelt and Hedlíková 1983]. In fact, it can be reduced to the case of finite median algebras (cf. Lemma 6.1 here).

We note that Lemma 4.1 is equivalent to asserting that M can be embedded in a hypercube. Indeed, Lemma 4.1 tells us that the natural homomorphism from M to $I^{\mathcal{W}}$ (after arbitrarily assigning a direction to each wall) is injective.

Let S be any finite set. The *free median algebra*, $M(S)$, on S can be constructed as follows. First note that we can embed S in a hypercube Q such that the coordinate projections to I are precisely the set of all functions from S to I. Thus, Q has dimension $2^{|S|}$. Now let $M(S)$ be the subalgebra of Q generated by S. Note that S naturally embeds in $M(S)$. It has the property that any function of S to any median algebra, M, extends uniquely, to $M(S)$. Indeed, this property determines $M(S)$ uniquely up to isomorphism fixing S.

Little seems to be known about the general structure of free median algebras, though some discussion can be found in [Roller 1998]. Here we just note that $|M(S)| < 2^{2^{|S|}}$.

Suppose that M is a median algebra, and $A \subseteq M$ with $|A| \leq p$. The inclusion of A in M extends uniquely to a homomorphism of the free median algebra, $M(A)$, into M. It's image is a subalgebra of M containing A. (In fact it is precisely the subalgebra, $\langle A \rangle$, generated by A.) Thus, $|\langle A \rangle| \leq |M(A)| \leq 2^{2^p}$. We have therefore shown:

Lemma 4.2. *Suppose that $A \subseteq M$ with $|A| \leq p < \infty$, then $|\langle A \rangle| < 2^{2^p}$.*

Given $A \subseteq M$, write $G(A) = \{\mu(a, b, c) \mid a, b, c \in A\}$. Define $G^i(A)$ inductively by $G^0(A) = A$ and $G^i(A) = G(G^{i-1}(A))$. From the above, it follows that $\langle A \rangle = G^q(A)$ where $q = 2^{2^p}$.

5. Finite median algebras

We observed in Section 2 that the vertex set of a finite CAT(0) cube complex has a median algebra structure. (See, for example, [Bridson and Haefliger 1999], for a discussion of CAT(0) spaces.)

Conversely, suppose that M is a finite median algebra.

Definition. A *cube* in M is a convex subset isomorphic to a hypercube. If it has dimension $\nu < \infty$, then we refer to it as a *ν-cube*.

The set of all cubes in M gives M the structure of the vertex set, $V(\Upsilon)$, of a finite cube complex Υ. One way to view this is to embed M in the hypercube, $I^{\mathcal{W}}$, where $\mathcal{W}$ is the set of walls of M. The complex Υ is then the full subcomplex of $I^{\mathcal{W}}$ with vertex set M. One can verify that Υ is simply connected, and that the link of every cube is a flag complex. Thus, Υ is CAT(0). Moreover, the median structure induced by Υ (as described in Section 2) agrees with the original. We can look at this as follows. Given $a, b \in M$, let $\mathcal{W}(a,b) \subseteq \mathcal{W}$ be the set of walls separating a and b. We write $\rho_\Upsilon(a,b) = |\mathcal{W}(a,b)|$. Then, ρ_Υ is the same as the combinatorial metric on $M = V(\Upsilon)$ induced from the 1-skeleton of Υ. In fact, if α is any shortest path in the 1-skeleton from a to b, then the edges of α are in bijective correspondence with the elements of $\mathcal{W}(a,b)$ — the endpoints of each edge are separated by a unique element of $\mathcal{W}(a,b)$.

In other words, we see that $\Upsilon = \Upsilon(M)$ is canonically determined by M. We can define the "rank" of M as the dimension of $\Upsilon(M)$. Since this description is only applicable to finite median algebras, we describe some equivalent formulations below.

Let $W \in \mathcal{W}$. It's sometimes helpful to view W geometrically as a closed totally geodesic codimension-1 subset, $\Upsilon^0(W)$, of Υ. This slices in half every cube of Υ which meets both $H^-(W)$ and $H^+(W)$. Geometrically this is closed and convex and has itself a natural structure of a cube complex (one dimension down). There is a natural nearest point retraction of Υ to $\Upsilon^0(W)$, which induces a median epimorphism. We will describe this more combinatorially later.

Suppose $W, W' \in \mathcal{W}$. There is a natural homomorphism, $\phi : M \to W \times W'$, to the square $W \times W'$.

Definition. We say that W and W' *cross* if ϕ is surjective.

In other words, each of the four sets $H^-(W) \cap H^-(W')$, $H^-(W) \cap H^+(W')$, $H^+(W) \cap H^-(W')$ and $H^+(W) \cap H^+(W')$ is nonempty. (It is also equivalent to saying that $\Upsilon^0(W) \cap \Upsilon^0(W') \neq \varnothing$.)

Lemma 5.1. *Suppose that P is a finite-dimensional hypercube, and that $A \subseteq P$ is a median subalgebra such that $\phi_F(A) = F$ for the projection ϕ_F to each square face, F. Then $A = P$.*

Proof. Suppose that $F \subseteq P$ is a square face. First note that if $A \cap F$ contains two opposite corners, a, b of F, then $F \subseteq A$. (Since, if $c \in F$, then $c = \mu(a,b,d)$ for any $d \in \phi_F^{-1}(c)$, and by assumption, $A \cap \phi_F^{-1}(c) \neq \varnothing$.) Now we proceed by induction on the dimension $\nu \geq 2$. Let $Q \subseteq P$ be any $(\nu-1)$-face. Applying the inductive hypothesis to $\phi_Q(A) \subseteq Q$, we see that $\phi_Q(A) = Q$. Now, by the (diagonal) observation above, we see easily that there must be some $a \in Q$ with $\phi_Q^{-1}(a) \subseteq A$. Again using the same observation, we see that if $b \in Q$ is adjacent to a (i.e., $\{a, b\}$ is a 1-face) then $\phi^{-1}(b) \subseteq Q$. Proceeding outwards from a, we eventually see that this holds for all elements of Q, and so $A = P$ as required. $\square$

One immediate consequence of this is the following. Suppose that $\mathcal{W}_0 \subseteq \mathcal{W}$ is a collection of pairwise crossing walls. Then the natural homomorphism, $M \to \prod \mathcal{W}_0 \equiv I^{\mathcal{W}_0}$, is surjective. In other words, the sets $\bigcap_{W \in \mathcal{W}_0} H^{\epsilon(W)}(W)$ are nonempty for all functions $\epsilon : \mathcal{W}_0 \to I$.

(In terms of CAT(0) complexes, this can be interpreted as the statement that if the subspaces $\Upsilon^0(W)$ pairwise intersect, then $\bigcap_{W \in \mathcal{W}_0} \Upsilon^0(W) \neq \varnothing$.)

Suppose now that $\phi : M \to Q$ is an epimorphism of M to a hypercube, Q. (This corresponds to a collection of pairwise intersecting walls as above.) We say that a ν-cube, P, of M is *transverse* to ϕ if $\phi(P) = Q$, that is, $\phi|P$ is an isomorphism. Let $\mathcal{F} = \mathcal{F}(\phi)$ be the set of such faces, and write $F(\phi) = \bigcup \mathcal{F}(\phi)$. It's not hard to see that $F(\phi)$ is convex in M, and is isomorphic to the product $\mathcal{F}(\phi) \times Q$, where $\phi|F(\phi)$ is projection to the second factor, and where each $\{a\} \times Q$ is a transverse face. Note that the sets $\mathcal{F}(\phi) \times \{b\} \subseteq F(\phi)$ are all convex in $F(\phi)$ and so also in M. (Reinterpreting in terms of CAT(0) cube complexes, this corresponds to saying that the "walls" all intersect in a codimension ν subspace, which intrinsically has the structure of a cube complex naturally isomorphic to $\mathcal{F}(\phi)$.)

Proposition 5.2. *If $\phi : M \to Q$ is an epimorphism to a hypercube, then $\mathcal{F}(\phi) \neq \varnothing$.*

Proof. One can proof this by induction on the dimension, ν, of Q.

If $\nu = 1$, we have a single wall W. We can choose $a \in H^-(W)$ and $b \in H^+(W)$ so as to minimise $|\mathcal{W}(a, b)|$. In this case, one can verify that $\mathcal{W}(a, b) = \{W\}$, and so $\{a, b\}$ is a transverse face.

If $\nu > 1$, write $Q = P \times I$, and let $\psi : W \to I$ be the composition of ϕ with projection of Q to I. Given $a \in P$, note that $M(a) = \phi^{-1}(\{a\} \times I)$ is a convex subset of M. Now $\psi|M(a)$ is an epimorphism, so (by the case $\nu = 1$), $\mathcal{F}(\psi|M(a)) \neq \varnothing$. But $\mathcal{F}(\psi)$ is the disjoint union of the sets $\mathcal{F}(\psi|M(a))$ as a ranges over P. The natural epimorphism from $\mathcal{F}(\psi)$ to P is therefore surjective, so by induction, there must be a transverse $(\nu - 1)$-face, say R, to this epimorphism. We see that $\bigcup R$ is now a transverse ν-cube to the original map ϕ. □

Proposition 5.3. *Let M be a finite median algebra. The following are equivalent.*

(1) *There is a ν-hypercube embedded in M.*

(2) *There is an epimorphism of M to a ν-hypercube.*

(3) *There is a set of ν pairwise crossing walls in M.*

(4) *There is a ν-cube embedded in M.*

Proof. (1) implies (3): Let $Q \subseteq M$ be a ν-hypercube. If $\{a, b\}$ is any 1-face of Q, then any wall of M separating a and b will also separate the $(\nu - 1)$-faces of Q containing a and b. In this way, we get a collection, $\mathcal{W}_0$, of ν pairwise intersecting walls — one for each factor of Q.

(3) implies (2): As observed above, using Lemma 5.1, the map from M to the product, $\prod \mathcal{W}_0$ is surjective.

(2) implies (4): By Proposition 5.2.

(4) implies (1): Trivial. □

Definition. We say that M has *rank at least* ν if any (hence all) the conditions of Proposition 5.3 are satisfied. We say that M has *rank* ν if it has rank at least ν but not at least $\nu+1$.

Note that the cubes of M correspond exactly to the cubical cells of the complex $\Upsilon(M)$, so in view of (4), the definition is equivalent to that given earlier in Section 2.

Lemma 5.4. *Suppose that $A, B \subseteq M$ are disjoint nonempty convex subsets. Then there is a wall separating A and B.*

Proof. Choose $a \in A$ and $b \in B$ so as to minimise $|\mathcal{W}(a,b)|$. One can check that any $W \in \mathcal{W}$ will separate A and B. □

In the case where $A = \{a\}$, there is unique $b \in B$ which minimises $\mathcal{W}(a,b)$. We write $\mathrm{proj}_B(a) = b$. If $a \in B$, then we set $\mathrm{proj}_B(a) = a$. This gives us a "nearest point" projection map $\mathrm{proj}_B : M \to B$ to any nonempty convex subset, B, of M.

Now suppose $W \in \mathcal{W}$. We write $\mathcal{F}(W)$ for the set of transverse 1-faces. Note that $F(W) = \bigcup \mathcal{F}(W) \cong \mathcal{F}(W) \times I$. In particular, it follows that $\mathrm{rank}(\mathcal{F}(W)) \leq \mathrm{rank}(M)-1$. Write $S^{\pm} = P \times \{\pm 1\} \subseteq H^{\pm}(W)$. If $a \in H^{\pm}(W)$, then $\mathrm{proj}_{H^{\mp}(W)}(a) \in S^{\mp}(W)$. We set ψ_W to be the unique element of $\mathcal{F}(W)$ containing $\mathrm{proj}_{H^{\mp}(W)}(a)$. This gives a map $\psi_W : M \to \mathcal{F}(W)$ which one can verify is a median epimorphism. (Geometrically, this corresponds to the nearest point projection of Υ the totally geodesic subspace $\Upsilon^0(W)$.)

Definition. The *convex hull*, $\mathrm{hull}(A)$, of a subset $A \subseteq M$ is the smallest convex subset of M containing A.

One can verify that $a \notin A$ if and only if there is a wall of M separating a from A. We also note that if $a, b \in M$, then $\mathrm{hull}\{a,b\} = [a,b]$.

Definition. If $A \subseteq M$, the *join*, $J(A)$, of A is defined by $J(A) = \bigcup_{a,b \in A} [a,b]$.

We define $J^i(A)$ iteratively by $J^0(A) = A$, and $J^i(A) = J(J^{i-1}(A))$. Clearly this must stabilise for some $p \in \mathbb{N}$, and we see that $\mathrm{hull}(A) = J^p(A)$. In fact:

Lemma 5.5. *If* $\mathrm{rank}(M) \leq \nu$, *and* $A \subseteq M$, *then* $\mathrm{hull}(A) = J^{\nu}(A)$.

Proof. Clearly, $J^{\nu}(A) \subseteq \mathrm{hull}(A)$. Suppose that $a \in \mathrm{hull}(A) \setminus J^{\nu}(A)$. Choose $b \in A$ so as to minimise $|\mathcal{W}(a,b)|$. Choose $W \in \mathcal{W}(a,b)$ so that $a \in S^-(W)$ and $b \in H^+(W)$ (for example, corresponding to the first edge in the 1-skeleton of Υ in a shortest path from a to b). Since $a \in \mathrm{hull}(A)$, A must meet both $H^-(W)$ and $H^+(W)$. Let $\psi_W : M \to \mathcal{F}(W)$ be the projection defined above. We see that $\bigcup \psi_W(A) \subseteq J(A)$. Now one can check (since ψ_W is an epimorphism) that $\psi_W(\mathrm{hull}(A)) = \mathrm{hull}(\psi_W(A))$. Now $\mathrm{rank}\, \mathcal{F}(W) \leq \mathrm{rank}\, M - 1 \leq \nu - 1$, so inductively, we have $\mathrm{hull}(\psi_W(A)) = J_W^{\nu-1}(\psi_W(A))$ (where J_W denotes join in $\mathcal{F}(W)$). But $\bigcup J_W(\psi_W(A)) = J(\bigcup \psi_W(A))$, and so $\bigcup \mathrm{hull}(\psi_W(A)) \subseteq \bigcup J_W^{\nu-1}(\psi_W(A)) = J^{\nu-1}(\bigcup \psi_W(A)) \subseteq J^{\nu-1}(J(A)) = J^{\nu}(A)$. Thus, $\bigcup \psi_W(\mathrm{hull}(A)) \subseteq J^{\nu}(A)$. But

$a \in \mathrm{hull}(A)$, and since $a \in S^-(W)$, we have $a \in \psi_W(a) \subseteq \psi_W(\mathrm{hull}(A)) \subseteq J^\nu(A)$. Technically, this is a contradiction. In any case, we deduce that $\mathrm{hull}(A) \subseteq J^\nu(A)$ as required. ◻

This is all we need from Section 5 up until Section 9. We conclude this section with some observations relevant to the discussion of the mapping class group in Section 10.

Suppose that $N \subseteq M$ is a subalgebra of M. We write hull_N and J_N for the intrinsic hulls and joins in N. For future reference, we note that the following does not make any use of finiteness.

Lemma 5.6. *Suppose* $A \subseteq N$, *then* $\mathrm{hull}_N(A) = N \cap \mathrm{hull}_M(A)$.

Proof. Since $\mathrm{hull}(A) = \bigcup_{i=0}^\infty J^i(A)$ and $\mathrm{hull}_N(A) = \bigcup_{i=0}^\infty J_N^i(A)$, it is enough to show that $J_N^q(A) = N \cap J^q(A)$ for any q. Clearly $J_N^p(A) \subseteq J^q(A)$. Conversely, suppose that $a \in N \cap J^q(A)$. Then $a \in [b_0, b_1]$ where $b_0, b_1 \in J^{q-1}(A)$. (Here, $[\,,\,]$ denotes an interval in M.) Similarly, $b_0 \in [b_{00}, b_{01}]$, $b_1 \in [b_{10}, b_{11}]$, where $b_{00}, b_{01}, b_{10}, b_{11} \in J^{q-2}(A)$. Continuing in this way, we get points $b_w \in J^{q-j}$, where w is a word of length j in $\{0, 1\}$, so that $b_w \in [b_{w0}, b_{w1}]$. Let $B_j \subseteq J^{q-j}(A)$ be the set of such b_w. We terminate with a set $B_q \subseteq A$.

We now work backwards, to give us points $c_w \in \mathrm{hull}_N(A)$, as follows. If w has length q, we set $c_w = b_w \in A$. If w has length less than q, we set $c_w = \mu(a, c_{w0}, c_{w1}) \in [c_{w0}, c_{w1}]_N$. By reverse induction, we end up with a point $c = \mu(a, c_0, c_1)$. We claim that $c = a$.

For suppose not. Then there is a wall $W \in \mathcal{W}(M)$ of M, with $a \in H^+(W)$ and $c \in H^-(W)$. Since $a \notin H^+(W)$, we cannot have $B_q \subseteq H^-(W)$. Thus, without loss of generality, we have $c_{0^q} = b_{0^q} \in H^+(W)$, where 0^j is the word consisting of j 0s. Working backwards, we see that $c_{0^j} \in H^+(W)$ for all j. Finally, when $j = 0$, we arrive at the contradiction that $c \in H^+(W)$.

This shows that $a = c \in \mathrm{hull}_N(A)$. ◻

Recall the notation $(A|B)_M$ to mean that subsets $A, B \subseteq M$ are separated by a wall in M. Note that, in view of Lemma 5.4 this is equivalent to saying that $\mathrm{hull}(A) \cap \mathrm{hull}(B) \neq \varnothing$. In fact, we note that:

Lemma 5.7. *Suppose* $\mathrm{hull}(A) \cap \mathrm{hull}(B) \neq \varnothing$, *then* $\mathrm{hull}(A) \cap \mathrm{hull}(B) \cap \langle A \cup B \rangle \neq \varnothing$.

Proof. Let $P(A) = \mathrm{hull}(A) \cap \langle A \cup B \rangle$ and $P(B) = \mathrm{hull}(B) \cap \langle A \cup B \rangle$. Suppose that $P(A) \cap P(B) = \varnothing$. Choose $a \in P(A)$ and $b \in P(B)$ so as to minimise $\rho(a, b) = |\mathcal{W}(a, b)|$. Choose any $W \in \mathcal{W}(a, b)$ with $a \in H^-(W)$ and $b \in H^+(W)$. Since $\mathrm{hull}(A) \cap \mathrm{hull}(B) = \varnothing$, we cannot have both $A \subseteq H^-(W)$ and $B \subseteq H^+(W)$, so without loss of generality, we can find $c \in B \cap H^-(W)$. Let $d = \mu(a, b, c)$. Since $d \in [a, b]$ we have $\rho(a, d) < \rho(a, b)$. But $d \in P(B)$, so we contradict the minimality of $\rho(a, b)$. ◻

Lemma 5.8. *Let $N \subseteq M$ be a subalgebra of a finite median algebra M. If $A, B \subseteq N$, then $(A|B)_N$ if and only if $(A|B)_M$.*

Proof. Clearly $(A|B)_M$ implies $(A|B)_N$, so suppose that $(A|B)_M$ fails. By Lemma 5.7, $\mathrm{hull}_M(A) \cap \mathrm{hull}_M(B) \cap N \neq \varnothing$, so by Lemma 5.6, $\mathrm{hull}_N(A) \cap \mathrm{hull}_N(B) \neq \varnothing$, so $(A|B)_N$ fails. □

If M, N are median algebras, then there are natural inclusions of $\mathcal{W}(M)$ and $\mathcal{W}(N)$ into $\mathcal{W}(M \times N)$ — by taking inverse images under the co-ordinate projections. In fact, under this identification, we have:

Lemma 5.9. $\mathcal{W}(M \times N) = \mathcal{W}(M) \sqcup \mathcal{W}(N)$.

Proof. This is best seen using the geometric description in terms of CAT(0) complexes. □

This result extends to finite (and indeed infinite) direct products.

6. Infinite median algebras

We now drop the assumption that M be finite. Let $\mathcal{M}$ be the set of all finite median subalgebras of M, which we view as a directed set under inclusion. By Proposition 5.2, $\mathcal{M}$ is cofinal in the directed set of all finite subsets of M.

The definition of *convex*, *wall*, *crossing* etc. remain unchanged from Section 5. However, we don't have such an immediate geometrical interpretation in terms of complexes. (If M is discrete, that is, all intervals are finite, then it is again the vertex set of a CAT(0) cube complex. However, we are not assuming discreteness here.) Let $\mathcal{W}$ be the set of walls. The following was proven in [Nieminen 1978].

Lemma 6.1. *If $A, B \subseteq M$ are disjoint convex subsets, then there is some wall, $W \in \mathcal{W}$, separating A from B.*

Proof. For finite median algebras, this was Lemma 5.4. For the general case, we use a compactness argument.

We identify the power set, $\mathcal{P}$, of M with the Tychonoff cube, $\{-1, 1\}^M$, of all functions from M to $\{-1, 1\}$. Here, a function, f, is identified with $f^{-1}(1)$. In particular, $\mathcal{P}$ is compact in this topology.

Suppose that $C \subseteq \mathcal{M}$. Let $\mathcal{S}(C) \subseteq \mathcal{P}$ be the set of subsets, $C \subseteq \mathcal{P}$ with the property that $C \cap H$ and $C \setminus H$ are both convex in C and such that $C \cap H \subseteq A$ and $C \cap H \cap B = \varnothing$. In other words, $(C \cap H, C \setminus H)$ is an intrinsic wall in A which separates $C \cap A$ from $C \cap B$. By Lemma 5.4, $\mathcal{S}(C) \neq \varnothing$. Moreover, $\mathcal{S}(C)$ is closed in $\mathcal{P}$.

Note that if $C \subseteq D$, then $\mathcal{S}(D) \subseteq \mathcal{S}(C)$. Since $\mathcal{M}$ is cofinal in the set of all finite subsets, it follows that $\{\mathcal{S}(C) \mid C \in \mathcal{M}\}$ has the finite intersection property. By compactness, $\bigcap_{C \in \mathcal{M}} \mathcal{S}(C) \neq \varnothing$. Let $H \in \bigcap_{C \in \mathcal{M}} \mathcal{S}(C)$.

If $a \in A$ and $b \in B$, then there is some $C \in \mathcal{M}$ with $a, b \in C$. Since $C \cap A \subseteq H$, we have $a \in H$, and since $C \cap H \cap B = \varnothing$, we have $b \notin H$. This shows that $B \subseteq H$ and $B \cap H = \varnothing$.

Also, H, and $M \setminus H$ are both convex. Suppose, for example, that $c, d \in H$, and $e \in [c, d]$ (the interval in M). Choose $C \in \mathcal{M}$ with $c, d, e \in C$. Now $[c, d] \cap A$ is an interval in C. Also, $c, d \in A \cap H$, which is convex in C. Thus, $e \in C \cap H \subseteq H$. This shows that H is convex. Similarly $M \setminus H$ is convex.

We have shown that $\{H, M \setminus H\}$ is a wall in M separating A and B. □

In particular, any pair of distinct points of M are separated by a wall. (This shows how Lemma 4.1 can be reduced to the finite case.)

Proposition 6.2. *Let M be a median algebra. The following are equivalent.*

(1) *There is a ν-hypercube embedded in M.*

(2) *There is an epimorphism of M to a ν-hypercube.*

(3) *There is a set of ν pairwise crossing walls in M.*

Proof. (1) implies (3): As in Proposition 5.3, this time using Lemma 6.1.

(3) implies (2): As in Proposition 5.3.

(2) implies (1): Let $\phi : M \to Q$ be an epimorphism to an ν-hypercube. There is some $A \in \mathcal{M}$ with $\phi(A) = Q$. By Proposition 5.2, A contains a ν-cube. This gives us a ν-hypercube in M. □

Definition. We say that the rank of M is at least ν if any (hence all) the conditions of Proposition 6.2 hold. We say that it has rank ν if it has rank at least ν but not at least $\nu + 1$. We write $\operatorname{rank}(M) \in \mathbb{N} \cup \{\infty\}$ for the rank of M.

Clearly the above agrees with the definition already given in the finite case. Also, using Lemma 4.2 and Proposition 6.2, we see that it is consistent with the descriptions of median algebras and rank as given in Section 2.

Let $A \subseteq M$. We define $\operatorname{hull}(A)$, $J(A)$ and $J^i(A)$ in the same way as before. This time, $\operatorname{hull}(A) = \bigcup_{i=1}^{\infty} J^i(A)$.

If $B \subseteq M$ is a finite median algebra, we write J_B for the intrinsic join in A, that is, $J_B(A) = B \cap J(A)$ for $A \subseteq B$. Note also that, by Lemma 5.6, $B \cap \operatorname{hull}(A \cap B)$ is the intrinsic convex hull of $A \cap B$ in B.

Lemma 6.3. *If $A \subseteq M$, then* $\operatorname{hull}(A)$ *is the union of the sets $B \cap \operatorname{hull}(A \cap B)$ as B ranges over $\mathcal{M}$.*

Proof. Note that $\operatorname{hull}(A) = \bigcup_{i=1}^{\infty} J^i(A)$. We prove inductively on i that $J^i(A) = \bigcup_{B \in \mathcal{M}}(J_B^i(A \cap B))$. First note that $J^0(A) = A = J_B^0(A)$ for any $B \in \mathcal{M}$ containing A. Suppose that $a \in J^i(A)$. Then $a \in [b, c]$ where $b, c \in J^{i-1}(A)$. By the inductive hypothesis, $b \in J_B^{i-1}(A \cap B)$ and $c \in J_C^{i-1}(A \cap C)$ for $B, C \in \mathcal{M}$. Now let $D \in \mathcal{M}$

with $\{a\} \cup B \cup C \subseteq D$. We see that $b, c \in J_D^{i-1}(A \cap D)$, so $a \in J_D(J_D^{i-1}(A \cap D)) = J_D^i(A \cap D)$. This proves the inductive statement. Now note that if $B \in \mathcal{M}$ then $J_B^i(A \cap B) \subseteq B \cap \mathrm{hull}(A \cap B)$, proving the result. $\square$

Lemma 6.4. *Suppose that M has rank at most ν. Then for any $A \subseteq M$, we have* $\mathrm{hull}(A) = J^\nu(A)$.

Proof. If $a \in \mathrm{hull}(A)$, then by Lemma 6.3, $a \in B \cap \mathrm{hull}(A \cap B)$ for some $B \in \mathcal{M}$. But $B \cap \mathrm{hull}(A \cap B)$ is the intrinsic convex hull of $A \cap B$ in B. (Indeed, in the proof of Lemma 6.3, we saw directly that $a \in \bigcup_{i=0}^{\infty} J_B^i(A \cap B)$.) Thus, by Lemma 5.5, we see that $a \in J_B^\nu(A \cap B) \subseteq J^\nu(A)$ as required. $\square$

Finally we note the following generalisation of Lemma 5.8 to arbitrary median algebras.

Lemma 6.5. *Let $N \subseteq M$ be a subalgebra of the median algebra M. If $A, B \subseteq N$, then $(A|B)_N$ if and only if $(A|B)_M$.*

Proof. First note that, by Lemma 5.6, for any $A \subseteq N$, we have $\mathrm{hull}_N(A) = N \cap \mathrm{hull}_M(A)$ (this did not make use of finiteness). We are therefore claiming that $N \cap \mathrm{hull}_M(A) \cap \mathrm{hull}_M(B) = \varnothing$ implies $\mathrm{hull}_M(A) \cap \mathrm{hull}_M(B) = \varnothing$. This was shown by Lemma 5.8, when M was finite. In the general case, suppose, for contradiction that there is some $c \in \mathrm{hull}_M(A) \cap \mathrm{hull}_M(B)$. It follows that $c \in \mathrm{hull}_\Pi(A \cap \Pi) \cap \mathrm{hull}_\Pi(B \cap \Pi)$ for some finite subalgebra, Π, of M. Now, $N \cap \Pi$ is a subalgebra of Π, and so, from the finite case, we have $N \cap \mathrm{hull}_\Pi(A \cap \Pi) \cap \mathrm{hull}_\Pi(A \cap \Pi) \neq \varnothing$. But this is contained in $N \cap \mathrm{hull}_M(A) \cap \mathrm{hull}_M(B)$, so we get a contradiction. $\square$

7. Topological median algebras

In this section we define the terms relevant to Theorem 2.2, and give a proof.

By a *topological median algebra* we mean a hausdorff topological space, M, together with a continuous ternary operation, $\mu : M^3 \to M$ such that (M, μ) is a median algebra.

Definition. We say that M is *locally convex* if every point has a base of convex neighbourhoods.

Put another way, if $a \in M$ and $U \ni a$ is open, then there is another open set $V \ni a$ with $\mathrm{hull}(V) \subseteq U$.

Definition. We say that M is *weakly locally convex* if, given any $a \in M$, and any open $U \ni a$, there is an open set $V \ni a$ such that $[b, c] \subseteq U$ for all $b, c \in V$.

In other words, $J(V) \subseteq U$.

Lemma 7.1. *If M has finite rank and is weakly locally convex, then it is locally convex.*

Proof. Let $a \in U$, where $U \subseteq M$ is open. We inductively construct open sets U_i with $J^i(U_i) \subseteq U$. By Lemma 6.4 if $\nu = \mathrm{rank}(M)$, then $\mathrm{hull}(U_\nu) = J^\nu(U_\nu) \subseteq U$, so we can set $V = U_\nu$. □

Given a set $C \subseteq M$, we write $\bar{C}$ for its topological closure. The following is an elementary observation:

Lemma 7.2. *If C is convex, then so is $\bar{C}$.*

Suppose $W \in \mathcal{W}$. By Lemma 7.2, the closures, $\bar{H}^-(W)$ and $\bar{H}^+(W)$ are both convex. We write $L(W) = \bar{H}^-(W) \cap \bar{H}^+(W)$. It follows that $L(W)$ is also convex. Let $O^\pm(W) = M \setminus \bar{H}^\mp(W)$. Note that $O^\pm(W)$ is contained in the interior of $H^\pm(W)$.

Definition. We say that W *strongly separates* two points $a, b \in M$ if $a \in O^-(W)$ and $b \in O^+(W)$, or vice versa.

For the rest of this section, we will assume that M is locally convex.

Lemma 7.3. *Any two distinct points of M are strongly separated by a wall.*

Proof. Let $a, b \in M$ be distinct. Let $A \ni a$ and $B \ni b$ be disjoint convex neighbourhoods. By Lemma 6.1, there is a wall $W \in \mathcal{W}$ with $A \subseteq H^-(W)$ and $B \subseteq H^+(W)$. It now follows that $a \in O^-(W)$ and $b \in O^+(W)$. □

Lemma 7.4. *Suppose that $Q \subseteq M$ is a finite dimensional hypercube, and that $\{P^-, P^+\}$ is an intrinsic wall of Q (i.e., a partition of Q into two codimension-*1 *faces). Then there is a wall $W \in \mathcal{W}$ with $P^- \subseteq O^-(W)$ and $P^+ \subseteq O^+(W)$.*

Proof. Choose $a \in P^-$ and $b \in P^+$ so that $\{a, b\}$ is a 1-face of Q. Let $W \in \mathcal{W}$ be a wall as given by Lemma 7.3. Suppose $c \in P^-$. Then $a \in [b, c]$. Since $\bar{H}^+(W)$ is convex, if $c \in \bar{H}^+(W)$, we would arrive at the contradiction that $a \in \bar{H}^+(W)$. It follows that $c \in O^-(W)$. Thus $P^- \subseteq O^-(W)$. Similarly, $P^+ \subseteq O^+(W)$. □

Lemma 7.5. *If* $\mathrm{rank}(M) \leq \nu$ *and* $W \in \mathcal{W}$, *then* $\mathrm{rank}(L(W)) \leq \nu - 1$.

Proof. Suppose, for contradiction, that $Q \subseteq L(W)$ is a ν-hypercube. Let $a : I^\nu \to Q$ be an isomorphism. Given $\epsilon \in I^\nu$, we write $\epsilon_i \in I = \{-1, +1\}$ for the i-th coordinate. For each $i \in \{1, \dots, n\}$, we can partition Q as $P_i^- \sqcup P_i^+$, where P_i^- and P_i^+ correspond to $\epsilon_i = -1$ and $\epsilon_+ = +1$. By Lemma 7.4, there is a wall, $W_i \in \mathcal{W}$ with $P_i^- \subseteq O^-(W_i)$ and $P_i^+ \subseteq O^+(W_i)$. Given $\epsilon \in I^\nu$, let $O(\epsilon) = \bigcap_{i=1}^\nu O^{\epsilon_i}$. Thus $O(\epsilon)$ is an open subset of M containing $a(\epsilon)$. Now $a(\epsilon) \in L(W) = \bar{H}^-(W) \cap \bar{H}^+(W)$. Thus, there are points, $a^\pm(\epsilon) \in O(\epsilon) \cap H^\pm(W)$. In particular, $a^\pm(\epsilon) \in H^{\epsilon_i}(W_i)$ for all i. It now follows that the walls, $W_1, W_2, \dots, W_\nu, W$, all pairwise intersect. We derive the contradiction that $\mathrm{rank}(M) \geq \nu + 1$. □

We also note that $L(W)$ is intrinsically a locally convex median algebra.

We now move on to our definition of "separation dimension". (One can find related ideas in [Behrstock and Minsky 2008].)

Let $\mathfrak{D}$ be a collection of (homeomorphism classes) of hausdorff topological spaces. Let Θ be a hausdorff topological space. We say two points $x, y \in \Theta$ are $\mathfrak{D}$-separated if there are closed sets, $X, Y \subseteq M$ with $x \notin Y$, $y \notin X$, $X \cup Y = M$ and $X \cap Y \in \mathfrak{D}$.

Define $\mathfrak{D}(n)$ inductively as follows. Set $\mathfrak{D}(-1) = \{\varnothing\}$. We say $\Theta \in \mathfrak{D}(n+1)$ if any two distinct points of Θ are $\mathfrak{D}(n)$-separated.

Definition. A space is has *separation dimension n* if it lies in $\mathfrak{D}(n) \setminus \mathfrak{D}(n-1)$.

Note that a space has separation dimension 0 if and only if it is nonempty and totally disconnected (in contrast to covering dimension [Erdös 1940]).

Suppose that $\Theta \in \mathfrak{D}(n)$ and that $\Phi \subseteq \Theta$. Then $\Phi \in \mathfrak{D}(n)$. This can be seen by induction on n as follows. Suppose that $x, y \in \Phi$ with $x \neq y$. There are closed sets $X, Y \subseteq \Theta$, with $x \notin Y$, $y \notin X$, $X \cup Y = \Theta$ and $X \cap Y \in \mathfrak{D}(n-1)$. Inductively, $X \cap Y \cap \Phi \in \mathfrak{D}(n-1)$. But $X \cap \Phi$ and $Y \cap \Phi$ are closed in Φ, $x \notin Y \cap \Phi$, $y \notin X \cap \Phi$, and $(X \cap \Phi) \cup (Y \cap \Phi) = \Phi$, so x and y are $\mathfrak{D}(n-1)$-separated in Φ.

We claim that if $x, y \in \Theta \in \mathfrak{D}(n)$, then there are open sets, $U \ni x$ and $V \ni y$ with $\bar{U} \cup \bar{V} = \Theta$ and $\bar{U} \cap \bar{V} \in \mathfrak{D}(n-1)$. To see this, let X, Y be as in the definition of $\mathfrak{D}(n)$. Let $U = \Theta \setminus Y$ and $V = \Theta \setminus X$. Now $U \subseteq X$, so $\bar{U} \subseteq X$. Thus, $\Theta \setminus X \subseteq \Theta \setminus \bar{U} = V$. Similarly, $\Theta \setminus \bar{V} \subseteq U$. In particular, $x \in U$ and $y \in V$. Also $\bar{U} \cup \bar{V} = \Theta$. We similarly have $\bar{V} \subseteq Y$, and so $\bar{U} \cap \bar{V} \subseteq X \cap Y \in \mathfrak{D}(n-1)$. Thus, by the preceding paragraph, we have $\bar{U} \cap \bar{V} \in \mathfrak{D}(n-1)$, thereby proving the claim.

Conversely, if U, V are as above, then $\bar{U}$ and $\bar{V}$ are as in the inductive definition of $\mathfrak{D}(n)$. This therefore gives rise to an equivalent formulation of separation dimension.

Finally, putting together Lemmas 7.3 and 7.5, we see by induction on n that if $\operatorname{rank}(M) \leq n$, then M has separation dimension at most n, thereby proving Theorem 2.2.

The usual notion of inductive dimension is similar — replacing separation of points with separation of disjoint closed sets. These notions are equivalent for locally compact spaces (see for example Section III(6) of [Hurewicz and Wallman 1941]). In particular, we note:

Lemma 7.6. *If Θ is a hausdorff topological space of separation dimension at most ν, then every locally compact subset has* (*covering*) *dimension at most ν.*

In particular, such a space does not admit any continuous injective map of $\mathbb{R}^{\nu+1}$.

We note that the conclusion of Lemma 7.6 suggests another notion of dimension for a topological space, namely the maximal dimension of a locally compact subspace. Indeed this was the notion that was used in [Behrstock and Minsky 2008].

8. Coarse median spaces

We establish some basic facts about coarse median spaces. We show that such a space satisfies certain quadratic isoperimetric inequality (Proposition 8.2).

Let (Λ, ρ) be a geodesic space. (A path-metric space would be sufficient.) Suppose that $\mu : \Lambda^3 \to \Lambda$ is (for the moment) any ternary operation on Λ.

Definition. If (Π, μ_Π) is a median algebra then a h-quasimorphism of Π into Λ is a map $\lambda : \Pi \to \Lambda$ satisfying

$$\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h$$

for all $x, y, z \in \Pi$.

Definition. We say that (Λ, ρ, μ) is a coarse median space if it satisfies:

(C1) There are constants, $k, h(0)$, such that for all $a, b, c, a', b', c' \in \Lambda$,

$$\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).$$

(C2) There is a function $h : \mathbb{N} \to [0, \infty)$ such that $1 \leq |A| \leq p < \infty$, then there is a finite median algebra and a $h(p)$-quasimorphism, $\lambda : \Pi \to \Lambda$ such that for all $a \in A$, $\rho(a, \lambda\pi a) \leq h(p)$.

We therefore have one multiplicative constant, k, and a sequence, $h(p)$, of additive constants. We can assume that $h(p)$ is increasing in p.

In (C2), we note that we can always assume that $\Pi = \langle \pi(A) \rangle$, so by Lemma 4.2, $|\Pi| \leq 2^{2^p}$. In particular, we can take Π to be finite. Our definition therefore agrees with that given in Section 2.

Remark. Note that in defining a coarse median space, there would be no loss in taking $\Pi = M(A)$ to be the free median algebra on A (since this will admit an epimorphism to any such Π). Also in (C2), there would be no loss in assuming that $\lambda\pi a = a$ for all $a \in A$. However, when we define a "coarse median space of rank ν" below, we can no longer assume these things.

Definition. If we can always take Π to have rank at most ν, then we say that (Λ, ρ, μ) has *rank* at most ν.

Here, of course, the function h is fixed independently of ν.

Lemma 8.1. *Suppose that* (Λ, ρ) *and* (Λ', ρ') *are quasi-isometric geodesic spaces. Then* (Λ, ρ) *admits a coarse median* (*of rank* ν) *if and only if* (Λ', ρ') *does.*

Proof. Let $f : \Lambda \to \Lambda'$ and $g : \Lambda' \to \Lambda$ be quasi-inverse quasi-isometries. (That is, $f \circ g$ and $g \circ f$ are each a bounded distance from the respective identity maps) We define μ' on Λ' by setting $\mu'(a, b, c) = f\mu(ga, gb, gc)$. $\square$

Definition. A finitely generated group Γ is *coarse median* (of *rank* ν) if and only if its Cayley graph with respect to any finite generating set admits a coarse median.

Any two such Cayley graphs are quasi-isometric, so this is well defined by Lemma 8.1.

Returning to Λ, suppose $a, b, c \in \Gamma$. Let $A = \{a, b, c\}$, and let $\pi : A \to \Pi$ and $\lambda : \Pi \to \Lambda$ be as in (C2). From the second part of (C2), we see that $\rho(a, \lambda\pi a)$, $\rho(b, \lambda\pi b)$ and $\rho(c, \lambda\pi c)$ are all bounded above by $h(3)$. Applying (C1), it follows that

$$\rho(\mu(a, b, c), \mu(\lambda\pi a, \lambda\pi b, \lambda\pi c)) \leq 3kh(3) + h(0).$$

Also from the first part of (C2),

$$\rho(\mu(\lambda\pi a, \lambda\pi b, \lambda\pi c), \lambda\mu_\Pi(\pi a, \pi b, \pi c)) \leq h(3),$$

and so

$$\rho(\mu(a, b, c), \lambda\mu_\Pi(\pi a, \pi b, \pi c)) \leq (3k + 1)h(3) + h(0).$$

The same holds for any permutation of a, b, c, and since μ_Π is invariant under such permutation, we deduce

$$\rho(\mu(a, b, c), \mu(b, c, a)) \leq (6k + 2)h(3) + 2h(0),$$

$$\rho(\mu(a, b, c), \mu(b, a, c)) \leq (6k + 2)h(3) + 2h(0).$$

Since $\mu_\Pi(\pi a, \pi a, \pi b) = \pi a$, a similar argument gives

$$\rho(\mu(a, a, b), a) \leq (3k + 2)h(3) + h(0).$$

In view of this, there is no essential loss in assuming (M1) and (M2), namely, $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$ and $\mu(a, a, b) = a$. We have already implicitly used this in Section 3.

Given this, we note that (C1) could be replaced by the assumption that

$$\rho(\mu(a, b, c), \mu(a, b, d))$$

is uniformly bounded above in terms of $\rho(c, d)$. Given that (Λ, ρ) is a geodesic space, it is easy to see that such a bound can always be taken to be linear.

Next, we discuss the quadratic isoperimetric inequality. Suppose, $l, L > 0$.

Definition. An l-cycle is a cyclically ordered sequence of points, $a_0, a_1, \ldots, a_p = a_0$ in Λ, with $\rho(a_i, a_{i+1}) \leq l$ for all i.

Definition. An L-disc consists of a triangulation of the disc, together with a map $b : V \to \Lambda$ of the vertex set, V, into Λ such that $\rho(b(x), b(y)) \leq L$ whenever $x, y \in V$ are adjacent in the 1-skeleton.

Definition. We say that b *spans* an l-cycle, $(a_i)_i$ if we can label the vertices on the boundary as x_i such that x_{i+1} is adjacent to x_i and with $a_i = b(x_i)$ for all i.

Proposition 8.2. *Suppose that Λ is a coarse median space. Given any $l > 0$, there is some $L > 0$, depending only on l and the parameters such that for any $p \in \mathbb{N}$, any l-cycle of length at most p bounds an L-disc with at most p^2 2-simplices.*

In fact, all we require of μ is (M1) and (M2) and the statement that

$$\rho(\mu(a,b,c),\mu(a,b,d)) \leq L/2$$

whenever $a,b,c,d \in \Lambda$ with $\rho(c,d) \leq l$.

To see this, we construct a triangulation of the disc as follows. Let

$$V = \{\{0\}\} \cup \{\{i,j\} \mid 1 \leq i, j \leq p-1\}.$$

We define the edge set by deeming $\{i,j\}$ to be adjacent to $\{i+1,j\}$ and to $\{i+1,j+1\}$ for all $1 \leq i, j \leq p-2$, and deeming $\{0\}$ to be adjacent to $\{1,i\}$ and to $\{p-1,i\}$ for all $1 \leq i \leq p-1$. Note that $\{i,i\} = \{i\}$, so $\{i\}$ is adjacent to $\{i+1\}$ for all $0 \leq i \leq p-2$, and $\{p-1\}$ is adjacent to $\{0\}$. Filling in every 3-cycle with a 2-simplex, we can see that this defines a triangulation of the disc whose boundary is the circuit with vertices $(\{i\})_i$. In total, it has $\frac{1}{2}(p^2-p+2)$ vertices, $\frac{p}{2}(3p-5)$ edges and p^2-2p triangles.

(We can realise this in the euclidean plane, $\mathbb{R}^2$, as follows. We make the identification $V \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$, by identifying $\{i,j\}$ with the ordered pair, (i,j), for $1 \leq j \leq i \leq p-1$, and identifying $\{0\}$ with $(p,0)$. We can triangulate the convex hull of $\{(1,1),(p-1,p-1),(p-1,1)\}$ by cutting along straight lines with slope 0, 1, and ∞ through the integer lattice points. We then connect $(p,0)$ by a geodesic segment to each of the points $(i,1)$ and $(p-1,i)$ for $1 \leq i \leq p-1$. This gives us a triangulation of the convex hull, Δ, of $\{(1,1),(p-1,p-1),(p,0)\}$, with vertices $V \equiv \mathbb{Z}^2 \cap \Delta$. Note that $V \cap \partial\Delta \equiv \{(p,0)\} \cup \{(i,i) \mid 1 \leq i \leq p-1\} \equiv \{\{i\} \mid 0 \leq i \leq p-1\}$.)

Now suppose that $a_0, a_1, \ldots, a_p = a_0$ is an l-cycle in Λ. Define $b : V \to \Lambda$ by $b(\{i,j\}) = \mu(a_0, a_i, a_j)$ thus, $b(\{i\}) = a_i$ for all i. Now, if $\{i',j'\}$ is adjacent to $\{i,j\}$, then $|i-i'| \leq 1$ and $|j-j'| \leq 1$, and so $\rho(b(\{i,j\}), b(\{i',j'\})) \leq 2(L/2) = L$.

This proves Proposition 8.2.

Note that, if Λ is the Cayley graph of a finitely generated group, then this implies that Γ is finitely presented, and that the Dehn function for any finite presentation is at most quadratic. In other words:

Corollary 8.3. *Any coarse median group is finitely presented, and has Dehn function that is at most quadratic.*

The following observations will be needed in the next section.

Lemma 8.4. *Suppose that Π is a finite median algebra generated by $B \subseteq \Pi$, with $|B| \leq p$. Suppose that $\lambda : \Pi \to \Lambda$ is a h-quasimorphism. then* $\operatorname{diam}(\lambda\Pi) \leq K_0(\operatorname{diam}(\lambda B) + h(0) + h(p))$, *where the constant, K_0, depends only on k (the multiplicative constant of* (C1)) *and p.*

Proof. Given $C \subseteq \Pi$, let $G(C) = \{\mu(x, y, z) \mid x, y, z \in C\}$. Let $G^i(C)$ be the i-th iterate of G. Set $q = 2^{2^p}$. By Lemma 4.2, $|\Pi| \leq q$, so $\Pi = C^q(B)$.

Now suppose $x, y, z \in \Pi$ and set $w = \mu_\Pi(x, y, z)$. Now $\mu_\Pi(x, x, y) = x$, and so $\rho(\mu_\Pi(\lambda x, \lambda x, \lambda y), \lambda x) \leq h$. Also

$$\rho(\mu(\lambda x, \lambda y, \lambda z), \mu(\lambda x, \lambda x, \lambda y)) \leq k\rho(x, y) + h(0)$$

and $\rho(\lambda w, \mu(\lambda x, \lambda y, \lambda z)) \leq h$. Thus, $\rho(\lambda x, \lambda w) \leq k\rho(x, y) + h(0) + 2h$. It follows that if $C \subseteq \Pi$, then $\operatorname{diam}(\lambda G(C)) \leq k \operatorname{diam}(\lambda C) + h(0) + 2h$.

Now iterating this q times, starting with $B \subseteq \Pi$, we obtain $\operatorname{diam}(\lambda\Pi) \leq K_0(\operatorname{diam}(\lambda B) + h(0) + h)$ where $K_0 = k^q$. $\square$

Lemma 8.5. *Suppose that $A \subseteq \Lambda$ with $1 \leq |A| \leq p < \infty$ and that $\pi : A \to \Pi$ and $\lambda : \Pi \to \Lambda$ are as in $(C2)$, with $\Pi = \langle \pi A \rangle$. Then*

$$\operatorname{diam}(\lambda\Pi) \leq K(\operatorname{diam}(A) + h(0) + h(p)),$$

where K depends only on k and p.

Proof. By Lemma 8.4, we have $\operatorname{diam}(\lambda\Pi) \leq K_0(\operatorname{diam}(\pi A) + h(0) + h(p))$. But if $a \in A$, then $\rho(a, \lambda\pi a) \leq h(p)$, so $\operatorname{diam}(\lambda\pi A) \leq \operatorname{diam}(A) + 2h(p)$, and the result follows. $\square$

9. Ultralimits

In this section we discuss ultralimits of coarse median spaces. When the ultralimit is obtained through a sequence of rescalings of a given space, we will refer to the resulting space as an "asymptotic cone". Asymptotic cones of groups and metric spaces were introduced by Van den Dries and Wilkie [1984] and elaborated upon by Gromov [1993]. They now play a major role in geometric group theory. We will show that the asymptotic cone of a coarse median space of rank at most ν is a locally convex topological median algebra of rank at most ν. (This was stated as Theorem 2.3.)

First, we give a general discussion. We fix an indexing set, $\mathcal{I}$, with a nonprincipal ultrafilter. Throughout this section, if $(t_i)_{i\in\mathcal{I}}$ is a sequence of real numbers, we will write $t_i \to t$ to mean that t_i tends to t with respect to this ultrafilter. We refer to a sequence as *bounded* if it is bounded with respect to the ultrafilter (i.e., there is some $K \geq 0$ so that the set of indices, $i \in \mathcal{I}$ for which $|t_i| \leq K$ lies in the ultrafilter). Note that any bounded sequence has a unique limit. We recall the following (e.g., [Gromov 1993]). Let $((\Lambda_i, \rho_i))_{i\in\mathcal{I}}$ be a collection of metric spaces indexed by $\mathcal{I}$. We will write $\boldsymbol{a} = (a_i)_i \in \prod_i \Lambda_i$ for a typical sequence of elements. We fix some *basepoint* $\boldsymbol{e} = (e_i)_i \in \prod_i \Lambda_i$. Let $\mathcal{B}$ be the set of sequences $\boldsymbol{a}$ in $\prod_i \Lambda_i$ such that $\rho_i(e_i, a_i)$ is bounded (in the above sense). Given $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$, write $\boldsymbol{a} \sim \boldsymbol{b}$ to mean that $\rho_i(a_i, b_i)$ is bounded. This is an equivalence relation, and we write $\Lambda_\infty = \mathcal{B}/\sim$.

Given $\boldsymbol{a} \in \mathcal{B}$, and $a \in \Lambda_\infty$, we write $a_i \to a$ to mean that a is the equivalence class of $\boldsymbol{a}$. Given $a, b \in \Lambda_\infty$, choose any $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{B}$ with $a_i \to a$ and $b_i \to b$. Now $\rho_i(a_i, b_i)$ is bounded and we define $\rho_\infty(a, b)$ to be the limit of $\rho_i(a_i, b_i)$. One can easily check that this is well defined, and that ρ_∞ is a metric on Λ_∞. With a bit more work, one can see that $(\Lambda_\infty, \rho_\infty)$ is complete.

Now suppose that $((\Lambda_i, \rho_i, \mu_i))_{i \in \mathcal{I}}$ is a sequence of coarse median spaces. We write k_i and h_i for the constants featuring in (C1) and (C2). We suppose:

(U1) k_i is bounded, and $h_i(p) \to 0$ for all $p \in \mathbb{N}$.

We may as well fix $k_i = k$.

Also, we will suppose that the spaces also satisfy properties (M1) and (M2) of a median algebra (that is, with no additive constant). As discussed earlier, there is no essential loss of generality in doing this.

Now suppose that $a, b, c \in \Lambda_\infty$. Choose $a_i \to a$, $b_i \to b$ and $c_i \to c$. Now

$$\rho_i(e_i, \mu_i(a_i, b_i, c_i)) \leq k(\rho_i(e_i, a_i) + \rho_i(e_i, b_i) + \rho_i(e_i, c_i)) + h_i(0),$$

so $\rho_i(e_i, \mu_i(a_i, b_i, c_i))$ is bounded. Moreover, if $a'_i \to a$, $b'_i \to b$ and $c'_i \to c$, is another such sequence, then

$$\rho_i(\mu_i(a_i, b_i, c_i), \mu_i(a'_i, b'_i, c'_i)) \leq k(\rho_i(a_i, a'_i) + \rho_i(b_i, b'_i) + \rho_i(c_i, c'_i)) + h_i(0),$$

so $\rho_i(\mu_i(a_i, b_i, c_i), \mu_i(a'_i, b'_i, c'_i)) \to 0$. It follows that the limit of $\mu_i(a_i, b_i, c_i)$ in Λ_∞ is well defined, and we write it as $\mu_\infty(a, b, c)$.

Now the metric ρ_∞ defines a topology in Λ_∞. With respect to this topology, we claim:

Proposition 9.1. *$(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is a topological median algebra.*

Proof. For this, we only need to consider a finite subset $A \subseteq \Lambda_\infty$. (In view of fact that the median axioms only require sets of four points we could restrict to the case where $|A| \leq 4$ here, and hence only require that $h_i(4) \to 0$. We will however need sets of arbitrary finite cardinality later, when we need to bound the rank.)

Let $A \subseteq \Lambda_\infty$ be finite, and set $p = |A|$. We define maps $f_i : A \to \Lambda_i$ by choosing a sequence $a_i \to a$ for all $a \in \Lambda_\infty$, and setting $f_i(a) = a_i$. We write $A_i = f_i(A) \subseteq \Lambda_i$. Thus $|A_i| \leq p$. Let $\pi_i : A_i \to \Pi_i$ and $\lambda_i : \Pi_i \to \Lambda_i$ be as in (C2). Thus λ_i is an $h_i(p)$-quasimorphism, and we can assume that $\Pi_i = \langle \pi_i A_i \rangle$, so that $|\Pi_i| \leq 2^{2^p}$. There are only finitely many possibilities for the median algebra (Π_i, μ_{Π_i}) up to isomorphism, so we can assume that $\Pi_i = \Pi$ is fixed. We can now also assume that the compositions $\pi_i f_i : A \to \Pi$ are all equal to some fixed map $\pi : A \to \Pi$. Note again that $\Pi = \langle \pi A \rangle$.

Now $\operatorname{diam}(A_i)$ is bounded. By Lemma 8.5,

$$\operatorname{diam}(\lambda_i \Pi) \leq K(\operatorname{diam}(A_i) + h_i(0) + h_i(p))$$

is also bounded. (Here K depends only on k and p and is therefore constant.) If $a \in A$, recall that $a_i = f_i(a) \to a$. Also $\rho_i(a_i, \lambda_i \pi_i a_i) \leq h_i(p) \to 0$, so $\lambda_i \pi_i a_i \to a$. Now if $x \in \Pi$, then $\rho_i(a_i, \lambda_i x)$ is bounded, by the above. So $\rho_i(e_i, \lambda_i x)$ is bounded, and so $\lambda_i x \to b$ for some $b \in \Lambda_\infty$. This gives us a well defined map $\lambda : \Pi \to \Lambda_\infty$, with $\lambda_i x \to \lambda x$.

Now $\Lambda_i : \Pi \to \Lambda_i$ is a $h_i(p)$-quasimorphism where $h_i(p) \to 0$, so it follows that $\lambda : \Pi \to \Lambda$ is a homomorphism; that is, for all $x, y, z \in \Pi$, $\lambda \mu_\Pi(x, y, z) = \mu_\infty(\lambda x, \lambda y, \lambda z)$. Moreover, if $a \in A$, we have seen that $\lambda_i \pi a = \lambda_i \pi_i f_i a = \lambda_i \pi_i a_i \to a$. By definition of λ, we have $\lambda_i \pi a \to \lambda \pi a$, and so $\lambda \pi a = a$. Setting $B = \lambda \Pi$ we have $A \subseteq B$.

Now λ is a homomorphism, so it follows easily that B is closed under μ_∞. Also, since Π is a median algebra, it follows easily that (B, μ_∞) is intrinsically a median algebra.

In summary, we have shown that any finite subset, $A \subseteq \Lambda_\infty$, is contained in another finite subset $B \subseteq \Lambda_\infty$ that is closed under μ_∞ and intrinsically a median algebra. It follows that $(\Lambda_\infty, \mu_\infty)$ is a median algebra. Note in particular, that $\mu_\infty(a, b, c)$ is invariant under permuting a, b, c.

Suppose that $a, b, c, d \in \Lambda_\infty$. Let $a_i \to a$, $b_i \to b$, $c_i \to c$ and $d_i \to d$. Then

$$\rho_i(\mu_i(a_i, b_i, c_i), \mu_i(a_i, b_i, d_i)) \leq k\rho_i(c_i, d_i) + h_i(0),$$

and so

$$\rho_\infty(\mu_\infty(a, b, c), \mu_\infty(a, b, d)) \leq k\rho_\infty(c, d).$$

We see that $\mu_\infty : \Lambda_\infty^3 \to \Lambda_\infty$ is continuous. In other words, $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is a topological median algebra. □

In fact, we can say more. Suppose $a, b, c \in \Lambda_\infty$ with $c \in [a, b]$. Now $\rho_\infty(a, c) \leq \rho_\infty(\mu_\infty(a, a, c), \mu_\infty(a, b, c)) \leq k\rho_\infty(a, b)$. Therefore, $\operatorname{diam}([a, b]) \leq k\rho_\infty(a, b)$. We deduce:

Lemma 9.2. *$(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is weakly locally convex.*

Note that the conclusion of Lemma 9.2 is a consequence of the fact that

$$\rho_\infty(\mu_\infty(a, b, c), \mu_\infty(a, b, d)) \leq k\rho_\infty(c, d)$$

for all $a, b, c, d \in \Lambda_\infty$. This is a key property used in the embedding theorem in [Bowditch 2011a].

Suppose now that each $(\Lambda_i, \rho_i, \mu_i)$ is coarse median of rank at most ν. We now interpret property (U1) above to mean that the constants k_i and $h_i(p)$ of (C2) refer to median algebras Π_i of rank at most ν.

Following the proof of Proposition 9.1, we see that Π has rank at most ν. It follows that $B = \lambda \Pi$ also has rank at most ν (using, for example, condition (2) of Proposition 6.2. We deduce:

Proposition 9.3. *If the spaces* $(\Lambda_i, \rho_i, \mu_i)$ *all have rank at most* ν *and satisfy* $(U1)$, *then* $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ *has rank at most* ν.

Putting together these results with Lemma 7.1, we deduce that $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is locally convex.

This proves Theorem 2.3.

Now suppose that (Λ, ρ, μ) is a coarse median space. Let $\mathcal{I} = \mathbb{N}$ with any nonprincipal ultrafilter. Let t_i be any sequence of positive numbers with $t_i \to 0$ (with respect to the ultrafilter is enough). Let $\Lambda_i = \Lambda$, $\rho_i = t_i \rho$ and $\mu_i = \mu$. Let $e \in \Lambda$, and set $e_i = e$ for all i to give us a fixed basepoint. The sequence $(\Lambda_i, \rho_i, \mu_i)$ satisfies the condition of Proposition 9.1, and so we get a topological median algebra $(\Lambda_\infty, \rho_\infty, \mu_\infty)$.

Definition. We refer to a topological median algebra arising in this way as an *asymptotic cone* of (Λ, ρ, μ).

Thus $(\Lambda_\infty, \rho_\infty)$ is an asymptotic cone in the traditional sense. The following is an immediate consequence of the above:

Proposition 9.4. *If* (Λ, ρ, μ) *has rank at most* ν, *then any asymptotic cone is locally convex and has rank at most* ν.

We can now deduce Corollary 2.4 as explained in Section 2.

Finally, we note:

Lemma 9.5. *Any geodesic space which admits a structure as a rank-*1 *topological median algebra is an* $\mathbb{R}$*-tree.*

Proof. We see that any pair of distinct points are separated by a rank-0 subalgebra, in other words, a point. This implies that a geodesic connecting any pair of point must in fact be the unique arc connecting those points. In other words, any two points are connected by a unique arc which is isometric to a real interval. This is one of the standard definitions of an $\mathbb{R}$-tree. $\square$

Using Proposition 9.4, we deduce:

Lemma 9.6. *Let* (Λ, ρ) *be a geodesic space which admits a rank-*1 *coarse median. Then any asymptotic cone of* (Λ, ρ) *is an* $\mathbb{R}$*-tree.*

This now gives us what we need to complete the proof of Theorem 2.1 as explained in Section 3.

10. Projection maps

In this section, we explain how the existence of certain projection maps imply that a given ternary operation on a geodesic space is a coarse median. We first give the constructions in a formal manner. The main application we have in mind is to the mapping class group, as we explain in Section 11.

Let (Λ, ρ) be a geodesic space, and let $\mu : \Lambda^3 \to \Lambda$ be a ternary operation. Let $\mathscr{X}$ be an indexing set, and suppose that to each $X \in \mathscr{X}$, we have associated a uniformly coarse median space $(\Theta(X), \sigma_X, \mu_X)$, together with a uniformly lipschitz quasimorphism, $\theta_X : \Lambda \to \Theta(X)$. Here, "uniform" means that the various parameters are independent of X. In particular, we are assuming that $\theta_X : (\Lambda, \rho) \to (\Theta(X), \sigma_X)$ is k_0-lipschitz, and that $\theta_X : (\Lambda, \mu) \to (\Theta(X), \mu_X)$ is a h_0-quasimorphism for fixed k_0 and h_0.

We also assume:

(P1) For all l there is some l' such that if $a, b \in \Lambda$ satisfy $\sigma_X(\theta_X a, \theta_X b) \leq l$ for all $X \in \mathscr{X}$, then $\rho(a, b) \leq l'$.

Proposition 10.1. *A ternary operation μ satisfying the above is a coarse median on* (Λ, ρ). (In fact, we will see that the parameters of (Λ, ρ, μ) depend only on those arising in the hypotheses.)

Before giving the proof, we note how the hypotheses arise in nature. In Section 11, Λ will be the "marking complex" of a compact surface Σ. This is quasi-isometric to the mapping class group of Σ. The map μ will be the "centroid" map defined in [Behrstock and Minsky 2011]. The set $\mathscr{X}$ is the set of homotopy classes of essential subsurfaces of Σ. In this, we include annuli and Σ itself, but do not allow three-holed spheres. For a nonannular surface, the space $(\Theta(X), \sigma_X)$ will be the curve graph of X, which is hyperbolic by [Masur and Minsky 1999], and hence is coarse median of rank 1. If X is an annulus, then $(\Theta(X), \sigma_X)$ is a certain arc complex, which is quasi-isometric to the real line. In all cases, the maps $\theta_X : \Lambda \to \Theta(X)$ arises from the subsurface projection map described in [Masur and Minsky 2000]. The property (P1) can be shown using the distance formula used in the same reference. A consequence of Proposition 10.1, is that the mapping class group is coarse median. We recover the fact that it is finitely presented and has a quadratic Dehn function [Mosher 1995].

Proof of Proposition 10.1. We need to verify (C1) and (C2).

(C1) Let $a, b, c, a', b', c' \in \Lambda$, and write $e = \mu(a, b, c)$, $f = \mu(a', b', c')$. Let $X \in \mathscr{X}$. Write $t = \sigma(a, a') + \sigma(b, b') + \sigma(c, c')$. Since θ_X is a quasimorphism, we have

$$\sigma_X\big(\theta_X e, \mu_X(\theta_X a, \theta_X b, \theta_X c)\big) \leq h_0,$$
$$\sigma_X\big(\theta_X f, \mu_X(\theta_X a', \theta_X b', \theta_X c')\big) \leq h_0.$$

Since μ_X satisfies (C1) and θ_X is k_0-lipschitz, we have

$$\begin{aligned}\sigma_X\big(\mu_X(\theta_X a, \theta_X b, \theta_X c), \mu_X(\theta_X a', \theta_X b', \theta_X c')\big) \\ \leq k\big(\sigma_X(\theta_X a, \theta_X a') + \sigma_X(\theta_X b, \theta_X b') + \sigma_X(\theta_X c, \theta_X c')\big) \\ \leq k k_0\big(\rho(a, a') + \rho(b, b') + \rho(c, c')\big) = k k_0 t.\end{aligned}$$

Thus, $\sigma_X(\theta_X e, \theta_X f)$ is (linearly) bounded above in term of t.

Now since this holds uniformly for all $X \in \mathscr{X}$, it follows by (P1) that $\rho(e, f)$ is bounded above in terms of t. Since (Λ, ρ) is a geodesic space, this is sufficient to verify (C1) for μ (as observed in Section 8).

(C2): Let $A \subseteq \Lambda$, with $|A| \leq p < \infty$. Let $q = 2^{2^p}$. Let Π be the free median algebra on A, and write $\pi : A \to \Pi$ for the inclusion map. Note that $\Pi = \langle \pi A \rangle$, and recall from Section 4, that $\Pi = G^i$, where $G^i = G^i(\pi A)$ is defined by iterating the median operation, μ_Π.

We define $\lambda : \Pi \to \Lambda$ inductively as follows. Given $x \in G^0 = \pi A$, set $\lambda x = a$, where $x = \pi a$. Given $u \in G^{i+1} \setminus G^i$, choose any $x, y, z \in G^i$ with $u = \mu_\Pi(x, y, z)$ and set $\lambda u = \mu(\lambda x, \lambda y, \lambda z)$. By construction, we have $\lambda \pi a = a$ for all $a \in A$. We want to show that λ is a quasimorphism.

Let $X \in \mathscr{X}$. We have a quasimorphism $\theta_X : \Lambda \to \Theta(X)$. There is also a quasimorphism, $\omega_X : \Pi \to \Theta(X)$ such that $\omega_X \pi a = \theta_X a$ for all $a \in A$. (Certainly, such a quasimorphism exists from some median algebra to $\Theta(X)$, by (C2) applied to $\Theta(X)$. But since we have taken Π to be free on A, we can precompose this with a homomorphism from Π to the given median algebra which fixes A. Thus, we can take the domain to be Π.) By assumption, the additive constants depend only on the parameters and on p. In particular, they are independent of X.

In what follows, it will be convenient to adopt the following convention. Given points, x, y in a metric space (namely Λ or $\Theta(X)$), we will write $x \sim y$ to mean that, at any particular stage in the argument, the distance between x and y is bounded above by some explicit constant, depending only on the parameters and on p. The bound may increase as the argument proceeds, though we won't keep track of it explicitly here.

We first claim that $\theta_X \lambda x \sim \omega_X x$ for all $x \in \Pi$. We show this by induction on i, where $x \in G^{i+1} \setminus G^i$. Note first that if $x \in G^0$, then setting $x = \pi a$, we have $\theta_X \lambda x = \theta_X a = \omega_X \pi a = \omega_X a$.

Now suppose that $u \in G^{i+1} \setminus G^i$. Let $x, y, z \in G^i$ be the three points that were chosen in the definition of λ, so that $\lambda u = \mu(\lambda x, \lambda y, \lambda z)$. We now have

$$\begin{aligned}\theta_X \lambda u &= \theta_X \mu(\lambda x, \lambda y, \lambda z)\\ &\sim \mu_X(\theta_X \lambda x, \theta_X \lambda y, \theta_X \lambda z)\\ &\sim \mu_X(\omega_X x, \omega_X y, \omega_X z)\\ &\sim \omega_X u.\end{aligned}$$

(The above follow respectively from the fact that $\theta_X : \Lambda \to \Theta(X)$ is a quasimorphism; the inductive hypothesis; and the fact that $\omega_X : \Pi \to \Theta(X)$ is a quasimorphism.) This proves that $\theta_X \lambda x \sim \omega_X x$ for all $x \in \Pi = G^q$.

Now suppose that $x, y, z \in \Pi$ are any three points. We have

$$\begin{aligned}\theta_X \lambda \mu_\Pi(x, y, z) &\sim \omega_X \mu_\Pi(x, y, z)\\ &\sim \mu_X(\omega_X x, \omega_X y, \omega_X z)\\ &\sim \mu_X(\theta_X \lambda x, \theta_X \lambda y, \theta_X \lambda z)\\ &\sim \theta_X \mu(\lambda x, \lambda y, \lambda z).\end{aligned}$$

(These relations follow respectively from the claim already proven above; the fact that $\omega_X : \Pi \to \Theta(X)$ is a quasimorphism; the claim again, together with property (C1) applied to $(\Theta(X), \mu_X)$; and the fact that $\theta_X : \Lambda \to \Theta(X)$ is a quasimorphism.)

In other words, we have shown that

$$\theta_X \lambda \mu_\Pi(x, y, z) \sim \theta_X \mu(\lambda x, \lambda y, \lambda z)$$

for all $X \in \mathscr{X}$, and for all $x, y, z \in \Pi$. Applying (P1), we get

$$\lambda \mu_\Pi(x, y, z) \sim \mu(\lambda x, \lambda y, \lambda z).$$

Thus $\lambda : \Pi \to \Lambda$ is a quasimorphism. The constants depend only on p and the parameters inputted. This verifies (C2). $\square$

We have shown that (Λ, ρ, μ) is a coarse median space With some additional hypotheses (justified for the mapping class group in Section 11), we can control the rank. For this we will assume the spaces $\Theta(X)$ to be uniformly hyperbolic. In this regard, we introduce the following notation.

Suppose that (Θ, σ) is k_0-hyperbolic. Given $x, y, z, w \in \Theta$, we write

$$(x, y : z, w) = \tfrac{1}{2}\big(\max\{\sigma(x, z) + \sigma(y, w), \sigma(x, w) + \sigma(y, z)\} - (\sigma(x, y) + \sigma(z, w))\big).$$

Up to an additive constant, depending only on k_0, this "crossratio" is equal to the distance between any geodesic from x to y and any geodesic from z to w. Note that $(x, y : z, w) \leq \sigma(x, z)$, and that $(x, x : y, y) = \sigma(x, y)$. Also, $(x, y : z, z)$ is the "Gromov product" of x and y with respect to z. Again, up to an additive constant, this equals the distance from z to any geodesic from x to y.

We now make the following additional hypotheses. We suppose that $\mathscr{X}$ comes equipped with a symmetric relation, $\wedge$, with not $X \wedge X$ for all $X \in \mathscr{X}$. We suppose:

(P2) There is some $k_0 \geq 0$ such that each $(\Theta(X), \sigma_X)$ is k_0-hyperbolic.

(P3) There is some $\nu \in \mathbb{N}$ such that if we have a subset $\mathscr{Y} \subseteq \mathscr{X}$ with $X \wedge Y$ for all distinct $X, Y \in \mathscr{Y}$, then $|\mathscr{Y}| \leq \nu$.

(P4) There is some $l_0 \geq 0$ such that if $X, Y \in \mathscr{X}$ and there exist $a, b, c, d \in \Lambda$ with

$$(\theta_X a, \theta_X b : \theta_X c, \theta_X d) \geq l_0 \quad \text{and} \quad (\theta_Y a, \theta_Y c : \theta_Y b, \theta_Y d) \geq l_0,$$

then $X \wedge Y$.

In relation to the mapping class group, where Λ is the marking complex, these are interpreted as follows. The relation, $\wedge$, refers to disjointness of the subsurfaces in Σ. Thus, (P3) is a purely topological observation, where $\nu = \xi(\Sigma)$ as defined in Section 2. For (P2), we have already noted that curve complexes are hyperbolic [Masur and Minsky 1999]. Property (P4) follows from properties of subsurface projection as we discuss in Section 11.

Proposition 10.2. *Suppose that* (Λ, ρ, μ) *satisfies the above — in particular, conditions* (P1)–(P4). *Then* (Λ, ρ, μ) *is a coarse median space of rank at most* ν.

Here, ν is the constant featuring in (P3). As usual, the parameters outputted depend only on those of the hypotheses.

Before giving the proof, we need a general observation regarding hyperbolic spaces. Let (Θ, σ) be k_0-hyperbolic. Let μ be the median as defined in Section 3. We know that (Θ, σ, μ) is coarse median of rank 1. In fact:

Lemma 10.3. *Given* $k_0, l \geq 0$ *and* $p \in \mathbb{N}$, *there is some* $h \geq 0$ *with the following property. Suppose that* (Θ, σ) *is* k_0*-hyperbolic, and that* μ *is the median on* (Θ, σ). *Suppose that* A *is any set with* $|A| \leq p < \infty$ *and that* $\theta : A \to \Theta$ *is any map. Then there is a rank-*1 *median algebra*, Π, *and maps* $\pi : A \to \Pi$ *and* $\lambda : \Pi \to \Theta$, *satisfying*:

(L1) $\sigma(\theta a, \lambda\pi a) \leq h$ *for all* $a \in A$.

(L2) λ *is an* h*-quasimorphism.*

(L3) *If* $a, b, c, d \in A$ *with* $(\pi a, \pi b|\pi c, \pi d)_\Pi$, *then* $(\theta a, \theta b : \theta c, \theta d) \geq l$.

Here, of course, Π is just the vertex set, $V(\tau)$, of a simplicial tree, τ. As in Section 4, we use the notation $(x, y|z, w)_\Pi$ to mean that the sets $\{x, y\}$ and $\{z, w\}$ are separated by a wall in Π. Here, this is equivalent to saying that the arcs $[x, y]_\tau$ and $[z, w]_\tau$ are disjoint.

Proof. Let $\tau_0 \subseteq \Theta$ be the embedded tree arising from $\theta(A) \subseteq \Theta$, as given by Lemma 3.2. Thus, if $a, b \in A$, then $\sigma_{\tau_0}(\theta a, \theta b) \leq \sigma(\theta a, \theta b) + k_1$, where $k_1 = k_0 h_0(p)$. Let $t = l + 2k_1$ and let τ be the metric tree obtained from τ_0 by collapsing down each edge of length at most t. Let $\Pi = V(\tau)$. Given $x \in \Pi$, let $\tau(x) \subseteq \tau$ be the preimage of x under the collapsing map. Thus, $\tau(x)$ is a subtree of diameter at most $k_2 = pt$.

Now let $\pi : A \to \Pi$ be the postcomposition of θ with the collapsing map of τ_0 to τ, define $\lambda : \Pi \to \Theta$ by setting λx to be any vertex of $\tau(x)$.

If $a \in A$, then $\theta a, \lambda\pi a \in \tau(\theta a)$, so $\sigma(\theta a, \lambda\pi a) \leq k_2$. This gives (L1) provided $h \geq k_2$.

For (L2), suppose that $x, y, z \in \Pi$. By definition, $\lambda x \in \tau(x)$, $\lambda y \in \tau(y)$ and $\lambda z \in \tau(z)$. Let $w = \mu_\Pi(x, y, z)$. Let $w' = \mu_{\tau_0}(\lambda x, \lambda y, \lambda z) \in \tau_0 \subseteq \Theta$. Now $w', \lambda w \in \tau(w)$, and so $\sigma(w', \lambda w) \leq k_2$. Now, as in the proof of Lemma 3.1, the median $\mu_\Theta(\lambda x, \lambda y, \lambda z)$ in Θ is a bounded distance from the median w' in τ_0, where the bound depends only on p and k_0. This gives a bound on $\sigma(\lambda w, \mu_\Theta(\lambda x, \lambda y, \lambda z))$ as required.

Finally, suppose that $a, b, c, d \in A$ with $(\pi a, \pi b | \pi c, \pi d)_\Pi$. It follows that $[\pi a, \pi b]_{\tau_0} \cap [\pi c, \pi d]_{\tau_0} = \varnothing$, and so the crossratio $(\theta a, \theta b : \theta c, \theta d)$ defined intrinsically to τ_0 must be at least t. But this agrees with the crossratio defined in Θ up to an additive constant $2k_1$. This proves property (L3). □

Now let l_0 be the constant in property (P4). Suppose that $A \subseteq \Lambda$ with $|A| \leq p < \infty$. Let $X \in \mathscr{X}$. Property (P2) tells us that $(\Theta(X), \sigma_X)$ is k_0-hyperbolic, where k_0 depends only on $\xi(\Sigma)$. Let μ_X be the median operation on $\Theta(X)$. Lemma 10.3 now gives us a rank-1 median algebra $\Pi(X)$, and maps $\pi_X : A \to \Pi(X)$ as well as a h-quasimorphism, $\lambda_X : \Pi(X) \to \Theta(X)$, such that if $a, b, c, d \in A$, with $(\pi_X a, \pi_X b | \pi_X c, \pi_X d)_{\Pi(X)}$ then $(\theta_X a, \theta_X b : \theta_X c, \theta_X d) > l_0$.

Now let $\Pi_0 = \prod_{X \in \mathscr{X}} \Pi(X)$, and let $\psi_X : \Pi_0 \to \Pi(X)$ be the projection map. We define $\pi : A \to \Pi_0$ so that $\psi_X \pi a = \pi_X a$ for all $a \in A$. Let $\Pi = \langle \pi A \rangle \subseteq \Pi_0$, be the subalgebra generated by πA. Note that Π is finite.

(We note that, a-priori, Π_0 might be infinite. In fact, in the application to the mapping class group, we will see that $\Pi(X)$ is trivial for all but finitely many X, so in fact, Π_0, can be taken to be finite. We do not formally need that here.)

Recall that we can naturally identify the set of walls, $\mathcal{W}(\Pi_0)$, with $\bigsqcup_{X \in \mathscr{X}} \mathcal{W}(\Pi(X))$ via the projection maps, ψ_X. Also, by Lemma 6.5 any wall, W, in Π arises from a wall in Π_0, and hence from a wall in $\Pi(X)$ for some $X \in \mathscr{X}$. (In fact, Lemma 5.8 will suffice in the case of the mapping class group, where Π_0 is finite.) We write $X(W)$ for some such X. (It might not be uniquely determined by W.) Note that the map $[W \mapsto X(W)]$ is injective.

Lemma 10.4. *Suppose that $W, W' \in \mathcal{W}(\Pi)$ cross. Then $X(W) \wedge X(W')$.*

Proof. Write $X = X(W)$ and $Y = X(W')$. Since W and W' cross, there is a natural epimorphism of Π to the square $W \times W'$. Since $\Pi = \langle \pi A \rangle$, the restriction to πA is also surjective (since any subset of $W \times W'$ is a subalgebra). In other words, we can find $a, b, c, d \in A$ satisfying $(\pi_X a, \pi_X b \mid \pi_X c, \pi_X d)_{\Pi(X)}$ and $(\pi_Y a, \pi_Y c \mid \pi_Y b, \pi_Y d)_{\Pi(Y)}$. Thus, by the construction of $\Pi(X)$ and $\Pi(Y)$, we have

$$(\theta_X a, \theta_X b : \theta_X c, \theta_X d) \geq l_0 \quad \text{and} \quad (\theta_Y a, \theta_Y c : \theta_Y b, \theta_Y d) \geq l_0.$$

By (P4) it now follows that $X \wedge Y$. □

Corollary 10.5. Π *has rank at most* ν.

Proof. Suppose that $\mathcal{W} \subseteq \mathcal{W}(\Pi)$ is a set of pairwise crossing walls. By Lemma 10.4, we have $X(W) \wedge X(W')$ for all distinct $W, W' \in \mathcal{W}_0$. It now follows by (P3) that $|\mathcal{W}_0| \leq \nu$. □

Proof of Proposition 10.2. We proceed as in the proof of Proposition 10.1. We already have (C1).

For (C2), we need that the rank of Π is at most ν. Instead of taking the free median algebra on A, we take Π as constructed above. In the verification of (C2) we only used the fact that $\Pi = \langle \pi A \rangle$, together with the existence of uniform quasimorphisms $\omega_X : \Pi \to \Theta(X)$ with $\theta_X a \sim \omega_X \pi a$ for all $a \in A$. (In the proof of Proposition 10.1, we had $\theta_X a = \omega_X \pi a$, but we only need that these agree up to bounded distance.)

This time, we have $\Pi = \langle \pi A \rangle$ by construction. The quasimorphism ω_X can now be defined as the composition $\omega_X = \lambda_X \psi_X$.

The proof now proceeds as before. □

11. Surfaces

In this section we verify the hypotheses of Proposition 10.2 in the case where Λ is a connected locally finite graph on which the mapping class group, $\mathrm{Map}(\Sigma)$, acts properly discontinuously with finite quotient. This shows that $\mathrm{Map}(\Sigma)$ is a coarse median group of rank at most $\xi(\Sigma)$.

Here, Θ will be the curve graph $\mathscr{C} = \mathscr{C}(\Sigma)$, of Σ, $\mathscr{X}$ will be the set of subsurfaces of Σ, and $\Theta(X) = \mathscr{C}(X)$ will be the curve graph defined intrinsically to $X \in \mathscr{X}$ (appropriately interpreted if X is an annulus). Briefly, Property (P1) is a consequence of the distance formula of [Masur and Minsky 2000] (see Lemma 11.5), Property (P2) is the hyperbolicity of the curve complex proven in [Masur and Minsky 1999], Property (P3) is an elementary topological observation (see Lemma 11.1) and Property (P4) follows from a result in [Behrstock 2006] which is reformulated here as Lemma 11.3 (see Lemma 11.7).

For the graph, Λ, we could use a Cayley graph with respect to a finite generating set, though we will find it more convenient to work with a "marking complex"; compare [Masur and Minsky 2000].

We now give more formal definitions. Let Σ be a compact orientable surface with (possibly empty) boundary $\partial\Sigma$. Let $\xi(\Sigma) = 3g + p - 3$, where g is the genus, and p the number of boundary components. We assume that $\xi(\Sigma) > 1$. Let $\mathscr{C}^0 = \mathscr{C}^0(\Sigma)$ be the set of homotopy classes of essential nonperipheral simple closed curves in Σ, referred to here simply as "curves". Given $\alpha, \beta \in \mathscr{C}^0$, we write $\iota(\alpha, \beta)$ for their geometric intersection number, in other words, the minimal possible number of intersections taken over all representative curves in the respective homotopy classes. (We remark that given any finite set of curves in Σ, we can find realisations which simultaneously achieve these minima for all pairwise intersections — for example, take geodesic representatives with respect to any complete hyperbolic structure on the interior of Σ.) The *curve graph*, $\mathscr{C} = \mathscr{C}(\Sigma)$, is the graph with vertex set,

$V(\mathscr{C}) = \mathscr{C}^0$, where $\alpha, \beta \in \mathscr{C}^0$ are adjacent if $\iota(\alpha, \beta) = 0$. (This is the 1-skeleton of Harvey's curve complex.) We write σ for the combinatorial metric on $\mathscr{C}$. It was shown in [Masur and Minsky 1999] that $\mathscr{C}$ is hyperbolic. (A constructive proof can be found in [Bowditch 2006b].) It is not hard to see that $\sigma(\alpha, \beta)$ is bounded above in terms of $\iota(\alpha, \beta)$ (for example, $\sigma(\alpha, \beta) \leq \iota(\alpha, \beta) + 1$). We will write $\alpha \pitchfork \beta$ to mean that $\iota(\alpha, \beta) > 0$.

Given $a \subseteq \mathscr{C}^0$, we write $\iota(a) = \max\{\iota(\alpha, \beta) \mid \alpha, \beta \in a\}$ for the *self-intersection* of a. If $\iota(a) < \infty$ then a is finite. (In fact, $\sum\{\iota(\alpha, \beta) \mid \alpha, \beta \in a\}$ is bounded above in terms of $\iota(a)$ and $\xi(\Sigma)$.) We say that a *fills* Σ if, for all $\gamma \in \mathscr{C}^0$, there is some $\alpha \in a$ with $\alpha \pitchfork \gamma$. Given $p \in \mathbb{N}$, we write $L(p)$ for the set of subsets $a \subseteq \mathscr{C}^0$ with $\iota(a) \leq p$ and which fill Σ. Given $p, q \in \mathbb{N}$ we write $\Lambda(p, q)$ for the graph with vertex set $L(p)$ where $a, b \in L(p)$ are deemed to be adjacent if $\iota(a \cup b) \leq q$. Thus, $\Lambda(p, q)$ is locally finite, and $\mathrm{Map}(\Sigma)$ acts on $\Lambda(p, q)$ with finite quotient. For a "marking complex", we could take any connected $\mathrm{Map}(\Sigma)$-invariant subgraph of $\Lambda(p, q)$ for some p, q (which might be allowed to depend on $\xi(\Sigma)$). The notion is quite robust, so it doesn't much matter exactly what construction we use. For definiteness, we can set Λ to be the marking complex used in [Masur and Minsky 2000]. In this case, $\Lambda \subseteq \Lambda(4, 4)$. (We could also use $\Lambda(p, q)$ itself for sufficiently large p, q.)

We define a map $\chi : \Lambda \to \mathscr{C}$, which chooses some element $\chi(a) \in a$ from each $a \in V(\Lambda)$. Note that this is uniformly lipschitz with respect to the metrics ρ and σ on $V(\Lambda)$ and $V(\mathscr{C}) = \mathscr{C}^0$. (We can extend to a map $\Lambda \to \mathscr{C}$, by first collapsing each of Λ to an incident vertex.)

We now move on to consider subsurfaces.

Definition. By a *subsurface realised in* Σ we mean a compact connected subsurface $X \subseteq \Sigma$ such that each boundary component of X is either a component of $\partial\Sigma$, or else an essential nonperipheral simple closed curve in $\Sigma \setminus \partial\Sigma$, and such that X is not homeomorphic to a three-holed sphere.

Note that we are allowing Σ itself as a subsurface, as well as nonperipheral annuli.

Definition. A *subsurface* is a free homotopy class of realised subsurfaces.

We will sometimes abuse notation and use the same symbol for a subsurface and some realisation of it in Σ.

We write $\mathscr{X} = \mathscr{X}(\Sigma)$ for the set of subsurfaces of Σ. We write $\mathscr{X} = \mathscr{X}_A \sqcup \mathscr{X}_N$ where $\mathscr{X}_A$ and $\mathscr{X}_N$ are respectively the sets of annular and nonannular subsurfaces. Note that there is a natural bijective correspondence between $\mathscr{X}_A$ and the set of curves, $\mathscr{C}^0$. (We will, however, treat them as distinct from the point of view of the notation introduced below.)

Suppose $X \in \mathscr{X}_N$. We have $0 < \xi(X) \le \xi(\Sigma)$, and write $\mathscr{C}^0(X)$, $\mathscr{C}(X)$, $\Lambda(X)$ respectively for $\mathscr{C}^0$, $\mathscr{C}$, Λ defined intrinsically to X. (In the exceptional cases where $\xi(X) = 1$, $\mathscr{C}(X)$ is defined by deeming two curves to be adjacent if they have minimal possible intersection for that surface, that is, 1 for a one-holed torus, and 2 for a four-holed sphere. In both cases this gives us a Farey graph.) Note that we can identify $\mathscr{C}^0(X)$ as a subset of $\mathscr{C}^0$. We write σ_X and ρ_X for the combinatorial metrics on $\mathscr{C}(X)$ and $\Lambda(X)$. Let $\mathscr{C}^0(\Sigma, X)$ and $\mathscr{C}^0(\Sigma, \partial X)$ be the subsets of $\mathscr{C}^0$ consisting of curves of Σ homotopic into X or ∂X respectively. In this way, $\mathscr{C}^0(\Sigma, X) = \mathscr{C}^0(X) \sqcup \mathscr{C}^0(\Sigma, \partial X)$.

If $X \in \mathscr{X}_A$, the set $\mathscr{C}(X)$ is defined as an arc complex in the cover of Σ corresponding to X, as in [Masur and Minsky 2000]. This is quasi-isometric to the real line. We set $\Lambda(X) = \mathscr{C}(X)$.

Given $X, Y \in \mathscr{X}$, we distinguish five mutually exclusive possibilities denoted as follows:

(1) $X = Y$.

(2) $X \prec Y$: $X \neq Y$, and X can be homotoped into Y but not into ∂Y.

(3) $Y \prec X$: $Y \neq X$, and Y can be homotoped into X but not into ∂X.

(4) $X \wedge Y$: $X \neq Y$ and X, Y can be homotoped to be disjoint.

(5) $X \pitchfork Y$: none of the above.

In (2)–(4) one can find realisations of X, Y in Σ such that $X \subseteq Y$, $Y \subseteq X$, $X \cap Y = \varnothing$, respectively. (Note that $X \wedge Y$ covers the case where X is an annulus homotopic to a boundary component of Y, or vice versa.) We can think of (5) as saying that the surfaces "overlap".

Lemma 11.1. *Suppose $\mathscr{Y} \subseteq \mathscr{X}$ satisfies $X \wedge Y$ for all distinct $X, Y \in \mathscr{Y}$. Then $|\mathscr{Y}| \le \xi(\Sigma)$.*

Proof. For each $Y \in \mathscr{Y}$, choose an essential curve, α_Y in Y which is nonperipheral if $Y \in \mathscr{X}_N$ and the core curve if $Y \in \mathscr{X}_A$. The curves α_Y are all pairwise nonhomotopic in Σ, so there can be at most $\xi(\Sigma)$ of them. $\square$

Next we consider subsurface projections. These were defined in [Masur and Minsky 2000].

Let $X \in \mathscr{X}$. If $\alpha \in \mathscr{C}^0$, write $\alpha \pitchfork X$ to mean that either $\alpha \in \mathscr{C}^0(X)$ or $\alpha \pitchfork \gamma$ for some $\gamma \subseteq \partial X$. In other words, α cannot be homotoped to be disjoint from X. (This is consistent with the notation above if we identify α with an annular neighbourhood.) In this case, we write $\theta_X \alpha$ for a projection of α in $\mathscr{C}(X)$, as defined in [Masur and Minsky 2000]. There is some ambiguity in the definition, but it is well defined up to bounded distance. In fact, if $X \in \mathscr{X}_N$, we can take $\theta_X \alpha \in \mathscr{C}^0(X)$, and this case, it is well defined up to bounded intersection. Moreover, if $\alpha, \beta \pitchfork X$, then $\iota(\theta_X\alpha, \theta_X\beta)$ is bounded above in terms of $\iota(\alpha, \beta)$. Note that if a fills Σ, then at least one $\alpha \in a$ must satisfy $a \pitchfork X$. The resulting curve, $\theta_X\alpha \in \mathscr{C}^0(X)$, is well defined up to bounded intersection number in X, where the bound depends only on $\iota(a)$. This gives rise to a map $\theta_X : \Lambda \to \mathscr{C}(X)$, well defined up to bounded distance. Moreover, θ_X is uniformly lipschitz with respect to the metrics ρ and σ_X.

Suppose that $a \in L(p)$, for $p \geq 4$. Let $a_X \subseteq a$ be the set of curves, $\alpha \in a$, with $\alpha \pitchfork X$. This must be nonempty. Note that $\{\theta_X\alpha \mid \alpha \in a_X\}$ has bounded self-intersection. Moreover, if p is large enough it's not hard to see that this set must fill X. Given these observations, we see that we have also a map $\phi_X : \Lambda \to \Lambda(X)$, well defined up to bounded distance, and uniformly lipschitz with respect to the metrics ρ and ρ_X. (Namely, set $\phi_X(\alpha) = \theta_X\alpha$ for some $\alpha \in a_X$.) Moreover, writing $\chi_X : \Lambda(X) \to \mathscr{C}(X)$, for the map χ defined intrinsically to X, we see that we the map θ_X agrees up to bounded distance with the composition $\chi_X\phi_X$.

Suppose that $X, Y \in \mathscr{X}$ with $X \pitchfork Y$ or $Y \prec X$. We define a point $\theta_X Y \in \mathscr{C}(X)$ as follows. If $Y \in \mathscr{X}_A$, we set $\theta_X Y = \theta_X\alpha$, where $\alpha \in \mathscr{C}^0$ is the curve homotopic to Y. If $Y \in \mathscr{X}_N$, we choose any $\alpha \in \mathscr{C}^0(\Sigma, \partial X)$ with $\alpha \pitchfork Y$ and set $\theta_X Y = \theta_X\alpha$. Note that this is well defined up to bounded distance.

We list a few properties of subsurface projections.

First note that if $X \prec Y$, we have a subsurface projection, θ_{XY} defined intrinsically to Y. In other words, we can replace Σ by Y in the earlier discussion, and work intrinsically with Y. (Note that $Y \in \mathscr{X}_N$.)

Lemma 11.2. *If $\alpha \in \mathscr{C}^0(\Sigma)$ with $\alpha \pitchfork X$, then $\alpha \pitchfork Y$, and $\sigma_X(\theta_X\alpha, \theta_{XY}\theta_Y\alpha)$ is bounded in terms of $\xi(\Sigma)$.*

Proof. This is an easy consequence of the construction in [Masur and Minsky 2000]. □

In fact, using that same construction, we see that the intersection number between $\theta_X\alpha$ and $\theta_{XY}\theta_Y\alpha$ is also bounded. In view of this, we can henceforth drop the suffix "Y", and write θ_{XY} as θ_X.

Lemma 11.3. *There is some constant l_1, depending only on $\xi(\Sigma)$, with the following property. Suppose that $X, Y \in \mathscr{X}$ with $X \pitchfork Y$, and that $a \in V(\Lambda)$. Then* $\min\{\sigma_X(\theta_X a, \theta_X Y), \sigma_Y(\theta_Y a, \theta_Y X)\} \leq l_1$.

Proof. This is an immediate consequence of the result in [Behrstock 2006]; see also [Mangahas 2010]. This was stated for curves, namely that if $\alpha \in \mathscr{C}^0$ with $\alpha \pitchfork X$ and $\alpha \pitchfork Y$, then $\min\{\sigma_X(\theta_X\alpha, \theta_X Y), \sigma_Y(\theta_Y\alpha, \theta_Y X)\}$ is bounded above in terms on $\xi(\Sigma)$. To relate this to our statement, it is a simple exercise to find such a curve, $\alpha \in \mathscr{C}^0$, with $\iota(a \cup \{\alpha\})$ bounded in terms of $\xi(\Sigma)$. Thus, $\sigma_X(\theta_X\alpha, \theta_X a)$ and $\sigma_Y(\theta_Y\alpha, \theta_Y a)$ are bounded. □

Lemma 11.4. *There is some l_2, depending only on $\xi(\Sigma)$ with the following property. Suppose $X, Y \in \mathscr{X}$ with $Y \prec X$, and suppose that $a, b \in \Lambda$ with $(\theta_X a, \theta_X b : \theta_X Y, \theta_X Y) \geq l_2$. Then $\sigma_Y(\theta_Y a, \theta_Y b) \leq l_2$.*

Proof. Choosing $\alpha \in a$ and $\beta \in b$ with $\alpha \pitchfork Y$ and $\beta \pitchfork Y$, we will also have $\alpha \pitchfork X$ and $\beta \pitchfork X$. We can therefore interpret the lemma as a statement about curves rather than markings (perhaps with a different constant). Also, in view of Lemma 11.2, we may as well assume that $X = \Sigma$, so that $\alpha = \theta_\Sigma \alpha$ and $\beta = \theta_\Sigma \beta$, and we set $\gamma = \theta_\Sigma Y \in \mathscr{C}^0(\Sigma, Y)$. Now $\mathscr{C}^0(\Sigma, Y)$ has diameter at most 2 in $\mathscr{C}$. Thus, if the Gromov product $(\alpha, \beta \mid \gamma, \gamma)$ is sufficiently large in relation to the hyperbolicity constant of $\mathscr{C}$, then any geodesic from α to β in $\mathscr{C}$ will miss $\mathscr{C}^0(\Sigma, Y)$. By the bounded geodesic image theorem of Masur and Minsky [2000], it then follows that $\sigma_Y(\theta_Y \alpha, \theta_Y \beta)$ and hence $\sigma_Y(\theta_Y a, \theta_Y b)$ is bounded as required. □

The following two lemmas are both consequences of the distance formula in [Masur and Minsky 2000] (though can also be seen more directly). The first of these implies (P1).

Lemma 11.5. *Given any $l \geq 0$, there is some $l' \geq 0$, depending only on l and $\xi(\Sigma)$ with the following property. Suppose that $a, b \in \Lambda$ and that $\sigma_X(\theta_X a, \theta_X b) \leq l$ for all $X \in \mathscr{X}$, then $\rho(a, b) \leq l'$.*

Lemma 11.6. *There is some l_3 depending only on $\xi(\Sigma)$ such that if $a, b \in \Lambda$, then $\{X \in \mathscr{X} \mid \sigma_X(\theta_X a, \theta_X b) \geq l_3\}$ is finite.*

We can now verify property (P4) of Proposition 10.1.

Lemma 11.7. *There is some $l_0 \geq 0$, depending only on $\xi(\Sigma)$ such that if $X, Y \in \mathscr{X}$ and there exist $a, b, c, d \in \Lambda$ with*

$$(\theta_X a, \theta_X b : \theta_X c, \theta_X d) \geq l_0 \quad \textit{and} \quad (\theta_Y a, \theta_Y c : \theta_Y b, \theta_Y d) \geq l_0,$$

then $X \wedge Y$.

Proof. Since $\mathscr{C}(X)$ and $\mathscr{C}(Y)$ are hyperbolic, we must have $X \neq Y$, provided that l_0 is large enough in relation to the hyperbolicity constant. We will also assume that $l_0 \geq 2 \max\{l_1, l_2\}$ (the constants of Lemmas 11.3 and 11.4). If not $X \wedge Y$, then either $X \pitchfork Y$ or, without loss of generality, $Y \prec X$.

Note that the hypotheses on a, b, c, d remain unchanged if we simultaneously swap a with b and c with d. Since $(\theta_X a, \theta_X b : \theta_X c, \theta_X d) \geq l_0 > 2 \max\{l_1, l_2\}$, we can assume that $(\theta_X a, \theta_X b : \theta_X Y, \theta_X Y) \geq \max\{l_1, l_2\}$. In particular, this implies that $\sigma_X(\theta_X a, \theta_X Y) > l_1$ and $\sigma_X(\theta_X b, \theta_X Y) > l_1$. Now, if $X \pitchfork Y$, then Lemma 11.3 tells us that $\sigma_Y(\theta_Y a, \theta_Y X) \leq l_1$ and $\sigma_Y(\theta_Y b, \theta_Y X) \leq l_1$, so that $\sigma_Y(\theta_Y a, \theta_Y b) \leq 2l_1$, giving the contradiction that $(\theta_Y a, \theta_Y c : \theta_Y b, \theta_Y d) \leq l_1$. If $Y \prec X$, then by Lemma 11.4, we have $\sigma_Y(\theta_Y a, \theta_Y b) \leq l_2$ again giving a contradiction. □

We have now verified each of the hypotheses of Proposition 10.2 for the mapping class group, where $\nu = \xi(\Sigma)$. This proves Theorem 2.5.

12. Colourability

In this section we briefly describe the notion of colourability for median algebras and coarse median spaces. In general, this is a strengthening of the rank condition. This property is used in [Bowditch 2011a] to give embeddings of median algebras into products of trees.

Let M be a median algebra.

Definition. We say that M is *ν-colourable* if there is a map, $\chi : \mathcal{W}(M) \to \{1, 2, \ldots, \nu\}$, such that $\chi(W) \neq \chi(W')$ whenever $W \pitchfork W'$.

Clearly this implies that the rank of M is at most ν. The converse does not hold in general, but it does for intervals (see Lemma 12.4).

Proposition 12.1. *A median algebra is ν-colourable if and only if every finite subalgebra is.*

(In fact, it is the latter condition that is applied in practice, so in principle one could bypass this discussion by defining colourability in that way.)

Lemma 12.2. *Any subalgebra of a ν-colourable median algebra in ν-colourable.*

Proof. Let N be a subalgebra of a ν-colourable median algebra, M. Let $\nu : M \to \{1, \ldots, \nu\}$ be a ν-colouring. If $W \in \mathcal{W}(N)$, then by Lemma 6.1, there is a wall in M separating $H^-(W) \subseteq N$ from $H^+(W) \subseteq N$. Let W_M be any such wall. We write $\chi(W) = \chi(W_M)$. Now if $W, W' \in \mathcal{W}(N)$ cross in N, then certainly W_M and W'_M cross in M, and so $\chi(W) \neq \chi(W')$. Thus, $\chi : \mathcal{W}(N) \to \{1, \ldots, \nu\}$ is a ν-colouring of N. □

Lemma 12.3. *If every finite subalgebra of a median algebra M is ν-colourable median algebra M is ν-colourable.*

Proof. We first note that it's enough to show that for any finite subset, $\mathcal{W}_0 \subseteq \mathcal{W}(M)$, we can find a map $\chi : \mathcal{W}_0 \to \{1, \ldots, \nu\}$ such that $\chi(W) \neq \chi(W')$ whenever $W, W' \in \mathcal{W}_0$ with $W \pitchfork W'$. To deduce Lemma 12.3 from this, we recall the standard compactness result from graph theory, namely that a graph is vertex ν-colourable if and only if every finite subgraph is. Here we construct a graph, $\mathcal{G}$, with vertex set $\mathcal{W}(M)$, where $W, W' \in \mathcal{W}(M)$ are deemed adjacent if and only if $W \pitchfork W'$. Thus, colouring M is equivalent to vertex-colouring the graph $\mathcal{G}$. Our claim therefore says that every full subgraph of $\mathcal{G}$ is ν-colourable.

Let $\mathcal{W}_0 \subseteq \mathcal{W}(M)$ be finite. Given any pair, $W, W' \in \mathcal{W}_0$ with $W \pitchfork W'$, choose any $a \in H^-(W) \cap H^-(W')$, $b \in H^+(W) \cap H^-(W')$, $c \in H^-(W) \cap H^+(W')$ and $d \in H^+(W) \cap H^+(W')$. Let A be the union of all such $\{a, b, c, d\}$ as (W, W')

ranges over all such pairs. Let Π be a finite median algebra of M containing A. By hypothesis, there is a ν-colouring, $\chi : \mathcal{W}(\Pi) \to \{1, \dots, \nu\}$. Now each $W \in \mathcal{W}_0$ determines a wall, $\hat{W} = \{H^-(W) \cap \Pi, H^+(W) \cap \Pi\}$ in $\mathcal{W}(\Pi)$. Clearly, if W, W' cross in M, then $\hat{W}, \hat{W}'$ cross in Π, and so we can set $\chi(W) = \chi(\hat{W})$ for any such W to prove the claim. □

Lemmas 12.2 and 12.3 now give Proposition 12.1

Suppose Δ is a metric median algebra with points $a, b \in \Delta$ such that $\Delta = [a, b]$. We can orient any wall, $W \in \mathcal{W}(\Delta)$, so that $a \in H^-(W)$ and $b \in H^+(W)$. Given $W, W' \in \mathcal{W}(\Delta)$, we write $W \leq W'$ to mean that $H^-(W) \subseteq H^-(W')$, or equivalently, $H^+(W') \subseteq H^+(W)$. This is a partial order on $\mathcal{W}(\Delta)$. In fact, given any $W, W' \in \mathcal{W}(\Delta)$, exactly one of $W = W'$, $W < W'$, $W' < W$ or $W \pitchfork W'$ holds. It follows that the rank of Δ is exactly the maximal cardinality of any antichain in $(\mathcal{W}(\Delta), <)$. Dilworth's lemma [Dilworth 1950] now tells us that we can partition $\mathcal{W}(\Delta)$ into ν disjoint chains (compare [Brodzki et al. 2009]). This defines a ν-colouring of Δ. We deduce:

Lemma 12.4. *Let M be a median algebra of rank a most ν. If $a, b \in M$, then the interval $[a, b]$ is intrinsically ν-colourable as a median algebra.*

The definition for coarse median spaces is now a simple variation on that for rank:

Definition. A coarse median space is ν-colourable, if in (C2), we can always take the finite median algebra Π to be ν-colourable.

Suppose now that $(\Lambda_i, \rho_i, \mu_i)$ is a directed set of coarse median space as in Theorem 2.3 (where the additive constants tend to 0, and the multiplicative constants are bounded with respect to the ultrafilter). Let $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ be the ultralimit constructed as in Proposition 9.1

Proposition 12.5. *If each of the $(\Lambda_i, \rho_i, \mu_i)$ is ν-colourable (for the given parameters) then $(\Lambda_\infty, \rho_\infty, \mu_\infty)$ is ν-colourable (as a median algebra).*

Proof. Substituting colourability for rank in the proof of Theorem 2.3 in Section 9, exactly the same argument shows that every finite subalgebra of Λ_∞ is ν-colourable. We now apply Lemma 12.3. □

Again the notion is quasi-isometry invariant, so we can apply it to finitely generated groups via their Cayley graphs. We note:

Theorem 12.6. *The mapping class group* $\mathrm{Map}(\Sigma)$ *is ν-colourable for some $\nu = \nu(\Sigma)$.*

In fact, we can get an explicit bound on $\nu(\Sigma)$ from the statement in [Bestvina et al. 2010] which gives us a map: $\chi : \mathcal{X} \to \{1, \dots, \nu(\Sigma)\}$ such that if $\chi(X) = \chi(Y)$, then $X \pitchfork Y$.

The proof of Lemma 11.2 now only requires a slight modification of that of Theorem 2.5. Recall that the median algebra Π used for Property (C2) was constructed using projection maps, before the statement of Lemma 10.4. We now need to check that this is $\nu(\Sigma)$-colourable — a slight modification of Corollary 10.5. For this we need a variation of Property (P3), namely:

(P3′) If $X, Y \in \mathscr{X}$ with $X \wedge Y$, then $\chi(X) \neq \chi(Y)$.

In the present situation, Property (P3′) is an immediate consequence of the definition of the relation $\pitchfork$ in Section 11, and the construction of [Bestvina et al. 2010] mentioned above.

Now let Π be the median algebra defined before Lemma 10.4. We define a map $\chi : \mathscr{W}(\Pi) \to \{1, \dots, \nu(\Sigma)\}$ by setting $\chi(W) = \chi(X(W))$. We claim that this is a $\nu(\Sigma)$-colouring of Π. To see this, suppose that $W, W' \in \mathscr{W}(\Pi)$ cross. Lemma 10.4 then tells us that $X(W) \wedge X(W')$ and so, by (P3′), $\chi(W) \neq \chi(W')$, as required. We can thus replace Corollary 10.5 by the statement that Π is $\nu(\Sigma)$-colourable, and so Theorem 12.6 follows.

As a consequence, from [Bowditch 2011a] we recover the result of Behrstock, Druțu and Sapir [Behrstock et al. 2011] that any asymptotic cone of $\mathrm{Map}(\Sigma)$ admits a bilipschitz embedding in a finite product of $\mathbb{R}$-trees. Moreover, using Lemma 12.3, any interval in the asymptotic cone is compact, and admits a bilipschitz embedding in $\mathbb{R}^{\xi(\Sigma)}$. From this one can recover the fact that $\mathrm{Map}(\Sigma)$ has rapid decay [Behrstock and Minsky 2011].

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GEOMETRIZATION OF CONTINUOUS CHARACTERS OF $\mathbb{Z}_p^\times$

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We define the p-adic trace of certain rank-one local systems on the multiplicative group over p-adic numbers, using Sekiguchi and Suwa's unification of Kummer and Artin–Schreier–Witt theories. Our main observation is that, for every nonnegative integer n, the p-adic trace defines an isomorphism of abelian groups between local systems whose order divides $(p-1)p^n$ and ℓ-adic characters of the multiplicative group of p-adic integers of depth less than or equal to n.

Introduction. Let p and ℓ be distinct primes and let q be a power of p. Let G be a connected commutative algebraic group over $\mathbb{F}_q$; that is, a smooth commutative group scheme of finite type over a field. To geometrize a character $\psi : G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ one pushes forward the Lang central extension

$$0 \to G(\mathbb{F}_q) \to G \xrightarrow{\text{Lang}} G \to 0, \quad \text{Lang}(x) = \text{Fr}(x) - x,$$

by ψ^{-1} and obtains a local system $\mathcal{L}_\psi$ on G. The trace of Frobenius of $\mathcal{L}_\psi$ equals ψ; which is to say that $\mathcal{L}_\psi$ and ψ correspond under the functions–sheaves dictionary. Thus, we think of $\mathcal{L}_\psi$ as *the geometrization of* ψ. Let $\mathsf{C}(G)$ be the abelian group (under tensor product) consisting of $\mathcal{L}_\psi$ as ψ ranges over $\text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times)$; in other words, $\mathsf{C}(G)$ is the group of irreducible summands of $\text{Lang}_! \overline{\mathbb{Q}}_\ell$. Trace of Frobenius defines an isomorphism of abelian groups

$$t_{\text{Fr}} : \mathsf{C}(G) \xrightarrow{\simeq} \text{Hom}(G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell^\times); \tag{1}$$

see [Deligne 1977, Sommes Trig.] and [Laumon 1987, Example 1.1.3].

Here we obtain an analogue of this isomorphism for $\mathbb{G}_m$ over p-adic numbers.

Theorem. *The work of Sekiguchi and Suwa, on unification of Kummer with Artin–Schreier theories, provides an isomorphism between the abelian group of rank-one*

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local systems on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and the abelian group of characters of $\mathbb{Z}_p^\times$ of depth less than or equal to n, for every nonnegative integer n.

Motivation and relation to character sheaves. Before proving the theorem, we take a moment to explain our motivation. Deligne used the local systems $\mathcal{L}_\psi$, appearing above, to prove bounds on the trigonometric sums over finite fields. A key fact used by Deligne in his computation is Grothendieck trace formula. An analogue of this trace formula is missing over p-adic fields. This is the main hurdle for pursuing an analogue of Deligne's results. We hope that the local systems we study here will be of use in obtaining bounds for corresponding sums over p-adic fields.

According to [Lusztig 1985, Section 2], character sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ are perverse sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ (cohomologically) concentrated in degree 1 where they are rank-one Kummer local systems. We restrict our attention to those character sheaves on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ whose order divides $(p-1)p^n$ and find that these are precisely those that admit a $\mathbb{Q}_p(\mu_{p^n})$-rational structure; that is, they can be defined on $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$. In this language, the above theorem states the following: *The p-adic trace* (*defined below*) *of every $\mathbb{Q}_p(\mu_{p^\infty})$-rational character sheaf on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ is a continuous character $\mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_\ell^\times$ and*, *moreover*, *every continuous ℓ-adic character of $\mathbb{Z}_p^\times$ is obtained in this manner*, *each one from a unique character sheaf of $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$*.

Our idea for defining a function from a $\mathbb{Q}_p(\mu_{p^n})$-rational character sheaf $\mathcal{K}$ on $\mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ is to consider $\mathbb{Z}_p[\mu_{p^n}]$-models for $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$ such that $\mathcal{K}$ extends to a local system on the model; then, after restriction to the special fibre of the model, we recover a local system to which we may apply the trace of Frobenius function, as above. Using the work of Sekiguchi and Suwa we find that this idea can be realized if one additional step is introduced: we must consider $\mathbb{Z}_p[\mu_{p^n}]$-models for $\mathbb{G}^{n+1}_{m,\mathbb{Q}_p(\mu_{p^n})}$, rather than $\mathbb{G}_{m,\mathbb{Q}_p(\mu_{p^n})}$. We believe that this strategy for passing from character sheaves on p-adic groups with rational structure to smooth characters by judicious use of integral models may be of wider applicability in establishing a relationship between character sheaves on p-adic groups and admissible characters. This note is meant to illustrate a case of this strategy.

Unification of Kummer with Artin–Schreier–Witt. Henceforth, we assume that p is an *odd* prime. Fix a nonnegative integer n and a primitive p^n-th root of unity $\zeta \in \overline{\mathbb{Q}}_p$. Set $R = \mathbb{Z}_p[\zeta]$, $K = \mathbb{Q}_p(\zeta)$. The main theorem of Sekiguchi and Suwa on the unification of Kummer and Artin–Schreier–Witt theories provides us with

- an exact sequence
$$0 \to \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \to \mathcal{Y} \xrightarrow{f} \mathcal{X} \to 0$$
of commutative group schemes over R,

- isomorphisms $\mathcal{Y}_K := \mathcal{Y} \otimes_R K \xrightarrow{\simeq} \mathbb{G}_{m,K}^{n+1}$ and $\mathcal{X}_K \to \mathbb{G}_{m,K}^{n+1}$,
- isomorphisms $\mathcal{Y}_{\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$ and $\mathcal{X}_{\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}$,

where $\mathbb{W}_{n,\mathbb{F}_p}$ is the Witt ring scheme of dimension n over $\mathbb{F}_p$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{G}_{m,K} \arrow[d, "\theta"'] & \mathbb{G}_{m,K}^{n+1} \arrow[l, "m"'] \arrow[d, "\gamma"] & \mathcal{Y}_K \arrow[l] \arrow[r] \arrow[d, "f_K"] & \mathcal{Y} \arrow[d, "f"'] & \mathcal{Y}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \arrow[l] \arrow[d, "f_{\mathbb{F}_p}"] & \\
\mathbb{G}_{m,K} & \mathbb{G}_{m,K}^{n+1} \arrow[l, "\alpha"] & \mathcal{X}_K \arrow[l] \arrow[r] & \mathcal{X} & \mathcal{X}_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}. \arrow[l] &
\end{tikzcd}$$

Here, $\theta(x) = x^{(p-1)p^n}$, m denotes the multiplication map, γ and α are defined by

$$\gamma(x_0, \dots, x_n) = \left(x_0^{p-1}, \frac{x_1^p}{x_2}, \frac{x_2^p}{x_3}, \dots, \frac{x_n^p}{x_{n-1}}\right),$$

$$\alpha(x_0, x_1, \dots, x_n) = \frac{(x_0 x_1 x_2 x_3 \cdots x_n)^{p^n}}{x_1 x_2^p x_3^{p^2} \cdots x_n^{p^{n-1}}},$$

and f_K and $f_{\mathbb{F}_p}$ are the restrictions of f to the generic and special fibre, respectively. The theorem in question was announced in [Suwa and Sekiguchi 1995] and a proof appeared in the preprint [Sekiguchi and Suwa 1999]. According to Sekiguchi, the main tools of this preprint have been published in [Sekiguchi and Suwa 2003]. For a general overview see [Tsuchiya 2003].

The p-adic trace function. Let $\mathsf{K}(\mathbb{G}_{m,K})$ denote the group (under tensor product) of local systems that are irreducible summands of $\theta_! \overline{\mathbb{Q}}_\ell$. One can easily check that all the squares in the above diagram are Cartesian; moreover, it is clear that all the vertical arrows are Galois covers of order $(p-1)p^n$. It follows that the diagram above determines a canonical isomorphism of groups

$$S : \mathsf{K}(\mathbb{G}_{m,K}) \xrightarrow{\simeq} \mathsf{C}(\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p}). \tag{2}$$

We define the *p-adic trace function* by

$$\begin{aligned} \mathfrak{Tr}_n : \mathsf{K}(\mathbb{G}_{m,K}) &\longrightarrow \mathrm{Hom}\big(\mathbb{G}_m(\mathbb{F}_p) \times \mathbb{W}_n(\mathbb{F}_p), \overline{\mathbb{Q}}_\ell^\times\big) \\ \mathcal{K} &\mapsto t_{\mathrm{Fr}}(S(\mathcal{K})). \end{aligned} \tag{3}$$

It follows at once from (1) and (2) that $\mathfrak{Tr}_n$ is a canonical isomorphism.

Relationship to continuous characters of $\mathbb{Z}_p^\times$. Since p is odd, the exponential map defines an isomorphism of algebraic $\mathbb{F}_p$-groups

$$\mathbb{G}_{m,\mathbb{F}_p} \times \mathbb{W}_{n,\mathbb{F}_p} \xrightarrow{\simeq} \mathbb{W}^*_{n+1,\mathbb{F}_p} \tag{4}$$

where $\mathbb{W}^*_{n+1,\mathbb{F}_p}$ refers to the group scheme of units in the Witt ring scheme $\mathbb{W}_{n+1,\mathbb{F}_p}$ (see [Greenberg 1962]) and therefore an isomorphism

$$\mathbb{G}_m(\mathbb{F}_p)\times\mathbb{W}_n(\mathbb{F}_p)=\mathbb{Z}/(p-1)\times\mathbb{Z}/p^n \xrightarrow{\simeq} \mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p). \tag{5}$$

Accordingly, we can think of the p-adic trace as a character of $\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p)$. Composing with the quotient $\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p)$, we see that the p-adic trace can be interpreted as a continuous ℓ-adic character of $\mathbb{Z}_p^\times$.

Conversely, for every continuous character $\chi:\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_\ell^\times$, there is a nonnegative integer n such that $\chi(\mathbb{Z}_p^\times/(1+p^{n+1}\mathbb{Z}_p))=\{1\}$. The smallest such n is known as the depth of χ. We propose to think of $\mathcal{K}_\chi:=\mathfrak{Tr}_n^{-1}(\chi)$ as *the geometrization of* χ, when $\chi:\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_\ell^\times$ is a continuous character of depth n. We do not discuss how to vary n in the present text.

We note that choosing an isomorphism of the form (5) is unappetizing. We hope, in time, to give a construction which does not depend on this choice.

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A NOTE ON LAGRANGIAN COBORDISMS BETWEEN LEGENDRIAN SUBMANIFOLDS OF $\mathbb{R}^{2n+1}$

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We study the relation of an embedded Lagrangian cobordism between two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. More precisely, we investigate the behavior of the Thurston–Bennequin number and (linearized) Legendrian contact homology under this relation. The result about the Thurston–Bennequin number can be considered as a generalization of the result of Chantraine which holds when $n = 1$. In addition, we provide a few constructions of Lagrangian cobordisms and prove that there are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$.

1. Introduction

Basic definitions. A *contact manifold* (M, ξ) is a $(2n+1)$-dimensional manifold M equipped with a smooth maximally nonintegrable hyperplane field $\xi \subset TM$, that is, locally $\xi = \ker \alpha$, where α is a 1-form which satisfies $\alpha \wedge (d\alpha)^n \neq 0$. ξ is a *contact structure* and α is a *contact* 1*-form* which locally defines ξ. The *Reeb vector field* R_α of a contact form α is uniquely defined by the conditions $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, \cdot) = 0$. The most basic contact manifold is $(\mathbb{R}^{2n+1}, \xi)$, where $\mathbb{R}^{2n+1}$ has coordinates $(x_1, y_1, \dots, x_n, y_n, z)$, and ξ is given by $\alpha = dz - \sum_{i=1}^n y_i\, dx_i$. Note that $R_\alpha = \partial_z$. From now on, for ease of notation, we write $\mathbb{R}^{2n+1}$ instead of $(\mathbb{R}^{2n+1}, \xi)$.

A *Legendrian submanifold* of $\mathbb{R}^{2n+1}$ is an n-dimensional submanifold Λ which is everywhere tangent to ξ, that is, $T_x\Lambda \subset \xi_x$ for every $x \in \Lambda$. The *Lagrangian projection* is a map $\Pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ defined by

$$\Pi(x_1, y_1, \dots, x_n, y_n, z) = (x_1, y_1, \dots, x_n, y_n).$$

Moreover, for Λ in an open dense subset of all Legendrian submanifolds with C^∞ topology, the self-intersection of $\Pi(\Lambda)$ consists of a finite number of transverse double points. Legendrian submanifolds which satisfy this property are called

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chord generic. A *Reeb chord* of Λ is a path along the flow of the Reeb vector field which begins and ends on Λ. Since $R_\alpha = \partial_z$, there is a one-to-one correspondence between Reeb chords of Λ and double points of $\Pi(\Lambda)$. From now on we assume that all Legendrian submanifolds of $\mathbb{R}^{2n+1}$ are connected and chord-generic.

The *symplectization* of $\mathbb{R}^{2n+1}$ is the symplectic manifold $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha))$, where t is a coordinate on $\mathbb{R}$.

Definition 1.1. Let Λ_- and Λ_+ be two Legendrian submanifolds of $\mathbb{R}^{2n+1}$. We say that Λ_- is cobordant to Λ_+ if there exists a smooth cobordism $(L; \Lambda_-, \Lambda_+)$, and an embedding from L to $(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha))$ such that

$$L|_{(-\infty,-T_L]\times\mathbb{R}^{2n+1}} = (-\infty, -T_L] \times \Lambda_-,$$
$$L|_{[T_L,\infty)\times\mathbb{R}^{2n+1}} = [T_L, \infty) \times \Lambda_+$$

for some $T_L \gg 0$ and $L^c := L|_{[-T_L-1,T_L+1]\times\mathbb{R}^{2n+1}}$ is compact. In the case of a Lagrangian (exact Lagrangian) embedding, we say that Λ_- is Lagrangian (exact Lagrangian) cobordant to Λ_+. We will in general not distinguish between L and L^c and call both L.

From now on we assume that all embedded cobordisms in the symplectization of $\mathbb{R}^{2n+1}$ are orientable.

We next define some notations. If L is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism from Λ_- to Λ_+, we write

$$\Lambda_- \prec_L \Lambda_+, \quad \Lambda_- \prec_L^{\mathrm{lag}} \Lambda_+, \quad \text{or } \Lambda_- \prec_L^{\mathrm{ex}} \Lambda_+,$$

respectively. If L_Λ is a filling, Lagrangian filling, or exact Lagrangian filling of Λ in the symplectization of $\mathbb{R}^{2n+1}$, that is, L_Λ is an embedded, embedded Lagrangian, or embedded exact Lagrangian cobordism with empty $-\infty$-boundary and $+\infty$-boundary Λ, then we write $\varnothing \prec_{L_\Lambda} \Lambda$, $\varnothing \prec_{L_\Lambda}^{\mathrm{lag}} \Lambda$ or $\varnothing \prec_{L_\Lambda}^{\mathrm{ex}} \Lambda$, respectively.

For the discussion about Lagrangian cobordisms between Legendrian knots, we refer to [Chantraine 2010; Ekholm et al. $\geq$ 2013], and for the obstructions to the existence of Lagrangian cobordisms defined using the theory of generating families, we refer to [Sabloff and Traynor 2010; Sabloff and Traynor 2011].

Legendrian contact homology. Legendrian contact homology was independently introduced by Eliashberg, Givental, and Hofer [Eliashberg et al. 2000] and, for Legendrian knots in $\mathbb{R}^3$, by Chekanov [2002]. We now briefly remind the reader of the definition of the linearized Legendrian contact homology complex of a closed, orientable, chord-generic Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$; for more details see [Ekholm et al. 2005a].

Let $\mathscr{C}$ be the set of Reeb chords of Λ. Since Λ is generic, $\mathscr{C}$ is a finite set. Let A_Λ be the vector space over $\mathbb{Z}_2$ generated by the elements of $\mathscr{C}$ and $\mathscr{A}_\Lambda$ the unital tensor algebra over A_Λ, that is,

$$\mathcal{A}_\Lambda = \bigotimes_{k=0}^{\infty} A_\Lambda^{\otimes k}.$$

$\mathcal{A}_\Lambda$ is a differential graded algebra whose grading is denoted by $|\cdot|$ and whose differential is denoted by ∂_Λ. $\mathcal{A}_\Lambda$ is called a Legendrian contact homology differential graded algebra of Λ. For the definitions of $|\cdot|$ and ∂_Λ we refer to Section 2 of [Ekholm et al. 2005b].

Note that it is difficult to use Legendrian contact homology in practical applications, as it is the homology of an infinite dimensional noncommutative algebra with a nonlinear differential. One of the ways to extract useful information from the Legendrian contact homology differential graded algebra is to follow Chekanov's [2002] linearization method, which uses an augmentation $\varepsilon : \mathcal{A}_\Lambda \to \mathbb{Z}_2$ to produce a finite-dimensional chain complex $\mathrm{LC}^\varepsilon(\Lambda)$ whose homology is denoted by $\mathrm{LCH}^\varepsilon(\Lambda)$. More precisely, ε is a graded algebra map $\varepsilon : \mathcal{A}_\Lambda \to \mathbb{Z}_2$ that satisfy the following two conditions:

(1) $\varepsilon(1) = 1$;

(2) $\varepsilon \circ \partial_\Lambda = 0$.

Consider the graded isomorphism $\varphi^\varepsilon : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ defined by $\varphi^\varepsilon(c) = c + \varepsilon(c)$. This map defines a new differential $\partial^\varepsilon(c) := \varphi^\varepsilon \circ \partial_\Lambda \circ (\varphi^\varepsilon)^{-1}(c)$ and $\mathrm{LC}^\varepsilon(\Lambda) := (A_\Lambda, \partial_1^\varepsilon)$, where $\partial_1^\varepsilon : A_\Lambda \to A_\Lambda$ is a 1-component of ∂^ε. We let $\mathrm{LCH}_\varepsilon(\Lambda)$ be the homology of the dual complex $\mathrm{LC}_\varepsilon(\Lambda) := \mathrm{Hom}(\mathrm{LC}^\varepsilon(\Lambda), \mathbb{Z}_2)$.

Following Ekholm [2008], we observe that exact Lagrangian cobordism between two Legendrian submanifolds can be used to define a map between the Legendrian contact homology algebras.

In this paper, we establish the following two long exact sequences.

Theorem 1.2. *Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$ such that $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$. Then from the condition $\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+$ it follows that there is an exact sequence*

$$\begin{aligned} \text{(1-1)} \quad \to H_i(\Lambda_-) \to H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \\ \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to . \end{aligned}$$

In addition, $\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+$ implies that there is an exact sequence

$$\begin{aligned} \text{(1-2)} \quad \to \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \\ \to H_i(L, \Lambda_-) \to \mathrm{LCH}^{n-i+3}_{\varepsilon_-}(\Lambda_-) \to . \end{aligned}$$

Here $\mathrm{LCH}^i_{\varepsilon_\pm}(\Lambda_\pm)$ *is the linearized Legendrian contact cohomology of* $\Lambda_\pm$ *over* $\mathbb{Z}_2$, *linearized with respect to the augmentation* $\varepsilon_\pm$. ε_- *is the augmentation induced by* L_{Λ_-}, *and* ε_+ *is the augmentation induced by* L *and* ε_-.

We thank Joshua Sabloff and Lisa Traynor for pointing out how to get the second long exact sequence in Theorem 1.2.

The Thurston–Bennequin invariant. The Thurston–Bennequin invariant (number) of a closed, orientable, connected Legendrian submanifold Λ of $\mathbb{R}^{2n+1}$ was independently defined for $n = 1$ by Bennequin [1983] and by Thurston, and was generalized to the case when $n \geq 1$ by Tabachnikov [1988].

Pick an orientation on $\Lambda \subset \mathbb{R}^{2n+1}$. Push Λ slightly off of itself along $R_\alpha = \partial_z$ to get another oriented submanifold Λ' disjoint from Λ. The Thurston–Bennequin invariant of Λ is the linking number

$$\mathrm{tb}(\Lambda) = \mathrm{lk}(\Lambda, \Lambda').$$

Note that $\mathrm{tb}(\Lambda)$ is independent of the choice of orientation on Λ, since changing it also changes the orientation of Λ'.

Our goal is to prove the following theorem.

Theorem 1.3. *Let* Λ_- *and* Λ_+ *be two closed, orientable Legendrian submanifolds of* $\mathbb{R}^{2n+1}$.

(1) *If* n *is even and* $\Lambda_- \prec_L \Lambda_+$,

$$\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L).$$

(2) *If* n *is odd*, $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$, *and* $\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+$,

$$\mathrm{tb}(\Lambda_+) - \mathrm{tb}(\Lambda_-) = (-1)^{((n-2)(n-1))/2+1}\chi(L).$$

Constructions and examples. Chantraine [2010] described the way to construct Lagrangian cobordisms from Legendrian isotopies of Legendrian knots. We show that the construction of Chantraine works in high dimensions. More precisely, we prove the following:

Proposition 1.4. *Let* Λ_-, Λ_+ *be two closed, orientable Legendrian submanifolds of* $\mathbb{R}^{2n+1}$ *that are Legendrian isotopic. Then there exists an exact Lagrangian cobordism* L *such that*

$$\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+.$$

Front spinning is a procedure invented by Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b] to construct a closed, orientable Legendrian submanifold $\Sigma\Lambda \subset \mathbb{R}^{2n+3}$ from a closed, orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$. We will provide a detailed description of this procedure in Section 4, and prove the following property of it.

Proposition 1.5. *Let Λ_-, Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. If $\Lambda_- \prec_L^{\mathrm{lag}} \Lambda_+$, there exists a Lagrangian cobordism ΣL such that*

$$\Sigma\Lambda_- \prec_{\Sigma L}^{\mathrm{lag}} \Sigma\Lambda_+.$$

In addition, if $\Lambda_- \prec_L^{\mathrm{ex}} \Lambda_+$, there exists an exact Lagrangian cobordism ΣL such that $\Sigma\Lambda_- \prec_{\Sigma L}^{\mathrm{ex}} \Sigma\Lambda_+$.

Finally, we apply Proposition 1.5 to the exact Lagrangian cobordisms from [Ekholm et al. ≥ 2013] and construct exact Lagrangian cobordisms between the nonisotopic Legendrian tori described in [Ekholm et al. 2005b].

Proposition 1.6. *There are infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$.*

2. Proof of Theorem 1.2

Proof. In this section, we prove the existence of the long exact sequences described in Theorem 1.2. We first construct an exact Lagrangian filling of Λ_+.

Since Λ_- is connected, and L, L_{Λ_-} are exact Lagrangian cobordisms in the symplectization of $\mathbb{R}^{2n+1}$ such that the $(-\infty)$-boundary of L, which is Λ_-, agrees with the $(+\infty)$-boundary of L_{Λ_-}, L and L_{Λ_-} can be joined to the exact Lagrangian cobordism L_{Λ_+} in the symplectization of $\mathbb{R}^{2n+1}$, where L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L. Since the $-\infty$-boundary of L_{Λ_-} is empty, the $-\infty$-boundary of L_{Λ_+} is also empty.

We now use the Mayer–Vietoris long exact sequence for $L_{\Lambda_-}, L \subset L_{\Lambda_+}$. We extend L_{Λ_-} and L in such a way that $L_{\Lambda_-} \cap L$ is diffeomorphic to $\mathbb{R} \times \Lambda_-$. Hence the Mayer–Vietoris long exact sequence can be written as

$$\to H_i(\mathbb{R} \times \Lambda_-) \to H_i(L) \oplus H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_{i-1}(\mathbb{R} \times \Lambda_-) \to .$$

Now we note that $H_i(\mathbb{R} \times \Lambda_-) \simeq H_i(\Lambda_-)$ for all i. Hence we can rewrite the Mayer–Vietoris long exact sequence as

$$\to H_i(\Lambda_-) \to H_i(L) \oplus H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_{i-1}(\Lambda_-) \to . \tag{2-1}$$

We now remind the reader of the following fact, which comes from certain observations of Seidel in wrapped Floer homology [Abouzaid and Seidel 2010; Fukaya et al. 2009].

Fact 2.1 [Ekholm 2012]. Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$ and $\varnothing \prec_{L_\Lambda}^{\mathrm{ex}} \Lambda$. Then

$$H_{n-i+2}(L_\Lambda) \simeq \mathrm{LCH}_\varepsilon^i(\Lambda). \tag{2-2}$$

Here ε is the augmentation induced by L_Λ.

For the definition of the augmentation induced by a filling, we refer to Section 3 of [Ekholm 2008]. Also, [Ekholm 2012] provides a fairly complete sketch of a proof of Fact 2.1.

We change the indices in (2-2) and write it as

$$H_i(L_{\Lambda_\pm}) \simeq \mathrm{LCH}^{n-i+2}_{\varepsilon_\pm}(\Lambda_\pm). \tag{2-3}$$

Using (2-3), we rewrite the Mayer–Vietoris long exact sequence (2-1) as

$$\begin{aligned}\to H_i(\Lambda_-) \to H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-)&\\ \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to .&\end{aligned} \tag{2-4}$$

We now write the long exact sequence for the pair $(L_{\Lambda_-}, L_{\Lambda_+})$

$$\to H_i(L_{\Lambda_-}) \to H_i(L_{\Lambda_+}) \to H_i(L_{\Lambda_+}, L_{\Lambda_-}) \to H_{i-1}(L_{\Lambda_-}) \to . \tag{2-5}$$

Using (2-3) and the excision theorem for $L_{\Lambda_+}, L \subset L_{\Lambda_+}$, we write the long exact sequence (2-5) as

$$\begin{aligned}\to \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-) \to \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+)&\\ \to H_i(L, \Lambda_-) \to \mathrm{LCH}^{n-i+3}_{\varepsilon_-}(\Lambda_-) \to .&\end{aligned} \tag{2-6}$$

$\square$

Remark 2.2. Under the conditions of Theorem 1.2, if $H_i(\Lambda_-) = H_{i-1}(\Lambda_-) = 0$ for some i, say when $\Lambda_- = S^n$ and $i, i-1 \neq 0, n$, then long exact sequence (2-4) implies that

$$\mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+) \simeq H_i(L) \oplus \mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-).$$

Hence, for such i, we get

$$H_i(L) \simeq \mathrm{LCH}^{n-i+2}_{\varepsilon_+}(\Lambda_+)/\,\mathrm{LCH}^{n-i+2}_{\varepsilon_-}(\Lambda_-).$$

Remark 2.3. We can rewrite the long exact sequences (2-4) and (2-6) using the relative symplectic field theory of $((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha)), L_{\Lambda_\pm})$, since

$$E^i_1\big((\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t\alpha)), L_{\Lambda_\pm}\big) \simeq \mathrm{LCH}^i_{\varepsilon_\pm}(\Lambda_\pm) \tag{2-7}$$

over $\mathbb{Z}_2$. For the definition of the relative symplectic field theory, we refer to [Ekholm 2008], and for the details about the isomorphism described in (2-7), we refer to [Ekholm 2012]. (We observe that since $L_{\Lambda_\pm}$ are connected, the associated spectral sequences have only one level.)

3. Proof of Theorem 1.3

Let n be even. We recall the following result:

Proposition 3.1 [Eliashberg 1990]. *Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$, where n is even. Then*

$$\mathrm{tb}(\Lambda) = (-1)^{n/2+1}\tfrac{1}{2}\chi(\Lambda).$$

We now note that

$$\chi(\partial L) = 2\chi(L), \tag{3-1}$$

since the Euler characteristic of an even-dimensional boundary is twice the Euler characteristic of its bounded manifold; see Chapter 21 of [May 1999]. We now observe that $\partial L = \Lambda_+ \sqcup \Lambda_-$ and hence, from (3-1), we get that

$$2\chi(L) = \chi(\partial L) = \chi(\Lambda_+) + \chi(\Lambda_-). \tag{3-2}$$

Then we use Proposition 3.1 and rewrite (3-2) as

$$2\chi(L) = \chi(\Lambda_+) + \chi(\Lambda_-) = 2(-1)^{-n/2-1}(\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-)). \tag{3-3}$$

From (3-3) it follows that

$$\mathrm{tb}(\Lambda_+) + \mathrm{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L). \tag{3-4}$$

This finishes the proof of Theorem 1.3 in the case when n is even.

We now prove case (2) of the theorem. First we provide an alternate definition of the Thurston–Bennequin number, found in [Ekholm et al. 2005a].

Let Λ be a closed, orientable, connected, chord-generic Legendrian submanifold of $\mathbb{R}^{2n+1}$ and let c be a Reeb chord of Λ with end points a and b such that $z(a) > z(b)$. We define $V_a := d\Pi(T_a\Lambda)$ and $V_b := d\Pi(T_b\Lambda)$. Given an orientation on Λ, V_a and V_b are oriented n-dimensional transverse subspaces of $\mathbb{R}^{2n}$. If the orientation of $V_a \oplus V_b$ agrees with that of $\mathbb{R}^{2n}$, we say that the sign of c, denoted by $\mathrm{sign}(c)$, is $+1$, otherwise we say that it is -1. Then

$$\mathrm{tb}(\Lambda) = \sum_c \mathrm{sign}(c), \tag{3-5}$$

where the sum is taken over all Reeb chords c of Λ.

The following proposition was proven using (3-5):

Proposition 3.2 [Ekholm et al. 2005b]. *If $\Lambda \subset \mathbb{R}^{2n+1}$ is a closed, orientable, connected, chord generic Legendrian submanifold,*

$$\mathrm{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2}\sum_{c\in\mathscr{C}}(-1)^{|c|}.$$

We now construct an exact Lagrangian filling of Λ_+. We do it the same way as in the proof of Theorem 1.2, namely L_{Λ_+} is obtained by gluing the positive end of L_{Λ_-} to the negative end of L in the symplectization of $\mathbb{R}^{2n+1}$.

By using Proposition 3.2 and taking Euler characteristics of the long exact sequence (1-2), we get

$$\operatorname{tb}(\Lambda_+) - \operatorname{tb}(\Lambda_-) = (-1)^{((n-2)(n-1))/2+1}\chi(L). \tag{3-6}$$

This finishes the proof of Theorem 1.3 when n is odd. □

Remark 3.3. When $n = 1$ we can write (3-6) as

$$\operatorname{tb}(\Lambda_+) - \operatorname{tb}(\Lambda_-) = -\chi(L),$$

which coincides with the formula from Theorem 1.2 of [Chantraine 2010].

Remark 3.4. Observe that the condition of Theorem 1.3 in the case when n is odd is much stronger than the condition of Theorem 1.3 in the case when n is even. If n is even, $\varnothing \prec^{\mathrm{ex}}_{L_{\Lambda_-}} \Lambda_-$ and $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$, then, taking Euler characteristics of the long exact sequence (1-2) and using Proposition 3.2, we get that

$$\operatorname{tb}(\Lambda_+) + \operatorname{tb}(\Lambda_-) = (-1)^{n/2+1}\chi(L).$$

The proof of Theorem 1.3 can be easily modified to become a proof of the following remark.

Remark 3.5. Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$.

(1) If n is even and $\varnothing \prec_{L_\Lambda} \Lambda$,

$$\operatorname{tb}(\Lambda) = (-1)^{n/2+1}\chi(L_\Lambda).$$

(2) If n is odd and $\varnothing \prec^{\mathrm{ex}}_{L_\Lambda} \Lambda$,

$$\operatorname{tb}(\Lambda) = (-1)^{((n-2)(n-1))/2+1}\chi(L_\Lambda).$$

4. Examples

In this section, we describe a few examples of Lagrangian cobordisms. These examples are based on [Chantraine 2010; Ekholm et al. 2005b] and the work of Ekholm, Honda, and Kálmán [Ekholm et al. ≥ 2013]. For the constructions of Lagrangian cobordisms based on the generating families technique, we refer to [Bourgeois et al. ≥ 2013].

Example 4.1. *Proof of Proposition 1.4.* Let Λ_- and $\Lambda_+ \subset \mathbb{R}^{2n+1}$ be two closed, orientable Legendrian submanifolds which are Legendrian isotopic. Then there is a smooth isotopy of a closed manifold Λ to $\mathbb{R}^{2n+1}$ given by $\varphi : \Lambda \times [0, 1] \to \mathbb{R}^{2n+1}$ such that $\Lambda_\nu := \varphi(\Lambda, \nu)$ is Legendrian for all $\nu \in [0, 1]$, $\Lambda_- = \Lambda_0$ and $\Lambda_+ = \Lambda_1$. We now construct L such that $\Lambda_- \prec^{\mathrm{ex}}_{L} \Lambda_+$. Observe that in the construction below one can omit the assumption that Λ_-, Λ_+, L are connected. In the case of Legendrian knots in $\mathbb{R}^3$, the construction of L was described in [Chantraine 2010, Theorem 1.1]. In our case, the construction of Chantraine can be described in the following way.

(1) Note that $\mathbb{R}\times\Lambda_-$ is an exact Lagrangian submanifold of $(\mathbb{R}\times\mathbb{R}^{2n+1}, d(e^t\alpha))$.

(2) Theorem 2.6.2 of [Geiges 2008] implies that there is a compactly supported one-parameter family of contactomorphisms f_ν which realizes the isotopy $(\Lambda_\nu)_{\nu\in[0,1]}$.

(3) Proposition 2.2 from [Chantraine 2010] implies that a contactomorphism of $\mathbb{R}^{2n+1}$ lifts to a Hamiltonian diffeomorphism of the symplectization
$$(\mathbb{R}\times\mathbb{R}^{2n+1}, d(e^t\alpha)).$$

(4) Let H be a Hamiltonian on $\mathbb{R}\times\mathbb{R}^{2n+1}$ whose flow realizes the lifts of f_νs. The existence of H follows from (3). Following Chantraine, we construct
$$H' : \mathbb{R}\times\mathbb{R}^{2n+1}\times[0,1]\to\mathbb{R}$$
such that
$$H'(t,x,\nu)=\begin{cases} H(t,x,\nu) & \text{for } t>T;\\ 0 & \text{for } t<-T.\end{cases}$$
Here $T\gg 0$.

(5) Let ϕ^ν be the Hamiltonian flow of H'. We now observe that $\phi^1(\mathbb{R}\times\Lambda_-)$ coincides with $\mathbb{R}\times\Lambda_-$ near $-\infty$ and with $\mathbb{R}\times\Lambda_+$ near ∞.

(6) Since $\mathbb{R}\times\Lambda_-$ is exact and ϕ^1 a Hamiltonian diffeomorphism, $L:=\phi^1(\mathbb{R}\times\Lambda_-)$ is exact. □

Remark 4.2. Eliashberg and Gromov [1998] provided another proof of the fact that Legendrian isotopy implies Lagrangian cobordism.

Example 4.3. *Proof of Proposition 1.5.* The following construction is based on the front spinning method invented in [Ekholm et al. 2005b].

First we recall the notion of the front projection. The *front projection* is a map Π_F from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{n+1}$ defined by
$$\Pi_F(x_1,y_1,\dots,x_n,y_n,z)=(x_1,x_2,\dots,x_n,z).$$
Let Λ be a closed, orientable Legendrian submanifold of $\mathbb{R}^{2n+1}$ parametrized by $f_\Lambda:\Lambda\to\mathbb{R}^{2n+1}$. We write
$$f_\Lambda(p)=(x_1(p),y_1(p),\dots,x_n(p),y_n(p),z(p))$$
for $p\in\Lambda$. The front projection of Λ is parametrized by $\Pi_F\circ f_\Lambda$, and we have
$$\Pi_F\circ f_\Lambda(p)=(x_1(p),x_2(p),\dots,x_n(p),z(p)).$$

Without loss of generality we can assume that $x_1(p) > 0$ for all $p \in \Lambda$. We now embed $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+2}$ via

$$(x_1, \dots, x_n, z) \to (x_0 = 0, x_1, \dots, x_n, z)$$

and construct the suspension of Λ, denoted by $\Sigma\Lambda$, such that $\Pi_F(\Sigma\Lambda)$ is obtained from $\Pi_F(\Lambda)$ by rotating it around the subspace $x_0 = x_1 = 0$. $\Pi_F(\Sigma\Lambda)$ can be parametrized by $(x_1(p)\sin\theta, x_1(p)\cos\theta, x_2(p), \dots, x_n(p), z(p))$ with $\theta \in S^1$ and is the front projection of a Legendrian embedding $\Lambda \times S^1 \to \mathbb{R}^{2n+3}$. For the properties of $\Sigma\Lambda$ we refer to Lemma 4.16 of [Ekholm et al. 2005b].

Let Λ_- and Λ_+ be two closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$ such that

$$\Lambda_\pm \subset \{(x_1, y_1, \dots, x_n, y_n, z) \in \mathbb{R}^{2n+1} \mid x_1 > 0\} \tag{4-1}$$

and $\Lambda_- \prec_L^{\mathrm{lag}} \Lambda_+$. Let L be parametrized by $f_L : L \to \mathbb{R}^{2n+2}$

$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)).$$

Without loss of generality we assume that $x_1(p) > 0$ for all p. (Formula (4-1) implies that

$$\{f_L(p) \mid x_1(p) \le 0\}$$

is compact and we can translate L so that $x_1(p) > 0$ for all p.) Then we construct a Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$ that we call ΣL. We define ΣL to be parametrized by

$$f_{\Sigma L} : L \times S^1 \to \mathbb{R} \times \mathbb{R}^{2n+3}$$

with

$$\begin{aligned} &f_{\Sigma L}(p, \theta) \\ &\quad = (t(p), x_1(p)\sin\theta, y_1(p)\sin\theta, x_1(p)\cos\theta, y_1(p)\cos\theta, x_2(p), \dots, z(p)). \end{aligned}$$

Here $p \in L$ and $\theta \in S^1$.

We now show that ΣL is really a Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$. Let

$$\begin{aligned} \Lambda_+^{T_L} &:= \{(x_0, \dots, y_n, z) \mid (T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{T_L\} \times \mathbb{R}^{2n+3})\}, \\ \Lambda_-^{T_L} &:= \{(x_0, \dots, y_n, z) \mid (-T_L, x_0, \dots, y_n, z) \in f_{\Sigma L}(\Sigma L) \cap (\{-T_L\} \times \mathbb{R}^{2n+3})\}. \end{aligned}$$

From the definition of T_L, it follows that

$$\begin{aligned} f_{\Sigma L}(\Sigma L) \cap ([T_L, \infty) \times \mathbb{R}^{2n+3}) &= [T_L, \infty) \times \Lambda_+^{T_L}, \\ f_{\Sigma L}(\Sigma L) \cap ((-\infty, -T_L] \times \mathbb{R}^{2n+3}) &= (-\infty, -T_L] \times \Lambda_-^{T_L}. \end{aligned}$$

In addition, we observe that $\Lambda_{\pm}^{T_L} \subset \mathbb{R}^{2n+3}$ can be parametrized by

$$f_{\Lambda_{\pm}^{T_L}} : \Lambda_{\pm} \times S^1 \to \mathbb{R}^{2n+3}$$

such that

$$f_{\Lambda_{\pm}^{T_L}}(p,\theta) = (x_1(p)\sin\theta, y_1(p)\sin\theta, x_1(p)\cos\theta, y_1(p)\cos\theta, x_2(p), \dots, z(p)).$$

Here $p \in \Lambda_{\pm} \subset \partial L$ and $\theta \in S^1$. We now prove that $\Lambda_{\pm}^{T_L}$ coincides with $\Sigma\Lambda_{\pm}$. It is clear that $\Pi_F(\Lambda_{\pm}^{T_L}) = \Pi_F(\Sigma\Lambda_{\pm})$. It remains to prove that $\Lambda_{\pm}^{T_L}$ is a Legendrian submanifold of $\mathbb{R}^{2n+3}$.

It is easy to see that

$$\begin{aligned}(4\text{-}2)\quad f^*_{\Lambda_{\pm}^{T_L}}\Big(dz - \sum_{i=0}^{n} y_i\,dx_i\Big) &= dz(p) - \sum_{i=2}^{n} y_i(p)\,dx_i(p)\\ &\quad - y_1(p)(\sin^2\theta + \cos^2\theta)\,dx_1(p) + (y_1(p)x_1(p)\sin\theta\cos\theta\\ &\qquad - y_1(p)x_1(p)\sin\theta\cos\theta)\,d\theta.\end{aligned}$$

Since $\Lambda_{\pm}$ is a Legendrian submanifold of $\mathbb{R}^{2n+1}$ and so $f^*_{\Lambda_{\pm}}(dz - \sum_{i=1}^{n} y_i\,dx_i) = 0$, we have

$$(4\text{-}3)\qquad y_1(p)\,dx_1(p) = dz(p) - \sum_{i=2}^{n} y_i(p)\,dx_i(p).$$

Hence (4-2) and (4-3) imply that

$$(4\text{-}4)\qquad f^*_{\Lambda_{\pm}^{T_L}}\Big(dz - \sum_{i=0}^{n} y_i\,dx_i\Big) = 0.$$

Since

$$f_{\Lambda_{\pm}}(p) := (x_1(p), \dots, y_n(p), z(p)),$$

where $p \in \Lambda_{\pm} \subset \partial L$ is a parametrization of an embedded submanifold of dimension n, and $x_1(p) > 0$ for $p \in \Lambda_{\pm} \subset \partial L$, one easily sees that

$$f_{\Lambda_{\pm}^{T_L}}(p) = (x_1(p)\sin\theta, y_1(p)\sin\theta, x_1(p)\cos\theta, y_1(p)\cos\theta, x_2(p), \dots, z(p)),$$

where $p \in \Lambda_{\pm}$, $\theta \in S^1$, is a parametrization of an embedded submanifold of dimension $n+1$. Thus, using (4-4), we see that $\Lambda_{\pm}^{T_L}$ is an embedded Legendrian submanifold of $\mathbb{R}^{2n+3}$ whose front projection coincides with $\Pi_F(\Sigma\Lambda_{\pm})$. Thus we get that $\Lambda_{\pm}^{T_L} = \Sigma\Lambda_{\pm}$.

We now note that

$$f^*_{\Sigma L}\left(d\left(e^t\left(dz-\sum_{i=0}^n y_i dx_i\right)\right)\right) = e^t(dt(p)\wedge dz(p) - \sum_{i=2}^n dy_i(p)\wedge dx_i(p) \tag{4-5}$$
$$- \sum_{i=2}^n y_i(p)dt(p)\wedge dx_i(p) - (y_1(p)(\sin^2\theta+\cos^2\theta)dt(p)\wedge dx_1(p)$$
$$+ (\sin^2\theta+\cos^2\theta)dy_1(p)\wedge dx_1(p) + (\sin^2\theta+\cos^2\theta)x_1(p)y_1(p)d\theta\wedge d\theta$$
$$+ (y_1(p)x_1(p)\sin\theta\cos\theta - y_1(p)x_1(p)\sin\theta\cos\theta)dt(p)\wedge d\theta$$
$$+ (y_1(p)\sin\theta\cos\theta - y_1(p)\sin\theta\cos\theta)d\theta\wedge dx_1(p)$$
$$+ (x_1(p)\sin\theta\cos\theta - x_1(p)\sin\theta\cos\theta)dy_1(p)\wedge d\theta)).$$

In addition, observe that

$$e^t(dt(p)\wedge dz(p) - \sum_{i=2}^n dy_i(p)\wedge dx_i(p) - \sum_{i=2}^n y_i(p)dt(p)\wedge dx_i(p)) \tag{4-6}$$
$$= e^t(y_1(p)dt(p)\wedge dx_1(p) + dy_1(p)\wedge dx_1(p)).$$

Hence (4-5) and (4-6) imply that

$$f^*_{\Sigma L}\left(d\left(e^t\left(dz - \sum_{i=0}^n y_i dx_i\right)\right)\right) = 0. \tag{4-7}$$

Since

$$f_L(p) = (t(p), x_1(p), y_1(p), \dots, x_n(p), y_n(p), z(p)),$$

where $p\in L$, is a parametrization of an embedded cobordism of dimension $n+1$ and $x_1(p) > 0$ for $p \in L$, one easily sees that

$$f_{\Sigma L}(p,\theta)$$
$$= (t(p), x_1(p)\sin\theta, y_1(p)\sin\theta, x_1(p)\cos\theta, y_1(p)\cos\theta, x_2(p), \dots, z(p)),$$

where $p\in L$ and $\theta\in S^1$, is a parametrization of an embedded cobordism of dimension $n+2$. Hence we use (4-7) and see that ΣL is really an embedded Lagrangian cobordism from $\Sigma\Lambda_-$ to $\Sigma\Lambda_+$.

We now assume that $\Lambda_- \prec^{\mathrm{ex}}_L \Lambda_+$. Then there is a function $h_L \in C^\infty(f_L(L),\mathbb{R})$ such that

$$dh_L = e^t\left(dz - \sum_{i=1}^n y_i dx_i\right).$$

From a calculation similar to (4-2) it follows that

$$f^*_{\Sigma L}\left(e^t\left(dz - \sum_{i=0}^n y_i\, dx_i\right)\right) = e^{t(p)}\left(dz(p) - \sum_{i=1}^n y_i(p)\, dx_i(p)\right). \tag{4-8}$$

Since $f_{\Sigma L}$ is an embedding, we can define $h_{\Sigma L} \in C^\infty(f_{\Sigma L}(\Sigma L), \mathbb{R})$ by setting

$$(f^*_{\Sigma L} h_{\Sigma L})(p,\theta) := (f^*_L h_L)(p).$$

Hence we use (4-8) and get

(4-9)
$$d(f^*_{\Sigma L} h_{\Sigma L}) = e^{t(p)}\Big(dz(p) - \sum_{i=1}^{n} y_i(p)\, dx_i(p)\Big) = f^*_{\Sigma L}\Big(e^t\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big)\Big).$$

Therefore, since $f_{\Sigma L}$ is an embedding, (4-9) implies that

$$d(h_{\Sigma L}) = e^t\Big(dz - \sum_{i=0}^{n} y_i\, dx_i\Big).$$

Hence, ΣL is an exact Lagrangian cobordism. □

Note that the proof of Proposition 1.5 can be easily modified to become a proof of the following remark.

Remark 4.4. Let Λ be a closed, orientable Legendrian submanifolds of $\mathbb{R}^{2n+1}$. If $\varnothing \prec^{\text{lag}}_{L_\Lambda} \Lambda$, there exists a Lagrangian filling $L_{\Sigma\Lambda}$ such that $\varnothing \prec^{\text{lag}}_{L_{\Sigma\Lambda}} \Sigma\Lambda$. In addition, if $\varnothing \prec^{\text{ex}}_{L_\Lambda} \Lambda$, there exists an exact Lagrangian filling $L_{\Sigma\Lambda}$ such that $\varnothing \prec^{\text{ex}}_{L_{\Sigma\Lambda}} \Sigma\Lambda$.

Before we discuss the next example, we briefly recall a few facts about exact Lagrangian cobordisms between Legendrian knots in $\mathbb{R}^3$.

Theorem 4.5 [Ekholm et al. ≥ 2013; Ekholm et al. 2007]. *There exists an exact Lagrangian cobordism for the following*:

(1) *Legendrian isotopy*,

(2) 0-*resolution at a contractible crossing in the Lagrangian projection*,

(3) *capping off a* $\text{tb} = -1$ *unknot with a disk.*

See Figure 1 for the 0-resolution on the Lagrangian projection.

Following Ekholm, Honda, and Kálmán, we say that a *contractible crossing* of Λ is a crossing so that $z_1 - z_0$ can be shrunk to zero without affecting the other crossings. (Here z_1 is the z-coordinate on the upper strand and z_0 is the z-coordinate on the lower strand.)

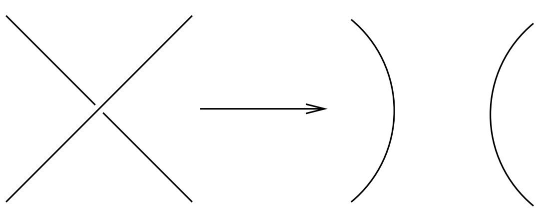

Figure 1. The 0-resolution on the Lagrangian projection.

Remark 4.6. Chantraine [2010] proved the first part of Theorem 4.5.

Remark 4.7. Note that the second part of Theorem 4.5 can be proven using the model from Section 3.3 of [Rizell 2012].

Conjecture 4.8 [Ekholm et al. ≥ 2013; Ekholm et al. 2007]. If $\varnothing \prec^{\mathrm{ex}}_{L_\Lambda} \Lambda$, then L_Λ is obtained by stacking exact Lagrangians cobordisms described in Theorem 4.5.

Example 4.9. *Proof of Proposition 1.6.* We now use Example 4.3 to get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$. We first recall that Theorem 4.5 says that 0-resolution at a contractible crossing in the Lagrangian projection can be realized as an exact Lagrangian cobordism. Let T_{2k+1} be the Legendrian torus knot from Example 4.18 of [Ekholm et al. 2005b]; see Figure 2 for the Lagrangian projection of T_{2k+1}. One observes that all the crossings in the middle part of the Lagrangian projection are contractible (see [Ekholm et al. 2007] for the case of T_3) and hence one can get T_{2k-1} from T_{2k+1} by contracting c_{2k+1} and then c_{2k}. Let L^{2k+1}_{2k} be an exact Lagrangian cobordism which corresponds to the 0-resolution at c_{2k+1} and L^{2k}_{2k-1} an exact Lagrangian cobordism from T_{2k-1} to T_{2k} which corresponds to the resolution of c_{2k}. Then we stack L^{2k+1}_{2k} and L^{2k}_{2k-1} and get an exact Lagrangian cobordism that we call L^{2k+1}_{2k-1} such that

$$T_{2k-1} \prec^{\mathrm{ex}}_{L^{2k+1}_{2k-1}} T_{2k+1}.$$

If we stack L^{2i+1}_{2i-1}s we get an exact Lagrangian cobordism L^{2k+1}_{2j+1} such that

$$T_{2j+1} \prec^{\mathrm{ex}}_{L^{2k+1}_{2j+1}} T_{2k+1}$$

for $k > j$. We use the construction described in Example 4.3 and get

$$\Sigma^n T_{2j+1} \prec^{\mathrm{ex}}_{\Sigma^n L^{2k+1}_{2j+1}} \Sigma^n T_{2k+1}$$

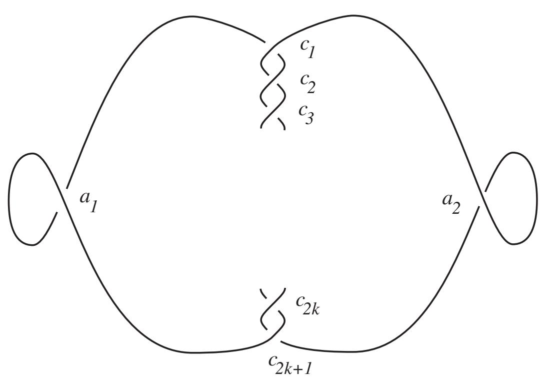

Figure 2. The knot T_{2k+1}; cf. Figure 13 of [Ekholm et al. 2005b] .

for $k > j$. We now recall that Ekholm, Etnyre, and Sullivan [Ekholm et al. 2005b, Theorem 4.19] proved that $\Sigma^n T_{2j+1}$ is not Legendrian isotopic to $\Sigma^n T_{2k+1}$ for $k > j+1$ and $j \in \mathbb{N}$.

Hence we get infinitely many pairs of exact Lagrangian cobordant and not pairwise Legendrian isotopic Legendrian n-tori in $\mathbb{R}^{2n+1}$. □

Remark 4.10. Given $n \geq 1$, we observe that Theorem 4.19 of [Ekholm et al. 2005b] implies that all the Legendrian n-tori from Proposition 1.6 are not distinguished by the classical invariants.

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ON SLOPE GENERA OF KNOTTED TORI IN 4-SPACE

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We investigate genera of slopes of a knotted torus in the 4-sphere analogous to the genus of a classical knot. We compare various formulations of this notion, and use this notion to study the extendable subgroup of the mapping class group of a knotted torus.

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1. Introduction

In classical knot theory, the genus of a knot in the 3-sphere is a basic numerical invariant which has been well-studied. In this note, we investigate some analogous notions for the slopes of a knotted torus in the 4-sphere S^4. These reflect certain essential differences between knotted tori and knotted spheres. Similar phenomena arise in the case of knotted surfaces in S^4, but the discussion would require more general treatments. We focus on the torus case in this note for the sake of simplicity.

A knotted torus in S^4 is a locally flat subsurface homeomorphic to the torus. Without loss of generality, we may fix a choice of marking (see Section 2B). Throughout this note, a *knotted torus* in S^4 means a locally flat embedding

$$K : T^2 \hookrightarrow S^4$$

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from the torus to the 4-sphere. By slightly abusing the notation, we often write the image of K still as K. For any slope (that is, an essential simple closed curve) $c \subset K$, it makes sense to define the *genus*

$$g_K(c)$$

of c as the smallest possible genus of all the locally flat, orientable, compact subsurfaces $F \hookrightarrow S^4$ whose image bounds c and meets K exactly in c. The genus of a slope is clearly an isotopy invariant of the knotted torus, and indeed, it is invariant under *extendable automorphisms*. More precisely, if τ is an automorphism (that is, an orientation-preserving self-homeomorphism up to isotopy) of T^2 that can be extended over S^4 as an orientation-preserving self-homeomorphism, then c and $\tau(c)$ must have the same genus for any slope $c \subset K$. It is clear that all such automorphisms form a subgroup

$$\mathscr{E}_K \le \mathrm{Mod}(T^2)$$

of the mapping class group $\mathrm{Mod}(T^2)$, called the *extendable subgroup* with respect to K. See Section 3 for more details. A primary motivation of our study is to understand $\mathscr{E}_K$ with the aid of the slope genera.

Natural as it is, the genus of a slope of a knotted torus is usually hard to capture. In contrast, two weaker notions yield much more interesting applications. One of them is called the *singular genus* of a slope c, denoted $g_K^{\star}(c)$. It is defined by loosening the locally flat embedding condition on the bounding surface F above, only requiring $F \to S^4$ to be continuous. Another is called the induced *seminorm* on $H_1(T^2)$, denoted $\|\cdot\|_K$. This is an analogue to the (singular) Thurston norm in the classical context. In Section 4, we prove an inequality relating the seminorms associated with the satellite construction, which is analogous to the classical Schubert inequality for knots in S^3.

A simple observation at this point is that both the singular genus and the seminorm of a slope are group-theoretic notions, which can be rephrased in terms of the commutator length and the stable commutator length in the fundamental group of the exterior of the knotted torus, respectively (Remarks 3.3, 4.5).

As an application of these results, we study braid satellites in Section 5. In particular, this allows us to obtain examples of knotted tori with finite extendable subgroups. In Section 6, we exhibit examples where the singular genus is positive for a slope with vanishing seminorm. This implies the singular genus is strictly stronger than the seminorm as an invariant associated to slopes. We also relate the vanishing of the singular genus for a slope $c \subset K$ to the extendability of the Dehn twist $\tau_c \in \mathrm{Mod}(T^2)$ along c in a stable sense (Lemma 6.2).

Section 2 surveys results relevant to our discussion. A few questions for further study related to slope genera and the extendable subgroups are raised in Section 7.

2. Background

This section briefly surveys the history relevant to our topic in several aspects. We hope that it will supply the reader some context for our discussion. However, the reader may safely skip this part for the moment, and perhaps come back later for further references. We thank the referee for suggesting us to include some of these materials.

2A. ***Genera of knots.*** For a classical knot k in S^3, one of the most important numerical invariants is its genus $g(k)$, introduced by Herbert Seifert [1935]. It is naturally defined as the smallest genus among that of all possible Seifert surfaces of k; recall that a Seifert surface of k is an embedded compact connected surfaces in S^3 whose boundary is k. In other words, if k is not the unknot, the smallest possible complexity of a Seifert surface is $2g(k) - 1 > 0$.

In 3-dimensional topology, a suitable generalization of this notion for any orientable compact 3-manifold M is the Thurston norm. It was introduced by William Thurston [1986]. Thurston discovered that the smallest possible complexity of properly embedded surface representatives for elements of $H_2(M, \partial M; \mathbb{Z})$ can be linearly continuously extended over $H_2(M, \partial M; \mathbb{R})$ to be a seminorm. It is actually a norm in certain cases, for example, if M is hyperbolic of finite volume. Thurston then asked if this notion coincides with the one defined similarly using properly immersed surfaces, which was later known as the singular Thurston norm. The question was answered affirmatively by David Gabai [1983] using his sutured manifold hierarchy. As an immediate consequence, it was made clear that there is only one notion of genus (or complexity) for classical knots, whether we consider connected or disconnected, properly immersed or embedded Seifert surfaces.

Generally speaking, the genus of a knot is quite accessible. For a (p, q)-torus knot, where p, q are coprime positive integers, the genus is well known to be $(p-1)(q-1)/2$. For a satellite knot, the Schubert inequality yields a lower bound $(\hat{g}_{\mathrm{p}} + |w| \cdot g_{\mathrm{c}})$ of the genus in terms of the genus g_{c} of the companion knot, the genus $\hat{g}_{\mathrm{p}}$ of the desatellite knot, and the winding number w of the pattern [Schubert 1953]. Furthermore, the genus of a knot is known to be algorithmically decidable [Schubert 1961]. In fact, certifying an upper bound is NP-complete [Agol et al. 2006]. The genus can also be bounded and detected in terms of other more powerful algebraic invariants, such as the knot Floer homology [Ozsváth and Szabó 2004] and twisted Alexander polynomials [Friedl and Vidussi 2012].

2B. ***Knotting and marking.*** One of the classical problems in topology is the knotting problem, namely, "Are two embeddings of a given space into n-space isotopic?" Usually, the given space is a connected closed m-manifold M where $m < n$, the embedding is locally flat, and the question can be made precise most naturally in the piecewise-linear or the smooth category. When the codimension is high enough, for example, if $n = 2m+1$ and $m > 1$, all embeddings are isotopic to one another so they "unknot" in this sense [Wu 1958]. However, below the stable range, the knotting problem becomes very interesting, as we have already seen in the classical knot case.

Regarding an embedding of M^m into $\mathbb{R}^n$ as a marking of its image, the knotting problem may be phrased to identify or distinguish knotting types (that is, isotopy classes) of marked submanifolds. Somewhat more naturally, one can ask if two unmarked knotted submanifolds are isotopic to each other, or precisely, if two embeddings are isotopic up to precomposing with an automorphism of M in the given category. Suppose we have already solved the knotting problem. Then, the latter question amounts to asking whether two markings differ only by an extendable automorphism; see [Ding et al. 2012, Lemma 2.5]. Therefore, marking does not make a difference if M has a trivial mapping class group in the category, for example, in the cases of classical knots and 2-knots, but it does in general if the extendable subgroup is a proper subgroup of the mapping class group; see [Ding et al. 2012; Hirose 1993; 2002; Montesinos 1983].

We refer the reader to the survey [Skopenkov 2008] for the embedding problem and the knotting problem in general dimensions.

2C. ***Knotted surfaces.*** The study of knotted surfaces can considered to be the mid-dimensional knot theory. In this transitional zone between the low-dimensional case and the high-dimensional (2-codimensional) case, we find geometric-topological and algebraic-topological methods to have an interesting interaction. For extensive references on this topic, see the books [Kawauchi 1996; Hillman 1989; Carter and Saito 1998; Carter et al. 2004; Kamada 2002].

With an auxiliary choice of marking, let us write a knotted surface as a locally flat embedding $K : F \hookrightarrow \mathbb{R}^4$, where F is a closed surface. We can visualize a knotted surface by drawing a diagram obtained via a generic projection of K onto a 3-subspace, or by displaying a motion picture of links in $\mathbb{R}^3$, obtained via a generic line projection that is Morse when restricted to K; see [Carter and Saito 1998; Kawauchi et al. 1982]. The fundamental group of the exterior is called the knot group of K, denoted as π_K. Similar to the classical case, π_K has a Wirtinger-type presentation in terms of its diagram [Yajima 1962], and π_K can be isomorphically characterized by having an Artin-type presentation, described in terms of 2-dimensional braids [Kamada 2002].

Exteriors of knotted surfaces form an interesting family of 4-manifolds. The fundamental group of any such manifold is nontrivial, and it contains much information about the topology. For instance, it has been suspected for orientable knotted surfaces that having an infinite cyclic knot group implies unknotting, namely, that K bounds an embedded handlebody [Hosokawa and Kawauchi 1979]. By deep methods of 4-manifold topology, this has been confirmed for knotted spheres in the topological category [Freedman and Quinn 1990, Theorem 11.7A]. In earlier studies of knotted surfaces, researchers frequently looked for examples with prescribed properties of the knot group, such as required deficiency [Fox 1962; Levine 1978; Kanenobu 1983], or required second homology [Brunner et al. 1982; Gordon 1981; Litherland 1981; Maeda 1977]. In some other constructions of particular topological significance, combinatorial group theory again plays an important role in verification [Gordon 1976; Kamada 1990; Livingston 1985; 1988].

Many of these constructions implement satellite knotting on various stages. The idea of such an operation is to replace a so-called companion knotted surface with another one that is embedded in the regular neighborhood of the former, often in a more complicated pattern. Basic examples of satellite knotting include the knot connected sum of knotted surfaces, and Artin's spinning construction [1925], as well as its twisted generalizations [Zeeman 1965; Litherland 1979]. Generally speaking, satellite knotting would lead to an increase of genus under certain natural assumptions such as nonzero winding number. However, this can be avoided if we are just concerned with knotted spheres or tori (see Section 4B). Like in the classical case, satellite knotting only changes the knot group by a van Kampen-type amalgamation. Therefore, it is usually an approach worth considering if one wishes to maintain some control on the group level during the construction. As far as we are concerned, the first explicit formulation of the satellite construction of n-knots in literature was due to Yaichi Shinohara [1971] in his paper about generalized Alexander polynomials and signatures; the satellite construction of knotted tori in $\mathbb{R}^4$ first appeared in Richard Litherland's paper [1981], where he studied the second homology of the knot group.

3. Genera of slopes

In this section, we introduce the genus and the singular genus for any slope of a knotted torus K in S^4. We provide criteria about finiteness associated to the extendable subgroup $\mathscr{E}_K$ and the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ of $\mathrm{Mod}(T^2)$ in terms of these notions.

3A. ***Genus and singular genus.*** Let $K : T^2 \hookrightarrow S^4$ be a knotted torus in S^4, that is, a locally flat embedding of the torus into the 4-sphere. Let $X_K = S^4 - K$ be the exterior of K obtained by removing an open regular neighborhood of K.

Lemma 3.1. *Let F^2_g be the closed orientable surface of genus g, and Y be a simply connected closed* 4*-manifold. Suppose $K : F^2_g \hookrightarrow Y$ is a null-homologous, locally flat embedding. Write $X = Y - K$ for the exterior of K in Y. Then ∂X is canonically homeomorphic to $F^2_g \times S^1$, up to isotopy, such that the homomorphism $H_1(F^2_g) \to H_1(X)$ induced by including F^2_g as the first factor $F^2_g \times \mathrm{pt}$ is trivial. In* *particular, every essential simple closed curve $c \subset F_g^2$ bounds a locally flat, properly embedded, orientable compact surface $S \hookrightarrow X_K$ with ∂S embedded as $c \times \mathrm{pt}$.*

Proof. This is well-known, following from an easy homological argument. In fact, since K is null-homologous, the normal bundle of K in Y is trivial, so ∂X has a natural circle bundle structure $p : \partial X \to F_g^2$ over F_g^2, which splits. The splitting is given by framings of the normal bundle, which are in natural bijection with all the homomorphisms $\iota : H_1(F_g^2) \to H_1(\partial X)$ such that $p_* \circ \iota : H_1(F_g^2) \to H_1(F_g^2)$ is the identity. Using Poincaré duality and excision, it is easy to see $H^1(X) \cong \mathbb{Z}$ and $H^1(X, \partial X) = 0$. Thus the homomorphism $H^1(X) \to H^1(\partial X)$ is injective, and the generator of $H_1(X)$ induces a homomorphism $\alpha : H_1(\partial X) \to \mathbb{Z}$. It is straightforward to check that α sends the circle-fiber of ∂X to ± 1, so the kernel of α projects isomorphically onto $H_1(F_g^2)$ via p_*. This gives rise to the canonical splitting $\partial X = F_g^2 \times S^1$. It follows clearly from the construction that $H_1(F_g^2) \to H_1(X)$ is trivial. Moreover, if $c \times \mathrm{pt}$ is an essential simple closed curve on $K \times \mathrm{pt}$, it is homologically trivial in X, so it represents an element $[a_1, b_1] \cdots [a_k, b_k]$ in the commutator subgroup of $\pi_1(X)$. We take a compact orientable surface S' of genus k with exactly one boundary component, and there is a map $j : S' \to X$ sending $\partial S'$ homeomorphically onto $c \times \mathrm{pt}$. By a general position argument we may assume j to be a locally flat proper immersion, and doing surgeries at double points yields a locally flat, properly embedded, orientable compact surface $S \hookrightarrow X$ bounded by $c \times \mathrm{pt}$. □

This allows us to make the following definition:

Definition 3.2. Let $K : T^2 \hookrightarrow S^4$ be a knotted torus. For any slope, that is, an essential simple closed curve, $c \subset K$, the *genus* $g_K(c)$ of c is defined to be the minimum of the genus of F, as F runs over all the locally flat, properly embedded, orientable, compact subsurfaces of X_K bounded by $c \times \mathrm{pt} \subset \partial X_K$; see Lemma 3.1. The *singular genus* $g_K^\star(c)$ of c is defined to be the minimum of the genus of F, as F runs over all the compact orientable surfaces with connected nonempty boundary such that there is a continuous map $F \to X_K$ sending ∂F homeomorphically onto $c \times \mathrm{pt}$.

Remark 3.3. Recall that for a group G and any element u in the commutator subgroup $[G, G]$, the *commutator length* $\mathrm{cl}(u)$ of u is the smallest possible integer $k \geq 0$ such that u can be written as a product of commutators $[a_1, b_1] \cdots [a_k, b_k]$, where $a_i, b_i \in G$, and $i = 1, \ldots, k$. Note that elements of $[G, G]$ that are conjugate in G have the same commutator length. As indicated in the proof of Lemma 3.1, it is clear that the singular genus $g_K^\star(c)$ is the commutator length $\mathrm{cl}(c)$, regarding c as an element of the commutator subgroup of $\pi_1(X_K)$.

3B. *Extendable subgroup and stable extendable subgroup.* Let $\mathrm{Mod}(T^2)$ be the mapping class group of the torus, which consists of the isotopy classes of orientation-preserving self-homeomorphisms of T^2. Fixing a basis of $H_1(T^2)$, one can naturally identify $\mathrm{Mod}(T^2)$ as $\mathrm{SL}(2, \mathbb{Z})$. We often refer to the elements of $\mathrm{Mod}(T^2)$ as *automorphisms* of T^2, and do not distinguish elements of $\mathrm{Mod}(T^2)$ and their representatives.

For any knotted torus $K : T^2 \hookrightarrow S^4$, an automorphism $\tau \in \mathrm{Mod}(T^2)$ is said to be *extendable* with respect to K if τ can be extended as an orientation-preserving self-homeomorphism of S^4 via K. Note that this notion does not depend on the choice of the representative of τ; see [Ding et al. 2012, Lemma 2.4]. It is also clear that all the extendable automorphisms form a subgroup of $\mathrm{Mod}(T^2)$.

Definition 3.4. For a knotted torus $K : T^2 \hookrightarrow S^4$, the *extendable subgroup* with respect to K is the subgroup of $\mathrm{Mod}(T^2)$ consisting of all the extendable automorphisms, denoted as $\mathscr{E}_K \leq \mathrm{Mod}(T^2)$.

The extendable subgroup $\mathscr{E}_K$ reflects some essential differences between knotted tori and knotted spheres (that is, 2-knots) in S^4. For instance, it is known that $\mathscr{E}_K$ is always a proper subgroup of $\mathrm{Mod}(T^2)$, of index at least three [Ding et al. 2012]; see [Montesinos 1983] for the diffeomorphism extension case. Moreover, index three is realized by any unknotted embedding, namely, one which bounds an embedded solid torus $S^1 \times D^2$ in S^4 [Montesinos 1983]; see [Hirose 2002] for the general case of trivially embedded surfaces. In [Hirose 1993], $\mathscr{E}_K$ has been computed for the so-called spun T^2-knots and twisted spun T^2-knots. It is also clear that taking the connected sum with a knotted sphere in S^4 does not change the extendable subgroup. However, for a general knotted torus in S^4, the extendable subgroup $\mathscr{E}_K$ is poorly understood. In the following, we introduce a weaker notion called the stable extendable subgroup. From our point of view, the stable extendable subgroup is more closely related to the singular genera than the extendable subgroup is; see Section 6B.

Suppose $K : T^2 \hookrightarrow S^4$ is a knotted torus in S^4, and Y is a closed simply connected 4-manifold. There is a naturally induced embedding $K[Y] : T^2 \hookrightarrow Y$ obtained by regarding Y as the connected sum $S^4 \# Y$ and embedding T^2 into the first summand via K. This is well defined up to isotopy, and we call $K[Y]$ the *Y-stabilization* of K. An automorphism $\tau \in \mathrm{Mod}(T^2)$ is said to be *Y-stably extendable* if τ extends over Y as an orientation-preserving self-homeomorphism via $K[Y]$. All such automorphisms clearly form a subgroup of $\mathrm{Mod}(T^2)$. An automorphism $\tau \in \mathrm{Mod}(T^2)$ is said to be *stably extendable* if τ is Y-stably extendable for some closed simply connected 4-manifold Y. Note that if τ_1 is Y_1-stably extendable and τ_2 is Y_2-stably extendable, they are both $(Y_1 \# Y_2)$-stably extendable. This means stably extendable automorphisms also form a subgroup of $\mathrm{Mod}(T^2)$.

Definition 3.5. For a knotted torus $K : T^2 \hookrightarrow S^4$, the *stable extendable subgroup* with respect to K is the subgroup of $\mathrm{Mod}(T^2)$ consisting of all the stably extendable automorphisms, denoted as $\mathscr{E}^{\mathrm{s}}_K \leq \mathrm{Mod}(T^2)$.

Proposition 3.6. *Let $K : T^2 \hookrightarrow S^4$ be a knotted torus.*

(1) *If the singular genus $g^\star_K(c)$ takes infinitely many distinct values as c runs over all the slopes of K, then the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ is of infinite index in* $\mathrm{Mod}(T^2)$.

(2) *If there are at most finitely many distinct slopes $c \subset K$ with the singular genus $g^\star_K(c)$ at most C for every $C > 0$, then the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ is finite.*

Remark 3.7. Hence the same holds for the extendable subgroup $\mathscr{E}_K$. Using a similar argument, one can also show that the statements remain true when replacing $g^\star_K$ with g_K, and $\mathscr{E}^{\mathrm{s}}_K$ with $\mathscr{E}_K$.

Proof. First observe that the singular genus of a slope is invariant under the action of a stably extendable automorphism, namely, if $\tau \in \mathscr{E}^{\mathrm{s}}_K$, then $g^\star_K(c) = g^\star_K(\tau(c))$ for every slope $c \subset K$. This is clear because by the definition, τ extends over $X'_K = X_K \# Y$ as a homeomorphism $\tilde{\tau} : X'_K \to X'_K$ for some simply connected closed 4-manifold Y. This induces an automorphism of $\pi_1(X'_K) \cong \pi_1(X_K)$, which preserves the commutator length of c, or equivalently, the singular genus $g^\star_K(c)$ (Remark 3.3).

To see (1), note that $\mathrm{Mod}(T^2)$ acts transitively on the space $\mathscr{C}$ of all the slopes on T^2. It follows immediately from the invariance of singular genera above that the cardinality of value set of $g^\star_K$ is at most the index $[\mathrm{Mod}(T^2) : \mathscr{E}^{\mathrm{s}}_K]$. Thus if the range of $g^\star_K$ is infinite, the index of $\mathscr{E}^{\mathrm{s}}_K$ in $\mathrm{Mod}(T^2)$ is also infinite.

To see (2), suppose $\tau \in \mathscr{E}^{\mathrm{s}}_K$. By the assumption and the invariance of the singular genus under τ, for any slope $c \subset K$ there are at most finitely many distinct slopes in the sequence $c, \tau(c), \tau^2(c), \ldots$. Thus for some integers $k > l \geq 0$, $\tau^k(c)$ is isotopic to $\tau^l(c)$, or in other words, $\tau^d(c)$ is isotopic to c, where $d = k - l$. As c is arbitrary, τ is a torsion element in $\mathrm{Mod}(T^2)$, so $\mathscr{E}^{\mathrm{s}}_K$ is a subgroup of $\mathrm{Mod}(T^2)$ consisting purely of torsion elements. It follows immediately that $\mathscr{E}^{\mathrm{s}}_K$ is a finite subgroup from the well-known fact that $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z})$ is virtually torsion-free. Indeed, the index of any finite-index torsion-free normal subgroup of $\mathrm{Mod}(T^2)$ yields an upper bound of the size of $\mathscr{E}^{\mathrm{s}}_K$. □

4. Induced seminorms on $H_1(T^2; \mathbb{R})$

In this section, we introduce the seminorm $\|\cdot\|_K$ on $H_1(T^2; \mathbb{R})$ induced from any knotted torus $K : T^2 \hookrightarrow S^4$. This may be regarded as a generalization of the (singular) Thurston norm in 3-dimensional topology. We prove a Schubert-type inequality in terms of seminorms associated with satellite constructions.

4A. ***The induced seminorm.*** There are various ways to formulate the induced seminorm, among which we shall take a more topological one. Suppose $K : T^2 \hookrightarrow S^4$

is a knotted torus in S^4. We shall first define the value of $\|\cdot\|_K$ on $H_1(T^2;\mathbb{Z})$ then extend linearly and continuously over $H_1(K;\mathbb{R})$.

Recall that for a connected orientable compact surface F, the complexity of F is defined as $\chi_-(F) = \max\{-\chi(F), 0\}$. In general, for an orientable compact surface $F = F_1 \sqcup \cdots \sqcup F_s$, the *complexity* of F is defined as

$$x(F) = \sum_{i=1}^{s} \chi_-(F_i).$$

For any $\gamma \in H_1(T^2)$, identified as an element of $H_1(\partial X_K)$, there exists a smooth immersion of pairs $(F, \partial F) \looparrowright (X_K, \partial X_K)$ such that F is a (possibly disconnected) oriented compact surface, and that ∂F represents γ. We define the *complexity* of γ as

$$x(\gamma) = \min_F x(F),$$

where F runs through all the possible immersed surfaces as described above.

The fact below follows immediately from the definition.

Lemma 4.1. *With the notation above,*

(1) $x(n\gamma) \leq n x(\gamma)$ *for any* $\gamma \in H_1(T^2)$ *and any integer* $n \geq 0$.

(2) $x(\gamma' + \gamma'') \leq x(\gamma') + x(\gamma'')$ *for any* $\gamma', \gamma'' \in H_1(T^2)$.

Definition 4.2. Let $K : T^2 \hookrightarrow S^2$ be a knotted torus. For any $\gamma \in H_1(T^2)$, we define

$$\|\gamma\|_K = \inf_{m \in \mathbb{Z}_+} \frac{x(m\gamma)}{m}.$$

Lemma 4.3. (1) $\|n\gamma\|_K = n\|\gamma\|_K$ *for any* $\gamma \in H_1(T^2)$ *and any integer* $n \geq 0$.

(2) $\|\gamma' + \gamma''\|_K \leq \|\gamma'\|_K + \|\gamma''\|_K$ *for any* $\gamma', \gamma'' \in H_1(T^2)$.

Proof. This follows from Lemma 4.1 and some elementary arguments. For any $\epsilon > 0$, there is some $m > 0$ such that $\|\gamma\|_K > (x(m\gamma)/m) - \epsilon$, and by Lemma 4.1,

$$\frac{x(m\gamma)}{m} - \epsilon \geq \frac{x(nm\gamma)}{nm} - \epsilon \geq \frac{\|n\gamma\|_K}{n} - \epsilon.$$

Letting $\epsilon \to 0$, we see $\|\gamma\|_K \geq \|n\gamma\|_K/n$. Moreover, for any $\epsilon > 0$, there exists $m > 0$ such that $\|n\gamma\|_K > (x(mn\gamma)/m) - \epsilon \geq n\|\gamma\|_K - \epsilon$. Letting $\epsilon \to 0$, we see $\|n\gamma\|_K \geq n\|\gamma\|_K$. This proves the first statement. To prove the second statement, for any $\epsilon > 0$, there are $m', m'' > 0$ such that $\|\gamma'\|_K > (x(m'\gamma')/m') - \epsilon$ and $\|\gamma''\|_K > (x(m''\gamma'')/m'') - \epsilon$, so using Lemma 4.1,

$$\begin{aligned}\|\gamma'\|_K + \|\gamma''\|_K &> \frac{x(m'\gamma')}{m'} + \frac{x(m''\gamma'')}{m''} - 2\epsilon \geq \frac{x(m'm''\gamma')}{m'm''} + \frac{x(m'm''\gamma'')}{m'm''} - 2\epsilon \\ &\geq \frac{x(m'm''(\gamma'+\gamma''))}{m'm''} - 2\epsilon \geq \|\gamma' + \gamma''\|_K - 2\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$, we see the second statement. □

By Lemma 4.3, we can extend $\|\cdot\|_K$ radially over $H_1(T^2; \mathbb{Q})$, then extend continuously over $H_1(T^2; \mathbb{R})$. This uniquely defines a seminorm

$$\|\cdot\|_K : H_1(T^2; \mathbb{R}) \to [0, +\infty).$$

Recall a seminorm on a real vector space V is a function $\|\cdot\| : V \to [0, +\infty)$ such that $\|rv\| = |r|\,\|v\|$ for any $r \in \mathbb{R}$, $v \in V$, and that $\|v' + v''\| \le \|v'\| + \|v''\|$ for any $v', v'' \in V$. It is a norm if it is in addition positive-definite, namely $\|v\| = 0$ if and only if $v \in V$ is zero.

Definition 4.4. Let $K : T^2 \hookrightarrow S^4$ be a knotted torus, and $c \subset T^2$ be a slope. Then the seminorm $\|c\|_K$ is defined as $\|[c]\|_K$, where $[c] \in H_1(T^2)$.

Remark 4.5. Recall that for a group G and any element u in the commutator subgroup $[G, G]$, the *stable commutator length* is

$$\mathrm{scl}(u) = \lim_{n\to+\infty} \frac{\mathrm{cl}(u^n)}{n},$$

where $\mathrm{cl}(\cdot)$ denotes the commutator length (Remark 3.3). It is not hard to see that for any slope $c \subset K$, the seminorm $\|c\|_K$ equals $\mathrm{scl}(c)$, regarding c as an element of the commutator subgroup of $\pi_1(X_K)$; see [Calegari 2009, Proposition 2.10].

The lemma below follows immediately from the definition and Proposition 3.6:

Lemma 4.6. *If $c \subset K$ is a slope with $\|c\|_K > 0$, then $g^\star_K(c) \ge (\|c\|_K + 1)/2$. Hence the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ is finite if $\|\cdot\|_K$ is nondegenerate. The same holds if we replace $g^\star_K$ with g_K and $\mathscr{E}^{\mathrm{s}}_K$ with $\mathscr{E}_K$.*

4B. ***The satellite construction.*** The satellite construction for knotted tori is analogous to that of classical knots in S^3; see Section 2C for historical remarks.

Fix a product structure of $T^2 \cong S^1 \times S^1$. We shall denote the thickened torus with the standard parametrization as

$$\Theta^4 = S^1 \times S^1 \times D^2.$$

The standard unknotted torus $T_{\mathrm{std}} : T^2 \subset S^4$ is a smoothly embedded torus such that T_{std} bounds two smoothly embedded solid tori $D^2 \times S^1$ and $S^1 \times D^2$ in S^4, respective to factors. It is unique up to diffeotopy of S^4. Let $K_{\mathrm{c}} : T^2 \hookrightarrow S^4$ be a knotted torus. There is a natural trivial product structure on a compact tubular neighborhood $\mathcal{N}(K_{\mathrm{c}}) \cong T^2 \times D^2$ of K_{c}, so that $c \times *$ is homologically trivial in the complement $X_{K_{\mathrm{c}}}$ for any slope $c \subset T^2$. Thus there is a natural isomorphism $\mathcal{N}(K_{\mathrm{c}}) \cong \Theta^4$, up to isotopy, as we fixed the product structure on T^2.

Definition 4.7. A *pattern* knotted torus is a smooth embedding $K_{\mathrm{p}} : T^2 \hookrightarrow \Theta^4$. The *winding number* $w(K_{\mathrm{p}})$ of K_{p} is the algebraic intersection number of $[K_{\mathrm{p}}] \in H_2(\Theta^4)$ and the fiber disk $[\mathrm{pt} \times \mathrm{pt} \times D^2] \in H_2(\Theta^4, \partial\Theta^4)$.

Definition 4.8. Let $K_{\mathrm{c}} : T^2 \hookrightarrow S^4$ be a knotted torus and $K_{\mathrm{p}} : T^2 \hookrightarrow \Theta^4$ be a pattern knotted torus. After fixing a product structure on T^2, the *satellite* knotted torus, denoted as $K = K_{\mathrm{c}} \cdot K_{\mathrm{p}}$, is the composition

$$T^2 \xrightarrow{K_{\mathrm{p}}} \Theta^4 \xrightarrow{\cong} \mathcal{N}(K_{\mathrm{c}}) \xrightarrow{\subset} S^4.$$

We call K_{c} the *companion* knotted torus. The *desatellite* $\hat{K}_{\mathrm{p}} : T^2 \hookrightarrow S^4$ of K is the knotted torus $\hat{K}_{\mathrm{p}} = T_{\mathrm{std}} \cdot K_{\mathrm{p}}$.

For any element $\gamma \in H_1(T^2)$ and a pattern $K_{\mathrm{p}} : T^2 \hookrightarrow \Theta^4$, there is a push-forward element $\gamma_{\mathrm{c}} \in H_1(T^2)$ under the composition:

$$T^2 \xrightarrow{K_{\mathrm{p}}} \Theta^4 \xrightarrow{\cong} T^2 \times D^2 \to T^2,$$

where the isomorphism respects the choice of the product structure on T^2, and the last map is the projection onto the T^2 factor. If $K = K_{\mathrm{c}} \cdot K_{\mathrm{p}}$ is a satellite with pattern K_{p}, one should regard γ as an element of $H_1(K)$, and γ_{c} as an element of $H_1(K_{\mathrm{c}})$.

4C. ***A Schubert-type inequality.*** The theorem below is analogous to the Schubert inequality in classical knot theory [Schubert 1953, Kapitel II, §12].

Theorem 4.9. *Suppose $K = K_{\mathrm{c}} \cdot K_{\mathrm{p}}$ is a satellite knotted torus in S^4. Then for any $\gamma \in H_1(T^2; \mathbb{R})$, $\|\gamma\|_K \geq \|\gamma\|_{\hat{K}_{\mathrm{p}}}$. Moreover, if the winding number $w(K_{\mathrm{p}})$ is nonzero, then $\|\gamma\|_K \geq \|\gamma\|_{\hat{K}_{\mathrm{p}}} + \|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}}$.*

We prove Theorem 4.9 in the rest of this subsection.

Let X_K be the complement of the satellite knot $K = K_{\mathrm{c}} \cdot K_{\mathrm{p}}$ in S^4. The satellite construction gives a decomposition $X_K = Y \cup X_{K_{\mathrm{c}}}$, glued along the image of $\partial\Theta^4$. Y is diffeomorphic to the complement of K_{p} in Θ^4, so it has two boundary components, namely the *satellite boundary* $\partial_{\mathrm{s}} Y$, which is ∂X_K, and the *companion boundary* $\partial_{\mathrm{c}} Y$ which is the image of $\partial\Theta^4$.

Similarly, the complement $X_{\hat{K}_{\mathrm{p}}}$ can be decomposed as $Y \cup X_{T_{\mathrm{std}}}$.

The first inequality is proved in the following lemma:

Lemma 4.10. $\|\gamma\|_K \geq \|\gamma\|_{\hat{K}_{\mathrm{p}}}$.

Proof. We equip $X_{K_{\mathrm{c}}}$ with a finite CW complex structure such that there is only one 0-cell and the 0-cell is contained in $\partial X_{K_{\mathrm{c}}}$, which is a subcomplex of $X_{K_{\mathrm{c}}}$. Let $X_{K_{\mathrm{c}}}^{(q)}$ be the union of $\partial X_{K_{\mathrm{c}}}$ and the q-skeleton of $X_{K_{\mathrm{c}}}$. We may extend the identity map on Y to a continuous map $f : Y \cup X_{K_{\mathrm{c}}}^{(2)} \to X_{\hat{K}_{\mathrm{p}}}$. To see this, note that the inclusion map $\partial X_K \to X_K$ induces a surjective map on H_1 for any $K : T^2 \to S^4$, so the identity map on ∂X_{K_c} induces a natural isomorphism $H_1(X_{K_c}) \cong H_1(X_{T_{std}})$. Since every 1-cell in X_{K_c} represents a 1-cycle, we can extend $\mathrm{id}_{\partial_c Y}$ to a map $f| : X^{(1)}_{K_c} \to X_{T_{std}}$, so that the induced map $H_1(X^{(1)}_{K_c}) \to H_1(X_{T_{std}})$ agrees with the map on the first homology induced by $X^{(1)}_{K_c} \hookrightarrow X_{K_c}$. It is easy to see $X_{T_{std}} \simeq S^1 \vee S^2 \vee S^2$, so $\pi_1(X_{T_{std}}) \cong \mathbb{Z}$. Hence the previous $f|$ can be further extended as $f| : X^{(2)}_{K_c} \to X_{T_{std}}$ since the boundary of any 2-cell is mapped to a null-homotopic loop in $X_{T_{std}}$ by the construction.

Thus we obtain a map $f : Y \cup X^{(2)}_{K_c} \to X_{\hat{K}_p}$ by the map above and the identity on Y. Let $j : F \looparrowright X_K$ be an immersed compact orientable surface such that $j(\partial F) \subset \partial X_K$. We may assume F meets $\partial_c Y$ transversely. We homotope j to $j' : F \to Y \cup X^{(2)}_{K_c}$. Then we obtain a map $f \circ j' : F \to X_{\hat{K}_p}$ which may be homotoped to an immersion. As F is arbitrary, this implies $\|\gamma\|_K \geq \|\gamma\|_{\hat{K}_p}$ by the definition of the seminorm. □

Now we consider the case when $w(K_p) \neq 0$. The image of $\mathrm{pt} \times \mathrm{pt} \times \partial D^2 \subset Y$ under the natural inclusion $Y \subset X_K$ will be denoted μ_c. We call μ_c the *companion meridian*. The following lemma follows immediately from the construction:

Lemma 4.11. *Identify $H_1(X_{K_c}) \cong \mathbb{Z}$ and $H_1(X_K) \cong \mathbb{Z}$. Then $H_1(X_{K_c}) \to H_1(X_K)$ is multiplication by $w(K_p)$.*

Proof. Note μ_c represents a generator of $H_1(X_{K_c})$. By definition of $w(K_p)$, μ_c is homologous to $w(K_p)$ times the meridian of K. The lemma follows as the meridian of K generates $H_1(X_K) \cong \mathbb{Z}$ by Alexander duality. □

Lemma 4.12. *If $w(K_p) \neq 0$, then the inclusion map $\partial_c Y \subset Y$ induces an injective homomorphism $H_1(\partial_c Y) \to H_1(Y)$. In particular, the inclusion map $\partial_c Y \subset Y$ is π_1-injective.*

Proof. By the long exact sequence

$$\cdots \to H_2(Y, \partial_c Y) \to H_1(\partial_c Y) \to H_1(Y) \to \cdots,$$

it suffices to show $H_2(Y, \partial_c Y)$ is finite, since $H_1(\partial_c Y) \cong H_1(\partial \Theta^4)$ is torsion-free. By the Poincaré–Lefschetz duality and excision,

$$H_2(Y, \partial_c Y) \cong H^2(Y, \partial_s Y) \cong H^2(\Theta^4, K_p).$$

The long exact sequence

$$\cdots \to H^1(\Theta^4) \to H^1(K_p) \to H^2(\Theta^4, K_p) \to H^2(\Theta^4) \to H^2(K_p) \to \cdots$$

is induced by the inclusion $K_p \subset \Theta^4$, (or equivalently by $K_p : T^2 \hookrightarrow \Theta^4$). Since $\Theta^4 \simeq T^2$, K_p induces a map $h : T^2 \to T^2$. It is also clear that $w(K_p)$ is the degree of h. Since $w(K_p) \neq 0$, it is clear that the map $h^* : H^*(T^2) \to H^*(T^2)$ is injective on all dimensions, so must be $H^*(\Theta^4) \to H^*(K_p)$. Thus $H^2(\Theta^4, K_p)$ is finite from the long exact sequence. We conclude $H_2(Y, \partial_c Y)$ is finite as desired. □

Note it suffices to prove Theorem 4.9 for $\gamma \in H_1(T^2;\mathbb{Z})$. Remember that we regard γ as in $H_1(K)$, identified as the kernel of $H_1(\partial X_K) \to H_1(X_K)$. For any $\epsilon > 0$, let $j : F \looparrowright X_K$ be a properly immersed orientable compact (possibly disconnected) surface, that is, $j^{-1}(\partial X_K) = \partial F$, such that $j_*[\partial F] = m\,\gamma$ for some integer $m > 0$, and that

$$\|\gamma\|_K \leq \frac{x(F)}{m} < \|\gamma\|_K + \epsilon.$$

We may assume F has no disk or closed component, so $x(F) = -\chi(F)$. We may also assume F intersects $\partial_c Y$ transversely, so $j^{-1}(\partial_c Y)$ is a disjoint union of simple closed curves on F. Write F_p, F_c for $j^{-1}(Y)$, $j^{-1}(X_{K_c})$, respectively.

Lemma 4.13. *Suppose $w(K_p) \neq 0$. If V is a component of F_p where $j(\partial V) \subset \partial_c Y$, then there is a map $j'| : V \to \partial_c Y$, such that $j'|_{\partial V} = j$.*

Proof. We may take a collection of embedded arcs $u_1, \ldots, u_n$ whose endpoints lie on ∂V, cutting V into a disk D. This gives a cellular decomposition of V. We may first extend the map $j|_{\partial V} : \partial V \to \partial_c Y$ to a map $j'|_{V^{(1)}}$ over the 1-skeleton of V. Let $\phi : \partial D \to V^{(1)}$ be the attaching map. We have $j'_*\phi_*[\partial D] = j_*[\partial V]$ in $H_1(\partial_c Y)$ by the construction. As $w(K_p) \neq 0$, by Lemma 4.12, $H_1(\partial_c Y) \to H_1(Y)$ is an injective homomorphism, so $j_*[\partial V] = 0$ in $H_1(\partial_c Y)$ since it is bounded by $j_*[V]$. Thus $j'_*\phi_*[\partial D] = 0$ in $H_1(\partial_c Y)$, and hence ∂D is null-homotopic in $\partial_c Y$ under $j' \circ \phi$ as $\pi_1(\partial_c Y) \cong H_1(\partial_c Y)$, (remember $\partial_c Y \cong \partial \Theta^4$ is a 3-torus). Therefore, we may extend $j'|_{V^{(1)}}$ further over D to obtain $j' : V \to \partial_c Y$ as desired. □

Lemma 4.14. *We may modify $j : F \looparrowright X_K$ within the interior of F so that every component of $j^{-1}(\partial_c Y)$ that is inessential on F bounds a disk component of $j^{-1}(X_{K_c})$.*

Proof. Let $a \subset j^{-1}(\partial_c Y)$ be a component inessential on F, and $D \subset F$ be an embedded disk whose boundary is a. If D is not contained in F_c, then $D \cap F_p \neq \varnothing$. Any component of $D \cap F_p$ must have all its boundary components lying on $j^{-1}(\partial_c Y)$. By Lemma 4.13, we may redefine j on these components relative to boundary so that they are all mapped into X_c. After this modification and a small perturbation, either a disappears from $j^{-1}(\partial_c Y)$ (if $\partial D \subset D \cap F_p$), or at least one component of $j^{-1}(\partial_c Y)$ in the interior of D disappears (if $\partial D \subset D \cap F_c$). Thus the number of inessential components of $j^{-1}(\partial_c Y)$ decreases strictly under this modification. Therefore, after at most finitely many such modifications, every inessential component of $j^{-1}(\partial_c Y)$ bounds a disk component of F_c. □

Without loss of generality, we assume that $j : F \looparrowright X_K$ satisfies the conclusion of Lemma 4.14.

Lemma 4.15. *There is a finite cyclic covering $\kappa : \tilde{F} \to F$ such that for every essential component $a \in j^{-1}(\partial_c Y)$ with $[j(a)] \neq 0$ in $H_1(X_K)$, and every component $\tilde{a}$ of $\kappa^{-1}(a)$, the image $j(\kappa(\tilde{a}))$ represents the same element in $H_1(X_K) \cong \mathbb{Z}$ up to sign.*

Proof. Let $a_1, \dots, a_s$ be all the essential components $j^{-1}(\partial_c Y)$ such that $[j(a_i)] \neq 0$ in $H_1(X_K) \cong \mathbb{Z}$. Let $d > 0$ be the least common multiple of all the $[j(a_i)]$. Consider the covering $\kappa : \tilde{F} \to F$ corresponding to the preimage of the subgroup $d \cdot H_1(X_K)$ under $\pi_1(F) \to \pi_1(X_K) \to H_1(X_K)$. It is straightforward to check that κ satisfies the conclusion. □

Let $\kappa : \tilde{F} \to F$ be a covering as obtained in Lemma 4.15. Let $d > 0$ be the degree of κ, so $x(\tilde{F}) = d\,x(F)$. Clearly $j_* \kappa_* [\partial \tilde{F}] = md\,\gamma$, and also

$$\|\gamma\|_K \leq \frac{x(\tilde{F})}{md} < \|\gamma\|_K + \epsilon.$$

Moreover, as any inessential component of $j^{-1}(\partial_c Y)$ bounds a disk component of F_c, it is clear that any inessential component of $(j \circ \kappa)^{-1}(\partial_c Y)$ bounds a disk component of $\tilde{F}_c = \kappa^{-1}(F_c)$.

Therefore, instead of using $j : F \looparrowright X_K$, we may use $j \circ \kappa : \tilde{F} \looparrowright X_K$ as well. From now on, we rewrite $j \circ \kappa$ as j, $\tilde{F}$ as F, and md as m, so $j : F \looparrowright X_K$ satisfies the conclusions of Lemmas 4.14, 4.15.

Let $Q \subset F_c$ be the union of the disk components of F_c. Let F'_c be $F_c - Q$, and F'_p be $F_p \cup Q$ (glued up along adjacent boundary components). We have the decompositions

$$F = F_p \cup F_c = F'_p \cup F'_c.$$

Moreover, there is no inessential component of $\partial F'_c$ by our assumption on F, so F'_c and F'_p are essential subsurfaces of F (that is, whose boundary components are essential).

Lemma 4.16. *Suppose F is a compact orientable surface with no disk or sphere component, and E_1, E_2 are essential compact subsurfaces of F with disjoint interiors such that $F = E_1 \cup E_2$. Then $x(F) = x(E_1) + x(E_2)$.*

Proof. Note $\chi(F) = \chi(E_1) + \chi(E_2)$. As each E_i is essential, there is no disk component of E_i, and by the assumption there is no sphere component, either. Thus, for each component C of E_i, $x(C) = -\chi(C)$. We have $x(F) = x(E_1) + x(E_2)$. □

The desatellite term in Theorem 4.9 comes from the following construction.

Lemma 4.17. *Under the assumptions above, there is a properly immersed compact orientable surface $\hat{j} : \hat{F}'_p \looparrowright X_{\hat{K}_p}$ such that $x(\hat{F}'_p) \leq x(F'_p)$, and that $\hat{j}_*[\partial \hat{F}'_p] = m\gamma$ in $H_1(T^2)$.*

Proof. As F has been assumed to satisfy the conclusion of Lemma 4.15, there is an $\omega \in H_1(X_K)$ such that every component of $\partial_c F'_p$ (that is, $F'_p \cap j^{-1}(\partial_c Y)$) represents either $\pm\omega$ or 0, and the algebraic sum over all the components is zero since they bound $j(F'_c) \subset X_K$. Thus we may assume there are s components representing 0, t components representing ω, and t components representing $-\omega$, where $s, t \geq 0$. We construct $\hat{F}'_p$ by attaching s disks and t annuli to $\partial_c F'_p$, such that each disk is attached to a component representing 0, and each annulus is attached to a pair of components representing opposite $\pm\omega$-classes. Let $\mathscr{D}$ be the union of attached disks, and $\mathscr{A}$ be the union of attached annuli. The result is a compact orientable surface $\hat{F}'_p = F'_p \cup \mathscr{D} \cup \mathscr{A}$ such that $\partial \hat{F}'_p \cong \partial F$. It is clear that $x(\hat{F}'_p) \leq x(F'_p \cup \mathscr{A}) = x(F'_p)$, (see Lemma 4.16).

To construct $\hat{j}$, we extend the map

$$j| : F_p \to Y \subset X_{\hat{K}_p} = Y \cup X_{T_{\mathrm{std}}}$$

over $\hat{F}_p = F_p \cup Q \cup \mathscr{D} \cup \mathscr{A}$, using the fact that $\pi_1(X_{T_{\mathrm{std}}}) \cong H_1(X_{T_{\mathrm{std}}}) \cong \mathbb{Z}$. Specifically, to extend the map over Q, let s be a component of $\partial_c F_p$ bounding a disk component of Q. Then $j_*[s] = 0$ in $H_1(X_K)$. Hence it lies in the subgroup $H_1(T^2 \times \mathrm{pt})$ of $H_1(\partial\Theta^4) \cong H_1(\partial_c Y)$, and by the desatellite construction, $\hat{j}(s)$ should also be null-homologous in $X_{T_{\mathrm{std}}}$. We can extend $\hat{j}$ over the disk $D \subset Q$ bounded by s. After extending for every component of Q, we obtain

$$\hat{j}| : F_p \cup Q \to X_{\hat{K}_p}.$$

Similarly, we may extend $\hat{j}|$ over $\mathscr{D}$. To extend over $\mathscr{A}$, let $A \subset \mathscr{A}$ be an attached annulus component as in the construction. Let $\partial A = s_+ \sqcup s_-$ such that $j_*[s_\pm] = \pm\omega$ in $H_1(X_K)$. By the desatellite construction, $\hat{j}_*[s_\pm] = \pm\omega$ in $H_1(X_{T_{\mathrm{std}}})$. Since $\pi_1(X_{T_{\mathrm{std}}}) \cong H_1(X_{T_{\mathrm{std}}})$, $\hat{j}(s_+)$ is free-homotopic to the orientation-reversal of $\hat{j}(s_-)$. In other words, we can extend $\hat{j}|$ over A. After extending for every attached annulus, we obtain $\hat{j} : \hat{F}'_p \to X_{\hat{K}_p}$.

Since $\hat{j}|_{\partial \hat{F}'_p}$ is the same as $j|_{\partial F}$ under the natural identification $\hat{j}_*[\partial \hat{F}'_p] = m\gamma$ in $H_1(T^2)$ (where $H_1(T^2)$ may be regarded as either $H_1(K)$ or $H_1(\hat{K}_p)$ under the natural identification), after homotoping $\hat{j} : \hat{F}'_p \to X_{\hat{K}_p}$ to a smooth immersion, we obtain the map as desired. □

The contribution of the companion term in Theorem 4.9 basically comes from F'_c. However, $j_*[F'_c]$ does not necessarily represent $m\gamma_c$, but may differ by a term of zero $\|\cdot\|_{K_c}$-seminorm.

To be precise, note the image of any component of $\partial Q \subset \partial_c Y$ under j lies in the kernel of $H_1(\partial_c Y) \to H_1(X_{K_c})$, which we may identify with $H_1(K_c)$. Thus $\alpha = j_*[\partial Q] \in H_1(\partial_c Y)$ lies in $H_1(K_c)$. Also, $j_*[\partial F_c] = m\,\gamma_c \in H_1(K_c) < H_1(\partial_c Y)$.

Thus $\beta = m\,\gamma_{\mathrm{c}} - \alpha$ in $H_1(K_{\mathrm{c}}) < H_1(\partial_{\mathrm{c}} Y)$ is represented by $j_*[F'_{\mathrm{c}}]$. We have

$$m\gamma_{\mathrm{c}} = \alpha + \beta.$$

Lemma 4.18. *With the notation above,* $\|\alpha\|_{K_{\mathrm{c}}} = 0$, *and hence* $m\|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}} = \|\beta\|_{K_{\mathrm{c}}}$.

Proof. For any component $s \subset \partial Q$, s bounds an embedded disk component D of $Q \subset F_{\mathrm{c}}$ by the definition of Q. It follows that $j(s)$ is null-homotopic in $X_{K_{\mathrm{c}}}$, and hence $\|j_*[s]\|_{K_{\mathrm{c}}} = 0$. As this works for any component of ∂Q, we see $\|\alpha\|_{K_{\mathrm{c}}} = \|j_*[\partial Q]\|_{K_{\mathrm{c}}} = 0$. The "hence" part follows from that $\|\cdot\|_{K_{\mathrm{c}}}$ is a seminorm on $H_1(K_{\mathrm{c}}; \mathbb{R})$. □

Proof of Theorem 4.9. The first inequality follows from Lemma 4.10. In the rest, we assume $w(K_{\mathrm{p}}) \neq 0$. Let $j : F \looparrowright X_K$ be a surface that ϵ-approximates $\|\gamma\|_K$ as before. We may assume j satisfies the conclusion of Lemma 4.14 possibly after a modification. Possibly after passing to a finite cyclic covering of F, we may further assume j satisfies the conclusion of Lemma 4.15 as we have explained. We have the decomposition $F = F'_{\mathrm{p}} \cup F'_{\mathrm{c}}$ of F into essential subsurfaces, so by Lemma 4.16, $x(F) = x(F'_{\mathrm{p}}) + x(F'_{\mathrm{c}})$. By Lemma 4.17, there is an immersed surface $\hat{j} : \hat{F}'_{\mathrm{p}} \looparrowright X_{\hat{K}_{\mathrm{p}}}$ representing $m\gamma$ in $H_1(\hat{K}_{\mathrm{p}})$, with $x(\hat{F}'_{\mathrm{p}}) \leq x(F'_{\mathrm{p}})$, so

$$x(F'_{\mathrm{p}}) \geq x(\hat{F}'_{\mathrm{p}}) \geq m\|\gamma\|_{\hat{K}_{\mathrm{p}}}.$$

By Lemma 4.18, since $j| : F'_{\mathrm{c}} \looparrowright X_{\mathrm{c}}$ is an immersed surface representing β in $H_1(K_{\mathrm{c}})$,

$$x(F'_{\mathrm{c}}) \geq \|\beta\|_{K_{\mathrm{c}}} = m\|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}}.$$

Combining the estimates above, $x(F) \geq m\big(\|\gamma\|_{\hat{K}_{\mathrm{p}}} + \|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}}\big)$, thus,

$$\|\gamma\|_{\hat{K}_{\mathrm{p}}} + \|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}} \leq \frac{x(F)}{m} < \|\gamma\|_K + \epsilon.$$

We conclude that $\|\gamma\|_{\hat{K}_{\mathrm{p}}} + \|\gamma_{\mathrm{c}}\|_{K_{\mathrm{c}}} \leq \|\gamma\|_K$, as $\epsilon > 0$ is arbitrary. □

5. Braid satellites

In this section, we introduce and study braid satellites.

5A. *Braid patterns.* We shall fix a product structure on $T^2 \cong S^1 \times S^1$ throughout this section. By a *braid* we shall mean an embedding $b : S^1 \hookrightarrow S^1 \times D^2$, whose image is a simple closed loop transverse to the fiber disks. We usually write k_b for the classical knot in S^3 associated to b, namely, the "satellite" knot with the trivial companion and the pattern b.

There is a family of patterns arising from braids:

Definition 5.1. Let $b : S^1 \hookrightarrow S^1 \times D^2$ be a braid. Define the *standard braid pattern* P_b associated to b as $P_b = \mathrm{id}_{S^1} \times b : S^1 \times S^1 \hookrightarrow \Theta^4$, where $\Theta^4 = S^1 \times S^1 \times D^2$ is the thickened torus. The *standard braid torus* K_b associated to b is defined as the desatellite $T_{\mathrm{std}} \cdot P_b$.

Remark 5.2. The standard braid torus K_b is sometimes called the *spun T^2-knot* obtained from the associated knot k_b. In [Hirose 1993], the extendable subgroup $\mathscr{E}_{K_b}$ has been explicitly computed.

Lemma 5.3. *If $b : S^1 \hookrightarrow S^1 \times D^2$ is a braid with winding number $w(b)$, then $w(P_b) = w(b)$. In particular, $w(P_b) \neq 0$.*

Proof. This follows immediately from the construction and the definition of winding numbers. □

Proposition 5.4. *Suppose b is a braid whose associated knot k_b is nontrivial. Then*

$$\|\mathrm{pt} \times S^1\|_{K_b} = 2g(k_b) - 1 \quad \textit{and} \quad \|S^1 \times \mathrm{pt}\|_{K_b} = 0,$$

where $g(k_b)$ denotes the genus of k_b.

Proof. For simplicity, we write K_b and k_b as K and k, respectively.

To see $\|\mathrm{pt} \times S^1\|_K \geq 2g(k) - 1$, the idea is to construct a map between the complements $f : X_K \to M_k$, where $X_K = S^4 - K$ and $M_k = S^3 - k$. Let $Y \subset X_K$ be the image of the complement $\Theta^4 - P_b$, and $N \subset M_k$ be the image of the complement $S^1 \times D^2 - b$. There is a natural projection map $f| : Y \cong S^1 \times N \to N$. As $M_k - N$ is homeomorphic to the solid torus, which is an Eilenberg–MacLane space $K(\mathbb{Z}, 1)$, it is not hard to see that $f|$ extends as a map $f : X_K \to M_k$.

Provided this, for any properly immersed compact orientable surface $j : F \looparrowright X_K$ whose boundary represents $m[c]$, the norm of $[f \circ j(F)]$ is bounded below by the singular Thurston norm of k. As the singular Thurston norm equals the Thurston norm (see [Gabai 1983]), which further equals $2g(k) - 1$ for nontrivial knots, we obtain $\|\mathrm{pt} \times S^1\|_K \geq 2g(k) - 1$.

To see $\|\mathrm{pt} \times S^1\|_K = 2g(k) - 1$, it suffices to find a surface realizing the norm. In fact, one may first take an inclusion $\iota : \Theta^4 \to S^1 \times D^3$, where $\iota = \mathrm{id}_{S^1} \times \iota'$ and where $\iota' : S^1 \times D^2 \to D^3$ is a standard unknotted embedding, that is, whose core is unknotted in D^3 and $S^1 \times \mathrm{pt} \subset S^1 \times \partial D^2$ is the longitude. Then K_b factorizes through a smooth embedding $S^1 \times D^3 \hookrightarrow S^4$ (unique up to isotopy) via $\iota \circ P_b$. This allows us to put a minimal genus Seifert surface of k into X_K so that it is bounded by the slope $\mathrm{pt} \times S^1$. Thus $\|\mathrm{pt} \times S^1\|_K = 2g(k) - 1$.

From the factorization above, we may also free-homotope $(\iota \circ P_b)(S^1 \times \mathrm{pt})$ to $S^1 \times \{\mathrm{pt}'\}$, where pt' is a point on ∂D^3, via an annulus $S^1 \times [\mathrm{pt}, \mathrm{pt}']$ where $[\mathrm{pt}, \mathrm{pt}']$ is an arc whose interior lies in $D^3 - k$. As $S^1 \times \{\mathrm{pt}'\}$ bounds a disk outside the image of $S^1 \times D^3$ in S^4, we see that $\|S^1 \times \mathrm{pt}\|_K = 0$. □

5B. *Braid satellites.* As an application of the Schubert inequality for seminorms, we estimate $\|\cdot\|_K$ for braid satellites of braid tori. We need the following notation.

Definition 5.5. Let $K : T^2 \hookrightarrow S^4$ be a knotted torus in S^4, and $\tau : T^2 \to T^2$ be an automorphism of T^2. We define the *τ-twist* K^τ of K to be the knotted torus $K \circ \tau : T^2 \hookrightarrow S^4$.

It follows immediately that the seminorm changes under a twist according to the formula $\|\gamma\|_{K^\tau} = \|\tau(\gamma)\|_K$.

Fix a product structure $T^2 \cong S^1 \times S^1$ as before. We denote the basis vectors $[S^1 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^1]$ on $H_1(T^2; \mathbb{R})$ as ξ, η, respectively. A *braid satellite* is known as some knotted torus of the form $K_b^\tau \cdot P_{b'}$, where b, b' are braids with nontrivial associated knots, and $\tau \in \mathrm{Mod}(T^2)$. It is said to be a *plumbing* braid satellite if $\tau(\xi) = \eta$ and $\tau(\eta) = -\xi$.

Proposition 5.6. *Suppose b, b' are braids with nontrivial associated knots, and τ is an automorphism of T^2. Let K be the satellite knotted torus $K_b^\tau \cdot P_{b'}$. Then for any $\gamma = x\,\xi + y\,\eta$ in $H_1(T^2; \mathbb{R})$,*

$$\|\gamma\|_K \ \geq\ (2g' - 1) \cdot |y| + (2g - 1) \cdot |rx + sw'y|.$$

Here $g, g' > 0$ are the genera of the associated knots of b, b', respectively, and w' is the winding number of b', and r, s are the intersection numbers $\xi \cdot \tau(\xi)$, $\xi \cdot \tau(\eta)$, respectively. Moreover, the equality is achieved if $K_b^\tau \cdot P_{b'}$ is a plumbing braid satellite.

We remark that one should not expect the seminorm lower bound be realized in general. For instance, in the extremal case when τ is the identity, $\pi_1(K)$ is exactly the knot group of the satellite of classical knots $k_b \cdot b'$, and the lower bound for the longitude slope is given by the classical Schubert inequality, which is not realized in general. However, the plumbing case is a little special. It provides examples of slopes on which the seminorm is not realized by the singular genus. In fact, when $c \subset K$ is a slope representing $x\,\xi + y\,\eta \in H_1(T^2)$, where x, y are coprime odd integers, the formula yields that $\|c\|_K$ is an even number, so the integer $g_K^\star(c)$ can never be $(\|c\|_K + 1)/2$. We shall give some estimate of the singular genus and the genus for plumbing braid satellites in Section 5C.

The corollary below follows immediately from Proposition 5.6 and Lemma 4.6:

Corollary 5.7. *With the notation of Proposition 5.6, if τ is an automorphism of T^2 not fixing ξ up to sign, then the stable extendable subgroup $\mathscr{E}_K^{\mathrm{s}}$ of $\mathrm{Mod}(T^2)$ with respect to K, and hence the extendable subgroup $\mathscr{E}_K$, is finite.*

In the rest of this subsection, we prove Proposition 5.6.

Lemma 5.8. *With the notation of Proposition 5.6,*

$$\|\gamma\|_K \ \geq\ (2g'-1)\cdot|y| + (2g-1)\cdot|rx+sw'y|.$$

Proof. By Lemma 5.3 and Theorem 4.9, $\|\gamma\|_K \geq \|\gamma\|_{K_{b'}} + \|\tau(\gamma_c)\|_{K_b}$. Note that we are writing γ_c with respect to $K_b \cdot P_{b'}$, so the second term equals the corresponding term in Theorem 4.9 with respect to the twisted satellite $K_b^\tau \cdot P_{b'}$ via an obvious transformation. By Proposition 5.4, $\|\gamma\|_{K_{b'}} = (2g'-1)\cdot|y|$. As b' is a braid, $P_{b'}: T^2 \to \Theta^4 \simeq T^2$ implies $\gamma_c = x\xi + w'y\eta$. Write τ as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{Z})$ under the given basis ξ,η. Note it agrees with the notation r,s in the statement. Then it is easy to compute $\tau(\gamma_c) = (px+qw'y)\xi + (rx+sw'y)\eta$. By Proposition 5.4 again, $\|\tau(\gamma_c)\|_{K_b} = (2g-1)\cdot|rx+sw'y|$. Combining these calculations, we obtain the estimate as desired. □

Lemma 5.9. *With the notation of Proposition 5.6, if K is a plumbing braid satellite,*

$$\|\gamma\|_K \leq (2g'-1)\cdot|y| + (2g-1)\cdot|x|.$$

Proof. Because $\|\cdot\|_K$ is a seminorm (Lemma 4.3), it suffices to prove $\|\xi\|_K \leq 2g-1$ and $\|\eta\|_K \leq 2g'-1$. The complement X_K is the union of the companion piece $X_{K_b} = S^4 - K_b$ and the pattern piece $Y = \Theta^4 - P_{b'}$. Note that $\pi_1(X_{K_b}) = \pi_1(M_{k_b})$ where $M_{k_b} = S^3 - k_b$ is the knot complement, and $\pi_1(Y) = \mathbb{Z}\times\pi_1(R_{b'})$ where $R_{b'} = S^1 \times D^2 - b'$ is the braid complement. From the construction it is clear that $\pi_1(Y) \to \pi_1(X_K)$ factors through the desatellite on the first factor, namely, $\mathbb{Z}\times\pi_1(M_{k_{b'}})$, so the commutator length of η in $\pi_1(X_K)$ is at most that of η in $\pi_1(M_{k_{b'}})$, which is $2g'$. Moreover, the slope $\xi \in \partial X_K$ can be free-homotoped to a slope ξ_c on ∂X_{K_b} since it is a fiber of $Y = S^1 \times R_{b'}$, and by the construction, it is clear that ξ_c represents the longitude slope of $\pi_1(\partial M_{k_b})$ in $\pi_1(M_{k_b}) \cong \pi_1(X_{K_b})$, so the commutator length of ξ in $\pi_1(X_K)$ is at most that of ξ_c in $\pi_1(M_{k_b})$, which is $2g$. This proves the lemma because the commutator length equals the singular genus $g_K^\star$, which gives upper bounds for the seminorm $\|\cdot\|_K$ on slopes (Remark 3.3 and Lemma 4.6). □

Now Proposition 5.6 follows from Lemmas 5.8, 5.9.

Remark 5.10. For plumbing braid satellites, since the norm is given by

$$\|\gamma\|_K = (2g'-1)|y| + (2g-1)|x|,$$

the unit ball of the norm of plumbing satellite is the rhombus on the plane with the vertices $(\pm 1/(2g-1), 0)$ and $(0, \pm 1/(2g'-1))$.

5C. ***On genera of plumbing braid satellites.*** In this subsection, we estimate the singular genera and the genera of slopes for plumbing braid satellites. While we obtain a pretty nice estimate for the singular genera, with the error at most one, we are not sure how close our genera upper bound is to being the best possible.

Proposition 5.11. *Suppose b, b' are braids with nontrivial associated knots, and K is the plumbing braid satellite $K_b^\tau \cdot P_{b'}$. Then for every slope $c \subset K$, we have:*

(1) *The singular genus satisfies*
$$\frac{\|c\|_K+1}{2} \le g_K^\star(c) \le \frac{\|c\|_K+3}{2}.$$
In particular, if c represents $x\xi + y\eta$ with both x and y odd, then
$$g_K^\star(c) = \frac{\|c\|_K}{2} + 1.$$

(2) *If c represents $x\xi + y\eta$ in $H_1(T^2)$, where x, y are coprime integers, then the genus satisfies*
$$g_K(c) \le g \cdot |x| + g' \cdot |y| + \frac{(|x|-1)(|y|-1)}{2},$$
where $g, g' > 0$ denote the genera of the associated knots $k_b, k_{b'}$ in S^3, respectively.

We prove Proposition 5.11 in the rest of this subsection. We shall rewrite the slopes $S^1 \times \mathrm{pt},\ \mathrm{pt} \times S^1 \subset T^2$ as c_ξ, c_η, respectively.

We need the notion of Euler number to state the next lemma. Let Y be a simply connected, closed oriented 4-manifold, and let $K : T^2 \hookrightarrow Y$ be a null-homologous knotted torus embedded in Y. Let $X = Y - K$ be the compact exterior of the knotted torus. For any locally flat, properly embedded compact oriented surface with connected boundary, $F \hookrightarrow X$, such that ∂F is mapped homeomorphically onto a slope $c \times \mathrm{pt}$ of $K \times \mathrm{pt}$ (which exists by Lemma 3.1), we may take a parallel copy $c \times \mathrm{pt}' \subset K \times \mathrm{pt}'$ of the slope, and perturb F to be another locally flat, properly embedded copy $F' \hookrightarrow X$ bounded by $c \times \mathrm{pt}'$, so that F, F' are in general position. The algebraic sum of the intersections between F and F' gives rise to an integer
$$e(F; K) \in \mathbb{Z},$$
which is known as the *Euler number* of the normal framing of F induced from K. In fact, one can check that $e(F; K)$ only depends on the class $[F] \in H_2(X, K \times \mathrm{pt})$. If Y is orientable but has no preferable choice of orientation, we ambiguously speak of the Euler number up to sign.

Lemma 5.12. *There exist two disjoint, properly embedded, orientable compact surfaces $E, E' \hookrightarrow X_K$, bounded by the slopes $c_\xi \times p$, $c_\eta \times p'$ in two parallel copies* *of the knotted torus $K \times p, K \times p' \subset \partial X$, respectively. Moreover, the genera of E, E' are g, g', respectively, and the Euler number of the normal framing is $e(E; K) = e(E'; K) = 0$.*

Proof. Regarding K as $T_{\text{std}} \cdot P_b^\tau \cdot P_{b'}$, there is a natural decomposition

$$X_K = X_0 \cup Y \cup Y',$$

where X_0 is the compact complement of the unknotted torus T_{std} in S^4, and Y, Y' are the exteriors of $P_b, P_{b'}$ in the thickened torus Θ^4, respectively. Moreover, Y and Y' have natural product structures $c_\eta \times R_b$ and $c_\xi \times R_{b'}$, respectively, where R_b and $R_{b'}$ denote the exteriors of the braids b and b', respectively, in the solid torus $S^1 \times D^2$. As before, ∂Y and $\partial Y'$ each have two components: ∂Y has $\partial_{\text{c}} Y$ and $\partial_{\text{s}} Y$, $\partial Y'$ has $\partial_{\text{c}} Y'$ and $\partial_{\text{s}} Y'$. Thus ∂X_0 is glued to $\partial_{\text{c}} Y$, and $\partial_{\text{s}} Y$ is glued to $\partial_{\text{c}} Y'$, and $\partial_{\text{s}} Y'$ is exactly ∂X_K.

The knot complement $M_{k_b} = S^3 - k_b$ is the union of R_b with a solid torus $S^1 \times D^2$. From classical knot theory, there is a genus g Seifert surface S of k_b properly embedded in $M_{k_b} = S^3 - k_b$, and one can arrange S so that it intersects $S^1 \times D^2$ in a finite collection of $n \geq w$ disjoint parallel fiber disks. Thus $S_b = S \cap R_b$ is a connected properly embedded orientable compact surface, so that ∂S_b has one component on $\partial_{\text{s}} R_b$ parallel to the longitude s, and n components $c_1, \ldots, c_n$ on $\partial_{\text{c}} R_b$ parallel to $\text{pt} \times \partial D^2$. Similarly, take a connected subsurface $S_{b'} \subset R_{b'}$ with n' boundary components $c'_1, \ldots, c'_{n'}$ on the companion boundary, and one boundary component s' on the satellite boundary.

Construct a properly embedded compact annulus $E_{Y'}$ in $Y' = c_\xi \times R_{b'}$ by taking the product of c_ξ with some arc $\alpha \subset R_{b'} - S_{b'}$, so that the two endpoints lie on $\partial_{\text{c}} R_{b'}$ and $\partial_{\text{s}} R_{b'}$, respectively. Construct a properly embedded compact surface $E'_{Y'} \subset Y'$ by taking the product of $S_{b'}$ with some point in c_ξ. Similarly, construct a properly embedded compact surface E_Y in $Y = c_\eta \times R_b$ by taking a product of S_b with some point in c_η, and construct a union of n' annuli E'_Y by taking the product of c_η with n' disjoint arcs $\alpha'_1, \ldots, \alpha'_{n'}$ in $R_b - S_b$, each of whose endpoints lie on $\partial_{\text{c}} R_b$ and $\partial_{\text{s}} R_b$, respectively. Under the gluing, we obtain two disjoint properly embedded surfaces $E_Y \cup E_{Y'}$ and $E'_Y \cup E'_{Y'}$ in $Y \cup Y'$, whose boundaries on $\partial_{\text{s}} Y' = \partial X_K \cong K \times S^1$ are $c_\xi \times \text{pt}$ and $c_\eta \times \text{pt}$, respectively. Moreover, it is clear that $\partial(E_Y \cup E_{Y'})$ has n other boundary components on $\partial_{\text{c}} Y = \partial X_0 \cong T_{\text{std}} \times S^1$ parallel to $c_\eta \times \text{pt}$, and $\partial(E'_Y \cup E'_{Y'})$ has n' other boundary components on $\partial_{\text{c}} Y$ parallel to $c_\xi \times \text{pt}$.

It is not hard to see that one can cap off these other boundary components with disjoint properly embedded disks in X_0. In fact, we may regard $T_{\text{std}} : T^2 \hookrightarrow S^4$ as the composition

$$T^2 \cong c_\xi \times c_\eta \hookrightarrow c_\xi \times D^3 \hookrightarrow S^4,$$

where c_η is a trivial knot in D^3. Thus the components of $\partial(E'_Y \cup E'_{Y'})$ that lie on ∂X_0 can be capped off in $c_\xi \times D^3$ disjointly. Moreover, the components of $\partial(E_Y \cup E_{Y'})$ lying on ∂X_0 can be isotoped to the boundary of $c_\xi \times D^3$, so that they are all c_ξ-fibers. Because $S^4 - c_\xi \times D^3$ is homeomorphic to $D^2 \times S^2$, we may further cap off these fibers in the complement of $c_\xi \times D^3$ in S^4.

It is straightforward to check that capping off $E_Y \cup E_{Y'}$ and $E'_Y \cup E'_{Y'}$ yields the surfaces E and E', as desired. Note that $e(E; K)$ vanishes because we can perturb the construction above to obtain a surface disjoint from E bounding a slope parallel to $c_\xi \times \mathrm{pt}$ in $K \times \mathrm{pt}$. For the same reason, $e(E'; K) = 0$ as well. □

Proof of Proposition 5.11. (1) It suffices to show the upper bound. By Lemma 5.12, there are properly embedded surfaces E, E' in X_K bounded by $c_\xi \times \mathrm{pt}$, $c_\eta \times \mathrm{pt}$, respectively, and the complexity of E and E' realizes $\|c_\xi\|_K$ and $\|c_\eta\|_K$, respectively (Proposition 5.6). Suppose $c \subset K$ is a slope representing $x\xi + y\eta$. By the main theorem of [Massey 1974], there exists an $|x|$-sheet connected covering space $\tilde{E}$ of E, which has exactly one boundary component if x is odd, or two boundary components if x is even. By the same method, there is also $\tilde{E}'$, which is connected $|y|$-sheet covering E' with one or two boundary components. Since x and y are coprime, at most one of them is even, so $\tilde{E} \cup \tilde{E}'$ have at most three components. Then there are immersions of these surfaces into X_K, and by homotoping the image of their boundaries to $K \times \mathrm{pt}$ and taking the band sum to make them connected, we obtain an immersed subsurface $F \looparrowright X_K$ bounding the slope c. Since we need to add up to two bands to make the boundary of F connected, this yields

$$2\, g^\star_K(c) - 1 \le -\chi(F) \le (-\chi(E)) \cdot |x| + (-\chi(E')) \cdot |y| + 2 = \|c\|_K + 2.$$

Note that the last equality follows from Proposition 5.6 as we assumed K is the plumbing braid satellite. This proves the first statement. The "in particular" part is also clear because when x, y are both odd, $\|c\|_K$ is an even number by the formula, so $(\|c\|_K/2) + 1$ is the only integer satisfying our estimation.

(2) In this case, we take $|x|$ copies of the embedded surface E, and $|y|$ copies of the embedded surface E', in X_K. Because the Euler numbers of the normal framing are zero for E and E', we may assume these copies to be disjoint. Isotope their boundaries to $K \times \mathrm{pt}$ in ∂X_K; we see $|x|$ slopes parallel to c_ξ, and $|y|$ slopes parallel to c_η. As there are $|xy|$ intersection points, we take $|xy|$ band sums to obtain a properly embedded surface $F \hookrightarrow X_K$ bounding the slope c. There are $|x| + |y| - 1$ bands that contribute to making the boundary of F connected, and each of the other $|xy| - |x| - |y| + 1$ bands contributes one half to the genus of F. This implies

$$g_K(c) \le g(F) = g \cdot |x| + g' \cdot |y| + \frac{(|x|-1)(|y|-1)}{2},$$

as desired. □

6. Miscellaneous examples

In this section, we exhibit examples to show difference between concepts introduced in this note.

6A. ***Slopes with vanishing seminorm but positive singular genus.*** Note that we have already seen slopes whose singular genus do not realize nonvanishing seminorm in plumbing braid satellites; see Proposition 5.6. There are also examples where the seminorm vanishes on some slope with positive singular genus, as follows. Our construction is based on the existence of incompressible knotted Klein bottles.

Denote the Klein bottle as Φ^2. A *knotted Klein bottle* in S^4 is a locally flat embedding $K : \Phi^2 \hookrightarrow S^4$. We usually denote its image also as K, and the exterior $X_K = S^4 - K$ is obtained by removing an open regular neighborhood of K from S^4 as before in the knotted torus case. We say a knotted Klein bottle K is *incompressible* if the inclusion $\partial X_K \subset X_K$ induces an injective homomorphism between the fundamental groups. There exist incompressible Klein bottles in S^4; see [Kamada 1990, Lemma 4].

Incompressible knotted Klein bottles give rise to examples of slopes on knotted tori which have vanishing seminorm but positive singular genus.

Specifically, let $K : \Phi^2 \hookrightarrow S^4$ be an incompressible knotted Klein bottle. Suppose $\kappa : T^2 \to \Phi^2$ is a two-fold covering of the Klein bottle Φ^2. Perturbing $K \circ \kappa : T^2 \to S^4$ in the normal direction of K gives rise to a knotted torus $\tilde{K} : T^2 \hookrightarrow S^4$.

Lemma 6.1. *With the notation above, $\tilde{K}$ has a slope c such that $\|c\|_{\tilde{K}} = 0$, but $g^{\star}_{\tilde{K}}(c) > 0$.*

Proof. Let $\alpha \subset \Phi^2$ be an essential simple closed curve on K so that $\kappa^{-1}(\alpha)$ has two components $c, c' \subset T^2$. Then c, c' are parallel on T^2. We choose orientations on c, c' so that they are parallel as oriented curves. Let $\mathcal{N}(K)$ be a compact regular neighborhood of K so that $Y = \mathcal{N}(K) - \tilde{K}$ is a pair-of-pants bundle over K. Then c is freely homotopic to the orientation-reversal of c' within Y. This implies that $2\,[c \times \mathrm{pt}] \in H_1(X_{\tilde{K}})$ is represented by a properly immersed annulus $A \looparrowright X_{\tilde{K}}$ whose boundary with the induced orientation equals $c \cup c'$. Therefore, $\|c\|_K$ equals zero. However, note that $X_{\tilde{K}} = X_K \cup Y$, glued along $\partial X_K = \partial \mathcal{N}(K)$. Since K is incompressible, ∂X_K is π_1-injective in X_K. It is also clear that both components of ∂Y are π_1-injective in Y. It follows that $\pi_1(Y)$ injects into $\pi_1(X_{\tilde{K}})$, and also that $\pi_1(\partial X_{\tilde{K}})$ injects into $\pi_1(X_{\tilde{K}})$. Therefore, the slope $c \times \mathrm{pt}$ in $\partial X_{\tilde{K}} \cong \tilde{K} \times S^1$ is homotopically nontrivial in $\pi_1(X_{\tilde{K}})$, so $g^{\star}_{\tilde{K}}(c)$ cannot be zero. □

6B. ***Stably extendable but not extendable automorphisms.*** It is clear that the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ contains the extendable subgroup $\mathscr{E}_K$ for any knotted torus $K : T^2 \hookrightarrow S^4$. They are in general not equal. In fact, we show that the Dehn twist along a slope with vanishing singular genus is stably extendable (Lemma 6.2). In particular, it follows that for any unknotted embedded torus K, the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$ equals $\mathrm{Mod}(T^2)$. However, in this case, the extendable subgroup $\mathscr{E}_K$ is a proper subgroup of $\mathrm{Mod}(T^2)$ of index three [Ding et al. 2012; Montesinos 1983]. Thus there are many automorphisms that are stably extendable but not extendable for the unknotted embedding.

Fix an orientation of the torus T^2. For any slope $c \subset T^2$ on the torus, we denote the (right-hand) Dehn twist along c as $\tau_c : T^2 \to T^2$. More precisely, the induced automorphism on $H_1(T^2)$ is given by $\tau_{c*}(\alpha) = \alpha + I([c], \alpha)[c]$ for all $\alpha \in H_1(T^2)$, where $I : H_1(T^2) \times H_1(T^2) \to \mathbb{Z}$ denotes the intersection form. Note that the expression is independent from the choice of the direction of c.

The criterion below is inspired from techniques of Susumu Hirose and Akira Yasuhara. However, the reader should beware that our notion of stabilization in this paper does not change the fundamental group of the complement, so it is slightly different from the definition in [Hirose and Yasuhara 2008].

Lemma 6.2. *Let $K : T^2 \hookrightarrow S^4$ be a knotted torus. Suppose $c \subset T^2$ is a slope with the singular genus $g^\star_K(c) = 0$. Then the Dehn twist $\tau_c \in \mathrm{Mod}(T^2)$ along c belongs to the stable extendable subgroup $\mathscr{E}^{\mathrm{s}}_K$.*

Proof. The idea of this criterion is that, for a closed simply connected oriented 4-manifold Y, to have the Dehn twist τ_c extendable over Y via the Y-stabilization $K[Y] : T^2 \hookrightarrow Y$, we need c to bound a locally flat, properly embedded disk of Euler number ± 1 in the complement of $K[Y]$ in Y. Such a Y can always be chosen to be the connected sum of copies of $\mathbb{C}\mathrm{P}^2$ or $\overline{\mathbb{C}\mathrm{P}^2}$.

Recall that we introduced the Euler number of a surface bounding a slope in Section 5C before the statement of Lemma 5.12. Suppose D is a locally flat, properly embedded disk in $X = Y - K[Y]$ bounded by a slope $c \times \mathrm{pt}$ on $K[Y] \times \mathrm{pt} \subset \partial X$ with $e(D; K[Y]) = \pm 1$. We claim in this case the Dehn twist $\tau_c \in \mathrm{Mod}(T^2)$ along c can be extended as an orientation-preserving self-homeomorphism of Y. In fact, following the arguments in the proof of [Hirose and Yasuhara 2008, Theorem 4.1], we may take the compact normal disk bundle ν_D of D, identified as embedded in X such that $\nu_D \cap (K[Y] \times \mathrm{pt})$ is an interval subbundle of ν_D over ∂D. Then $e(D; K[Y]) = \pm 1$ implies that $\nu_D \cap (K[Y] \times \mathrm{pt})$ is a (positive or negative) Hopf band in the 3-sphere $\partial \nu_D$, whose core is $c \times \mathrm{pt}$. Thus τ_c extends over Y as a self-homeomorphism by [Hirose and Yasuhara 2008, Proposition 2.1].

Now it suffices to find a Y fulfilling the assumption of the claim above. Suppose $c \subset K$ is a slope with the singular genus $g^\star_K(c) = 0$. Then there is a map $j : D^2 \to X_K$ so that ∂D^2 is mapped homeomorphically onto $c \times \mathrm{pt}$ in $\partial X_K \cong K \times S^1$. We may also assume j to be an immersion by the general position argument. Blowing up all the double points of $j(D^2)$, we obtain an embedding

$$j' : D^2 \hookrightarrow X_K \,\#\, (\overline{\mathbb{CP}^2})^{\#r}$$

for some integer $r \geq 0$. Suppose $e(j'(D); K[(\overline{\mathbb{CP}^2})^{\#r}])$ equals $s \in \mathbb{Z}$. If $s > 1$, we may further blow up $s-1$ points in $j'(D) \subset X_K \,\#\, (\overline{\mathbb{CP}^2})^{\#r}$. This gives rise to

$$j'' : D^2 \hookrightarrow X_K \,\#\, (\overline{\mathbb{CP}^2})^{\#(r+s-1)}$$

satisfying the assumption of the claim, so the Dehn twist τ_c is extendable over $X = X_K \,\#\, (\overline{\mathbb{CP}^2})^{\#(r+s-1)}$, or in other words, it is Y-stably extendable, where $Y = (\overline{\mathbb{CP}^2})^{\#(r+s-1)}$. If $s < 1$, a similar argument using negative blow-ups shows that τ_c is Y-stably extendable, where $Y = (\mathbb{CP}^2)^{\#(1-s)} \,\#\, (\overline{\mathbb{CP}^2})^{\#r}$. □

7. Further questions

In conclusion, for a knotted torus $K : T^2 \hookrightarrow S^4$, the seminorm and the singular genus of a slope are meaningful numerical invariants which are sometimes possible to control using group theoretic methods. However, the genera of slopes seem to be much harder to compute. It certainly deserves further exploration how to combine the group-theoretic methods with the classical 4-manifold techniques when the fundamental group comes into play.

We propose several further questions about genera, seminorm and extendable subgroups. Suppose $K : T^2 \hookrightarrow S^4$ is a knotted torus.

Question 7.1. When is the unit disk of the seminorm $\|\cdot\|_K$ a finite rational polygon, that is, bounded by finitely many segments of rational lines? (See Remark 5.10.)

Question 7.2. If the index of the extendable subgroup $\mathscr{E}_K$ in $\mathrm{Mod}(T^2)$ equals three, is K necessarily the knot connected sum of the unknotted torus with a knotted sphere?

Question 7.3. If the stable extendable subgroup $\mathscr{E}_K^{\mathrm{s}}$ equals $\mathrm{Mod}(T^2)$, does the singular genus $g_K^\star$ vanish for every slope?

Question 7.4. If K is incompressible, that is, ∂X_K is π_1-injective in the complement X_K, is the stable extendable subgroup $\mathscr{E}_K^{\mathrm{s}}$ finite?

Question 7.5. For plumbing knotted satellites, does the upper bound in Proposition 5.11(2) realize the genus of the slope?

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FORMAL GROUPS OF ELLIPTIC CURVES WITH POTENTIAL GOOD SUPERSINGULAR REDUCTION

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Let L be a number field and let E/L be an elliptic curve with potentially supersingular reduction at a prime ideal $\wp$ of L above a rational prime p. In this article we describe a formula for the slopes of the Newton polygon associated to the multiplication-by-p map in the formal group of E, depending only on the congruence class of p mod 12, the $\wp$-adic valuation of the discriminant of a model for E over L, and the valuation of the j-invariant of E. The formula is applied to prove a divisibility formula for the ramification indices in the field of definition of a p-torsion point.

1. Introduction

Let L be a number field with ring of integers $\mathcal{O}_L$, let $p \geq 2$ be a prime, let $\wp$ be a prime ideal of $\mathcal{O}_L$ lying above p, and let $L_\wp$ be the completion of L at $\wp$. Let E be an elliptic curve defined over L with potential good (supersingular) reduction at $\wp$. Let us fix an embedding $\iota : \overline{L} \hookrightarrow \overline{L}_\wp$. Via ι, we may regard E as defined over $L_\wp$. Let $L_\wp^{\mathrm{nr}}$ be the maximal unramified extension of $L_\wp$, and let K_E be the extension of $L_\wp^{\mathrm{nr}}$ of minimal degree such that E has good reduction over K_E (see Section 3 for more details). Let $K = K_E$, and let ν_K be a valuation on K such that $\nu_K(p) = e$ and $\nu_K(\pi) = 1$, where π is a uniformizer for K. Let A be the ring of elements of K with nonnegative valuation. We fix a minimal model of E over A with good reduction, given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in A$. In particular, the discriminant Δ is a unit in A. Let $\hat{E}/A$ be the formal group associated to E/A, with formal group law given by a power series $F(X, Y) \in A[\![X, Y]\!]$, as defined in [Silverman 2009, Chapter IV]. Let

$$[p](Z) = \sum_{i=1}^{\infty} s_i Z^i$$

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be the multiplication-by-p homomorphism in $\hat{E}$, for some $s_i \in A$ for all $i \geq 1$. Since E/K has good supersingular reduction, the formal group $\hat{E}/A$ associated to E has height 2; see [Silverman 2009, Chapter V, Theorem 3.1]. Thus, $s_1 = p$ and the coefficients s_i satisfy $\nu_K(s_i) \geq 1$ if $i < p^2$ and $\nu_K(s_{p^2}) = 0$. Let $q_0 = 1$, $q_1 = p$ and $q_2 = p^2$, and put $e_i = \nu_K(s_{q_i})$. In particular $e_0 = \nu_K(s_1) = \nu_K(p) = e$ and $e_2 = \nu_K(s_{p^2}) = 0$. Let $e_1 = \nu_K(s_p)$. Then, the multiplication-by-p map can be expressed as

$$[p](Z) = pf(Z) + \pi^{e_1} g(Z^p) + h(Z^{p^2}),$$

where $f(Z)$, $g(Z)$ and $h(Z)$ are power series in $Z \cdot A[\![Z]\!]$, with

$$f'(0) = g'(0) = h'(0) \in A^{\times}.$$

In this article, we are interested in determining the value of e_1. In the next section we discuss three examples that will be used during the rest of the paper to fix ideas. In Section 3, we prove consecutive refinements of a formula for e_1 that culminate in Theorem 3.9 and Corollary 3.12, where we show a formula that only depends on the congruence class of p mod 12, the $\wp$-adic valuation of the discriminant of a model for E over L, and the valuation of the j-invariant of E. In Section 4 we use the formula to calculate the value of e_1 for several interesting examples, and we show that if $p > 3$, the ramification index of $\wp$ in $L/\mathbb{Q}$ is $e(\wp, L) = 1$, and $e_1 < e$, then the numbers e_1 and $e - e_1$ can only take the values 1, 2, or 4 (see Corollary 4.7). Finally, in Section 5, we apply our formula to prove the following divisibility formulas for the ramification indices in the field of definition of a p-torsion point (see Theorem 5.2 and Corollary 5.4):

Theorem 1.1. *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above a prime $p > 3$, and let e and e_1 be defined as above. Let $P \in E[p]$ be a nontrivial p-torsion point.*

(1) *Suppose $e_1 \geq pe/(p+1)$. Then the ramification index of any prime over $\wp$ in the extension $L(P)/L$ is divisible by $(p^2-1)/\gcd(p^2-1, e)$.*

(2) *Suppose $e_1 < pe/(p+1)$.*
   - *There are p^2-p points P in $E[p]$ such that the ramification index of a prime above $\wp$ in $L(P)/L$ is divisible by $(p-1)p/\gcd(p(p-1), e_1)$.*
   - *There are $p-1$ points P in $E[p]$ such that the ramification index of any prime above $\wp$ in $L(P)/L$ is divisible by $(p-1)/\gcd(p-1, e-e_1)$.*

In particular, suppose that $e(\wp, L) = 1$.

- *If $e_1 < e$, then $e_1 < pe/(p+1)$ and the ramification index of any prime over $\wp$ in $L(P)/L$ is divisible by $(p-1)/\gcd(p-1, 4)$.*
- *If $p \equiv 1 \bmod 12$, then $e_1 \geq e$ and the ramification index of any prime over $\wp$ in $L(P)/L$ is divisible by $(p^2-1)/\gcd(p^2-1, e)$.*

2. First examples

Before we dive deeper into the theory, let us exhibit two examples of elliptic curves over $L = \mathbb{Q}$ and one curve defined over a quadratic field $L = \mathbb{Q}(\sqrt{13})$, together with their minimal fields of good reduction (over $L_{\wp}^{\mathrm{nr}}$), and the values of e and e_1. The calculations have been completed with the aid of Sage [Stein et al. 2012] and Magma [Bosma et al. 2010].

Example 2.1. Let $E/\mathbb{Q}$ be the elliptic curve with Cremona label 121c2, with $j(E) = -11 \cdot 131^3$, given by a Weierstrass equation

$$y^2 + xy = x^3 + x^2 - 3632x + 82757.$$

The elliptic curve E has bad additive reduction at $p = 11$, but potentially good supersingular reduction at the same prime. The extension $K = K_E$ of $\mathbb{Q}_{11}^{\mathrm{nr}}$ is given by adjoining $\pi = \sqrt[3]{11}$, thus $e = 3$. The curve E has a minimal model with good supersingular reduction of the form

$$y^2 + \sqrt[3]{11}xy = x^3 + \sqrt[3]{11^2}x^2 + 3\sqrt[3]{11}x + 2$$

over $\mathbb{Q}_{11}^{\mathrm{nr}}(\pi)$, where $\pi = \sqrt[3]{11}$, and the discriminant of this model is $\Delta = -1$. The multiplication-by-11 map on the associated formal group $\hat{E}$ is given by a power series:

$$\begin{aligned}[11](Z) = 11Z - 55\pi Z^2 - 275\pi^2 Z^3 + 42350Z^4 - 181148\pi Z^5 - 659417\pi^2 Z^6 \\ +96265708Z^7 - 341161040\pi Z^8 - 1521191342\pi^2 Z^9 \\ +183261837077Z^{10} - 497606935519\pi Z^{11} + O(Z^{12}).\end{aligned}$$

Since $497606935519 = 17 \cdot 23 \cdot 151 \cdot 8428159$ is relatively prime to 11, we conclude that $e_1 = \nu_K(s_{11}) = \nu_K(-497606935519\pi) = 1$.

Example 2.2. Let $E/\mathbb{Q}$ be the elliptic curve with Cremona label 27a4, with $j(E) = -2^{15} \cdot 3 \cdot 5^3$, given by a Weierstrass equation

$$y^2 + y = x^3 - 30x + 63.$$

The elliptic curve E has bad additive reduction at $p = 3$, but potentially good supersingular reduction at the same prime. The extension $K = K_E$ of $\mathbb{Q}_3^{\mathrm{nr}}$ is given by adjoining $\alpha = \sqrt[4]{3}$ and a root β of $x^3 - 120x + 506 = 0$. The result is an extension $K = \mathbb{Q}_3^{\mathrm{nr}}(\alpha, \beta)$ of degree $e = 12$. For convenience we write $K = \mathbb{Q}_3^{\mathrm{nr}}(\gamma)$ where γ is a root of $p(x) = 0$, with

$$\begin{aligned}p(x) = x^{12} - 480x^{10} - 2024x^9 + 86391x^8 + 728640x^7 - 5378664x^6 \\ -87509664x^5 - 161677413x^4 + 2979983776x^3 \\ +22119216120x^2 + 62098532232x + 65301304309.\end{aligned}$$

The curve E has a minimal model with good supersingular reduction (which we will not write here, because the coefficients are unwieldy expressions in γ). The multiplication-by-3 map on the associated formal group $\hat{E}$ is given by a power series

$$[3](Z) = 3Z + s_3 Z^3 + O(Z^4),$$

where

$$s_3 = \tfrac{91366247104560778}{113527481110579959}\gamma^{11} - \tfrac{1556952329592412502}{340582443331739877}\gamma^{10} + \tfrac{3943076616393619924}{340582443331739877}\gamma^{9} + \cdots + \tfrac{4950136311175538 48}{340582443331739877}\gamma^{2} - \tfrac{544095024526171682}{113527481110579959}\gamma - \tfrac{3353034524919522230}{340582443331739877}.$$

The valuation we sought (computed with Sage) is $\nu_K(s_3) = 2$. Hence, $e_1 = 2$ in this case.

Example 2.3. Let j_0 be a root of the polynomial

$$x^2 - 6896880000x - 567663552000000,$$

and let $L = \mathbb{Q}(j_0) = \mathbb{Q}(\sqrt{13})$. Let $p = 13$ and let $\wp = (\sqrt{13})$ be the ideal above p in $\mathcal{O}_L$. Let E/L be the elliptic curve with j-invariant equal to j_0. The curve E has complex multiplication by $\mathbb{Z}[\sqrt{-13}]$, that is, $\mathrm{End}(E/\mathbb{C}) \cong \mathbb{Z}[\sqrt{-13}]$ and, in fact, all the endomorphisms are defined over $\mathbb{Q}(\sqrt{13}, i)$; see [Silverman 1994, Chapter 2, Theorem 2.2(b)]. Since 13 ramifies in L, it follows from Deuring's criterion (see [Lang 1987, Chapter 13, §4, Theorem 12]) that the reduction of E at $\wp$ is potentially supersingular. We choose a model for E/L given by

$$y^2 = x^3 + \frac{5231j_0 - 50692880808000}{3825792}x + \frac{-550711j_0 + 4485396184200000}{239112}.$$

The discriminant of this model is

$$\Delta_L = \frac{13546495176890000j_0 - 93429639900045292464000000}{29889}$$

and $\nu_\wp(\Delta_L) = 0$. Hence, E/L has good supersingular reduction at $\wp$. In particular $K_E = L_\wp^{\mathrm{nr}}$ and $e = 2$. The multiplication-by-13 map on the associated formal group $\hat{E}$ is given by a power series:

$$[13](Z) = 13Z + \frac{-8092357j_0 + 78421886609976000}{39852}Z^5 + \cdots + s_{13}Z^{13} + O(Z^{15}),$$

where

$$s_{13} = (-1939238152610407708754766400000j_0 + 13701099619974313634962780362896640000000)/29889.$$

Since $\nu_K(s_{13}) = \nu_\wp(s_{13}) = 1$, we conclude that $e_1 = 1$. The formal group and the valuation of s_{13} were calculated using Magma. Thanks to Harris Daniels for providing the polynomial that defines j_0.

Remark 2.4. Let N be the part of the Newton polygon of $[p](Z)$ that describes the roots of valuation > 0. Let $P_0 = (1, e)$, $P_1 = (p, e_1)$, and $P_2 = (p^2, 0)$. The slope of the segment P_0P_1 is $-(e - e_1)/(p - 1)$, while the slope of the segment P_0P_2 is $-e/(p^2 - 1)$. It follows from the theory of Newton polygons (see [Serre 1972, p. 272]) that:

(1) If $pe/(p+1) < e_1$, then N is given by a single segment P_0P_2.

(2) Otherwise, if $pe/(p+1) \geq e_1$, then N is given by two segments P_0P_1 and P_1P_2.

In particular, if $e_1 \geq e$, then N has one single segment. We will frequently focus on the case $e_1 < e$, in which case the Newton polygon may have two segments. In this case, we shall show later (Corollary 3.2) that e_1 is independent of the chosen minimal model for E/K.

3. A formula for e_1

In this section we prove a formula for e_1 in terms of the valuations of the constants c_4 and c_6 of a minimal model for E/A. We need a number of preliminary results before we state and prove our formulas in Theorem 3.9 and Corollary 3.12. Let us begin with some further details about the extension $K_E/L_{\wp}^{\mathrm{nr}}$ that was mentioned in the introduction. We follow [Serre and Tate 1968] (see in particular p. 498, Corollary 3 there) to define an extension K_E of $L_{\wp}^{\mathrm{nr}}$ of minimal degree such that E has good reduction over K_E. Let ℓ be any prime such that $\ell \neq p$, and let $T_\ell(E)$ be the ℓ-adic Tate module. Let $\rho_{E,\ell} : \mathrm{Gal}(\overline{L_{\wp}^{\mathrm{nr}}}/L_{\wp}^{\mathrm{nr}}) \to \mathrm{Aut}(T_\ell(E))$ be the usual representation induced by the action of Galois on $T_\ell(E)$. We define the field K_E as the extension of $L_{\wp}^{\mathrm{nr}}$ such that

$$\mathrm{Ker}(\rho_{E,\ell}) = \mathrm{Gal}(\overline{L_{\wp}^{\mathrm{nr}}}/K_E).$$

In particular, the field K_E enjoys the following properties:

(1) E/K_E has good (supersingular) reduction.

(2) K_E is the smallest extension of $L_{\wp}^{\mathrm{nr}}$ such that E/K_E has good reduction, that is, if $K'/L_{\wp}^{\mathrm{nr}}$ is another extension such that E/K' has good reduction, then $K_E \subseteq K'$.

(3) $K_E/L_{\wp}^{\mathrm{nr}}$ is finite and Galois. Moreover (see [Serre 1972, §5.6, p. 312] when $L = \mathbb{Q}$, but the same reasoning holds over number fields, as the work of Néron [1964, p. 124–125] is valid for any local field):
   - If $p > 3$, then $K_E/L_{\wp}^{\mathrm{nr}}$ is cyclic of degree 1, 2, 3, 4, or 6.
   - If $p = 3$, the degree of $K_E/L_{\wp}^{\mathrm{nr}}$ is a divisor of 12.
   - If $p = 2$, the degree of $K_E/L_{\wp}^{\mathrm{nr}}$ is 2, 3, 4, 6, 8, or 24.

As before, we will write $K = K_E$. Let ν_K be a valuation on K such that $\nu_K(p) = e$ and $\nu_K(\pi) = 1$, where π is a uniformizer for K. Let A be the ring of elements of K with valuation ≥ 0.

Proposition 3.1. *Let $\omega(Z) = \left(1 + \sum_{i=1}^{\infty} w_i Z^i\right) dZ$ be the unique normalized invariant differential associated to $\hat{E}$ (as in [Silverman 2009, IV, §4]), with $w_i \in A$ for all $i \geq 1$. Then,*

$$[p](Z) = \sum_{i=1}^{\infty} s_i Z^i \equiv w_{p-1} Z^p + O(Z^{p+1}) \bmod pA.$$

In particular, $s_p \equiv w_{p-1} \bmod pA$. Thus, if $\nu_K(w_{p-1}) < e$, then

$$e_1 = \nu_K(s_p) = \nu_K(w_{p-1}).$$

Otherwise, if $\nu_K(w_{p-1}) \geq e$, then $e_1 \geq e$.

Proof. The congruence is shown in [Katz 1973, Lemma 3.6.5], so here we just give the key ingredients in the proof. Let $\varphi(Z) = Z + \sum_{k=2}^{\infty} (w_{k-1}/k) Z^k$ so that $\omega = d(\varphi(Z))$, and let $\psi(Z)$ be the inverse series to $\varphi(Z)$, so that $\psi(\varphi(Z)) = Z$. Since ω is the normalized invariant differential for $\hat{E}$, it follows that $p\omega(Z) = (\omega \circ [p])(Z)$ (see [Silverman 2009, Chapter IV, Corollary 4.3]), therefore, $[p](Z) = \psi(p\varphi(Z))$. The desired congruence falls out from this and the equality $\psi(\varphi(Z)) = Z$.

The congruence implies that $s_p = w_{p-1} + p\alpha$, for some $\alpha \in A$. In particular,

$$\nu_K(s_p) \geq \min\{\nu_K(w_{p-1}), \nu_K(p\alpha)\} = \min\{\nu_K(w_{p-1}), e + \nu_K(\alpha)\}.$$

If we assume that $\nu_K(w_{p-1}) < e$, then $\nu_K(w_{p-1}) < e + \nu_K(\alpha)$, and the inequality is in fact an equality and $\nu_K(s_p) = \nu_K(w_{p-1})$. Otherwise, if $\nu_K(w_{p-1}) \geq e$, then $e_1 = \nu_K(s_p) \geq e$, as claimed. $\square$

Corollary 3.2. *Let*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad \textit{and} \quad y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6'$$

be two minimal models for an elliptic curve E/A and let $[p](Z) = \sum s_i Z$ and $[p]'(Z) = \sum s_i'(Z)$ be the multiplication-by-p maps for their respective formal groups. Then, there is a constant $u \in A^\times$ such that $s_p \equiv u^{p-1} s_p' \bmod pA$. In particular, if $e_1 < e$, then the number $e_1 = \nu_K(s_p)$ as defined above is independent of the chosen minimal model for the elliptic curve E/A.

Proof. Let

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \text{ and } y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6'$$

be two minimal models, with $a_i, a_i' \in A$, for the same elliptic curve E/A, and let $\hat{E}/A$ and $\hat{E}'/A$ be the formal groups associated to each model, with formal group laws given by $F(X, Y)$ and $F'(X, Y)$, respectively. Since these are minimal models for the same curve E/A, it follows that $(\hat{E}, F)$ and $(\hat{E}', F')$ are isomorphic formal groups; see [Silverman 2009, Chapter VII, Proposition 2.2]. Thus, there is a power series $f(Z) = uZ + O(Z^2)$, for some $u \in A^\times$, such that

$$f(F(X, Y)) = F'(f(X), f(Y)).$$

Let $\omega(Z) = \sum w_n Z^n$, $[p](Z) = \sum s_i Z$ and $\omega'(Z) = \sum w'_n Z^n$, $[p]'(Z) = \sum s'_i(Z)$ be the invariant differentials, and multiplication-by-p maps, for $\hat{E}$ and $\hat{E}'$, respectively. Then, by Proposition 3.1,

$$\begin{aligned} f([p](Z)) &= [p]'(f(Z)) \\ &= \sum s'_i(f(Z)) \equiv w'_{p-1}(f(Z))^p + \cdots \equiv u^p \cdot w'_{p-1} Z^p + O(Z^{p+1}), \\ f([p](Z)) &= u([p](Z)) + \cdots \equiv u(w_{p-1}Z^p + \cdots) + \cdots \equiv u \cdot w_{p-1} Z^p + O(Z^{p+1}). \end{aligned}$$

Therefore, $u^p \cdot w'_{p-1} \equiv u \cdot w_{p-1} \bmod pA$, or $w_{p-1} \equiv u^{p-1} w'_{p-1} \bmod pA$. Hence $s_p \equiv u^{p-1} s'_p \bmod pA$, as claimed.

In particular, if $e_1 < e$, and $e_1 = \nu_K(s_p)$ and $e'_1 = \nu_K(s'_p)$, then there is some $\alpha \in A$ such that $s_p = u^{p-1} s'_p + p\alpha$. Hence,

$$e_1 = \nu_K(s_p) = \nu_K(u^{p-1} s'_p + p\alpha) = \min\{\nu_K(s'_p), e + \nu_K(\alpha)\} = \nu_K(s'_p) = e'_1.$$

Thus, the valuation of s_p is independent of the chosen minimal model for E/A. □

Remark 3.3. Here is an alternative proof of Corollary 3.2 using the Hasse invariant $\mathcal{H}(E, \omega)$ as defined in [Katz 1973, Section 2.0]. Let E/A be given by a minimal model

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

with $a_i \in A$, and let $\omega = dx/(2y + a_1 x + a_3)$ be an invariant differential for E/A. Let $\mathcal{H}(E, \omega)$ be the Hasse invariant. Moreover, let $\hat{E}/A$ be the associated formal group, let

$$\omega(Z) = \left(1 + \sum_{n=1}^{\infty} w_n Z^n\right) dZ = (1 + a_1 Z + (a_1^2 + a_2)Z^2 + \cdots)\, dZ,$$

be the unique normalized invariant differential associated to $\hat{E}$ and write

$$[p](Z) = \sum_{i=1}^{\infty} s_i Z^i,$$

as before. Then, Lemmas 3.6.1 and 3.6.5 of [Katz 1973] imply that $a_p \equiv \mathcal{H}(E, \omega) \bmod pA$.

Now, if
$$y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6'$$
is another minimal model for E/A, then there is a constant $u \in A^\times$ such that the new invariant differential ω' and ω are related by $\omega' = u\omega$, and $\mathcal{H}(E, \omega) = u^{p-1}\mathcal{H}(E, u\omega)$; see [Katz 1973, p. Ka-29]. If $\hat{E}'/A$ is the formal group associated to this new minimal model, and $[p]'(Z) = \sum_{i=1}^{\infty} s_i' Z^i$, then
$$s_p \equiv \mathcal{H}(E, \omega) \equiv u^{p-1}\mathcal{H}(E, u\omega) \equiv u^{p-1}s_p' \bmod pA.$$
Since we have assumed that $e' = \nu(a_p) < e$, the coefficients s_p and s_p' have the same valuation.

Lemma 3.4. *Let E/A be given by a model $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, with $a_i \in A$, and let $\omega(Z) = (1 + \sum_{i=1}^{\infty} w_i Z^i)\, dZ$ be the unique normalized invariant differential associated to $\hat{E}$. Then, $w(Z) \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6][[Z]]$. Moreover, if $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is made into a graded ring by assigning weights* $\mathrm{wt}(a_i) = i$, *then $w_n \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is homogeneous of weight n.*

Proof. Let $f(x, y) = y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6)$ and let $v(Z) \in A[[Z]]$ be the unique power series such that $v(Z) = f(Z, v(Z))$. The existence of $v(Z)$ is shown in [Silverman 2009, Chapter IV, Proposition 1.1], and, moreover, it is also shown that $v(Z) = Z^3(1 + \sum_{k=1}^{\infty} A_k Z^k) \in \mathbb{Z}[a_1, \ldots, a_6][[Z]]$. When we assign weights $\mathrm{wt}(a_i) = i$, then A_n is homogeneous of weight n.

Now define $x(Z) = Z/v(Z)$ and $y(Z) = -1/v(Z)$. It follows that the coefficients of Z^n in $Z^2x(Z)$, $Z^3 \frac{d}{dZ}(x(Z))$, and $Z^3y(Z)$ are homogeneous of weight n. Since
$$\omega(Z) = \left(\frac{\frac{d}{dZ}(x(Z))}{2y(Z) + a_1X(Z) + a_3} \right) dZ = \left(\frac{Z^3 \frac{d}{dZ}(x(Z))}{2Z^3y(Z) + (a_1Z)(Z^2x(Z)) + a_3Z^3} \right) dZ,$$
it follows that w_n, the coefficient of Z^n in $\omega(Z)$, must be homogeneous of degree n, as claimed. $\square$

Lemma 3.5. *Let E/A be given by a model $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, with $a_i \in A$, with discriminant $\Delta(E)$ and j-invariant $j(E)$, and let $\omega(Z) = \sum w_n Z^n$ be the normalized invariant differential on $\hat{E}/A$. Define the constants b_2, b_4, b_6, b_8, c_4, and $c_6 \in A$ as usual, such that $y^2 = x^3 - 27c_4x - 54c_6$ is an alternative model for E/A (which is also minimal as long as $p \neq 2$ or 3), and such that*
$$1728\Delta(E) = c_4^3 - c_6^2 \quad \textit{and} \quad j(E) = \frac{c_4^3}{\Delta}.$$

(1) *With the grading* $\mathrm{wt}(a_k) = k$, *the constants $b_{2k}, c_4, c_6 \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ have weights $2k$, 4 and 6, respectively.*

(2) *We have $w_1^4 \equiv a_1^4 \equiv c_4 \bmod 2A$, and $w_2^2 \equiv (a_1^2 + a_2)^2 \equiv c_4 \bmod 3A$.*

(3) *Let $p > 3$ and let $R = \mathbb{Z}[X, Y]$ be a graded ring with* $\mathrm{wt}(X) = 4$ *and* $\mathrm{wt}(Y) = 6$. *Then, there is a constant $u \in A^\times$ and a homogeneous polynomial $P_p(X, Y) \in R$ of degree $p-1$ such that $w_{p-1} \equiv u^{p-1} P_p(c_4, c_6) \bmod pA$.*

Proof. Part (1) follows by inspection of the formulas that define $b_2, \ldots, b_8, c_4, c_6$ (see for instance [Silverman 2009, Chapter III.1], but notice that there is a typo in the formula for b_2: the correct formula is $b_2 = a_1^2 + 4a_2$).

Part (2) follows from the expression of $\omega(Z)$ in terms of $a_1, \ldots, a_6$,

$$\omega(Z) = (1 + a_1 Z + (a_1^2 + a_2)Z^2 + (a_1^3 + 2a_1a_2 + 2a_3)Z^3 + \cdots)\, dZ,$$

together with the fact that from the formulas one can easily check that $c_4 \equiv b_2^2 \bmod 6$, $b_2 = a_1^2 + 4a_2 \equiv a_1^2 \bmod 2$, and $b_2 \equiv a_1^2 + a_2 \bmod 3$.

To show part (3), let us assume that $p > 3$. Thus, E/A has a minimal model of the form $y^2 = x^3 - 27c_4x - 54c_6$. Let $\hat{E}'/A$ be the formal group associated to this model, and let $\omega'(Z) = \sum w_n' Z^n$ be its normalized invariant differential. By Lemma 3.4, w_{p-1} may be expressed as a homogeneous polynomial in $\mathbb{Z}[a_4', a_6']$, where $a_4' = -27c_4$ and $a_6' = -54c_6$. Hence, there is a polynomial $P_p \in R = \mathbb{Z}[X, Y]$ such that $w_{p-1} = P_p(c_4, c_6)$. Now, if E/A is given by any other minimal model, Proposition 3.1 and Corollary 3.2 combined say that there exists some $u \in A^\times$ such that, as claimed,

$$w_{p-1} \equiv s_p \equiv u^{p-1} s_p' \equiv u^{p-1} w_{p-1}' \equiv u^{p-1} P_p(c_4, c_6) \bmod pA. \qquad \square$$

Before we state the next result, we define quantities $r(p)$ and $s(p)$ for each prime $p > 3$, by

$$r(p) = \begin{cases} 1, & \text{if } p \equiv 5 \text{ or } 11 \bmod 12, \\ 0, & \text{if } p \equiv 1 \text{ or } 7 \bmod 12, \end{cases} \quad \text{and} \quad s(p) = \begin{cases} 1, & \text{if } p \equiv 3 \bmod 4, \\ 0, & \text{if } p \equiv 1 \bmod 4. \end{cases}$$

Equivalently, $r(p) = \frac{1}{2}\left(1 - \left(\frac{-3}{p}\right)\right)$ and $s(p) = \frac{1}{2}\left(1 - \left(\frac{-4}{p}\right)\right)$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

Lemma 3.6. *Let $p > 3$ be a prime, and let $R = \mathbb{Z}[X, Y]$ be a graded ring with* $\mathrm{wt}(X) = 4$ *and* $\mathrm{wt}(Y) = 6$. *Suppose $P(X, Y) \in R$ is homogeneous of degree $p-1$, and let Δ and j be two extra variables such that $1728\Delta = X^3 - Y^2$ and $\Delta \cdot j = X^3$. Then, there is some polynomial $Q(T) \in \mathbb{Z}[T]$ such that*

$$P(X, Y) = X^{r(p)} Y^{s(p)} \Delta^{\frac{p-\alpha}{12}} Q(j),$$

where $\alpha = 1, 5, 7$ or 11, and such that $p \equiv \alpha \bmod 12$.

Proof. Suppose that $p > 3$ is a prime with $p \equiv \alpha \bmod 12$, with $\alpha = 1, 5, 7$ or 11. Since $P(X, Y)$ is homogeneous of degree $p-1$, we can write

$$P(X, Y) = \sum c_{a,b} X^a Y^b$$

such that $a, b \geq 0$, $4a + 6b = p - 1$, and $c_{a,b} \in \mathbb{Z}$. Since $p \equiv \alpha \bmod 12$, there is some integer $t \geq 0$ such that $p = \alpha + 12t$. In particular, $4a + 6b = (\alpha - 1) + 12t$, or $2a + 3b = (\alpha - 1)/2 + 6t$. Notice that $2r(p) + 3s(p) = (\alpha - 1)/2$. It follows that $a, b > 0$, and we may write

$$P(X, Y) = \sum c_{a,b} X^a Y^b = X^{r(p)} Y^{s(p)} \sum c_{a,b} X^{a-r(p)} Y^{b-s(p)}$$

and $2(a - r(p)) + 3(b - s(p)) = 6t$. We conclude that $a - r(p) \equiv 0 \bmod 3$, and $b - s(p) \equiv 0 \bmod 2$. Let us write $a - r(p) = 3f$ and $b - s(p) = 2g$, so that

$$P(X, Y) = X^{r(p)} Y^{s(p)} \sum c_{3f+r(p),2g+s(p)} (X^3)^f (Y^2)^g,$$

where $f, g \geq 0$ and $f + g = t = (p - \alpha)/12$. Put $d_{f,g} = c_{3f+r(p),2g+s(p)}$. Then,

$$\begin{aligned} P(X, Y) &= X^{r(p)} Y^{s(p)} \sum d_{f,g} (X^3)^f (Y^2)^g \\ &= X^{r(p)} Y^{s(p)} \sum d_{f,g} (X^3)^f (X^3 - 1728\Delta)^{\frac{p-\alpha}{12} - f} \\ &= X^{r(p)} Y^{s(p)} \Delta^{\frac{p-\alpha}{12}} \sum d_{f,g} \Big(\frac{X^3}{\Delta}\Big)^f \Big(\frac{X^3 - 1728\Delta}{\Delta}\Big)^{\frac{p-\alpha}{12} - f} \\ &= X^{r(p)} Y^{s(p)} \Delta^{\frac{p-\alpha}{12}} \sum d_{f,g} j^f (j - 1728)^{\frac{p-\alpha}{12} - f}. \end{aligned}$$

Hence, if we define a polynomial

$$Q(T) = \sum d_{f,g} T^f (T - 1728)^{\frac{p-\alpha}{12} - f} \in \mathbb{Z}[T],$$

then $P(X, Y) = X^{r(p)} Y^{s(p)} \Delta^{\frac{p-\alpha}{12}} Q(j)$, as desired. □

Definition 3.7. Let $p > 3$ be a prime and let $P_p(X, Y)$ be the polynomial whose existence was shown in Lemma 3.5. We define $Q_p(T) \in \mathbb{Z}[T]$ as the unique polynomial with integer coefficients such that

$$P_p(X, Y) = X^{r(p)} Y^{s(p)} \Delta^{\frac{p-\alpha}{12}} Q_p(j),$$

where, as usual, $1728\Delta = X^3 - Y^2$ and $\Delta \cdot j = X^3$, and $\alpha = 1, 5, 7$ or 11 such that $p \equiv \alpha \bmod 12$.

Remark 3.8. Let $p > 3$. The polynomial $P_p(c_4, c_6)$ of Lemma 3.5 can be explicitly calculated (mod pA) as follows. Let E/A be given by

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

with $a_i \in A$, and let $\omega = dx/(2y + a_1 x + a_3)$ be an invariant differential for E/A. Let $\mathscr{H}(E, \omega)$ be the Hasse invariant (as in Remark 3.3). Then $w_{p-1} \equiv \mathscr{H}(E, \omega) \bmod pA$. The curve E/A is also given by a minimal model $E'/A : y^2 = x^3 - 27c_4 x - 54c_6$ and it is well known that the Hasse invariant $\mathscr{H}(E', \omega')$ of a curve given by $y^2 = f(x)$

is congruent to the coefficient of x^{p-1} in $f(x)^{(p-1)/2}$ modulo pA; see, for instance, [Silverman 2009, Chapter V, Theorem 4.1(a)]. Thus,

$$\begin{aligned} P_p(c_4,c_6) &\equiv \sum_{\frac{p-1}{6}\le k\le \frac{p-1}{4}} (-1)^k \binom{\frac{p-1}{2}}{k}\binom{k}{3k-\frac{p-1}{2}}(27c_4)^{3k-\frac{p-1}{2}}(54c_6)^{\frac{p-1}{2}-2k} \\ &\equiv \sum_{\substack{m,n\ge 0\\ 4m+6n=p-1}} (-1)^{m+n}\binom{\frac{p-1}{2}}{m+n}\binom{m+n}{m}(27c_4)^m(54c_6)^n \bmod pA. \end{aligned}$$

For instance, $P_5=-54c_4$, $P_7=-162c_6$, $P_{11}=29160c_4c_6$, and

$$P_{13}=-393660c_4^3+43740c_6^2=\Delta(E)(-349920j(E)-75582720).$$

Notice these polynomials satisfy the conclusions of Lemma 3.6, with $Q_5(T)=-54$, $Q_7(T)=-162$, $Q_{11}(T)=29160$, $Q_{13}(T)=-349920T-75582720$.

Theorem 3.9. *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above a prime p. Let $K=K_E$ be the extension of $L_\wp^{\mathrm{nr}}$ defined above, let A, $e=\nu_K(p)$, and e_1 be as before, and let $e(\wp,L)$ be the ramification index of $\wp$ in $L/\mathbb{Q}$. Let $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$ be a minimal model for E/A with good reduction, and let $c_4,c_6\in A$ be the usual quantities associated to this model.*

(1) *If $p=2$, and $(\nu_K(c_4))/4<e$, then*

$$e_1=\frac{\nu_K(c_4)}{4}=\frac{\nu_K(j(E))}{12}=\frac{e\cdot\nu_\wp(j(E))}{12e(\wp,L)}.$$

(2) *If $p=3$, and $(\nu_K(c_4))/2<e$, then*

$$e_1=\frac{\nu_K(c_4)}{2}=\frac{\nu_K(j(E))}{6}=\frac{e\cdot\nu_\wp(j(E))}{6e(\wp,L)}.$$

(3) *If $p>3$, and $\lambda=r(p)\nu_K(c_4)+s(p)\nu_K(c_6)+\nu_K(Q_p(j(E)))<e$, then*

$$\begin{aligned} e_1=\lambda &= r(p)\frac{\nu_K(j(E))}{3}+s(p)\frac{\nu_K(j(E)-1728)}{2}+\nu_K(Q_p(j(E))) \\ &=\frac{e}{e(\wp,L)}\cdot\Big(r(p)\frac{\nu_\wp(j(E))}{3}+s(p)\frac{\nu_\wp(j(E)-1728)}{2}+\nu_\wp(Q_p(j(E)))\Big). \end{aligned}$$

Otherwise, $e_1\ge e$.

Proof. Let $\hat{E}/A$ be the formal group associated to E and let $[p](Z)=\sum_{i=1}^\infty s_iZ^i$ be the multiplication-by-p map on $\hat{E}$. By definition, $e=\nu_K(p)$ and $e_1=\nu_K(s_p)$. Moreover, by Proposition 3.1, we know that if $\nu_K(w_{p-1})<e$, then $e_1=\nu_K(w_{p-1})$ where $\omega(Z)=\big(1+\sum_{i=1}^\infty w_iZ^i\big)\,dZ$ is the normalized invariant differential for $\hat{E}$, and $e_1\ge e$ otherwise. Let us assume that $\nu_K(w_{p-1})<e$. Now we can use Lemma 3.5:

(1) If $p = 2$, then $w_1^4 \equiv c_4 \bmod 2A$. Since we are assuming $\nu_K(2) = e > \nu_K(w_1)$, we must have $4\nu_K(w_1) = \nu_K(w_1^4) = \nu_K(c_4)$, and it follows that $e_1 = \nu_K(c_4)/4$.

(2) Similarly, if $p = 3$, then $w_2^2 \equiv c_4 \bmod 3A$. Hence, $e_1 = \nu_K(c_4)/2$.

(3) Suppose $p > 3$. Then, there is a constant $u \in A^\times$ and a homogeneous polynomial $P_p(X, Y) \in R$ of degree $p - 1$ (where $\mathrm{wt}(X) = 4$ and $\mathrm{wt}(Y) = 6$) such that $w_{p-1} \equiv u^{p-1} P_p(c_4, c_6) \bmod pA$. Let $\alpha = 1, 5, 7$, or 11, such that $p \equiv \alpha \bmod 12$. Then, by Lemma 3.6, there is a polynomial $Q_p(T) \in \mathbb{Z}[T]$ such that
$$w_{p-1} \equiv u^{p-1} c_4^{r(p)} c_6^{s(p)} \Delta(E)^{\frac{p-\alpha}{12}} Q_p(j(E)) \bmod pA.$$
Since E/L has potential good reduction, the j-invariant $j(E)$ is integral at $\wp$ (see [Silverman 2009, Chapter VII, Proposition 5.5]), thus via our fixed embedding ι, we have $j(E) \in A$. Since $j(E) \in A \cap L_\wp$, and $Q_p(T) \in \mathbb{Z}[T]$, it follows that $Q_p(j(E)) \in A \cap L_\wp$. Therefore, $\nu_K(Q_p(j(E))$ is a nonnegative multiple of $e/e(\wp, L)$. Define λ as in the statement of the theorem, so that λ equals $\nu_K(u^{p-1} c_4^{r(p)} c_6^{s(p)} \Delta(E)^{(p-\alpha)/12} Q_p(j(E)))$. Thus, if $\lambda < e$, it follows that $\nu_K(w_{p-1}) = \lambda$ and Proposition 3.1 implies that $e_1 = \lambda$, as desired. □

When $p \equiv 1 \bmod 12$, the quantities $r(p)$ and $s(p)$ vanish simultaneously and we obtain the following simpler formula.

Corollary 3.10. *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above a prime $p \equiv 1 \bmod 12$. Let K_E, A, e and e_1 be as before, and let $e(\wp, L)$ be the ramification index of $\wp$ in $L/\mathbb{Q}$. Let $Q_p(T) \in \mathbb{Z}[T]$ be as in Definition 3.7, and define an integer λ by*
$$\lambda = \nu_K(Q_p(j(E))) = \frac{e}{e(\wp, L)} \cdot \nu_\wp(Q_p(j(E))).$$
If $\lambda < e$, then $e_1 = \lambda \geq 1$. Otherwise, if $\lambda \geq e$, then $e_1 \geq e$. In particular, if $e(\wp, L) = 1$ or $\nu_\wp(Q_p(j(E))) = 0$, then $e_1 \geq e$.

The value of $e/e(\wp, L)$, and therefore the value of e, can be obtained directly from a model of E/L, thanks to the classification of Néron models. As a reference for the following theorem, the reader can consult [Néron 1964, p. 124–125] or [Serre 1972, §5.6, p. 312], where $\mathrm{Gal}(K_E/L_\wp^{\mathrm{nr}})$ is denoted by Φ_p, and therefore $e/e(\wp, L) = \mathrm{Card}(\Phi_p)$. Notice, however, that the section we cite of [Serre 1972] restricts its attention to the case $L = \mathbb{Q}$.

Theorem 3.11. *Let $p > 3$, let E/L be an elliptic curve with potential good reduction, and let Δ_L be the discriminant of any model of E defined over L. Let K_E be the smallest extension of $L_\wp^{\mathrm{nr}}$ such that E/K_E has good reduction. Then $e/e(\wp, L) = [K_E : L_\wp^{\mathrm{nr}}] = 1, 2, 3, 4,$ or 6. Moreover:*

- *$e/e(\wp, L) = 2$ if and only if $\nu_\wp(\Delta_L) \equiv 6 \bmod 12$,*

- $e/e(\wp, L) = 3$ *if and only if* $\nu_\wp(\Delta_L) \equiv 4$ *or* $8 \bmod 12$,
- $e/e(\wp, L) = 4$ *if and only if* $\nu_\wp(\Delta_L) \equiv 3$ *or* $9 \bmod 12$,
- $e/e(\wp, L) = 6$ *if and only if* $\nu_\wp(\Delta_L) \equiv 2$ *or* $10 \bmod 12$.

Therefore, our formula for e_1 only depends on the $\wp$-adic valuation of $j(E)$, $j(E) - 1728$, and Δ_L.

Corollary 3.12. *Let $p > 3$ be a prime and let E/L be an elliptic curve with potentially supersingular good reduction at a prime $\wp$ above p. Let $e(\wp, L)$ be the ramification index of $\wp$ in $L/\mathbb{Q}$. Let $j(E) \in L$ be its j-invariant, let Δ_L be the discriminant of a model for E over L, and define an integer λ as follows*:

- *If* $\nu_\wp(\Delta_L) \equiv 6 \bmod 12$, *then* $e/e(\wp, L) = 2$. *Let*
$$\lambda = \tfrac{2}{3} r(p)\nu_\wp(j(E)) + s(p)\nu_\wp(j(E) - 1728) + 2\nu_\wp(Q_p(j(E))).$$
- *If* $\nu_\wp(\Delta_L) \equiv 4$ *or* $8 \bmod 12$, *then* $e/e(\wp, L) = 3$. *Let*
$$\lambda = r(p)\nu_\wp(j(E)) + \tfrac{3}{2} s(p)\nu_\wp(j(E) - 1728) + 3\nu_\wp(Q_p(j(E))).$$
- *If* $\nu_\wp(\Delta_L) \equiv 3$ *or* $9 \bmod 12$, *then* $e/e(\wp, L) = 4$. *Let*
$$\lambda = \tfrac{4}{3} r(p)\nu_\wp(j(E)) + 2s(p)\nu_\wp(j(E) - 1728) + 4\nu_\wp(Q_p(j(E))).$$
- *If* $\nu_\wp(\Delta_L) \equiv 2$ *or* $10 \bmod 12$, *then* $e/e(\wp, L) = 6$. *Let*
$$\lambda = 2r(p)\nu_\wp(j(E)) + 3s(p)\nu_\wp(j(E) - 1728) + 6\nu_\wp(Q_p(j(E))).$$

If $\lambda < e$, then $e_1 = \lambda$. Otherwise, if $\lambda \geq e$, then $e_1 \geq e$.

4. More examples

In this section we provide a few examples of usage of the formula for e_1 developed in Theorem 3.9.

Example 4.1. Let us return to the curve $E/\mathbb{Q}$ with label 121c2. In Example 2.1 we showed a minimal model over $\mathbb{Q}_{11}^{\mathrm{nr}}(\sqrt[3]{11})$ and we proved that $e_1 = 1$. We can verify the value $e_1 = 1$ using the formula of Theorem 3.9. Here $p = 11$, so $r(11) = s(11) = 1$, and $L = \mathbb{Q}$, so $e(\wp, L) = 1$. Moreover, for the chosen minimal model we have quantities
$$c_4 = 131\sqrt[3]{11}, \quad \text{and} \quad c_6 = -4973.$$
Moreover, we saw in Remark 3.8 that $Q_{11}(T) = 29160 = 2^3 \cdot 3^6 \cdot 5$. Thus,
$$\begin{aligned}\lambda &= \nu_K(c_4) + \nu_K(c_6) + \nu_K(Q_p(j)) \\ &= \nu_K(131\sqrt[3]{11}) + \nu_K(-4973) + \nu_K(29160) = 1 + 0 + 0 = 1.\end{aligned}$$

Since $\lambda < e = 3$, we conclude that $e_1 = \lambda = 1$. We may also verify this value using the formula in Corollary 3.12. The discriminant of the model for $E/\mathbb{Q}$ given in Example 2.1 is $\Delta_{\mathbb{Q}} = -11^8$; we have $j(E) = -11 \cdot 131^3$ and $j(E) - 1728 = -4973^2$. Hence,

$$\lambda = r(p)\nu_p(j(E)) + \tfrac{3}{2}s(p)\nu_p(j(E) - 1728) + 3\nu_p(Q_p(j(E))) \\ = 1 \cdot 1 + \tfrac{3}{2} \cdot 1 \cdot 0 + 3 \cdot 0 = 1,$$

and so $e_1 = \lambda = 1$.

Example 4.2. Let $E'/\mathbb{Q}$ be the curve with label 121a1, given by a Weierstrass equation

$$y^2 + xy + y = x^3 + x^2 - 30x - 76.$$

The j-invariant of E' is $j(E') = -11 \cdot 131^3$, equal to $j(E)$, where E is curve 121c2 as in Examples 2.1 and 4.1. Thus, E' is a quadratic twist of E. Indeed, E' is the quadratic twist of E by -11. In particular, E and E' are isomorphic over $\mathbb{Q}(\sqrt{-11})$. Since $K_E = \mathbb{Q}_{11}^{\mathrm{nr}}(\sqrt[3]{11})$, it follows that

$$K_{E'} = \mathbb{Q}_{11}^{\mathrm{nr}}(\sqrt[3]{11}, \sqrt{-11}) = \mathbb{Q}_{11}^{\mathrm{nr}}(\sqrt[6]{-11}).$$

Thus, $e = e(E') = 6$, while $e = e(E) = 3$, and $\nu_{K_{E'}}(\kappa) = 2\nu_{K_E}(\kappa)$ for any $\kappa \in K_E \subseteq K_{E'}$. Moreover, since $K_E \subseteq K_{E'}$, the minimal model for E over K_E,

$$y^2 + \sqrt[3]{11}xy = x^3 + \sqrt[3]{11^2}x^2 + 3\sqrt[3]{11}x + 2,$$

is also a minimal model for E' over $K_{E'}$. It follows that

$$\lambda(E') = \nu_{K_{E'}}(c_4) + \nu_{K_{E'}}(c_6) + \nu_{K_{E'}}(Q_{11}(j)) \\ = 2\nu_{K_E}(c_4) + 2\nu_{K_E}(c_6) + 2\nu_{K_E}(Q_{11}(j)) = 2 \cdot 1 + 0 + 0 = 2,$$

where we have used the fact that $c_4, c_6 \in K_E$. Since $\lambda(E') < e(E') = 6$, we conclude that $e_1(E') = 2$.

Alternatively, we can verify $e_1(E') = 2$ using the formula of Corollary 3.12. The discriminant of the rational model for $E'/\mathbb{Q}$ listed above is $\Delta_{\mathbb{Q}} = -11^2$. Moreover, $j(E') = -11 \cdot 131^3$, and $j(E') - 1728 = -4973^2$. Hence

$$\lambda = 2r(p)\nu_p(j) + 3s(p)\nu_p(j - 1728) + 6\nu_p(Q_p(j)) = 2 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 0 + 6 \cdot 0 = 2,$$

and so $e_1 = \lambda = 2$.

Example 4.3. In Example 2.2 we looked at the elliptic curve $E/\mathbb{Q}$ with label 27a4, for $p = 3$, and concluded that $e_1 = 2$. The constant c_4 (which we will not write explicitly here due again to its unwieldy form in terms of γ) for the minimal model we used to compute e_1 has valuation $\nu_K(c_4) = 4$, in agreement with the formula

$e_1 = \nu_K(c_4)/2$ given by Theorem 3.9. Alternatively, and much easier to compute,

$$\lambda = \frac{e \cdot \nu_3(j(E))}{6} = \frac{12 \cdot \nu_3(-2^{15} \cdot 3 \cdot 5^3))}{6} = 2.$$

Since $2 = \lambda < e = 12$, we conclude that $e_1 = \lambda = 2$.

Example 4.4. Let $L = \mathbb{Q}(\sqrt{13})$, put $p = 13$ and $\wp = (\sqrt{13})$, and let E/L be the elliptic curve with j-invariant j_0 as described in Example 2.3. There we found that $K = L_\wp^{\mathrm{nr}}$. Thus, $e = e(\wp, L) = 2$, and we calculated directly that $e_1 = 1$. Since $p \equiv 1 \bmod 12$, we may use Corollary 3.10 to verify that indeed $e_1 = 1$. Here $e(\wp, L) = 2$, and we know from Remark 3.8 that $Q_{13}(T) = -349920T - 75582720$. One can verify (using Sage or Magma) that

$$\nu_\wp(Q_{13}(j_0)) = \nu_\wp(-349920 j_0 - 75582720) = 1.$$

Thus,

$$\lambda = \nu_K(Q_{13}(j(E)) = \frac{e}{e(\wp, L)} \nu_\wp(Q_{13}(j_0)) = \nu_\wp(Q_{13}(j_0)) = 1.$$

Since $1 = \lambda < 2 = e$, it follows from Corollary 3.10 that $e_1 = \lambda = 1$, as desired.

Example 4.5. In this example (see Table 1) we provide the values of e and e_1, calculated using our formula, and verified using the multiplication-by-p map on the formal group, for all those elliptic curves with potentially supersingular reduction that appear as rational points on modular curves $X_0(p)$ of genus > 0 (if the curve $X_0(p)$ has genus 0, then $p = 2, 3, 5, 7$, or 13, and there are infinitely many rational points given by a 1-parameter family; see [Maier 2009]). These points are well-known, but seem to be spread out across the literature. Our main references are [Birch and Kuyk 1975, pp. 78–80; Mazur 1978; Kenku 1982].

The reader may notice that in Table 1 the difference $e - e_1$, and the value e_1, are always 1 or 2, for all $p > 3$. In addition, in Example 4.2 we have seen an example of a curve with $e - e_1 = 6 - 2 = 4$. A priori, we know that $e = 1, 2, 3, 4$ or 6 for elliptic curves over $\mathbb{Q}$ (see [Serre 1972, §5.6, p. 312]), so if we assume $e_1 < e$, then e_1 and $e - e_1$ may take the values $1, 2, 3, 4$, or 5. In fact, we will show next that the difference $e - e_1$ and e_1 may only take the values 1, 2, or 4, when $L = \mathbb{Q}$ and more generally whenever $e(\wp, L) = 1$.

Corollary 4.6. *Let E/L be an elliptic curve with potentially supersingular reduction at a prime $\wp$ lying above a prime $p > 3$, and let e and e_1 be defined as in Section 1. Assume that $e_1 < e$, and also assume that $e(\wp, L) = 1$. Then e_1 and $e - e_1$ can only take the values* 1, 2, *or* 4. *Moreover, $j(E) \equiv 0$ or $1728 \bmod \wp$, and*

(1) *If $j(E) \equiv 0 \bmod \wp$, then $e = 3$ or 6, and $e_1 = ek/3$, where $k = \nu_\wp(j(E)) = 1$ or 2.*

(2) *If $j(E) \equiv 1728 \bmod \wp$, then $e = 2$ or 4, and $e_1 = e/2$.*

j-invariant	p	Cremona label(s)	Good reduction over	e	e_1
$-2^{15}\,3\cdot 5^3$	3	27A2, 27A4	L (see caption)	12	2
$-11\cdot 131^3$	11	121C2	$\mathbb{Q}(\sqrt[3]{11})$	3	1
-2^{15}		121B1, 121B2	$\mathbb{Q}(\sqrt[4]{11})$	4	2
-11^2		121C1	$\mathbb{Q}(\sqrt[3]{11})$	3	2
$-17^2\,101^3/2$	17	14450P1	$\mathbb{Q}(\sqrt[3]{17})$	3	2
$-17\cdot 373^3/2^{17}$		14450P2	$\mathbb{Q}(\sqrt[3]{17})$	3	1
$-2^{15}\,3^3$	19	361A1, 361A2	$\mathbb{Q}(\sqrt[4]{19})$	4	2
$-2^{18}\,3^3\,5^3$	43	1849A1, 1849A2	$\mathbb{Q}(\sqrt[4]{43})$	4	2
$-2^{15}\,3^3\,5^3\,11^3$	67	4489A1, 4489A2	$\mathbb{Q}(\sqrt[4]{67})$	4	2
$-2^{18}\,3^3\,5^3\,23^3\,29^3$	163	26569A1, 26569A2	$\mathbb{Q}(\sqrt[4]{163})$	4	2

Table 1. j-invariants with potentially supersingular reduction in $X_0(p)$. In the first row, $L=\mathbb{Q}(\sqrt[4]{3},\beta)$, where $\beta^3-120\beta+506=0$.

Proof. Let $p>3$ be a prime, assume that $e_1<e$, let K_E be the extension of degree e of $L_{\wp}^{\mathrm{nr}}$ defined above, and fix a minimal model of E over K_E with good supersingular reduction. Let Δ be its discriminant, and let c_4 and c_6 be the usual quantities. Let $\lambda=r(p)\nu_K(c_4)+s(p)\nu_K(c_6)+\nu_K(Q_p(j(E)))$ as in Theorem 3.9. If $\lambda\ge e$ then $e_1\ge e$, but we have assumed that $e_1<e$, and hence $e_1=\lambda$. Notice that we have assumed $e(\wp,L)=1$. In this case, $\nu_K(Q_p(j(E)))=e\cdot\nu_{\wp}(Q_p(j(E)))$ is a multiple of e. Since $e_1=\lambda<e$, it follows that $\nu_K(Q_p(j(E)))=0$, and under our assumptions

$$e_1=r(p)\nu_K(c_4)+s(p)\nu_K(c_6). \tag{4-1}$$

Since $\nu_K(\Delta)=0$ and $p\neq 2,3$, the equality $1728\Delta=c_4^3-c_6^2$ implies that $\nu_K(c_4)$ and $\nu_K(c_6)$ cannot be simultaneously positive. If both were zero, then our formula (4-1) would say $1\le e_1=0$, a contradiction, so one of the valuations must be positive and the other one must vanish.

If $\nu_K(c_4)>0$ and $\nu_K(c_6)=0$, then $\nu_K(j(E))=\nu_K(c_4^3/\Delta)=3\nu_K(c_4)>0$. Since $j(E)\in L$, it follows that $j(E)\equiv 0 \bmod \wp$. In particular, $\nu_K(j)$ is a multiple of $e/e(\wp,L)=e$, say $\nu_K(j)=ek$, for some $k\ge 1$. Theorem 3.9 says that $e_1=r(p)\nu_K(c_4)+s(p)\nu_K(c_6)=r(p)\nu_K(c_4)$. Thus, we must have $r(p)=1$ (in particular, $p\equiv 5 \bmod 6$ in this case) and $e_1=\nu_K(c_4)$, otherwise $0=e_1\ge 1$, a contradiction. Hence,

$$e_1=\nu_K(c_4)=\frac{\nu_K(j)}{3}=\frac{ek}{3}.$$

Since $e_1 < e$ by assumption, it follows that $1 \le k < 3$. In addition, e_1 is a positive integer, so $ek \equiv 0 \mod 3$, hence $e \equiv 0 \mod 3$. Finally, $e = 1, 2, 3, 4$, or 6, so $e = 3$ or 6 in this case, and $e_1 = 1, 2$, or 4, as claimed.

If instead we have $\nu_K(c_4) = 0$ and $\nu_K(c_6) > 0$, we have $e_1 = \nu_K(c_6)$ (we must have $p \equiv 3 \mod 4$ in this case). The equality $c_6^2 = \Delta \cdot (j(E) - 1728)$ implies that

$$e_1 = \nu_K(c_6) = \frac{\nu_K(j - 1728)}{2} > 0.$$

It follows that $j \equiv 1728 \mod \wp$ and $\nu_K(j - 1728) = eh$ for some $h \ge 1$. Since $e_1 < e$, we have $h < 2$ so $h = 1$, and since e_1 is an integer, we have $e \equiv 0 \mod 2$. Thus, $e = 2, 4$, or 6, and therefore, $e_1 = 1, 2$, or 3. However, we shall show next that $j \equiv 1728 \mod \wp$ and $e = 6$ is not possible. Thus, $e_1 = 1$, or 2, and the proof of the corollary would be finished.

Indeed, suppose $j \equiv 1728 \mod \wp$ and $e = 6$. Let Δ_L, $c_{4,L}$ and $c_{6,L}$ be the discriminant and the usual constants associated to the original model of E over L. By the work of Néron on minimal models (Theorem 3.11), the degree $e = 6$ if and only if $\nu_\wp(\Delta_L) \equiv 2$ or $10 \mod 12$. Since $\Delta_L \cdot j(E) = (c_{4,L})^3$, and $j \equiv 1728 \mod \wp$, with $p > 3$, it follows that $\nu_\wp(\Delta_L) = 3\nu_\wp(c_{4,L})$ and therefore $\nu_\wp(\Delta_L) \equiv 0 \mod 3$, and we cannot have $\nu_\wp(\Delta_L) \equiv 2$ or $10 \mod 12$. This is a contradiction, and therefore $e = 6$ and $j \equiv 1728 \mod \wp$ are incompatible. This ends the proof of the corollary. □

Corollary 4.7. *Under the notation and assumptions of Corollary 4.6, if $p > 3$ and $e_1 < e$, then $e_1 \le 2e/3$. In particular, $pe/(p+1) > e_1$.*

Proof. Let $p \ge 5$ and $e_1 < e$. It follows from Corollary 4.6 that, in all cases, we have $e_1 = e/3$, or $e_1 = 2e/3$ or $e_1 = e/2$. Thus, $e_1 \le 2e/3$. In particular,

$$\frac{pe}{p+1} \ge \frac{5e}{6} > \frac{2e}{3} \ge e_1.$$ □

5. Torsion points

Lemma 5.1 (Serre). *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above p. Let $K = K_E$ be the smallest extension of $L_\wp^{\mathrm{nr}}$ such that E/K has good (supersingular) reduction at $\wp$, and let $e = \nu_K(p)$ be its ramification index. Let A, $e_1 = \nu(s_p)$ and π be as above, so that $[p](Z) = pf(Z) + \pi^{e_1} g(Z^p) + h(Z^{p^2})$, where $f(Z)$, $g(Z)$ and $h(Z)$ are power series in $Z \cdot A[[Z]]$, with $f'(0) = g'(0) = h'(0) \in A^\times$.*

1. *If $pe/(p+1) \le e_1$, then $[p](Z) = 0$ has $p^2 - 1$ roots of valuation $e/(p^2 - 1)$.*
2. *If $pe/(p+1) > e_1$, then $[p](Z) = 0$ has $p - 1$ roots of valuation $(e - e_1)/(p - 1)$ and $p^2 - p$ roots with valuation $e_1/(p(p-1))$.*

Proof. This is shown in [Serre 1972, §1.10, pp. 271–272]. If $pe/(p+1) < e_1$, the Newton polygon for $[p](Z)$ has only one segment and if $pe/(p+1) \geq e_1$, then the polygon has two segments (see Remark 2.4). □

Theorem 5.2. *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above a prime $p > 3$, and let e and e_1 be defined as above. Let $P \in E[p]$ be a nontrivial p-torsion point.*

(1) *Suppose $e_1 \geq pe/(p+1)$. Then the ramification index of any prime over $\wp$ in the extension $L(P)/L$ is divisible by $(p^2-1)/\gcd(p^2-1, e)$.*

(2) *Suppose $e_1 < pe/(p+1)$.*
   - *There are p^2-p points P in $E[p]$ such that the ramification index of a prime above $\wp$ in $L(P)/L$ is divisible by $(p-1)p/\gcd(p(p-1), e_1)$.*
   - *There are $p-1$ points P in $E[p]$ such that the ramification index of any prime above $\wp$ in $L(P)/L$ is divisible by $(p-1)/\gcd(p-1, e-e_1)$.*

In particular, if $e(\wp, L) = 1$ and $e_1 < e$, then $e_1 < pe/(p+1)$ and the ramification index of any prime over $\wp$ in $L(P)/L$ is divisible by $(p-1)/\gcd(p-1, 4)$.

Proof. Let E/L be an elliptic curve with potentially supersingular reduction at $\wp$ above $p > 3$, and let $P \in E(\overline{L})[p]$ be a point of exact order p. Let $\iota : \overline{L} \hookrightarrow \overline{L}_\wp$ be a fixed embedding. Let $F = L(P)$ and let $\mathfrak{P}$ be the prime of F above $\wp$ associated to the embedding ι. Let K be the smallest extension of $L_\wp^{\mathrm{nr}}$ such that E/K has good (supersingular) reduction at $\wp$. Choose a model E'/K with good reduction and isomorphic to E over K, and let $T \in E'(K)[p]$ be the point that corresponds to $\iota(P)$ on $E(\overline{L}_\wp)$. Suppose that the degree of the extension $K(T)/K$ is g. Since $K/L_\wp^{\mathrm{nr}}$ is of degree $e/e(\wp, L)$, it follows that the degree of $K(T)/L_\wp^{\mathrm{nr}}$ is $eg/e(\wp, L)$.

Let $\mathscr{F} = \iota(F) \subseteq \overline{L}_\wp$. Since E and E' are isomorphic over K, it follows that $K(T) = K\mathscr{F}$ and, therefore, the degree of the extension $K\mathscr{F}/L_\wp^{\mathrm{nr}}$ is $eg/e(\wp, L)$. Since $K/L_\wp^{\mathrm{nr}}$ is Galois (see Section 1), $g = [K(T) : K] = [\mathscr{F}L_\wp^{\mathrm{nr}} : K \cap \mathscr{F}L_\wp^{\mathrm{nr}}]$, so the degree of $[\mathscr{F}L_\wp^{\mathrm{nr}} : L_\wp^{\mathrm{nr}}]$ equals $g \cdot k$ where $k = [K \cap \mathscr{F}L_\wp^{\mathrm{nr}} : L_\wp^{\mathrm{nr}}]$. Hence, the degree of $\mathscr{F}/L_\wp$ is divisible by gk and, in particular, the ramification index of the prime ideal $\mathfrak{P}$ over $\wp$ in the extension $L(P)/L$ is divisible by gk, where $g = [K(T) : K]$. Thus, we just need to show that $[K(T) : K]$ satisfies the divisibility properties that are claimed in the statement of the theorem.

Let $T \in E'[p]$ be an arbitrary point on $E'(\overline{K})$ of exact order p, and write t for the corresponding torsion point in the formal group, that is, $t = -x(T)/y(T) \in \hat{E}'(\mathcal{M}_p)$.

(1) Let us first assume that $e_1 \geq pe/(p+1)$. By Lemma 5.1, the valuation of $t \in \hat{E}'[p]$ is $e/(p^2-1)$. Hence, the ramification index in the extension $K(T)/K$ is divisible by the quantity $(p^2-1)/\gcd(p^2-1, e)$, as claimed.

(2) Now let us suppose that $e_1 < pe/(p+1)$. By Lemma 5.1, there are $p-1$ points in $\hat{E}'[p]$ with valuation $(e-e_1)/(p-1)$ and p^2-p points with valuation

$e_1/(p(p-1))$, respectively. Thus, the ramification index of $K(T)/K$ is divisible by $(p-1)/\gcd(p-1, e-e_1)$ or $p(p-1)/\gcd(p(p-1), e_1)$, respectively.

Finally, suppose that $e(\wp, L) = 1$ and $e_1 < e$. Then, Corollary 4.7 shows that $pe/(p+1) > e_1$. Moreover, we showed in Corollary 4.6 that, when $p > 3$ and $e_1 < e$, the numbers e_1 and $e-e_1$ can only take the values 1, 2, or 4. Thus, the ramification index in $K(T)/K$ is divisible by at least $(p-1)/\gcd(p-1, 4)$, as claimed. This concludes the proof of the theorem. □

Example 5.3. Let $E/\mathbb{Q}$ be the elliptic curve with Cremona label "121c2", which we already studied in Examples 2.1 and 4.1, and we calculated $e = 3$ and $e_1 = 1$. Hence, if P is any nontrivial 11-torsion point on $E(\overline{\mathbb{Q}})$, then the ramification of any prime above $p = 11$ in the extension $\mathbb{Q}(P)/\mathbb{Q}$ must be divisible by, at least, $(p-1)/\gcd(p-1, 4) = 10/2 = 5$. Let us show that there is a 11-torsion point where the ramification index is exactly 5.

Indeed, let $F = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_{11}$ is a primitive 11-th root of unity. Then, $E(F)_{\mathrm{tors}} \cong \mathbb{Z}/11\mathbb{Z}$ and there is a point $P \in E(F)$ of order 11 with coordinates

$$x(P) = 11\zeta^9 + 11\zeta^8 + 22\zeta^7 + 22\zeta^6 + 22\zeta^5 + 22\zeta^4 + 11\zeta^3 + 11\zeta^2 + 39,$$
$$y(P) = 44\zeta^9 - 55\zeta^8 - 66\zeta^7 - 99\zeta^6 - 99\zeta^5 - 66\zeta^4 - 55\zeta^3 + 44\zeta^2 + 85.$$

Notice, however, that $x(P)$ and $y(P)$ are stable under complex conjugation. Hence, $P \in E(\mathbb{Q}(\zeta)^+)$, and in fact $\mathbb{Q}(P) = \mathbb{Q}(x(P), y(P)) = \mathbb{Q}(\zeta)^+ = \mathbb{Q}(\zeta + \zeta^{-1})$. Thus, $\mathbb{Q}(P)/\mathbb{Q}$ is totally ramified at 11 and the ramification index is 5.

Corollary 3.10 implies that if $p \equiv 1 \mod 12$, and $e(\wp, L) = 1$, then $e_1 \geq e$. When we combine this with Theorem 5.2 we obtain:

Corollary 5.4. *Let E/L be an elliptic curve with potential good supersingular reduction at a prime $\wp$ above a rational prime $p \equiv 1 \mod 12$, let e be as above, and suppose $e(\wp, L) = 1$. Let $P \in E[p]$ be a nontrivial p-torsion point. Then the ramification index of any prime over $\wp$ in $L(P)/L$ is divisible by $(p^2-1)/\gcd(p^2-1, e)$.*

However, the conclusion of the previous corollary is not valid when $e(\wp, L) > 1$.

Example 5.5. Let $L = \mathbb{Q}(\sqrt{13})$, and let E/L be the elliptic curve with j-invariant j_0 as described in Example 2.3 and 4.4. There is a point $P \in E(\overline{L})$ such that $L(P)$ is given by $L(\alpha)$, where α is a root of a polynomial $q(x) \in L[x] = \mathbb{Q}(j_0)[x]$,

$$q(x) = x^{12} + \frac{34960589\,j_0 - 2813426633070000000}{478224}x^{10} + \cdots$$

of degree 12, and such that $L(P)/L$ is totally ramified above $\wp$. Recall that we have calculated $e = 2$ and $e_1 = 1$ for this curve, so the ramification in this extension agrees with the conclusion of Theorem 5.2 which predicts the existence of 12 points in $E[p]$ such that the ramification index of any prime above $\wp$ in $L(P)/L$ is divisible by $12/\gcd(12, e-e_1) = 12/\gcd(12, 2-1) = 12$.

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CODIMENSION-ONE FOLIATIONS CALIBRATED BY NONDEGENERATE CLOSED 2-FORMS

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A class of codimension-one foliations has been recently introduced by imposing a natural compatibility condition with a closed maximally nondegenerate 2-form. In this paper we study for such foliations the information captured by a Donaldson-type submanifold. In particular we deduce that their leaf spaces are homeomorphic to leaf spaces of 3-dimensional taut foliations. We also introduce surgery constructions to show that this class of foliations is broad enough. Our techniques come mainly from symplectic geometry.

1. Introduction and statement of main results

Codimension-one foliations are too large a class of structures to obtain strong structure theorems for them. According to a theorem of Thurston [1976] a closed manifold admits a codimension-one foliation if and only if its Euler characteristic vanishes. In order to draw significant results it is necessary to assume the existence of other structures compatible with the foliation.

From the point of view of symplectic geometry it is natural to consider the following class of codimension-one foliations:

Definition 1 [Ibort and Martínez Torres 2004a]. A codimension-one foliation $\mathcal{F}$ of M^{2n+1} is said to be 2-calibrated if there exists a closed 2-form ω such that ${\omega_{\mathcal{F}}}^n$ is nowhere-vanishing (we also say that ω^n is nowhere-vanishing on $\mathcal{F}$).

The 2-calibrated foliation is said to be integral if $[\omega] \in H^2(M; \mathbb{Z})$.

The notation ${\omega_{\mathcal{F}}}^n$ in Definition 1 stands for the restriction of ω^n to the leaves of $\mathcal{F}$. We will be using the subscripts $\mathcal{F}$ and W, if W is a submanifold of M, to denote the restriction of a form, connection, etc, to the leaves of $\mathcal{F}$ and to W,

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respectively. In what follows the manifolds will always be closed and oriented, the codimension-one foliations cooriented and all the structures and maps smooth.

In the next paragraphs we are going to describe how the 2-calibrated condition appears naturally when looking at the problem of constructing submanifolds transverse to a codimension-one foliation.

Recall that a codimension-one foliation $\mathcal{F}$ is said to be taut if every leaf meets a transverse 1-cycle. Tautness in codimension-one can be characterized in several ways using forms, metrics and currents [Sullivan 1976; Rummler 1979; Harvey and Lawson 1982]. The characterization we are interested in says that a rank p codimension-one foliation $\mathcal{F}$ is taut if and only if there exists a closed p-form ξ nowhere vanishing on $\mathcal{F}$ (and furthermore according to Proposition 2.7 in [Harvey and Lawson 1982], it is possible to construct a metric g so that ξ is a calibration for $(M, \mathcal{F})$). Note in particular that a 2-calibrated foliation $(M, \mathcal{F}, \omega)$ is always taut, since $\xi := \omega^n$ is nowhere-vanishing on $\mathcal{F}$. In dimension three, 2-calibrated foliations are the same as taut foliations.

Let us analyze one direction of the aforementioned characterization: the existence of a closed p-form whose restriction to each leaf is a volume form is equivalent to a reduction of the structural pseudogroup of $(M, \mathcal{F})$ to $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p}) \times \mathrm{Diff}(\mathbb{R})$, where

$$\Xi_{\mathbb{R}^p} := dx_1 \wedge \cdots \wedge dx_p,$$

$x_1, \ldots, x_p$ are coordinates on $\mathbb{R}^p$, and $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p})$ and $\mathrm{Diff}(\mathbb{R})$ are the pseudogroups of local diffeomorphisms of $\mathbb{R}^p$ and $\mathbb{R}$, respectively, preserving the volume form $\Xi_{\mathbb{R}^p}$. Let U be any open subset of a leaf of $\mathcal{F}$. The Poincaré recurrence theorem implies that the flow of any vector field spanning $\ker \xi$ defines a first return map from $U' \subset U$ to $U'' \subset U$. A straightforward consequence is that closed transverse 1-cycles through any given $x \in M$ can be constructed by slightly deflecting integral curves of $\ker \xi$.

The first return map belongs to the pseudogroup $\mathrm{Vol}(\mathbb{R}^p, \Xi_{\mathbb{R}^p})$. If $p = 2$, that is, if we have a taut foliation on a 3-manifold, then under certain circumstances we can deduce interesting geometric information about the existence of more closed orbits (Poincaré–Birkhoff theorem). If $p > 2$ we have little geometric control on the return map because, assuming for simplicity that U' and U'' are diffeomorphic to a ball, the only invariant is the total volume [Greene and Shiohama 1979, Theorem 1]. Therefore problems such as the existence of transverse submanifolds of dimension bigger than one seem difficult to attack.

It has been known for some time that the right setting to obtain higher dimensional generalizations of Poincaré–Birkhoff theorem is not volume geometry but symplectic geometry [Hofer and Zehnder 1994, Chapter 6; McDuff and Salamon 1998, Chapter IV]. It can be checked (see Section 2) that the existence of a closed 2-form ω which makes the leaves of $(M, \mathcal{F})$ symplectic manifolds amounts to a reduction of the structural pseudogroup of $(M, \mathcal{F})$ to $\mathrm{Symp}(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}}) \times \mathrm{Diff}(\mathbb{R})$, where $\mathrm{Symp}(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}})$ is the pseudogroup of local diffeomorphisms of $\mathbb{R}^{2n}$ preserving the standard symplectic form

$$\Omega_{\mathbb{R}^{2n}} := \sum_{i=1}^{n} dx_i \wedge dy_i .$$

Thus, return maps associated to the flow of vector fields generating $\ker\omega$ belong to $\mathrm{Symp}(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}})$. Symplectomorphisms are much more rigid than transformations preserving the volume form $\Omega^n_{\mathbb{R}^{2n}} = n!\, \Xi_{\mathbb{R}^{2n}}$. They preserve the symplectic invariants of subsets of $\mathbb{R}^{2n}$, so for example these cannot be squeezed along symplectic 2-planes [Hofer and Zehnder 1994, Chapters 2 and 3; McDuff and Salamon 1998, Section 12]. Naively, one might try to construct transverse 3-manifolds by choosing tiny 2-dimensional symplectic pieces Σ inside a leaf, whose image by the first return map is a small 2-dimensional symplectic manifold that can be isotoped to Σ through symplectic surfaces. The isotopy would be used to connect both symplectic surfaces in nearby leaves, and thus get a piece of transverse 3-dimensional taut foliation. Of course this idea seems difficult to be carried out because different pieces should be combined to construct a closed 3-manifold. However, it provides some insight on why 2-calibrated foliations are expected to have embedded 3-dimensional taut foliations.

In [Ibort and Martínez Torres 2004a, Corollary 1.2], it was proved that for any 2-calibrated foliation $(M, \mathcal{F}, \omega)$ there exists an embedding of a 3-dimensional submanifold $W^3 \hookrightarrow M$, such that W^3 is transverse to $\mathcal{F}$ and ω_W is nowhere vanishing on $\mathcal{F}_W$; the 3-dimensional submanifold W^3, which inherits a taut foliation, is a Donaldson-type submanifold [Donaldson 1996; Auroux 1997]. Its existence is an elementary consequence of the extension to 2-calibrated foliations of the approximately holomorphic techniques for symplectic manifolds introduced by Donaldson [1996].

1.1. ***Statement of results.*** Take $(M, \mathcal{F}, \omega)$ to be a 2-calibrated foliation, and let $W \hookrightarrow (M, \mathcal{F})$ be a 3-dimensional Donaldson-type submanifold. In this paper we are mainly concerned with finding out which properties of $(M, \mathcal{F})$ are captured by W.

If F is a compact leaf of $(M, \mathcal{F}, \omega)$, an appropriate version of the Lefschetz hyperplane theorem [Donaldson 1996, Proposition 39] asserts that $W \cap F$ is connected. A codimension-one foliation $(M, \mathcal{F})$ has noncompact leaves unless it is a fibration over the circle (a mapping torus). If F is a noncompact leaf then describing global properties of $W \cap F$ seems very difficult. Our main result is a rather surprising and counterintuitive global property of such intersections for appropriate Donaldson-type submanifolds.

Theorem 2. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. Then there exist Donaldson-type submanifolds $W^3 \hookrightarrow (M, \mathcal{F})$, such that for every leaf F of $\mathcal{F}$ the intersection $W \cap F$ is connected.*

Remark 3. Any integral 2-calibrated foliation $(M, \mathcal{F}, \omega)$ admits embeddings in complex projective spaces $\mathbb{CP}^N$ of large dimension, with the property that the ambient Fubini–Study symplectic form restricts to a multiple of ω [Ibort and Martínez Torres 2004a, Corollary 1.3]. The 3-dimensional transverse submanifolds in Theorem 2 can be arranged to appear as intersections of $M \subset \mathbb{CP}^N$ with appropriate projective subspaces. Theorem 2 should be understood as a leafwise Lefschetz hyperplane-type result for π_0.

An important consequence of Theorem 2 is the following result:

Theorem 4. *Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. There exists a 3-dimensional embedded taut foliation such that the inclusion $(W^3, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F})$ descends to a homeomorphism of leaf spaces $W/\mathcal{F}_W \to M/\mathcal{F}$.*

Thus, leaf spaces of 2-calibrated foliations are no more complicated than those of 3-dimensional taut foliations.

A second goal of this paper is showing that 2-calibrated foliations are a broad enough class of foliations. In this respect there are three basic families of 2-calibrated foliations: products, cosymplectic foliations and symplectic bundle foliations.

In a product we cross a 2-calibrated foliation — typically a 3-dimensional taut foliation — with a (nontrivial) symplectic manifold, and put the product foliation and the obvious closed 2-form.

A cosymplectic foliation is a triple (M, α, ω), where α is a nowhere vanishing closed 1-form and $(M, \ker\alpha, \omega)$ is a 2-calibrated foliation.

A bundle foliation with fiber S^1 is by definition an S^1-fiber bundle $\pi : M \to X$ endowed with a codimension-one foliation $\mathcal{F}$ transverse to the fibers. If the base space admits a symplectic form σ, then $(M, \mathcal{F}, \pi^*\sigma)$ is a 2-calibrated foliation which we refer to as a symplectic bundle foliation.

The second topic of this paper concerns the introduction of two surgery constructions for 2-calibrated foliations: normal connected sum and generalized Dehn surgery or Lagrangian surgery. Using surgery we have obtained the following result:

Proposition 5. *There exist 2-calibrated foliations (of dimension bigger than three) which are neither products, nor cosymplectic foliations, nor symplectic bundle foliations.*

The paper is organized as follows. In Section 2 we introduce definitions and basic facts on 2-calibrated foliations, and address their relation to regular Poisson structures.

Section 3 describes how to adapt the normal connected sum for symplectic and Poisson manifolds to integral 2-calibrated foliations; this is the surgery used to prove Proposition 5.

In Section 4 we present a surgery based on generalized Dehn twists. Generalized Dehn surgery is the natural extension to 2-calibrated foliations of positive Dehn surgery along a curve in a leaf of a 3-dimensional taut foliation $(M^3, \mathcal{F})$.

It is a classical result of Lickorish [1965] that positive Dehn surgery along a curve γ has an alternative description: γ carries a canonical framing and therefore it determines an elementary cobordism from M^3 to M', which amounts to attaching a 2-handle to the trivial cobordism $M \times [0, 1]$. The "new" boundary component M' is endowed with a canonical foliation which coincides with positive Dehn surgery on $(M, \mathcal{F})$ along γ.

If $(M^{2n+1}, \mathcal{F}, \omega)$ is a 2-calibrated foliation, a parametrized Lagrangian n-sphere inside a leaf of $\mathcal{F}$ canonically determines the attaching of a $(n+1)$-handle. We show that the corresponding elementary $(2n+2)$-dimensional cobordism admits a symplectic structure, which induces a 2-calibrated foliation on the new boundary component of the cobordism. We call this construction Lagrangian surgery. In Theorem 26 we extend Lickorish's result by proving that generalized Dehn surgery and Lagrangian surgery produce equivalent 2-calibrated foliations. The importance of this result stems from the fact that the aforementioned symplectic elementary cobordisms do appear in a natural way associated to Lefschetz pencil structures. As a byproduct we get an application to contact geometry that we have included in an appendix: it is a proof of a result announced by Giroux and Mohsen [2003], relating generalized Dehn surgery along a parametrized Lagrangian sphere L in an open book decomposition compatible with a contact structure, and Legendrian surgery along L. Results in this section require a fine analysis of the symplectic monodromy about the singular fiber of the complex quadratic form.

In Section 5 we prove Theorems 2 and 4. The main tools are Lefschetz pencil structures for $(M, \mathcal{F}, \omega)$, which are appropriate analogs of leafwise complex Morse functions and whose existence is an application of approximately holomorphic geometry for 2-calibrated foliations. A regular fiber of a Lefschetz pencil structure is a Donaldson-type submanifold. A Lefschetz pencil structure admits a leafwise symplectic connection. Its associated leafwise symplectic parallel transport is the key ingredient to prove our main theorem relating the leaf space of any regular fiber of the pencil to the leaf space of $(M, \mathcal{F}, \omega)$. Symplectic parallel transport also allows us to compare the 2-calibrated foliations induced on different regular fibers. Namely, in Theorem 37 we show that any two regular fibers of a Lefschetz pencil structure for $(M, \mathcal{F}, \omega)$ are related by a sequence of symplectic handle attachings along Lagrangian spheres. By the symplectic analog of Lickorish's result proved in Section 4, we conclude that any two regular fibers of a Lefschetz pencil structure are related by a sequence of generalized Dehn surgeries. We finish the section by discussing some open problems.

2. Definitions and basic results

In this section we introduce some basic definitions, results and examples. We also address the relation of 2-calibrated foliations to Poisson structures.

Definition 6. Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation and let $l : N \hookrightarrow M$ be a submanifold. We say that N is a 2-calibrated submanifold if $(N, l^*\mathcal{F}, l^*\omega)$ is a 2-calibrated foliation.

The definition of a 2-calibrated foliation can be given locally.

Definition 7. A 2-calibration for $(M, \mathcal{F})$ is a reduction of its structural pseudogroup to $\mathrm{Symp}(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}}) \times \mathrm{Diff}(\mathbb{R})$.

Definitions 1 and 7 are equivalent. A standard Darboux-type result (see for example [McDuff and Salamon 1998, Chapter 3] for basic material on symplectic geometry) implies that about any point in M, there exists a foliated chart with coordinates $x_1, y_1, \dots, x_n, y_n, t$ (the image of $\mathcal{F}$ in $\mathbb{R}^{2n+1}$ is the foliation by affine hyperplanes with constant coordinate t), such that ω is the pullback of

$$\omega_{\mathbb{R}^{2n+1}} := \sum_{i=1}^{n} dx_i \wedge dy_i.$$

It is clear that on a given manifold, 2-calibrated foliations are an open subset of the set of codimension-one foliations in the C^0-topology. More precisely, in the product space of codimension-one foliations and closed 2-forms, pairs corresponding to 2-calibrated foliations are an open set in the C^0-topology.

The first examples of 2-calibrated manifolds are 3-dimensional taut foliations. In this paper we are concerned with higher dimensional 2-calibrated foliations. An elementary family is obtained by applying the product construction to 3-dimensional taut foliations and nontrivial symplectic manifolds.

Another important family of 2-calibrated foliations are cosymplectic foliations. Recall that they are given by a triple $(M^{2n+1}, \alpha, \omega)$, α a closed 1-form and ω a closed 2-form such that $\alpha \wedge \omega^n$ is a volume form. An example of a cosymplectic foliation is a 2-calibrated foliation whose leaves are the fibers of a fibration over the circle; the closed 1-form defining the foliation is the pullback of any volume form on the circle. Each fiber is a closed symplectic manifold and the first return map associated to the kernel of the calibrating 2-form is a symplectomorphism. We refer to such cosymplectic foliations as symplectic mapping tori. In fact, symplectic mapping tori are characterized as cosymplectic foliations whose defining 1-form has rank one period lattice. This characterization implies that symplectic mapping tori are C^0-dense in cosymplectic foliations. The reason is that the defining 1-form can be approximated by closed 1-forms with rational periods.

Cosymplectic foliations appear naturally in symplectic geometry as follows: recall that a vector field Y on a symplectic manifold (Z, Ω) is called symplectic if $L_Y\Omega = 0$. If Y is a symplectic vector field transverse to ∂Z, then its symplectic annihilator

$$\mathrm{Ann}(Y)^{\Omega} = \{v \in TZ \mid \Omega(Y, v) = 0\}$$

is an integrable codimension-one distribution. Since it contains the vector field Y, it induces a codimension-one foliation $\mathcal{F}$ on ∂M. Let $\alpha := i_Y\Omega$. It can be checked that $(\partial M, \alpha_{\partial M}, \Omega_{\partial M})$ is a cosymplectic foliation.

The previous construction leads to an analogy between cosymplectic foliations and contact structures. The reason is that on a symplectic manifold (Z, Ω) endowed with a vector field Y transverse to the boundary and satisfying $L_Y\Omega = \Omega$, the restriction of $i_Y\Omega$ to ∂M is a contact form. Following this analogy, we define the Reeb vector field R of a cosymplectic foliation (M, α, ω) to be the vector field characterized by the equations $i_R\omega = 0$, $i_R\alpha = 1$. The foliation is invariant under the flow of the Reeb vector field. In fact, a cosymplectic foliation can be defined as a 2-calibrated foliation endowed with a vector field R spanning the kernel of ω and whose flow preserves the foliation; we say that R is a Reeb vector field.

A third family of 2-calibrated foliations are symplectic bundle foliations,[1] which are defined as bundle foliations with fiber S^1 over symplectic manifolds. There is a very rough way of associating symplectic bundle foliations to any bundle foliation $\pi : M \to X$ with fiber S^1. The latter is characterized by a conjugacy class of representations of $\pi_1(X, x)$ in $\mathrm{Diff}(S^1)$. A result of Gompf [1995, Theorem 0.1] asserts that there exist closed symplectic manifolds (of dimension 4) whose fundamental group isomorphic to $\pi_1(X, x)$.

Example 8. Let x_1, y_1, x_2, y_2, t be coordinates on $\mathbb{R}^5$ and consider the canonical 2-form $\omega_{\mathbb{R}^5}$. It descends to a closed 2-form $\omega_{\mathbb{T}^5}$, where $\mathbb{T}^5 = \mathbb{R}^5/\mathbb{Z}^5$. Let $\mathcal{F}$ be any of the foliations on $\mathbb{T}^5$ induced by a constant 1-form α on $\mathbb{R}^5$ whose kernel is transverse to $\partial/\partial t$. Then $(\mathbb{T}^5, \alpha, \omega_{\mathbb{T}^5})$ is a 2-calibrated foliation. Its leaves are all diffeomorphic to $\mathbb{R}^i \times \mathbb{T}^{4-i}$, where $i \in \{0, \dots, 4\}$ depends on the slopes of the kernel of the 1-form.

By construction $(\mathbb{T}^5, \alpha, \omega_{\mathbb{T}^5})$ is both a cosymplectic foliation and a symplectic bundle foliation. It is a product (respectively a mapping torus) if and only if the leaves are diffeomorphic to $\mathbb{R}^i \times \mathbb{T}^{4-i}$, $i \leq 2$ (respectively $\mathbb{T}^4$).

Deciding which manifolds admit a 2-calibrated foliation can be divided in several subproblems which in general are very hard. A 2-calibrated foliation $(M, \mathcal{F}, \omega)$

[1] This family of 2-calibrated foliations was pointed out to the author by the referee.

is the superposition of several compatible structures. Firstly, there is the foliation. Secondly, the 2-form restricts to a closed nondegenerate foliated 2-form $\omega_{\mathcal{F}}$. The pair $(\mathcal{F}, \omega_{\mathcal{F}})$ defines a (regular) Poisson structure on M and as such it is also defined by an appropriate bivector field Π. And thirdly, the foliated symplectic form $\omega_{\mathcal{F}}$ admits a lift to a global closed 2-form ω.

Determining which codimension-one foliations are the symplectic foliations of a Poisson structure is very complicated; there exist partial results which use h-principles and only apply to open manifolds [Bertelson 2001; Bertelson 2002; Fernandes and Frejlich 2012]. The existence of a closed lift of a foliated 2-form $\omega_{\mathcal{F}}$ is controlled by three obstructions associated to the spectral sequence which relates basic cohomology, leafwise cohomology and the cohomology of the total space [El Kacimi-Alaoui 1983] (see [Alcalde-Cuesta and Hector 1993] for a treatment in the setting of Poisson geometry); if the foliation is defined by a closed 1-form, then the obstruction to the existence of a closed lift admits a simpler description [Guillemin et al. 2011, Section 2.2].

We would like to regard a 2-calibrated foliation as a codimension-one regular Poisson manifold with a lift of $\omega_{\mathcal{F}}$ to a closed 2-form ω. We are not fully interested in the 2-form ω, as the following definition reflects.

Definition 9. Let $(M_j, \mathcal{F}_j, \omega_j)$, $j = 1, 2$, be 2-calibrated foliations. They are said to be equivalent if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that

- ϕ is a Poisson morphism or equivalence (it preserves the foliations together with the leafwise 2-forms),
- $[\phi^* \omega_2] = [\omega_1] \in H^2(M_1; \mathbb{R})$ and ϕ preserves the coorientations.

For symplectic mapping tori, an equivalence is just a Poisson diffeomorphism preserving coorientations. Alternatively, equivalent symplectic mapping tori are those with the same symplectic leaf and isotopic first return maps (the isotopy being through symplectomorphisms).

As we shall see in the following sections, the notion of equivalence is the right one to remove the dependence on choices in our surgeries.

3. Normal connected sum

In the previous section we saw that deciding whether a manifold supports a 2-calibrated foliation is very complicated. It is thus natural to look for procedures to build new 2-calibrated foliations out of given ones. In this section we introduce the normal connected sum of integral 2-calibrated foliations, and we use it to give examples of 2-calibrated foliations which do not belong to either of the three elementary families, hence proving Proposition 5.

Symplectic normal connected sum is a surgery construction in which two symplectic manifolds are glued along two copies of the same codimension-two symplectic submanifold, which enters in the manifolds with opposite normal bundles [Gompf 1995, Theorem 1.3]. A parametric version of this surgery gives rise to an analogous construction for regular Poisson manifolds [Ibort and Martínez Torres 2003, Theorem 1]. We propose the following extension to integral 2-calibrated foliations.

Theorem 10. *Let* $(M_j^{2n+1}, \mathcal{F}_j, \omega_j)$, $j=1,2$, *be integral* 2*-calibrated foliations. Let* $(N^{2n-1}, \mathcal{F}_N, \omega_N)$ *be a* 2*-calibrated foliation which is a symplectic mapping torus. Assume that we have maps* $l_j : N \hookrightarrow M_j$, $j=1,2$, *embedding* N *as a* 2*-calibrated submanifold of* M_j (*Definition 6*), *such that the following properties hold*:

(i) *The* 2*-calibrated foliations induced by the embeddings are equivalent to the given one* $(N, \mathcal{F}_N, \omega_N)$ (*Definition 9*).

(ii) *The normal bundles of* $l_j(N) \subset M_j$, $j=1,2$, *are trivial.*

(iii) *The fiber of* $N \to S^1$ *is simply connected.*

Then there exist gluing maps ψ *such that the Poisson structure* Π *on* $M_1 \#_\psi M_2$ *characterized by matching on* $M_j \backslash l_j(N)$ *the Poisson structures* Π_j *associated to* $(M_j, \mathcal{F}_j, \omega_j)$, $j=1,2$, *admits a lift to a* 2*-calibrated structure.*

Proof. By assumptions (i) and (ii) Poisson surgery produces a Poisson structure Π on $M_1\#_\psi M_2$ [Ibort and Martínez Torres 2003]. Very briefly, there is a gluing map ψ identifying $A_1 \to A_2$ annular neighborhoods of $l_1(N)$ and $l_2(N)$ (by this we mean tubular neighborhoods from which we remove $l_j(N)$, $j=1,2$) defined as follows: by assumption (ii) the normal bundles are trivial and by Darboux–Weinstein theorem with parameters the (smooth) leaf space of N [McDuff and Salamon 1998, Chapter 3], there exist trivializations in which Π_j, $j=1,2$, split. One factor is the leafwise symplectic form on $l_j(N)$ and the other one is the standard symplectic form $dx \wedge dy$ on the normal disk with coordinates x, y. On each normal disk ψ is the unique rotationally independent symplectomorphism of the punctured disk of radius $\delta > 0$ which reverses the orientation of the radii.

Let $(\mathcal{F}, \omega_{\mathcal{F}})$ denote the foliation and leafwise symplectic form associated to Π. If there is a lift of $\omega_{\mathcal{F}}$ to an integral closed 2-form ω, then there must be a Hermitian line bundle L and a compatible connection ∇ such that $-2\pi i \omega = F_\nabla$, where F_∇ is the curvature of the connection.

Because the w_j, $j=1,2$, represent integral cohomology classes, there exist Hermitian line bundles $(L_j, \nabla_j) \to M_j$ with compatible connections such that

$$-2\pi i \omega_j = F_{\nabla_j}. \tag{1}$$

We look for a lift of ψ to a bundle isomorphism $\Psi : L_{1|A_1} \to L_{2|A_2}$ to define a (Hermitian) line bundle $L := L_1\#_\Psi L_2 \to M_1 \#_\psi M_2$. Let c_j, $j=1,2$, denote the

Chern classes of $L_{j|A_j}$, which are integral lifts of the restrictions of w_j to A_j. An isomorphism lifting ψ exists if and only if

$$\psi^* c_2 = c_1 \in H^2(A_1; \mathbb{Z}). \tag{2}$$

Because the fiber of $N \to S^1$ is simply connected, the Wang sequence for the mapping torus $A_1 \to S^1$ implies that $H^2(A_1; \mathbb{Z})$ is torsion free. Therefore (2) is equivalent to

$$[\psi^* w_{2|A_2}] = [w_{1|A_1}] \in H^2(A_1; \mathbb{R}). \tag{3}$$

Because the $w_j,\ j = 1, 2$, extend to $A_j \cup l_j$ and the cohomology of the tubular neighborhoods is concentrated in $l_j(N)$, (3) is equivalent to

$$[l_2^* w_2] = [l_1^* w_1] \in H^2(N; \mathbb{R}),$$

which holds true because by assumption (i) the 2-calibrations induced by l_1 and l_2 on N are equivalent.

Therefore we obtain $L \to M_1 \#_\psi M_2$ a Hermitian line bundle with two not everywhere defined compatible connections ∇_1, ∇_2, overlapping on $A_1 \subset M_1 \#_\psi M_2$. Note that by (1) the leafwise curvatures match on A_1. We are going to use the assumptions to modify ∇_1 and ∇_2 (the latter away from $l_2(N)$), so that we obtain the leafwise equality of connections on A_1. Then a convex combination of both connections associated to a partition of the unity subordinated to $M_j \backslash l_j(N)$, $j = 1, 2$, is a connection on $M_1 \#_\psi M_2$ whose leafwise curvature is $-2\pi i \omega_{\mathcal{F}}$.

The difference

$$l_1^* \nabla_1 - l_2^* \nabla_2 \tag{4}$$

is a leafwise closed 1-form on N (recall that N is a mapping torus and therefore all leaves are compact). By assumption (iii) it is leafwise exact and therefore we can modify say ∇_2, by adding a smooth leafwise primitive function so the 1-form in (4) is leafwise vanishing.

Triviality of the normal bundles implies the existence of normal forms for the leafwise connections on tubular neighborhoods of $l_j(N),\ j = 1, 2$, which only depend on the restrictions of the leafwise connections to $l_j(N)$; the normal forms amount to fixing a primitive 1-form for $dx \wedge dy$. The connections can be assumed to coincide with the normal forms. Finally the difference $\nabla_1 - \psi^* \nabla_2$ is not still leafwise vanishing; on each normal annulus it is the differential of an (explicit) function, and what we do is modify ∇_2 accordingly on $M_2 \backslash l_2(N)$.

As for dependence of the construction on choices, remark that the choice of isotopy classes of trivializations of the normal bundles (the framings), may affect the diffeomorphism class of $M_1 \#_\psi M_2$. For fixed isotopy classes of trivializations of the normal bundles, the underlying Poisson structure is unique up to Poisson diffeomorphism. The reason is that the leafwise symplectic form is unique up to isotopy supported near N. This follows from an elementary argument which is going to be used several times: because the leaves of N have no first cohomology group, the local path of symplectomorphisms provided by Moser's argument is Hamiltonian [McDuff and Salamon 1998, Chapter 3]. The choice of primitive Hamiltonian function can be done coherently for all leaves of N. By extending the corresponding function to a global one supported near N, we construct a path of transformations connecting both Poisson structures. Also, if we fix an isotopy class of lifts Ψ, the 2-calibrated structure provided by the normal connected sum is unique up to equivalence. This is because the cohomology class of the calibrating 2-form is the image in real cohomology of the first Chern class of the bundle L, which is fixed by the choice of isotopy class of lifts. □

Remark 11. The hypotheses needed to define normal connected sum of regular Poisson manifolds are much weaker than the requirements in Theorem 10. In particular the normal bundles $l_j(N)$, $j = 1, 2$, are not required to be trivial, just opposite. Triviality of the normal bundles is necessary if we want to produce an integral 2-calibrated foliation extending the given Poisson structures Π_j on $M_j \backslash l_j(N)$, $j = 1, 2$. The reason is that already in the symplectic setting, having nontrivial normal bundle gives rise to choices in the construction which result is symplectic forms with different volume; this is a well-known issue that appears when blowing up symplectic submanifolds [McDuff and Salamon 1998, Chapter 7].

Perhaps the assumptions in Theorem 10 can be weakened if we just require the existence of a 2-calibration on the normal connected sum.

The normal connected sum can be applied to construct integral 2-calibrated foliations that use as building blocks 2-calibrated foliations which are products and symplectic mapping tori, but which are neither products, nor cosymplectic foliations nor symplectic bundle foliations.

Proof of Proposition 5. Let (P^4, Ω) be an integral symplectic 4-manifold which contains a symplectic sphere S^2 with trivial normal bundle; let $A \in \mathbb{Z}$ be the induced area form on the sphere. Let $\varphi \in \mathrm{Symp}(P, \Omega)$ such that $\varphi_{|S^2} = \mathrm{Id}$; for example φ can be the identity. We define $(M_1, \mathscr{F}_1, \omega_1)$ to be the symplectic mapping torus associated to φ.

Let $(M_2, \mathscr{F}_2, \omega_2)$ be the product 2-calibrated foliation with factors any taut foliation $(Y^3, \mathscr{F}^3, \sigma)$ and the sphere (S^2, A); via a small perturbation and a rescaling of σ, we may take ω_2 to be integral. Let C be a fixed transverse cycle for $(Y^3, \mathscr{F}^3, \sigma)$ and $\theta : S^1 \to C$ any fixed positive parametrization with respect to the coorientation.

Let N^3 be the result of applying the mapping torus construction to

$$\mathrm{Id} \in \mathrm{Symp}(S^2, A), \quad \text{where } N \cong S^1 \times S^2.$$

Since $\varphi_{|S^2} = \mathrm{Id}$, there is an obvious embedding $l_1 : N \hookrightarrow M_1$. The embedding l_2 is the product map $\theta \times \mathrm{Id} : N \hookrightarrow M_2$.

By construction the embeddings fulfill the hypothesis of Theorem 10, so we obtain a 2-calibrated foliation $(M_1 \#_\psi M_2, \mathcal{F}, \omega)$.

We impose the following additional constraints on the summands to make sure that $(M_1 \#_\psi M_2, \mathcal{F}, \omega)$ does not belong to the three basic families:

- $(Y^3, \mathcal{F}^3)$ contains compact and noncompact leaves.
- There is a compact leaf Σ of $(Y^3, \mathcal{F}^3)$ which intersects C in exactly one point, and (P^4, Ω) is an odd Hirzebruch surface [McDuff and Salamon 1998, Chapter 4].
- The genus of Σ is greater than one, and $\pi_1(Y)$ is not isomorphic to $\pi_1(S^1 \times \Sigma)$.

Because $l_2(N)$ intersects each leaf of $(M_2, \mathcal{F}_2)$ in a unique connected component, there is a one to one correspondence between leaves of $(Y^3, \mathcal{F}^3)$ and leaves of $(M_1 \#_\psi M_2, \mathcal{F})$. This correspondence sends a leaf F of $(Y^3, \mathcal{F}^3)$ to the leaf which contains $(F \times S^2) \backslash (l_2(N) \cap (F \times S^2))$. Because the leaves of $(M_2, \mathcal{F}_2)$ are compact, the correspondence sends compact leaves to compact leaves and noncompact leaves to noncompact leaves. Since $(Y^3, \mathcal{F}^3)$ contains compact and noncompact leaves, so does $(M_1 \#_\psi M_2, \mathcal{F}, \omega)$, and hence it has nontrivial holonomy. Consequently, $(M_1 \#_\psi M_2, \mathcal{F}, \omega)$ cannot be a cosymplectic foliation.

Let Σ be a compact leaf of $\mathcal{F}^3$ which intersects C in one point. The correspondence between leaves described in the previous paragraph sends Σ to a compact leaf F_Σ, which is the symplectic normal connected sum of the odd Hirzebruch surface and $(\Sigma \times S^2, p_1^* \omega_{|\Sigma} + p_2^* A)$ along a symplectic sphere with trivial normal bundle. At the differentiable level F_Σ is the normal connected sum of the trivial S^2-fibration over Σ and the twisted S^2-fibration over S^2, and hence it is the twisted S^2-fibration over Σ (the fibers of our fibrations have a coherent orientation, since they are symplectic). If F_Σ is diffeomorphic to a product of surfaces then we can only have $F_\Sigma \cong S^2 \times \Sigma$; otherwise we could not have isomorphic fundamental groups. But then F_Σ would admit two different S^2-fibration structures, and this is in contradiction with [Melvin 1984]. Therefore $(M_1 \#_\psi M_2, \mathcal{F}, \omega)$ cannot be a product.

If the normal connected sum is a symplectic bundle foliation $\pi : M_1 \#_\psi M_2 \to X$, then F_Σ is a covering space of X. Because the fundamental group of F_Σ is the fundamental group of Σ, our assumption on the genus of Σ implies that the covering must be trivial. Therefore π sends F_Σ diffeomorphically onto X. This also implies that the principal S^1-bundle has a section, so $M_1 \#_\psi M_2$ is the trivial bundle $S^1 \times F_\Sigma$. Hence $\pi_1(M_1 \#_\psi M_2)$ is diffeomorphic to $\pi_1(S^1 \times \Sigma)$. But applying Seifert–Van Kampen theorem to the open subsets $M_1 \backslash l_1(N)$, $M_2 \backslash l_2(N)$ gives that $\pi_1(M_1 \#_\psi M_2)$ is diffeomorphic to $\pi_1(Y)$, and this contradicts the assumption on $\pi_1(Y)$. $\square$

4. Generalized Dehn surgery

In this section we introduce our second surgery, generalized Dehn surgery. We give a first definition which is the most natural one from the viewpoint of foliation theory. We present a second approach via handle-attaching along Lagrangian spheres; this is a very natural definition taking into account the description of Legendrian surgeries in contact geometry [Weinstein 1991, Elementary Cobordisms Section]. We prove the equivalence of both constructions in Theorem 26.

Generalized Dehn surgery is done, unlike normal connected sum, along a submanifold inside one of the leaves. Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. We orient M so that a positive transverse vector followed by a positive basis of the leaf with respect to the volume form $\omega^n_{\mathcal{F}}$ gives a positive basis.

Let $T := T^*S^n$ and $d\alpha_{\mathrm{can}}$ its canonical symplectic structure. Let $\tau : T \to T$ be a generalized Dehn twist. Recall that these are certain compactly supported symplectomorphisms of $(T, d\alpha_{\mathrm{can}})$ which induce the antipodal map on the zero section. Let $T(\lambda)$ be the subset of cotangent vectors of length at most λ with respect to the round metric. Generalized Dehn twists can be chosen to be supported in the interior of $T(\lambda)$ for any fixed λ, and any two with such property are isotopic in $\mathrm{Symp}^{\mathrm{comp}}(T(\lambda), d\alpha_{\mathrm{can}})$, the group of compactly supported symplectomorphisms [Seidel 2003, Lemma 1.10 in Section 1.2]. They are symplectic generalizations of Dehn twists on T^*S^1.

A parametrized Lagrangian sphere $L \subset (M, \mathcal{F}, \omega)$ is a submanifold of a leaf F_L such that $\omega_L \equiv 0$, together with a parametrization $l : S^n \to L$. By a theorem of Weinstein [McDuff and Salamon 1998, Chapter 3], there exists U a compact neighborhood of L inside F_L and $\lambda > 0$, such that $l^{-1} : L \to S^n$ extends to a symplectomorphism $\varphi : (U, \omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}})$. Let us assume that if $n = 1$ the loop L has trivial holonomy; if $n > 1$ the absence of holonomy is a consequence of Reeb's theorem. In a neighborhood of L the foliation is a product. We let R be a local positive Reeb vector field and we let Φ^R_t denote its time t flow, which by definition preserves $\mathcal{F}$. Let $\epsilon > 0$ small enough so that

$$\begin{aligned} \Phi^R : [-\epsilon, \epsilon] \times U &\to M, \\ (t, x) &\mapsto \Phi^R_t(x) \end{aligned}$$

is an embedding. We introduce the following notation:

$$\begin{aligned} U(\epsilon) &:= \Phi^R([-\epsilon, \epsilon] \times U), & U_t &:= \Phi^R_t(U), \\ U^+(\epsilon) &:= \Phi^R([0, \epsilon] \times U), & U^-(\epsilon) &:= \Phi^R([-\epsilon, 0] \times U). \end{aligned} \tag{5}$$

The result of cutting $U(\epsilon)$ along U is the manifold $U^-(\epsilon) \amalg U^+(\epsilon)$ whose boundary contains $U^- = U \times \{0\} \subset U^-(\epsilon)$ and $U^+ = U \times \{0\} \subset U^+(\epsilon)$.

Definition 12. Let $L \subset (M, \mathcal{F}, \omega)$ be a parametrized Lagrangian sphere. If $n = 1$, assume that L is a loop with trivial holonomy. Generalized Dehn surgery along L is defined by cutting M along U as above and then gluing back via the composition

$$\chi : (U^-, \omega_{\mathcal{F}}) \xrightarrow{\varphi} (T(\lambda), d\alpha_{\mathrm{can}}) \xrightarrow{\tau} (T(\lambda), d\alpha_{\mathrm{can}}) \xrightarrow{\varphi^{-1}} (U^+, \omega_{\mathcal{F}}), \tag{6}$$

where τ is any choice of generalized Dehn twist supported in the interior of $T(\lambda)$ and we use the canonical identifications of U^-, U^+ with U.

We denote the resulting foliated manifold by $(M^L, \mathcal{F}^L)$.

Proposition 13. *The foliation* $(M^L, \mathcal{F}^L)$ *admits calibrations* ω^L. *If* $n > 1$, *then*

(i) $(M^L, \mathcal{F}^L, \omega^L)$ *is unique up to equivalence*,

(ii) $[\omega]$ *is integral if and only if* $[\omega^L]$ *is integral*,

(iii) $\pi_i(M^L) \cong \pi_i(M)$ *and* $H_i(M^L; \mathbb{Z}) \cong H_i(M; \mathbb{Z})$, $0 \le i \le n - 1$.

Proof. We restrict our attention to $U(\epsilon)$. After cutting $U(\epsilon)$ along U and gluing back using the identification χ in (6), we obtain

$$U^L(\epsilon) := U^-(\epsilon) \#_\chi U^+(\epsilon) \subset M^L.$$

Since the flow of R preserves both ω and the foliation, the restriction of ω to $U^-(\epsilon)$ and $U^+(\epsilon)$ defines closed 2-forms ω^- and ω^+ independent of the coordinate t. When we glue U^- to U^+ using χ, since this map is a symplectomorphism the 2-forms ω^- and ω^+ induce a 2-form ω^L_ϵ on $U^L(\epsilon)$. Then

$$\omega^L := \begin{cases} \omega & \text{in } M^L \backslash U^L(\epsilon), \\ \omega^L_\epsilon & \text{in } U^L(\epsilon) \end{cases}$$

is the desired closed 2-form.

The 2-calibrated structure we obtain is unique up to equivalence. Firstly, different identifications $\varphi : (U, \omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}})$ are related by a global Poisson diffeomorphism. The reason is the same as in the proof of the uniqueness statement of Theorem 10: S^n is simply connected for $n > 1$. Secondly, generalized Dehn twists are symplectically isotopic by an isotopy supported in a neighborhood of the sphere. Thirdly, changing the Reeb vector field amounts to a change of variable in the coordinate t, and this does not modify the construction.

The calibration is a real cohomology class determined by its values on closed 2-chains (which by a theorem of Thom are always homologous to embedded surfaces). If $n > 1$ the 2-chains can be homotoped to avoid the neighborhood $U(\epsilon)$ of the Lagrangian sphere L, where ω^L coincides with ω. Hence the integrality of the 2-calibrated foliation is unaffected by the surgery.

The same general position arguments imply that maps from CW complexes of dimension less or equal than n can be homotoped to miss $U(\epsilon)$. Therefore homology and homotopy groups up to dimension $n - 1$ are unaffected by the surgery. $\square$

Remark 14. A "framed" Lagrangian n-sphere [Seidel 2000] is a parametrized n-sphere up to isotopy and the action of $\mathrm{O}(n+1)$. Generalized Dehn twists associated to two parametrizations defining the same "framed" Lagrangian n-sphere are isotopic, the isotopy by symplectomorphisms supported in a compact neighborhood of the Lagrangian sphere (Remark 5.1 in [Seidel 2000] or the paragraph after Lemma 1.10 in [Seidel 2003]). Therefore generalized Dehn surgery is well-defined for "framed" Lagrangian spheres.

Remark 15. The flow of the local Reeb vector field R can be used to displace the Lagrangian sphere L to a new Lagrangian sphere L' inside a nearby leaf. It follows that $(M^L, \mathcal{F}^L, \omega^L)$ and $(M^{L'}, \mathcal{F}^{L'}, \omega^{L'})$ are equivalent.

If, instead of τ, we use its inverse, we get a 2-calibrated foliation $(M^{L^-}, \mathcal{F}^{L^-}, \omega^{L^-})$ referred to as negative generalized Dehn surgery along L; negative generalized Dehn surgery is generalized Dehn surgery for the opposite coorientation.

Generalized Dehn surgery along L and negative generalized Dehn surgery along L are inverse to each other.

4.1. ***Lagrangian surgery.*** Let $L \subset (M, \mathcal{F}, \omega)$ be a parametrized Lagrangian sphere. Let $\nu(L)$ denote a tubular neighborhood of L, and $\nu_{\mathcal{F}}(L)$ a tubular neighborhood of L inside the leaf containing L. The parametrized Lagrangian sphere L carries a canonical framing μ_L: because L is Lagrangian $\nu_{\mathcal{F}}(L) \cong T^*L$ and we deduce

$$\nu(L) \cong \nu_{\mathcal{F}}(L) \oplus \underline{\mathbb{R}} \cong T^*S^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}_{|S^n}, \tag{7}$$

where in the last isomorphism in (7) a positive nowhere-vanishing section of $\underline{\mathbb{R}}_{|S^n}$ is sent to the outward normal unit vector field. Therefore L determines up to diffeomorphism an elementary cobordism Z, which amounts to attaching a $(n+1)$-handle to the parametrized sphere L with framing μ_L [Gompf and Stipsicz 1999, Chapter 4]. The boundary of the cobordism is $\partial Z = M \amalg M^{\mu_L}$.

This subsection addresses the construction of $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$, a 2-calibrated foliation which extends $(M, \mathcal{F}, \omega)$ on the complement of a neighborhood of L (the complement understood as a subset of both M and M^{μ_L}). We do it by using the relation between symplectic manifolds and cosymplectic foliations presented in Section 2: we have to endow the cobordism Z with a symplectic form Ω — at least in a neighborhood of the $(n+1)$-handle — and a symplectic vector field Y transverse to the boundary. This produces automatically a cosymplectic foliation on ∂Z, and that is how we obtain $(M^{\mu_l}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$. Remark that our strategy is the same one used in contact geometry to show that surgeries along Legendrian spheres give rise to new contact manifolds [Weinstein 1991, third paragraph on page 242].

The elementary cobordism Z is the result of gluing a $(n+1)$-handle to the trivial cobordism $P_1 := M \times [-\varepsilon, \varepsilon]$. We have to define symplectic structures and symplectic vector fields transverse to the boundary on both the trivial cobordism and the $(n+1)$-handle in a way that is compatible with the gluing.

We start with the trivial cobordism P_1: by the coisotropic embedding [Gotay 1982] there is a unique choice of symplectic structure on P_1 which extends the given closed 2-form ω on $M \times \{0\}$. We now give a specific normal form for it which is convenient for the purpose of describing a compatible gluing with the $(n+1)$-handle: let us denote $H_1 := \nu(L)$. Since the gluing between the trivial cobordism and the $(n+1)$-handle occurs near $\nu(L)$, we can assume without loss of generality that $P_1 = H_1 \times [-\varepsilon, \varepsilon]$. Let $(\mathcal{F}_1, \omega_1)$ denote the restriction of $(\mathcal{F}, \omega)$ to H_1. We select R_1 a positive Reeb vector field on H_1 with dual (closed) defining 1-form α_1 ($i_R\alpha_1 = 1$, $\ker \alpha_1 = \mathcal{F}_1$). We let v be the coordinate on the interval $[-\varepsilon, \varepsilon]$, and we extend α_1 and ω_1 to $H_1 \times [-\varepsilon, \varepsilon]$ independently of v.

On P_1, we define $\Omega_1 := \omega_1 + d(v\alpha_1)$, which is a symplectic form provided ε is small enough.

As symplectic vector field on (P_1, Ω_1) we take $Y_1 := \partial/\partial v$, which is transverse to $H \times \{-\varepsilon\}$ and $H \times \{\varepsilon\}$.

We let P_2 denote the $(n+1)$-handle. Before defining the symplectic form Ω_2 and a symplectic vector field Y_2 on (P_2, Ω_2), we address the problem of gluing symplectic cobordisms.

Lemma 16 [Gotay 1982, Extension theorem]. *Let (P_j, Ω_j), $j = 1, 2$, be symplectic manifolds, $H_j \subset P_j$ hypersurfaces and Y_j symplectic vector fields transverse to them, so that we have product structures $H_j \times [-\varepsilon, \varepsilon]$. Define*

$$\omega_j = \Omega_{j_{|H_j}}, \quad \alpha_j = i_{Y_j}\Omega_{j_{|H_j}}$$

and $\mathcal{F}_j$ the foliation integrating $\ker \alpha_j$, *$j = 1, 2$. Suppose that $\phi : H_1 \to H_2$ is a diffeomorphism such that $\phi^*\omega_2 = \omega_1$ and $\phi^*\alpha_2 = \alpha_1$ (and therefore $\phi^*\mathcal{F}_2 = \mathcal{F}_1$). Then*

$$\phi \times Id : (H_1 \times [-\varepsilon, \varepsilon], \Omega_1) \to (H_2 \times [-\varepsilon, \varepsilon], \Omega_2)$$

is a symplectomorphism (obviously compatible with the symplectic vector fields).

Lemma 16 is the analog of Proposition 4.2 in [Weinstein 1991].

In our specific situation of gluing near Lagrangian spheres, the amount of information needed to describe ϕ as in Lemma 16 is much smaller.

Corollary 17. *Let $(P_j, \Omega_j, H_j, Y_j)$, $j = 1, 2$, be as in Lemma 16 and assume further that $L_j \subset H_j$ are Lagrangian spheres and P_j small tubular neighborhoods of L_j.*

Let $\theta : L_1 \to L_2$ be a diffeomorphism. Then θ extends to an isomorphism of tuples $(P_1, \Omega_1, H_1, Y_1) \to (P_2, \Omega_2, H_2, Y_2)$.

Proof. The symplectic vector fields give rise by contraction to closed 1-forms defining the foliations, and therefore to Reeb vector fields. We extend θ to a symplectomorphism of neighborhoods of the spheres inside their leaves, and we further extend it to $\phi : (H_1, \alpha_1, \omega_1) \to (H_2, \alpha_2, \omega_2)$ by declaring it to be equivariant with respect to the Reeb flows. By construction ϕ is in the hypothesis of Lemma 16.

Notice that the only choice is the identification of the symplectic neighborhoods of L_j, $j = 1, 2$, inside their respective leaves. □

4.1.1. *The choice of symplectic form and symplectic vector field on the* $(n+1)$-*handle.* Let W be a neighborhood of $0 \in \mathbb{C}^{n+1}$. This neighborhood will contain our $(n+1)$-handle P_2.

Let us consider the complex Morse function

$$h : \mathbb{C}^{n+1} \to \mathbb{C},$$
$$(z_1, \dots, z_{n+1}) \mapsto z_1^2 + \cdots + z_{n+1}^2.$$

We take $\Omega_2 \in \Omega^2(W)$ to be any symplectic form of type $(1, 1)$ at the origin with respect to the standard complex structure of $\mathbb{C}^{n+1}$, and Y_2 to be the Hamiltonian vector field of $-\operatorname{Im} h$.

Let us explain the reason behind the choice of (Ω_2, Y_2). In the construction of the symplectic $(n+1)$-handle we have to reconcile several aspects:

The data (P_2, Ω_2, Y_2) has to determine the standard $(n+1)$-handle: if $\Omega_2 = \Omega_{\mathbb{R}^{2n+2}}$ then Y_2 is the gradient flow of $-\operatorname{Re} h$ with respect to the Euclidean metric, whose dynamics determine the standard $(n+1)$-handle. In Lemma 18 we are going to prove that for Ω_2 of type $(1, 1)$ at the origin, the Hamiltonian vector field Y_2 has a hyperbolic singularity at $0 \in \mathbb{C}^{n+1}$. Therefore the flow of Y_2 has both the right dynamical behavior to construct a standard $(n+1)$-handle about $0 \in \mathbb{C}^{n+1}$ and the right symplectic behavior.

The second aspect is that we want to define Lagrangian surgery along L so that it becomes equivalent to generalized Dehn surgery. Generalized Dehn twists appear in our current setting as follows: the origin $0 \in \mathbb{C}^{n+1}$ is an isolated critical point for h. Let h_z denote the fiber $h^{-1}(z) \cap W$, $z \in \mathbb{C}$, and let Ω be any closed 2-form on W for which the fibers h_z are symplectic. The annihilator with respect to Ω of the tangent space to the fibers is an Ehresmann connection for $h : W \backslash \{0\} \to \mathbb{C}$. Parallel transport over a path not containing the critical value $0 \in \mathbb{C}$ defines a symplectomorphism from the regular fiber over the starting point to the regular fiber over the ending point. Seidel proves [2003, Lemma 1.10 in Section 1.2] that for a certain choice of closed 2-form Ω_τ, which is Kähler near the origin, and for all $r \in \mathbb{R}^{>0} \subset \mathbb{C}$, parallel transport of the fiber h_r over the boundary of the disk $\overline{D}(r) \subset \mathbb{C}$ counterclockwise is conjugated to a generalized Dehn twist supported in a given $T(\lambda)$. An argument using Taylor expansions shows that for symplectic forms of type $(1, 1)$ at the origin, the fibers h_z are symplectic near the origin, and therefore there is an associated symplectic parallel transport with respect to Ω_2. Besides, symplectic parallel transport with respect to Ω_2 can be connected to symplectic parallel transport with respect to Ω_τ. The upshot is that symplectic parallel transport over $\overline{D}(r) \subset \mathbb{C}$ counterclockwise with respect to Ω_2 can be isotoped to a generalized Dehn twist, which is the property we need to prove the equivalence of generalized Dehn surgery and Lagrangian surgery.

The third aspect is that we need a flexible choice of symplectic form Ω_2 on the $(n+1)$-handle, so the cobordisms naturally associated to Lefschetz pencil structures to be described in Section 5.3 can be identified with Lagrangian surgery.

In the next lemma we collect some useful properties of parallel transport with respect to forms of type $(1, 1)$ at the origin:

Lemma 18. *Let $\Omega \in \Omega^2(W)$ be a symplectic form of type $(1, 1)$ at the origin. Let $Y \in \mathfrak{X}(W)$ be the Hamiltonian vector field of $-\mathrm{Im}\, h$ with respect to Ω. Then:*

(i) *Y is a section of $\mathrm{Ann}(Y)^\Omega$ which vanishes at $0 \in \mathbb{C}^{n+1}$.*

(ii) *$h_*Y(p)$ is a strictly negative multiple of $\partial/\partial x$, where $p \in W\backslash\{0\}$, $z = (x, y)$.*

(iii) *Y has a nondegenerate singularity at the origin with $n+1$ positive eigenvalues and $n+1$ negative eigenvalues.*

(iv) *For each $r \in \mathbb{R}\backslash\{0\}$ we have Lagrangian spheres $\Sigma_r \subset h_r$ characterized as the set of points contracting into the critical point by the parallel transport over the segment $[0, r]$; the spheres come with a parametrization up to isotopy and the action of $\mathrm{O}(n+1)$ (they are "framed"). More generally, for each z and γ an embedded curve joining z and the origin, the points in $h_{\gamma(0)}$ sent to the origin by parallel transport over γ are a Lagrangian sphere $\Sigma_{\gamma(0)}$. Their construction depends smoothly on Ω and γ.*

(v) *For any embedded curve γ through the origin parallel transport*

$$\rho_\gamma : (h_{\gamma(0)}\backslash\Sigma_{\gamma(0)}, \Omega) \to (h_{\gamma(1)}\backslash\Sigma_{\gamma(1)}, \Omega)$$

is a symplectomorphism possibly not everywhere defined.

Proof. This is a generalization of [Seidel 2003, Lemma 1.13] for local symplectic forms which are of type $(1, 1)$ at the origin; also — and very important for our applications — smooth dependence on the symplectic form and curve $\gamma \subset \mathbb{C}$ is proved.

Points (i) and (ii) are a straightforward calculation. Point (iii) is also elementary once we use Taylor expansions at the origin.

Point (iii) implies that $0 \in \mathbb{C}^{n+1}$ is a hyperbolic singular point for Y (see [Palis and de Melo 1982] for basic theory on dynamical systems). Let $W^s(Y)$ denote the stable manifold. Point (ii) implies that $[0, r_0) \subset h(W^s(Y))$ for some $r_0 > 0$, and that for any $r \in (0, r_0)$ the intersection $h_r \cap W^s(Y)$ is transverse. Since $\Sigma_r := h_r \cap W^s(Y)$ is a hypersurface of $W^s(Y)$ transverse to Y, it is diffeomorphic to a sphere. More precisely, the stable manifold theorem gives a parametrization $\Psi^{\mathrm{st}} : B^{n+1} \to W^s(Y)$ of a neighborhood of the origin inside $W^s(Y)$, which is unique up to isotopy and the action of $\mathrm{O}(n+1)$, the latter associated to the choice of an orthonormal basis of the tangent space of $W^s(Y)$ at the origin; such a parametrization induces a parametrization $l : S^n \to \Sigma_r$ unique up to isotopy and the action of $\mathrm{O}(n+1)$.

That Σ_r is Lagrangian follows from point (ii), exactly as in the proof of Lemma 1.13 in [Seidel 2003].

The result for any other point z and a curve γ joining it to the origin follows from the previous ideas applied to the Hamiltonian of $-\mathrm{Im}(F \circ h)$, where $F : \mathbb{C} \to \mathbb{C}$ is a diffeomorphism fixing the origin which sends γ to $[0, r]$, for some $r \in \mathbb{R}\backslash\{0\}$.

If Ω_u is a smooth family, the stable manifold theorem with parameters (the proof of Theorem 6.2 in [Palis and de Melo 1982], Chapter 2, is seen to depend smoothly on parameters) gives parametrizations $\Psi_u^{\mathrm{st}} : B^{n+1} \to W^s(Y_u)$ of neighborhoods of 0 inside the corresponding stable manifolds. This induces a smooth family of parametrizations of the Lagrangian spheres $l_u : S^n \to \Sigma_{u,r}$.

Clearly there is also smooth dependence on the path γ if we choose diffeomorphisms $F_\gamma : \mathbb{C} \to \mathbb{C}$ with such dependence.

Parallel transport is not defined for points in $h_{\gamma(0)}$ which converge to the singular point $0 \in \mathbb{C}^{n+1}$, which by definition are the Lagrangian sphere $\Sigma_{\gamma(0)}$. Parallel transport may send points of $h_{\gamma(0)} \backslash \Sigma_{\gamma(0)}$ away from W. For those points which do not leave W, which at least are those close enough to $\Sigma_{\gamma(0)}$, parallel transport is a well-known symplectomorphism, and this finishes the proof of the lemma. □

4.1.2. *The shape of the symplectic* $(n+1)$*-handle.* A parametrized sphere $L \subset M$ together with a framing determine a diffeomorphism $\phi : H \to S^n \times \overline{B^{n+1}}(1)$, where H is a compact neighborhood of L and $S^n \times \overline{B^{n+1}}(1)$ is seen as a subset of the boundary of the standard $(n+1)$-handle

$$\overline{B^{n+1}}(1) \times \overline{B^{n+1}}(1) \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} = \mathbb{C}^{n+1}.$$

The diffeomorphism determines the manifold with corners $M \#_\phi \overline{B^{n+1}}(1) \times \overline{B^{n+1}}(1)$. A way to smooth the corners uses the gradient flow Y of $-\mathrm{Re}\, h$ with respect to the Euclidean metric: let us consider a function

$$f : H\backslash L \cong S^n \times \overline{B^{n+1}}(1)\backslash S^n \times \{0\} \to \mathbb{R}^+$$

supported in the interior of H, and such that near the attaching sphere $L \cong S^n \times \{0\}$ its value is the time needed to flow from H to a neighborhood of

$$L' \cong \{0\} \times S^n \subset \overline{B^{n+1}}(1) \times S^n.$$

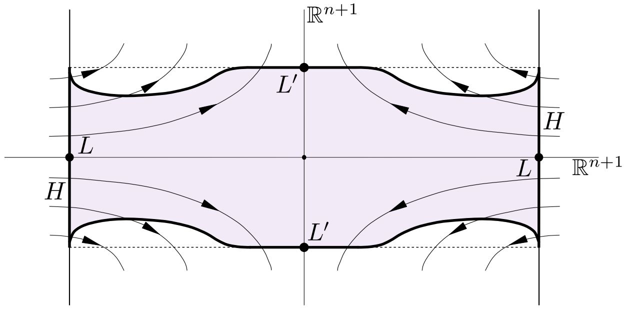

Figure 1. The modified handle is the shaded region, which is everywhere transverse to the gradient flow lines. The dotted segments are part of the boundary of the standard handle with corners.

Then $M' = M \backslash H \cup \Phi_1^{fY}(H \backslash L) \cup L'$ is a smoothing of the new boundary of the cobordism. Actually, one equally thinks of using as modified handle the region bounded by H and $\Phi_1^{fY}(H \backslash L) \cup L'$ (Figure 1).

We now proceed to define smoothings of the standard $(n+1)$-handle using Y_2, which is our symplectic replacement for the gradient flow of $-\mathrm{Re}\, h$. For the sake of flexibility in the definition of Lagrangian surgery we make the construction depend on a small enough parameter $r > 0$.

We start by introducing some notation: the complex coordinate of $\mathbb{C}$ is $z = (x, y)$. For any $r, a, b \in \mathbb{R}$, we let $y_r(a, b), x_r(a, b) \subset \mathbb{C}$ be the "vertical" and "horizontal" segments joining the points (r, a) and (r, b), and (a, r) and (b, r) respectively.

Let us consider $r_0 > 0$ small enough so that the neighborhood (W, Ω_2) of $0 \in \mathbb{C}^{n+1}$ contains all Lagrangian spheres Σ_r, $r \in [-r_0, 0) \cup (0, r_0]$, described in point (iv) in Lemma 18. We fix $\epsilon > 0$ small enough and define for all $r \in (0, r_0]$,

$$H_{2,r} := h^{-1}(y_r(-\epsilon, \epsilon)), \qquad H_{2,-r} := h^{-1}(y_{-r}(-\epsilon, \epsilon)).$$

By point (ii) in Lemma 18, $Y_2 \pitchfork H_{2,r}, H_{2,-r}$, so both hypersurfaces inherit 2-calibrated foliations. By definition of the symplectic connection, the leaves of these 2-calibrated foliations are exactly the symplectic fibers of $h : H_{2,r} \to y_r(-\epsilon, \epsilon)$ and $h : H_{2,-r} \to y_{-r}(-\epsilon, \epsilon)$.

The Lagrangian sphere Σ_r is going to be the attaching sphere of the $(n+1)$-handle, and therefore we need to specify an isotopy class of parametrizations (its framing is the Lagrangian framing): if $\Omega_2 = \omega_{\mathbb{R}^{2n+2}} = \sum_{i=1}^{n+1} dx_i \wedge dy_i$, then the Lagrangian sphere over $(r_0, 0)$ is the sphere radius $\sqrt{r_0}$ in the coordinates x:

$$\{(x, 0) \in \mathbb{R}^{2n+2} \mid x_1^2 + \cdots + x_{n+1}^2 = r_0\}.$$

Remark that the Lagrangian framing is the standard framing. The subset of forms of type $(1, 1)$ at the origin is convex and hence connected (the symplectic condition holds for the segment close enough to the origin). We choose any path ζ connecting $\omega_{\mathbb{R}^{2n+2}}$ to Ω_2, and Lemma 18 with parameter space ζ allows us to transfer the canonical parametrization of the sphere of radius $\sqrt{r_0}$ to a parametrization l of Σ_{r_0}. This completely determines the isotopy class of l.

To connect the hypersurfaces $H_{2,r}$ and $H_{2,-r}$ we want a careful parametrization of a neighborhood of Σ_r inside $H_{2,r}$, $r \in (0, r_0]$. Let us extend the parametrization of Σ_{r_0} to a neighborhood of Σ_{r_0} inside its leaf

$$\varphi_{r_0} : (U, \Omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}}).$$

Parallel transport over the horizontal segment $x_0(r_0, r)$ induces a parametrization of a neighborhood of Σ_r inside its leaf

$$\varphi_r := \varphi_{r_0} \circ \rho_{x_0(r,r_0)} : (\rho^{-1}_{x_0(r,r_0)}(U), \Omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}}),\ r \in (0, r_0]. \tag{8}$$

We define $T_r(\lambda) := \varphi_r^{-1}(T(\lambda))$, $r \in (0, r_0]$.

Let R_r be the (negative) Reeb vector field on $H_{2,r}$ determined by the equality $h_* R_r = \partial/\partial y$, $r \in (0, r_0]$. The neighborhood of Σ_r inside $H_{2,r}$ that we are going to consider is $T_r(\lambda, \epsilon)$, defined as in (5) using the flow of R_r on $H_{2,r}$. In fact we redefine $H_{2,r} := T_r(\lambda, \epsilon)$, $r \in (0, r_0]$.

Let

$$f_r \in C^\infty(T_r(\lambda, \epsilon)\backslash\Sigma_r, \mathbb{R}^+) \tag{9}$$

have the following properties:

- The support of f_r is contained in the interior of $T_r(\lambda, \epsilon)$.
- The time 1 flow of $f_r Y_2$ sends $T_r(\lambda/2, \epsilon/2)\backslash\Sigma_r$ into $H_{2,-r}$.

We use the hypersurface

$$H^{\mu_L}_{2,r} := \Phi_1^{f_r Y_2}(T_r(\lambda, \epsilon)\backslash\Sigma_r) \cup \Sigma_{-r} \tag{10}$$

to define the handle $P_{2,r}$ as the compact domain of $\mathbb{C}^{n+1}$ bounded by $H^{\mu_L}_{2,r}$ and $H_{2,r}$. The new boundary of the cobordism is $M^{\mu_L} = (M\backslash H_{2,r}) \cup H^{\mu_L}_{2,r}$.

4.1.3. *Lagrangian surgery.*

Proposition 19. *Any parametrized Lagrangian sphere $L \subset (M^{2n+1}, \mathcal{F}, \omega)$, $n > 1$, determines symplectic elementary cobordisms (Z, Ω) carrying a symplectic vector field transverse to the boundary, which induce 2-calibrated foliations*

$$(M, \mathcal{F}, \omega),\ (M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L}).$$

Proof. Any form of type $(1, 1)$ at the origin endows the $(n+1)$-handle $P_{2,r}$ with a symplectic structure Ω_2. The Hamiltonian vector field Y_2 is transverse to $\partial P_{2,r}$ and determines a parametrized Lagrangian sphere Σ_r. The parametrized

Lagrangian sphere Σ_r with its Lagrangian framing is isotopic to the standard sphere with its standard framing. Therefore applying Corollary 17 produces the elementary cobordism Z. Moreover, it gives rise to a symplectic structure Ω and a symplectic vector field Y transverse to ∂Z, which induce a 2-calibrated foliation on $\partial Z = M \amalg M^{\mu_L}$. By construction we recover $(\mathcal{F}, \omega)$ on M and obtain $(\mathcal{F}^{\mu_L}, \omega^{\mu_L})$ on M^{μ_L}, which coincides with $(\mathcal{F}, \omega)$ away from a neighborhood of L. □

Definition 20. Let $L \subset (M^{2n+1}, \mathcal{F}, \omega)$, $n > 1$, be a parametrized Lagrangian sphere. We define Lagrangian surgery along L as any of the 2-calibrated foliations $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$ in Proposition 19, obtained as the new boundary component of the symplectic elementary cobordism, which amounts to attaching a symplectic $(n+1)$-handle as described in 4.1.1, 4.1.2, to the trivial symplectic cobordism determined by $(M, \mathcal{F}, \omega)$.

Remark 21. Instead of gluing the $(n+1)$-handle to the trivial cobordism, we can proceed the other way around. This amounts to reversing the coorientation on $(M, \mathcal{F}, \omega)$ and hence considering the opposite symplectic vector field $\operatorname{Im} h$ on the $(n+1)$-handle. Actually, we can do things in an equivalent way: on the $(2n+2)$-dimensional $(n+1)$-handle we can use as attaching sphere Σ_{-r} instead of Σ_r, $r > 0$ (and also choosing an appropriate shape for the handle). We go from this second point of view to the first one by using the symplectic transformation $(z_1, \ldots, z_{n+1}) \mapsto (-iz_1, \ldots, -iz_{n+1})$. It can be checked that the new boundary is a 2-calibrated foliation

$$(M^{-\mu_L}, \mathcal{F}^{-\mu_L}, \omega^{-\mu_L}). \tag{11}$$

Surgery along L with framing μ_{L^-} gives (11) with opposite orientation.

4.1.4. *Independence from choices.* In the construction of $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$ there are several choices both in the symplectic handle and in the trivial cobordism, which in principle may result into nonequivalent 2-calibrations ω^{μ_L}. The choices in the symplectic handle are the symplectic form Ω_2, the parameter $r \in (0, r_0]$ (r_0 itself depends on Ω_2), the function f_r (this includes the choice of $\epsilon > 0$) and the parametrization φ_{r_0}. Choices in the trivial cobordism correspond to choices in H_1. There, we have a fixed $l^{-1} : L \to S^n$ and we choose an extension $\varphi : (U, \omega_{\mathcal{F}}) \to (T(\lambda), d\alpha_{\mathrm{can}})$ and a Reeb vector field R_1. When applying Corollary 17 to construct the elementary cobordism Z, the choice of extension φ_{r_0} is absorbed into the choice of extension φ.

In Theorem 26 we will show that for all $r > 0$ small enough, Lagrangian surgery produces a 2-calibrated foliation equivalent to generalized Dehn surgery. Since according to Proposition 13 generalized Dehn surgery is independent of the extension φ and of the Reeb vector field, we just need to prove independence of Lagrangian surgery on the function f_r and the parameter r. Note that these two choices do not matter for the diffeomorphism type of $(M^{\mu_L}, \mathcal{F}^{\mu_L})$. The key technical result that provides the required flexibility in our Poisson setting is an extension result for symplectomorphisms (Lemma 22).

Let us first address the case when all choices are the same except for the functions f_r, f_r' in (9). They give rise to two hypersurfaces $H_{2,r}^{\mu_L}(f_r)$, $H_{2,r}^{\mu_L}(f_r')$ as described in (10), transverse to Y_2 and matching near their boundary and near Σ_{-r}. Following the flow lines of Y_2 defines a compactly supported diffeomorphism from $H_{2,r}^{\mu_L}(f_r)$ to $H_{2,r}^{\mu_L}(f_r')$. The diffeomorphism is a Poisson equivalence because, by construction, it is symplectic parallel transport over horizontal segments. Therefore the extension by the identity is a Poisson equivalence between the 2-calibrated foliations associated to f_r and f_r'. The general position argument used in the proof of Theorem 10 implies that this is in fact an equivalence of 2-calibrated foliations.

The case where the only different choice is $r < r'$ is more delicate. We want to construct a Poisson equivalence

$$\phi : (M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega_r^{\mu_L}) \to (M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega_{r'}^{\mu_L}),$$

which extends the identity map in the complement of $H_{2,r}^{\mu_L} \subset M^{\mu_L}$. Let us define $\phi_1 : H_{2,r}^{\mu_L} \to H_{2,r'}^{\mu_L}$ to be the map given by the flow lines of Y_2, which we just saw corresponds to symplectic parallel transport over horizontal segments. It is well defined near Σ_{-r} because for points in $\Sigma_{-r} \subset H_{2,r}^{\mu_L}$ we make parallel transport over the segment $x_0(-r, -r')$, which does not contain the origin.

We need to introduce the following annular subsets around the Lagrangian sphere Σ_r, $r \in (0, r_0]$:

$$\begin{aligned} A_r(\lambda, \lambda') &:= T_r(\lambda)\backslash \mathrm{int} T_r(\lambda'), & \lambda > \lambda' > 0,\\ A_r(\lambda, \lambda', \epsilon, \epsilon') &:= T_r(\lambda, \epsilon)\backslash \mathrm{int} T_r(\lambda', \epsilon'), & \lambda > \lambda' > 0,\ \epsilon > \epsilon' > 0. \end{aligned}$$

The boundary of an annular subset is made of an inner and an outer connected component, according to their distance to the Lagrangian sphere (Figure 2).

Let $\lambda', \epsilon' > 0$ be such that the supports of f_r and $f_{r'}$ do not intersect $A_r(\lambda, \lambda', \epsilon, \epsilon')$ and $A_{r'}(\lambda, \lambda', \epsilon, \epsilon')$, respectively. Therefore $A_r(\lambda, \lambda', \epsilon, \epsilon') \subset H_{2,r}^{\mu_L} \cap H_{2,r}$ and $A_{r'}(\lambda, \lambda', \epsilon, \epsilon') \subset H_{2,r'}^{\mu_L} \cap H_{2,r'}$, and on $A_r(\lambda, \lambda', \epsilon, \epsilon)$,

$$\phi_1(p) = \rho_{x_{y(h(p))}}(r, r')(p). \tag{12}$$

Note that ϕ_1 does not extend to the identity map on

$$M^{\mu_L}\backslash T_r(\lambda', \epsilon') \subset M \to M^{\mu_L}\backslash T_r(\lambda', \epsilon') \subset M.$$

The problem is that according to the parametrizations of $T_r(\lambda, \epsilon)$ and $T_{r'}(\lambda, \epsilon)$ described in the paragraph following (8), the identity map corresponds to

$$\phi_2(p) := \rho_{y_{r'}(0, y(h(p)))} \circ \rho_{x_0(r, r')} \circ \rho_{y_r(y(h(p)), 0)}(p). \tag{13}$$

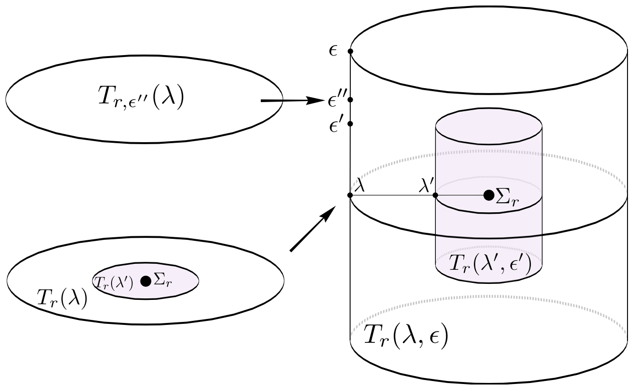

Figure 2. Right: the neighborhoods $T_r(\lambda, \epsilon)$ and $T_r(\lambda', \epsilon')$ of the Lagrangian sphere Σ_r. Horizontal slices correspond to intersections with leaves of the foliation. Left: the slice $t = 0$, which intersects both $T_r(\lambda, \epsilon)$ and $T_r(\lambda', \epsilon')$, and the slice $t = \epsilon''$, which does not intersect $T_r(\lambda', \epsilon')$.

In addition, ϕ_1 may not be everywhere defined since $\phi_1(A_r(\lambda, \lambda', \epsilon, \epsilon))$ can fail to be contained in $A_{r'}(\lambda, \lambda', \epsilon, \epsilon') \subset H_{2,r'} \cap H_{2,r}^{\mu_L} \subset M^{\mu_L}$.

Let us assume the existence of $[\lambda_1, \lambda_1'] \subset [\lambda, \lambda']$ and

$$\phi_3 : A_r(\lambda_1, \lambda_1', \epsilon, \epsilon') \to A_{r'}(\lambda, \lambda', \epsilon, \epsilon')$$

a Poisson diffeomorphism onto its image, which equals ϕ_1 near the inner boundary of $A_r(\lambda_1, \lambda_1', \epsilon, \epsilon')$ and ϕ_2 near the outer boundary. Then

$$\phi := \begin{cases} \phi_1 & \text{in } H_{2,r}^{\mu_L} \backslash A_r(\lambda_1, \lambda_1', \epsilon, \epsilon'), \\ \phi_3 & \text{in } A_r(\lambda_1, \lambda_1', \epsilon, \epsilon'), \\ \mathrm{Id} & \text{in } M^{\mu_L} \backslash (H_{2,r}^{\mu_L} \cap T_r(\lambda_1, \epsilon)), \end{cases}$$

is clearly an equivalence between $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega_r^{\mu_L})$ and $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega_{r'}^{\mu_L})$.

The construction of ϕ_3 requires the following basic result on extension of symplectic transformations, which is going to be also crucial to prove the equivalence of Lagrangian and generalized Dehn surgery.

Lemma 22. *Let* $\varsigma_j : A(\lambda, \lambda') \subset (T(\lambda), d\alpha) \to (T^*S^n, d\alpha)$, $j = 1, 2$, $n > 1$, *be symplectic diffeomorphisms onto their image with the following properties*:

(i) *There exists* $[\lambda_1, \lambda_1'] \subset [\lambda, \lambda']$ *such that* $\sigma_1 := \varsigma_2^{-1} \circ \varsigma_1$ *is defined on* $A(\lambda_1, \lambda_1')$ *and there exists* $\sigma_s : A(\lambda_1, \lambda_1') \to (T^*S^n, d\alpha)$, $s \in [0, 1]$, *an isotopy connecting the identity to* σ_1 *and satisfying* $\sigma_s(A(\lambda_1, \lambda_1')) \subset A(\lambda, \lambda')$ *for all* $s \in [0, 1]$.

(ii) *The isotopy* σ_s *is Hamiltonian.*

Then there exists $\varsigma : (A(\lambda, \lambda'), d\alpha) \to (T^*S^n, d\alpha)$ *a symplectic diffeomorphism onto its image, which coincides with* ς_1 *and* ς_2*, respectively, near the inner and outer boundaries of* $A(\lambda, \lambda')$*; moreover, if the* C^0*-norm of* σ_s *is small enough, then* ς *sends* $A(\lambda_1, \lambda_1')$ *into* $A(\lambda, \lambda)$*.*

In case ς_j*,* $j = 1, 2$*, the radii* λ, λ'*, the isotopy* σ_s *and the symplectic form* $d\alpha$ *depend on a smooth parameter,* ς *can be arranged to depend smoothly on the parameter.*

Proof. Let us define

$$V = \bigcup_{s\in[0,1]} \sigma_s(A(\lambda_1, \lambda_1')) \times \{s\} \subset T^*S^n \times [0, 1].$$

Its inner and outer boundaries are by definition the union of the inner and outer boundaries, respectively, of $\sigma_s(A(\lambda_1, \lambda_1'))$.

Condition (i) implies $V \subset A(\lambda, \lambda') \times [0, 1]$. Let X be the vector field on V whose flow after projection on $T^*S^n \times \{0\}$ gives the isotopy σ_s. Let $\beta_s = i_{X_s} d\alpha$. Since σ_s is Hamiltonian there exists a (time dependent) Hamiltonian $F \in C^\infty(V)$ such that $dF_s = \beta_s$ and $F_0 = 0$.

Because σ_s in an isotopy, for each $s \in [0, 1]$ the subset $A(\lambda, \lambda')\backslash\sigma_s(A(\lambda_1, \lambda_1'))$ has an outer connected component $C_{o,s}$ (containing the outer boundary of $A(\lambda, \lambda')$) and an inner connected component $C_{i,s}$. We define

$$\tilde{V} = \bigcup_{s\in[0,1]} (\sigma_s(A(\lambda_1, \lambda_1')) \cup C_{o,s}) \times \{s\} \subset A(\lambda, \lambda') \times [0, 1].$$

Let $\tilde{F} \in C^\infty(\tilde{V})$ be a function which coincides with F near the inner boundary of V is supported inside V and vanishes for $s = 0$. Then the time 1 flow of the path of Hamiltonian vector fields of $\tilde{F}$ composed with ς_2 is a symplectomorphism that coincides with ς_1 and ς_2, respectively, near the inner and outer boundaries of $A(\lambda, \lambda_1')$. The lemma is proved once we extend the symplectomorphism to $A(\lambda, \lambda')$ by using ς_1 on $A(\lambda_1', \lambda')$.

Also, if the C^0-norm of the isotopy is arbitrarily small, we can pick $\hat{\lambda}_1 < \lambda_1$ so that $\sigma_s(A(\hat{\lambda}_1, \lambda_1')) \subset A(\lambda_1, \lambda')$, and therefore $\varsigma(A(\lambda_1, \lambda_1')) \subset A(\lambda, \lambda')$. □

Remark 23. There is an analogous symplectic extension result when the ς_j, $j = 1, 2$, are defined on $T(\lambda)$. Under assumption (i) (with domain $T(\lambda_1)$ instead of $A(\lambda_1, \lambda_1')$), the outcome is ς is a symplectomorphism that matches ς_1 in a neighborhood of $T(\lambda')$ and ς_2 near the boundary of $T(\lambda)$. If the C^0-norm of the isotopy is small enough, then we can assume as well that $\varsigma(T(\lambda_1)) \subset T(\lambda)$.

We are going to apply Lemma 22 in several instances in which the isotopy σ_s is defined by symplectic parallel transport over curves γ_s. To that end, we are going to recall a straightforward result to control the C^0-norm of σ_s. Before that we need to introduce some notation. Given curves $\gamma_1, \ldots, \gamma_n \subset \mathbb{C}$ parametrized by the interval and such that $\gamma_l(1) = \gamma_{l+1}(0)$, $l = 1, \ldots, n-1$, their concatenation is the piecewise smooth curve

$$\gamma_1 * \cdots * \gamma_n, \quad v \in [(l-1)/n, l/n] \mapsto \gamma_l(n(v-(l-1)/n)), \quad l = 1, \ldots, n-1.$$

If we speak of a family of piecewise smooth curves, it is understood that all the curves can be written as a concatenation of the same number of curves and the family is smooth on each of the intervals.

Once we have fixed a symplectic form Ω on a neighborhood W of the origin which makes the fibers of the quadratic form h symplectic, any piecewise smooth curve $\gamma \subset \mathbb{C}$ inside the image of h induces by parallel transport a symplectomorphism ρ_γ, which in general is not everywhere defined on $h_{\gamma(0)}$ (both for points converging to the critical points and for points escaping W): we just need to pull back the symplectic fibration $f : (W \backslash \{0\}, \Omega) \to \mathbb{C} \backslash \{0\}$ and follow over each smooth piece of the curve the 1-dimensional kernel of the closed 2-form induced on the pullback fibration. From now on, unless otherwise stated, by a curve $\gamma \subset \mathbb{C}$ we will mean a piecewise smooth curve such that on each smooth interval it is either constant or embedded. In this way (i) we can define horizontal lifts of γ without using pullback bundles, and (ii) on each smooth interval γ is the integral curve of a locally defined vector field. These two properties will make our proofs more transparent.

We also recall that $A_{r,t}(\lambda, \lambda')$, $r \in (0, r_0]$, $t \in [-\epsilon, \epsilon']$, stands for the time t Reeb flow of $A_r(\lambda, \lambda')$, where the Reeb vector field is R_r. If we let $\tilde{Y}$ denote the horizontal lift of $\partial/\partial y$, then $R_r = \tilde{Y}$. Then we also define $A_{0,t}(\lambda, \lambda') := \Phi_t^{\tilde{Y}}(A_0(\lambda, \lambda'))$ ($A_0(\lambda, \lambda')$ itself well defined because $A_{r_0}(\lambda, \lambda') \cap \Sigma_{r_0}$ is empty).

Lemma 24. *Let $\kappa_{t,s} \subset \mathbb{C}$, $t \in [\delta, \delta']$, $s \in [0, 1]$, be a family of loops. Let $\gamma_{t,s,l}$ be a sequence of families of loops converging to $\kappa_{t,s}$ in the C^1-norm uniformly on t, s. If the horizontal lifts $\tilde{\kappa}_{t,s}$ starting at $A_{r,t}(\lambda, \lambda')$ are defined for all $v \in [0, 1]$ (the lift neither converges to $0 \in \mathbb{C}^{n+1}$ nor leaves W), then the following hold:*

(i) *As l tends to infinity we have convergence $\rho_{\gamma_{t,s,l}} \xrightarrow{C^0} \rho_{\kappa_{t,s}}$ on $A_{r,t}(\lambda, \lambda')$ uniformly on t, s.*

(ii) *For any fixed t, if $\rho_{\kappa_{t,0}}, \rho_{\gamma_{t,0,l}}$ are the identity map, $\gamma_{t,s,l}$ does not intersect the origin and the homotopies $\gamma_{t,s,l}$ converge to the homotopy $\kappa_{t,s}$ in the C^2-norm, then $\rho_{\kappa_{t,s}}$ is a Hamiltonian isotopy.*

Proof. Recall that $\rho_{\kappa_{t,s}} = \tilde{\kappa}_{t,s}(1)$. Let K be the union of the horizontal lifts $\tilde{\kappa}_{t,s}$ starting at all $p \in A_{r,t}(\lambda, \lambda')$ for all t, s. By assumption $K \subset W$ is a compact subset not containing the critical point $0 \in \mathbb{C}$. Then we can work inside $U_K \subset W$ a compact neighborhood of K missing the critical point, where the convergence in point (i) follows from basic ODE theory.

If $n = 2$ then $\kappa_{t,s}$ may not be Hamiltonian because $A_{r,t}(\lambda, \lambda')$ has nontrivial first Betti number. If $\gamma_{t,s,l}$ does not contain the origin, then parallel transport cannot converge to the critical point $0 \in \mathbb{C}^{n+1}$. It cannot scape W for connectivity reasons: for each fixed t and for l large enough, parallel transport $\rho_{t,s.l}$, $s \in [0, 1]$ is an isotopy sending $A_{r,t}(\lambda, \lambda')$ inside $h_{\kappa_{t,s}(0)} \cap W$. Then it must send $T_{r,t}(\lambda)$ inside $h_{\kappa_{t,s}(0)} \cap W$.

Because $T_{r,t}(\lambda)$ has trivial first Betti number, $\rho_{\gamma_{t,s,l}}$ is a Hamiltonian isotopy. Because convergence of the homotopies in the C^2-norm implies convergence of the isotopies in the C^1-norm, the closed 1-form β_s associated to the isotopy $\rho_{\kappa_{t,s}}$, $s \in [0, 1]$, can be C^0-approximated by exact ones, and therefore it is exact and $\rho_{\kappa_{t,s}}$ is Hamiltonian. □

Remark 25. A similar convergence result holds if the horizontal lifts start at all points in $T_{r,t}(\lambda)$.

We are ready to construct ϕ_3 on $A(\lambda_1, \lambda_1', \epsilon, \epsilon')$, which coincides with the Poisson morphism ϕ_1 of (12) near the inner boundary of $A(\lambda_1, \lambda_1', \epsilon, \epsilon')$ and with the morphism ϕ_2 of (13) near the outer boundary.

Recall that t is the coordinate on the interval $[-\epsilon, \epsilon]$ and fix $\epsilon'' \in (\epsilon', \epsilon)$. In a first stage we are going to apply Lemma 22 to the restrictions to the t-leaf $\phi_{1,t}$, $\phi_{2,t}$ with parameter space $t \in [-\epsilon'', \epsilon'']$: let us define

$$\gamma_{t,1} := x_t(r, r') * y_{r'}(t, 0) * x_0(r', r) * y_r(0, t).$$

By equations (12) and (13), $\sigma_t := \phi_{2,t}^{-1} \circ \phi_{1,t} = \rho_{\gamma_{t,1}}$. We let $\sigma_{t,s} := \rho_{\gamma_{t,s}}$, where $\gamma_{t,s}$ is a family of curves in $\mathbb{C}$ connecting the constant path (r, t) to $\gamma_{t,1}$, for example as depicted in Figure 3.

To get control on the C^0-norm of $\rho_{\gamma_{t,s}}$, we define

$$\kappa_{t,s} = y_r(t, (s-1)t) * y_r((s-1)t, t),$$

and we let the family $\gamma_{t,s}$ vary with r', so that when r' converges to r the curves $\gamma_{t,s}$ converge to the curves $\kappa_{t,s}$ in the C^1-norm. Since $\kappa_{t,s}$ does not contain the

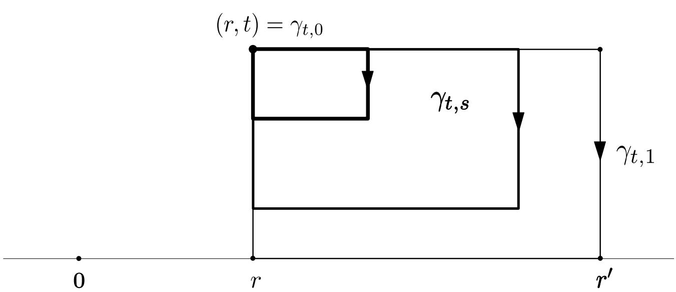

Figure 3. A family of curves shrinking $\gamma_{t,1}$ the boundary of the rectangle to the vertex (r, t).

origin and $\rho_{\kappa_{t,s}} = \mathrm{Id}$, by Lemma 24 if r' is close enough to r then $\rho_{\gamma_{t,s}}$ is as close as desired to the identity on $A_{r,t}(\lambda', \lambda)$ in the C^0-norm. Remark that here we do not use the full power of Lemma 24, as the curves $\kappa_{t,s}$ do not contain the origin.

The conclusion is that for $t \in [-\epsilon', \epsilon']$, hypothesis (i) in Lemma 22 is satisfied (it is understood that we conjugate the isotopy problem in $A_{r,t}(\lambda, \lambda')$ to an isotopy problem in $A(\lambda, \lambda')$, using minus the Reeb flow for time t and the chart φ_r). We can perform exactly the same construction for $|t| \in [\epsilon', \epsilon'']$ with the maps $\rho_{\kappa_{t,s}} = \mathrm{Id}$, $\rho_{\gamma_{t,s}}$ defined now on $T_{r,t}(\lambda)$, and conclude that the hypothesis of Remark 25 is also satisfied.

Because $\kappa_{s,t}$ does not contain $0 \in \mathbb{C}$, for r' close enough to r the isotopy $\gamma_{t,s}$ misses the origin, and therefore it is a Hamiltonian isotopy. Thus the hypotheses of Lemma 22 and Remark 23 are satisfied. Inspection of the proof of Remark 23 shows that the lemma and remark can be combined to produce $\phi_{3,t}$, $t \in [-\epsilon'', \epsilon'']$, depending smoothly on t and extending ϕ_1 and ϕ_2.

The extension for $|t| \in [\epsilon'', \epsilon]$ is straightforward: we let $\tilde{\sigma}_{t,s}$ be the isotopy corresponding to the Hamiltonian $\tilde{F}_t$ in the proof of Lemma 22 (rather in the proof of Remark 23). We have defined $\phi_{3,t} := \phi_{2,t} \circ \tilde{\sigma}_{t,1}$. Let $\beta : [\epsilon'', \epsilon] \to [0, 1]$ be orientation-reversing and constant near the boundary. For $t \in [\epsilon'', \epsilon]$ we set $\phi_{3,t} := \phi_{2,t} \circ \sigma_{\epsilon'', \beta(s)}$. For negative t we proceed analogously and this produces the required extension $\phi_{3,t}$, $t \in [-\epsilon, \epsilon]$.

We have showed that Lagrangian surgery produces equivalent 2-calibrations if r, r' are close enough, which obviously implies the independence of the construction on $r \in (0, r_0]$.

4.2. ***Generalized Dehn surgery is equivalent to Lagrangian surgery.*** The equivalence of the two surgeries will remove all dependences appearing in Lagrangian surgery. The proof of the equivalence bears much resemblance to the proof of the independence of Lagrangian surgery on the parameter $r > 0$, though it has additional technical complications. We give a brief overview in the following paragraphs.

To construct the equivalence between $(M^L, \mathcal{F}^L, \omega^L)$ and $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$, a Poisson diffeomorphism suffices. The morphism is defined to be the identity away from a neighborhood of the Lagrangian spheres; then — working already in the symplectic handle we used in the cobordism — following the flow lines of Y_2 extends the identity to a morphism

$$\phi_2 : H_{2,r} \backslash T_r(\lambda/2, \epsilon/2) \to H_{2,r}^{\mu_L}.$$

For some $\lambda' \in (\lambda/2, \lambda)$, $\epsilon' \in (\epsilon/2, \epsilon)$, ϕ_2 restricts in $A_r(\lambda', \lambda/2, \epsilon', \epsilon/2)$ to parallel transport over horizontal segments $x_t(r, -r)$.

Let us cut $T_r(\lambda', \epsilon')$ along $T_r(\lambda')$ and let $\chi : T_r^+(\lambda') \to T_r^-(\lambda')$ be conjugated to a generalized Dehn twist supported in the interior of $T(\lambda/2)$. We would be done if perhaps after modifying ϕ_2 near the inner boundary of $A_r(\lambda', \lambda/2, \epsilon', \epsilon/2)$, we can extend it to a morphism

$$\phi_3 : T_r^+(\lambda', \epsilon') \#_\chi T_r^-(\lambda', \epsilon') \to H_{2,-r} \subset H_{2,r}^{\mu_L}. \tag{14}$$

Equivalently, we need a pair of morphisms $\phi_3^\pm : T_r^\pm(\lambda', \epsilon') \to H_{2,-r}$ which satisfy

$$\phi_3^+(p) = \phi_3^- \circ \chi(p),\ \ p \in T_r^+(\lambda'), \tag{15}$$

and which are independent of t for $|t|$ small, so the induced morphism $\phi_3^+ \#_\chi \phi_3^-$ is smooth.

If Ω_2 is the closed 2-form Ω_τ, then χ can be taken to be $\rho_{\partial\bar{D}(r)}$, parallel transport over $\partial\bar{D}(r)$ counterclockwise.

Consider the positive half disks $\partial\overline{D}^+(r) := \{re^{i\theta\pi} \mid 0 \leq \theta \leq 1\} \subset \mathbb{C}$, and set $\zeta_t = y_r(t, 0) * \partial\overline{D}^+(r) * y_{-r}(0, t)$. Then define

$$\begin{aligned} \rho^+ : T_r^+(\lambda', \epsilon') &\to H_{2,-r}, \\ p &\mapsto \rho_{\zeta_{y(h(p))}}(p), \end{aligned}$$

and define ρ^- on $T_r^-(\lambda', \epsilon')$ by parallel transport over the reflection of ζ_t in the x-axis. Then $\rho^\pm$ satisfy (15) and therefore they induce a morphism as in (14). But this morphism does not match ϕ_2 because for the latter we do parallel transport over horizontal segments and for $\rho^\pm$ we use half disks (up to composition with vertical segments). So our problem reduces to define Poisson equivalences on $A_r^\pm(\lambda', \lambda/2, \epsilon', \epsilon/2)$, which extend parallel transport over ζ_t (and its reflection in the x-axis) near the inner boundary and parallel transport over $x_t(-r, r)$ near the outer boundary. Of course, the extensions $\phi_3^\pm$ have to be compatible on $A_r^\pm(\lambda', \lambda/2)$ with χ; because χ is supported in the interior of $T_r(\lambda/2)$ the extensions must coincide on $A_r^\pm(\lambda', \lambda/2)$. This compatibility condition is going to follow from a careful choice of the families of curves connecting $x_r(-t, t)$ to ζ_t. Since we will be doing parallel transport near the critical point, we will need the full power of Lemma 24 to argue that we can control the norm of the isotopies we construct and hence we are in the hypothesis of the interpolation lemma.

A further technical complication appears because the symplectic form Ω_2 in the handle is different from Ω_τ. So the extension of parallel transport over segments and half disks has to include a deformation from parallel transport with respect to Ω_2 to parallel transport with respect to Ω_τ.

Theorem 26. *Assume $n > 1$. We have equivalences of 2-calibrated foliations*

$$\phi : (M^L, \mathcal{F}^L, \omega^L) \to (M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L}) \tag{16}$$

for all $r > 0$ small enough.

Proof. Stage 1. The complement $(M, \mathcal{F}, \omega)\backslash T_r(\lambda, \epsilon)$ can be seen as a subset of both $(M, \mathcal{F}, \omega)$ and $(M^{\mu_L}, \mathcal{F}^{\mu_L}, \omega^{\mu_L})$. We may assume without loss of generality that for some $\lambda' > \lambda/2$, $\epsilon' > \epsilon/2$, the time 1 flow of $f_r Y_2$ sends $T_r(\lambda', \epsilon')\backslash\Sigma_r$ into $H_{2,-r} \subset H_{2,r}^{\mu_L} \subset M^{\mu_L}$. We define

$$\phi_0 = \begin{cases} \mathrm{Id} & \text{in } M\backslash T_r(\lambda, \epsilon), \\ \Phi_1^{f_r Y_2} & \text{in } A_r(\lambda, \lambda/2, \epsilon, \epsilon/2), \end{cases}$$

which is a Poisson morphism given on $A_r(\lambda, \lambda/2, \epsilon', \epsilon/2)$ by parallel transport over horizontal segments $x_t(-r, r)$, $t \in [-\epsilon', \epsilon']$.

Stage 2. In both $H_{2,r}$ and $H_{2,-r}$ we have Reeb vector fields R_r, R_{-r} defined near Σ_r and Σ_{-r} respectively (they are horizontal lifts of $\partial/\partial y$). Their flow parametrizes the leaf spaces by $t \in [-\epsilon, \epsilon]$. For the purpose of checking the smoothness of the morphism $\phi : M^L \to M^{\mu_L}$ in the statement of the theorem, in this stage we shall modify ϕ_0 near the inner boundary of $A_r(\lambda', \lambda/2, \epsilon', \epsilon/2)$ to make it t-invariant for $|t|$ small (equivariant with respect to the flows of R_r and R_{-r}).

Let $\beta : [-\epsilon', \epsilon'] \to [-\epsilon', \epsilon']$ be an odd monotone function which is the identity near the boundary and maps to zero exactly the interval $[-\delta, \delta]$, with $0 < \delta < \epsilon/2$. Set

$$\zeta_t := y_r(t, \beta(t)) * x_{\beta(t)}(r, -r) * y_{-r}(\beta(t), t), \quad t \in [-\epsilon', \epsilon'],$$

and define

$$\phi_1(p) = \rho_{\zeta_{y(h(p))}}(p), \quad p \in A_r(\lambda', \lambda/2, \epsilon', \epsilon/2),$$

which by construction is t-invariant for $t \in [-\delta, \delta]$.

We are going to construct $\phi_2' : A_r(\lambda', \lambda/2, \epsilon', \epsilon/2) \to H_{2,-r}$ extending ϕ_0 near the outer boundary and ϕ_1 near the inner boundary, by applying Lemma 22: let

$$\begin{aligned}(17)\quad &\gamma_{t,s} = \\ &y_r(t, (1-s)t+s\beta(t)) * x_{(1-s)t+s\beta(t)}(r, -r) * y_{-r}((1-s)t+s\beta(t), t) * x_t(-r, r),\end{aligned}$$

with $t \in [-\epsilon', \epsilon']$, $s \in [0, 1]$ and $r \in (0, r_0]$. Parallel transport over $\gamma_{t,s}$ defined on $A_{r,t}(\lambda', \lambda/2)$ connects the identity map to $\phi_{0,t}^{-1} \circ \phi_{1,t}$. To estimate the C^0-norm of $\rho_{\gamma_{t,s}}$ we define $\kappa_{t,s}$ by using the formula of $\gamma_{t,s}$ in (17) for $r = 0$, and consider $\rho_{\kappa_{t,s}}$ with domain $A_{0,t}(\lambda', \lambda/2)$. By construction $\rho_{\kappa_{t,s}}$ is the identity.

Let $\gamma'_{t,s}$ be the conjugation of $\gamma_{t,s}$ by $x_t(0, r)$ and let us consider $\rho_{\gamma'_{t,s}}$ defined on $A_{0,t}(\lambda', \lambda/2)$, the same domain as for $\kappa_{t,s}$.

We construct the extension $\phi'_{2,t}$ first for the leaves in $[-\epsilon/2, \epsilon/2]$: the union of the horizontal lifts $\tilde{\kappa}_{t,s}$ at $A_{0,t}(\lambda', \lambda/2)$ is exactly

$$(18)\qquad K = \bigcup_{t \in [-\epsilon/2, \epsilon/2]} A_{0,t}(\lambda', \lambda/2),$$

a compact subset not containing the critical point $0 \in \mathbb{C}^{n+1}$. The curves $\gamma'_{t,s}$ clearly converge in the C^1-norm to $\kappa_{t,s}$ as r goes to zero. Therefore by point (i) in Lemma 24 there is C^0-convergence of $\rho_{\gamma'_{t,s}}$ to the identity.

The same result holds for $\rho_{\gamma_{t,s}}$, though not automatically since parallel transport over $x_t(0,r)$ does not send $A_{0,t}(\lambda', \lambda/2)$ diffeomorphically into $A_{r,t}(\lambda', \lambda/2)$. This is the same situation as in the proof of independence of Lagrangian surgery on r. We define

$$\tau_{t,s} = x_t(0,r) * y_r((t,(s-1)t) * x_{(s-1)t}(r,0) * y_r((s-1)t,t).$$

For $r=0$ we get $x_t(0,0)*y_0(t,(s-1)t)*x_{(s-1)t}(0,0)*y_0((s-1)t,t)$. We consider parallel transport $\rho_{\tau_{t,s}}$ defined on $A_{r,t}(\lambda', \lambda/2)$, which for $r=0$ is the identity. Since for $r=0$ the union of the horizontal lifts of $\tau_{t,s}$ starting at $A_{0,t}(\lambda', \lambda/2)$ is again K in (18), by point (i) in Lemma 24 we conclude that $\rho_{x_t(0,r)}(A_{0,t}(\lambda', \lambda/2))$ converges to $A_{r,t}(\lambda', \lambda/2)$ in the C^0-norm as r tends to zero, and this finishes the proof of the estimate needed in point (i) of Lemma 22 for $t \in [-\epsilon/2, \epsilon/2]$.

For $|t| \in [\epsilon/2, \epsilon']$ the estimate holds by connectivity arguments already mentioned: the proof above shows that for some interval $[\lambda'_1, \lambda'_2] \subset [\lambda', \lambda/2]$, the isotopy $\rho_{\gamma_{t,s}}$ sends $A_{r,t}(\lambda'_1, \lambda'_2)$ into $A_{r,t}(\lambda', \lambda/2)$, for $|t| \in [\epsilon/2, \epsilon']$. Hence it must send $T_{r,t}(\lambda'_1)$ into $T_{r,t}(\lambda')$.

The isotopies $\rho_{\gamma_{t,s}}$ are Hamiltonian: if t is not in $[-\delta, \delta,]$, then $\rho_{\gamma_{t,s}}$ extends to $T_{r,t}(\lambda')$ because $\gamma_{t,s}$ does not contain the origin. For the remaining values of t it easy to check that the homotopy $\gamma_{t,s}$ can be approximated in the C^2-norm by a homotopy which does not contain $0 \in \mathbb{C}$. Therefore by point (ii) in Lemma 24 the isotopies are Hamiltonian. Hence we can apply Lemma 22 and Remark 23 in a compatible manner to produce ϕ'_2 on $A_r(\lambda', \lambda/2, \epsilon''\epsilon/2)$, $\epsilon'' \in (\epsilon/2, \epsilon')$, extending ϕ_0 and ϕ_1. For the t-leaves with $|t| \in [\epsilon'', \epsilon']$, we apply the same patching trick as in the construction of the extension ϕ_3 at the end of 4.1.4.

We define for $r > 0$ small enough

$$\phi_2 = \begin{cases} \phi_0 & \text{in } M \backslash T_r(\lambda', \epsilon'), \\ \phi'_2 & \text{in } A_r(\lambda', \lambda/2, \epsilon', \epsilon/2), \end{cases}$$

which is a Poisson morphism independent of $t \in [-\delta, \delta]$.

Stage 3. In this stage we cut M along a neighborhood of L inside its leaf F_L, and then define a Poisson morphism which extends ϕ_2 from Stage 2 and parallel transport over boundaries of half disks ("conjugated" by vertical segments); the latter parallel transport also includes a deformation from Ω_2 to Ω_τ.

Let us assume for the moment that Ω_2 equals $\Omega_{\mathbb{R}^{2n+2}}$. The closed 2-forms Ω_τ [Seidel 2003, Section 1.2] are written $\Omega_\tau = \Omega_{\mathbb{R}^{2n+2}} + d\alpha$, where $d\alpha$ vanishes on the tangent space to the fibers h_z and is zero in a neighborhood of the union of the stable and the unstable manifold of Y_2 with respect to $\Omega_{\mathbb{R}^{2n+2}}$. The first property implies that the fibers h_z are symplectic. The second property implies that symplectic parallel transport with respect to Ω_τ over $x_0(r,-r)$ is defined on $T_r(\lambda)\backslash\Sigma_r$.

We assume that α has been chosen so that parallel transport over $\partial\bar{D}(r)$ counterclockwise is conjugated by φ_r to a generalized Dehn twist supported in the interior of $T(\lambda/2)$. Let us define

$$\Omega_u = \Omega_{\mathbb{R}^{2n+2}} + u\, d\alpha, \quad u \in [0,1],$$

and let $u : [0,\epsilon'] \to [0,1]$ be a monotone function which attains the value 0 on $[2\delta/3,\epsilon']$ and the value 1 on $[0,\delta/3]$.

Let us consider the arcs $\partial\overline{D}_t^+(r) := \{(0,t) + re^{i\theta\pi}\} \subset \mathbb{C}$, and define the curves $\zeta_t = y_r(t,\beta(t)) * \partial\overline{D}^+_{\beta(t)}(r) * y_{-r}(\beta(t),t)$. Next we cut $T_r(\lambda',\epsilon')$ along $T_r(\lambda')$ and define on $T_r^+(\lambda',\epsilon')$

$$\phi_3(p) = \rho_{u(y(h(p))),\zeta_{y(h(p))}}(p), \tag{19}$$

which is t-invariant for $t \in [0,\delta/3]$ and on $T_{r,0}(\lambda')$ is parallel transport over $\partial\bar{D}(r)^+$ counterclockwise with respect to Ω_τ, and therefore conjugated to a Dehn twist supported in the interior of $T(\lambda/2)$. We stress that this is a Poisson morphism because the restriction of Ω_u to fibers of h is independent of u (of course what changes is the symplectic connection).

We address now the construction of ϕ_3^+ on $A_r^+(\lambda',\lambda/2,\epsilon',\epsilon/2)$, a Poisson morphism extending ϕ_2 and ϕ_3 and t-invariant for $t \in [0,\delta/3]$, using the same pattern as in Stage 2.

Let us define the curves

$$\gamma_{t,s} = y_r(t,\beta(t)) * x_{\beta(t)}(r,sr) * \partial\overline{D}^+_{\beta(t)}(sr) * x_{\beta(t)}(-sr,r) * y_r(\beta(t),t), \tag{20}$$

for $t \in [0,\epsilon']$ and $s \in [0,1]$ (see Figure 4). We have $\phi_{\gamma_{t,1}} = \phi_{2,t}^{-1} \circ \phi_{3,t}$, $\phi_{\gamma_{t,0}} = \mathrm{Id}$.

Smoothness of $\rho_{u(t),\gamma_{t,s}}$ for $s = 0$ may not be evident.

Lemma 27. *The map $\rho_{u(t),\gamma_{t,s}}$ depends smoothly on t, s.*

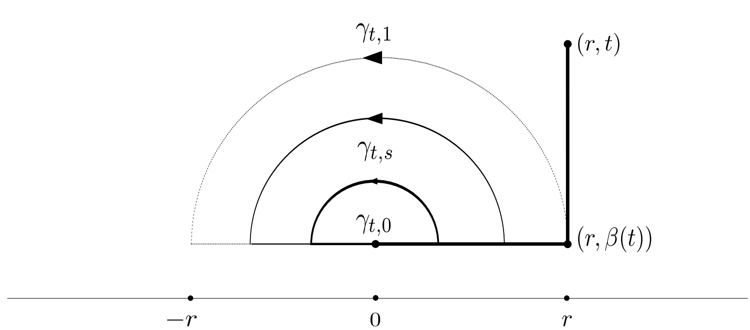

Figure 4. The curves $\gamma_{t,s}$ defined in (20).

Proof. We rewrite $\rho_{u(t),\gamma_{t,s}}$ using vector fields on $\mathbb{C}$ whose integral curves are the pieces whose concatenation defines $\gamma_{t,s}$. Let

$$X := \frac{\partial}{\partial x}, \quad Y := \frac{\partial}{\partial y}, \quad \Theta_{r,t} := rx\frac{\partial}{\partial y} - (r(y-t))\frac{\partial}{\partial x}, \quad t \in \mathbb{R},$$

be vector fields on $\mathbb{C}$. Let $\tilde{X}_u, \tilde{Y}_u, \tilde{\Theta}_{u,r,t} \in \mathfrak{X}(W\backslash\{0\})$ be their horizontal lifts with respect to the symplectic connection defined by Ω_u. The flows $\Phi_l^{\tilde{X}_u}$, $\Phi_l^{\tilde{Y}_u}$, $\varphi_l^{\tilde{\Theta}_{u,r,t}}$ are smooth in u, r, t, l. It follows that

$$\rho_{u(t),\gamma_{t,s}} = \Phi^{\tilde{Y}_{u(t)}}_{t-\beta(t)} \circ \Phi^{\tilde{X}_{u(t)}}_{(s+1)r} \circ \Phi^{\tilde{\Theta}_{u(t),sr,\beta(t)}}_{\pi} \circ \Phi^{-\tilde{X}_{u(t)}}_{(1-s)r} \circ \Phi^{-\tilde{Y}_{u(t)}}_{t-\beta(t)},$$

and thus $\rho_{u(t),\gamma_{t,s}}$ has smooth dependence on t, s. □

The estimate in Lemma 24 is written for parallel transport with respect to a fixed symplectic form, but it can be checked that it holds true as well in case the parallel transport is with respect $\Omega_{u(t)}$. Let $\kappa_{t,s}$ be as defined in (17) for $r = 0$. By comparing $\rho_{u(t),\gamma_{t,s}}$ with $\rho_{u(t),\kappa_{t,s}} = \mathrm{Id}$ as in the previous stage (first conjugating with $x_t(r, 0)$ to have common domain, and then showing that the estimate holds after undoing the conjugation), we get control on the C^0-norm of $\rho_{u(t),\gamma_{t,s}}$ for r small enough. The isotopies $\rho_{u(t),\gamma_{t,s}}$ are Hamiltonian since they can be C^1-approximated by Hamiltonian ones. Given that the isotopy $\rho_{u(t),\gamma_{t,s}}$ is t-invariant for $t \in [0, \delta/3]$, choices in the proof of Lemma 22 can be done to obtain, for all r small enough, an extension ϕ_3^+ on $A_r^+(\lambda', \lambda/2, \epsilon', \epsilon/2)$ which is t-invariant for $t \in [0, \delta/3]$.

For $t \in [-\epsilon', 0]$ we proceed as we did for positive values, but using the reflection of the curves $\gamma_{t,s}$ in the x-axis. It is possible to arrange the proof of Lemma 22 to produce an extension ϕ_3^- on $A_r^+(\lambda', \lambda/2, \epsilon', \epsilon/2)$ such that

- ϕ_3^- is t-invariant for $t \in [-\delta/3, 0]$,
- $\phi_3^+ = \phi_3^-$ on $A_{r,0}(\lambda', \lambda/2)$.

Then we extend ϕ_3^+ to $T_r^+(\lambda', \epsilon')$ by using on $T_r^+(\lambda/2, \epsilon/2)$ the same parallel transport over ζ_t as in (19). Likewise, we extend ϕ_3^- to $T_r^-(\lambda', \epsilon')$ by using on $T_r^-(\lambda/2, \epsilon/2)$ parallel transport over the reflection of ζ_t in the x-axis.

Because $\phi_3^+(p) = \phi_3^- \circ \chi(p)$ for $p \in T_r^+(\lambda')$, and ϕ_3^+, ϕ_3^- are t-invariant for $|t|$ small, they give rise to a Poisson morphism

$$\phi_3^+ \#_{\rho_{\partial \bar{D}(r)}} \phi_3^- : T_r^+(\lambda', \epsilon') \#_{\rho_{\partial \bar{D}(r)}} T_r^-(\lambda', \epsilon') \to H_{2,-r}.$$

The equivalence of 2-calibrated foliations for all $r > 0$ small enough is

$$\phi = \begin{cases} \phi_2 & \text{in } M^L\backslash(T_r^+(\lambda', \epsilon') \#_{\rho_{\partial \bar{D}(r)}} T_r^-(\lambda', \epsilon')), \\ \phi_3^+ \#_{\rho_{\partial \bar{D}(r)}} \phi_3^- & \text{in } T_r^+(\lambda', \epsilon') \#_{\rho_{\partial \bar{D}(r)}} T_r^-(\lambda', \epsilon'). \end{cases} \tag{21}$$

Let us now drop the assumption $\Omega_2 = \Omega_\tau$. Let Ω_u be a path which is constant near its boundary and which connects Ω_2 to $\Omega_{\mathbb{R}^{2n+2}}$.

Recall that the neighborhoods $T_r(\lambda, \epsilon)$ have been defined with respect to Ω_2. By the parametric version of Lemma 18 (more specifically by the parametric version of the stable manifold theorem), we have smooth parametrizations $\Sigma_{u,r}$, $r \in (0, r']$. By compactness we can extend $\varphi_{r'}$ to parametrizations

$$\varphi_{u,r'} : (T_{u,r'}(\tilde{\lambda}), \Omega_u) \to (T(\tilde{\lambda}), d\alpha_{\mathrm{can}}).$$

Then we define the subsets $T_{u,r}(\tilde{\lambda})$ by parallel transport of $T_{u,r'}(\tilde{\lambda})$ over $x_0(r', r)$ with respect to Ω_u, and their associated parametrizations $\varphi_{u,r} := \varphi_{u,r'} \circ \rho_{x_0(r,r')}$. The subsets $T_{u,-r}(\tilde{\lambda}) \backslash \Sigma_{u,-r}$ and their parametrizations are defined in the same manner.

We can assume without loss of generality that the inclusion

$$T_{u,r}(\tilde{\lambda}) \subset T_r(\lambda/2) \tag{22}$$

holds for all $r \in (0, r']$. This is because the parametric version of the stable manifold theorem implies that

$$\begin{aligned} T(\lambda) \times [0, r'] \times [0, 1] &\to h_0, \\ (q, r, u) &\mapsto \rho_{u,x_0(r',0)}(\varphi_{u,r}^{-1}(q)) \end{aligned}$$

is continuous, where by definition $\rho_{u,x_0(r,0)}(p) = 0 \in \mathbb{C}^{n+1}$ for $p \in \Sigma_{u,r}$.

We proceed to modify both $A_r^+(\lambda', \lambda/2, \epsilon', \epsilon/2)$ and ϕ_3 in (19) just for values of t in $[0, \delta]$: let $b_i : [0, \delta] \to [2\tilde{\lambda}/3, \lambda']$ and $b_o : [0, \delta] \to [\tilde{\lambda}/2, \lambda/2]$ be monotone increasing functions which are constant on $[0, 3\delta/4]$ and near δ.

Let $\upsilon : [0, \delta] \to [0, 1]$ be a orientation reversing smooth function which is constant on $[0, \delta/2]$ and on $[3\delta/4, \delta]$.

We substitute $A_{r,t}(b_o(t), b_i(t))$ for $A_{r,t}(\lambda', \lambda/2)$ and

$$\tilde{\phi}_{3,t} := \rho_{0,y_{-r}(0,t)} \circ \varphi_{0,-r}^{-1} \circ \varphi_{\upsilon(t),-r} \circ \rho_{\upsilon(t),\partial \bar{D}^+_{\beta(t)}(r)} \circ \varphi_{\upsilon(t),r}^{-1} \circ \varphi_{0,r} \circ \rho_{0,y_r(t,0)}$$

defined on $A_{r,t}(b_o(t), b_i(t))$ for $\phi_{3,t}$ in (19). Note that the modification of the symplectic form only occurs when the domain has been modified to $A_{r,t}(2\tilde{\lambda}/3, \tilde{\lambda}/2)$.

By the inclusion in (22), the image of $A_{r,t}(2\tilde{\lambda}/3, \tilde{\lambda}/2)$ under $\varphi_{\upsilon(t),r}^{-1} \circ \varphi_{0,r} \circ \rho_{0,y_r(t,0)}$ is contained in $T_r(\lambda/2)$. If in addition $r > 0$ is small enough, control on the C^0-norm of $\rho_{\upsilon(t),\partial \bar{D}^+_{\beta(t)}(r)}$ by r implies that

$$\rho_{\upsilon(t),\partial \bar{D}^+_{\beta(t)}(r)} \circ \varphi_{\upsilon(t),r}^{-1} \circ \varphi_{0,r} \circ \rho_{0,y_r(t,0)}$$

sends $A_{r,t}(2\tilde{\lambda}/3, \tilde{\lambda}/2)$ into $T_{u(t),-r}(\tilde{\lambda}) \backslash \Sigma_{u(t),-r}$, so we can compose with the chart $\varphi_{\upsilon(t),-r}$. Therefore $\tilde{\phi}_{3,t}$ is well defined and for $t \in [0, \delta/2]$ we are in the situation $\Omega_2 = \Omega_{\mathbb{R}^{2n+2}}$.

Then we have to choose Ω_τ whose conjugation by $\varphi_{0,r}$ ($\Omega_0 = \Omega_{\mathbb{R}^{2n+2}}$) is a Dehn twist supported in the interior of $T(\tilde{\lambda}/2)$.

We can use the same pattern to modify the isotopy needed to apply the extension lemma with parameters (this time the radii of the annuli vary with t). The result is an extension ϕ_3^+ which is t-invariant for $t \in [0, \delta/4]$, and for which on $A_{r,0}(2\tilde{\lambda}/3, \tilde{\lambda}/2)$ the chart $\varphi_{0,t}$ conjugates to a Dehn twist supported on $T(\tilde{\lambda}/2)$.

As we did in the previous stage, we construct the extension ϕ_3^- using as domain and curves the reflection of the previous data in the x-axis. Then ϕ, defined as in (21), is the equivalence of 2-calibrated foliations which proves the theorem. □

Remark 28. Similarly, for $n > 1$ and every $r > 0$ small enough one constructs equivalences

$$(-M^{L^-}, \mathcal{F}^{L^-}, \omega^{L^-}) \to (M^{-\mu_L}, \mathcal{F}^{-\mu_L}, \omega^{-\mu_L}),$$
$$(M^{-L}, \mathcal{F}^{-L}, \omega^{-L}) \to (M^{\mu_{L^-}}, \mathcal{F}^{\mu_{L^-}}, \omega^{\mu_{L^-}}).$$

5. Lefschetz pencil structures and transverse taut foliations

Let $(M^{2n+1}, \mathcal{F}, \omega)$ be an integral 2-calibrated foliation. In this section we gather information on the intersection of a Donaldson-type submanifold with the leaves of $\mathcal{F}$ using Lefschetz pencil structures. We also describe the relation between two Donaldson type submanifolds belonging to the same Lefschetz pencil.

We start by saying a few words about how Donaldson-type submanifolds W are constructed, and how the failure of standard Morse theoretic methods to describe the topology of $W \cap F$, $F \in \mathcal{F}$, leads to the use of Lefschetz pencil structures to address this problem.

Let us fix J a leafwise almost complex structure compatible with ω. If J is integrable then by definition $(M, \mathcal{F}, J)$ is a Levi-flat manifold, and the line bundle L_ω whose curvature is $-2\pi i\omega$ is a positive CR line bundle. According to [Ohsawa and Sibony 2000], large powers of L_ω (suitably twisted) have plenty of CR sections. In particular there exist CR sections leafwise transverse to the zero section of $L_\omega^{\otimes k}$. The zero set of any such section is a codimension-two CR submanifold, or a divisor, intersecting $\mathcal{F}$ transversely.

In general J is not integrable. However $L_\omega^{\otimes k} \otimes \mathbb{C}^l$ has sections s which are both close to being J-holomorphic in an appropriate sense and leafwise transverse to the zero section of $L_\omega^{\otimes k} \otimes \mathbb{C}^l$ [Ibort and Martínez Torres 2004a, Corollary 1.2]. As a consequence $W = s^{-1}(0)$ is a 2-calibrated submanifold of $(M, \mathcal{F}, \omega)$ of codimension $2l$, and it is what we call a Donaldson-type submanifold. The topology of W and the topology of M are related by a Lefschetz hyperplane-type result: the section s is chosen so that $\log s\bar{s}$ is a Morse function. By approximate J-holomorphicity the index of critical points is greater than $n - l$, from which the vanishing of $\pi_i(M, W)$, $0 \le i \le n - l - 1$, follows [ibid., Corollary 1.2]. In particular the common zero set of $n-1$ well-chosen such sections of $L_\omega^{\otimes k}$ is $W^3 \hookrightarrow M$, a connected Donaldson type 3-dimensional submanifold.

For any given leaf F, it is tempting to study the topology of $W^3 \cap F$ by the same Morse-theoretic methods. It is always possible to arrange the tuple $s = (s_1, \ldots, s_{n-1})$ so that the restriction of $\log s\bar{s}^2$ to F is a Morse function. The usual Morse theoretic argument [Donaldson 1996; Auroux 1997, Proposition 2] implies that critical points have index greater than one, and therefore if F is compact (and hence $W^3 \cap F$ is compact), then $W^3 \cap F$ is connected. If F is not compact then the restriction of $\log s\bar{s}^2$ to $W^3 \cap F$ is never proper, and it is not clear how the information on index of critical points can be translated into topological information about $W^3 \cap F$.

A second approach to studying the topology of complex manifolds is via holomorphic Morse functions and Picard–Lefschetz theory. In our setting these are Lefschetz pencil decompositions of $(M, \mathscr{F})$ provided by ratios of suitable pairs of sections s_1, s_2 of $L_\omega^{\otimes k}$. Very much as we did in the previous section with the complex quadratic function h, we are going to use the parallel transport associated to a Lefschetz pencil decomposition to "reconstruct" a leaf F from its intersection with a regular fiber of the pencil (the previous section contains the analysis around a critical point of the holomorphic Morse function). This will be enough to prove Theorem 2. Parallel transport is also the way to compare two regular fibers of a given Lefschetz pencil structure, showing that they differ by a sequence of generalized Dehn twists.

5.1. ***Lefschetz pencil structures.*** We recall the notion of Lefschetz pencil structure and the main existence result, and collect some necessary results regarding the associated leafwise parallel transport.

Definition 29. Let $x \in (M, \mathscr{F}, \omega)$. A chart $\varphi_x : (\mathbb{C}^n \times \mathbb{R}, 0) \to (M, x)$ is compatible with $(\mathscr{F}, \omega)$ if it is a foliated chart, and $\varphi_x^*\omega$ restricted to the leaf through the origin is of type $(1, 1)$ at the origin.

Definition 30 [Ibort and Martínez Torres 2004b]. A Lefschetz pencil structure for $(M, \mathscr{F}, \omega)$ is given by a triple (f, B, Δ), where $B \subset M$ is a codimension-four 2-calibrated submanifold and $f : M \backslash B \to \mathbb{CP}^1$ is a smooth map such that:

(i) f is a leafwise submersion away from Δ, a 1-dimensional manifold transverse to $\mathscr{F}$ where the restriction of the differential of f to $\mathscr{F}$ vanishes. The fibers of the restriction of f to $M \backslash (B \cup \Delta)$ are 2-calibrated submanifolds.

(ii) Around any critical point $c \in \Delta$ there exist Morse coordinates $z_1, \ldots, z_n, t$ compatible with $(\mathscr{F}, \omega)$, and a standard complex affine coordinate on $\mathbb{CP}^1$ such that

$$f(z, t) = z_1^2 + \cdots + z_n^2 + \sigma(t), \tag{23}$$

where $\sigma \in C^\infty(\mathbb{R}, \mathbb{C})$.

(iii) Around any base point $b \in B$ there exist coordinates $z_1, \dots, z_n, t$ compatible with $(\mathscr{F}, \omega)$, and a standard complex affine coordinate on $\mathbb{CP}^1$ such that $B \equiv z_1 = z_2 = 0$ and $f(z, t) = z_1/z_2$.

(iv) $f(\Delta)$ is an immersed curve in general position.

For each regular value $z \in \mathbb{CP}^1 \backslash f(\Delta)$, the regular fiber is the compactification $W_z := f^{-1}(z) \cup B$, which is a (compact) 2-calibrated submanifold.

Theorem 31 [Ibort and Martínez Torres 2004b, Theorem 1.2]. *Let $(M, \mathscr{F}, \omega)$ be an integral 2-calibrated foliation and let e be an integral lift of $[\omega]$. Then for all $k \gg 1$ there exist Lefschetz pencils (f_k, B_k, Δ_k) such that:*

(i) *The regular fibers are Poincaré dual to ke.*

(ii) *The inclusion $l_k : W_k \hookrightarrow M$ induces maps*

$$l_{k*} : \pi_i(W_k) \to \pi_i(M) \quad \textit{and} \quad l_{k*} : H_i(W_k; \mathbb{Z}) \to H_i(M; \mathbb{Z}),$$

which are isomorphisms for $i \leq n-2$ and epimorphisms for $i = n-1$.

5.1.1. *Leafwise symplectic parallel transport.* Let (f, B, Δ) be a Lefschetz pencil structure for $(M, \mathscr{F}, \omega)$. Away from the union of base points and critical points $B \cup \Delta$, the fibers of f are 2-calibrated submanifolds. In particular for any point $p \notin B \cup \Delta$ this is equivalent to the tangent space of the leaf through p and the tangent space to the fiber of f through p intersecting transversely in a symplectic subspace. Therefore the leafwise symplectic orthogonals to the fibers define an Ehresmann connection for f, which we denote by $\mathscr{H}$ and also refer to as the horizontal distribution.

The Ehresmann connection $\mathscr{H}$ is defined in the noncompact manifold $M \backslash (B \cup \Delta)$. We are going to show that we have good control on parallel transport near base points and critical points.

Let F be a leaf of the foliation. Let B_F denote the codimension-four submanifold of base points in F and let Δ_F denote the dimension zero submanifold of critical points in F. The image $f(\Delta_F)$ is a possibly countable collection of points in the immersed curve $f(\Delta)$. In particular it is easy to construct curves $\gamma \subset \mathbb{CP}^1$ which do not intersect $f(\Delta_F)$ (or to homotope curves to avoid $f(\Delta_F)$).

Lemma 32. *Let $\gamma \subset \mathbb{CP}^1$ be a curve not intersecting $f(\Delta_F)$. Then parallel transport $\rho_\gamma : f^{-1}_{\gamma(0)} \cap F \to f^{-1}_{\gamma(1)} \cap F$ is a well defined symplectomorphism. Moreover, it extends smoothly to a symplectomorphism $\rho_\gamma : W_{\gamma(0)} \cap F \to W_{\gamma(1)} \cap F$ which is the identity on B_F. In particular if γ misses $f(\Delta)$, it induces an equivalence of 2-calibrated foliations $\rho_\gamma : W_{\gamma(0)} \to W_{\gamma(1)}$ which is the identity on B.*

Proof. A standard procedure in this situation is to blow up B along its leafwise almost complex normal directions.

We consider the following model for the blow up as a submanifold of $M \times \mathbb{CP}^1$: we let $\tilde{M}$ be the union of the graph of f and $B \times \mathbb{CP}^1$. We need to show that $\tilde{M}$ is a submanifold around points in $B \times \mathbb{CP}^1$.

Around a point $b \in B$, Theorem 31 provides coordinates $z_1, \ldots, z_n, t$ and a standard affine coordinate on $\mathbb{CP}^1$ such that $B \cong z_1 = z_2 = 0$ and $f = z_1/z_2$. This is equivalent to saying that near b the graph of f is given by

$$((z_1, \ldots, z_n, t), [z_1 : z_2]) \subset M \times \mathbb{CP}^1. \tag{24}$$

In these coordinates $\tilde{M}$ coincides with the complex blow up in the first two coordinates, and therefore it is a submanifold.

The first projection restricts to the blow down map $\pi : \tilde{M} \to M$, which is the identity away from B and collapses each $\{b\} \times \mathbb{CP}^1 \subset \tilde{M}$ to $b \in B \subset M$. The restriction to $\tilde{M}$ of the second projection on $M \times \mathbb{CP}^1$ defines an extension of f, $\tilde{f} : \tilde{M} \to \mathbb{CP}^1$. Because we are blowing up directions inside leaves we have an induced foliation $\tilde{\mathcal{F}}$, and the blow down map is a map of foliated manifolds.

The fibers of $\tilde{f}$ are transverse to $\tilde{\mathcal{F}}$, and by construction the restriction of the projection $\pi : \tilde{f}_z \to W_z$ is a diffeomorphism of foliated manifolds. We let $\tilde{F}$ denote the leaf mapping into F.

Let $\tilde{\omega}$ denote the pullback of ω by the blow down map. We claim that the intersection of the fibers of $\tilde{f}$ with $\tilde{\mathcal{F}}$ are symplectic manifolds with respect to $\tilde{w}_{\tilde{\mathcal{F}}}$, and therefore there is an associated leafwise Ehresmann connection which extends $\mathcal{H}$. At a point $p = (b, [z_1 : z_2])$, say $z_2 \neq 0$, the tangent space $T_{[z_1:z_2]}\mathbb{CP}^1 \subset T_{(b,[z_1:z_2])}\tilde{F}$ is in the kernel of $\tilde{\omega}_{\mathcal{F}}$ because the blow down map collapses the $\mathbb{CP}^1$ factor into the point b. The subspace $T_{(b,[z_1:z_2])}\tilde{f} \cap T_{(b,[z_1:z_2])}\tilde{F}$ is complementary to $T_{[z_1:z_2]}\mathbb{CP}^1$ and it is mapped isomorphically into $T_b f_{z_1/z_2} \cap T_b F$, and the latter is symplectic with respect to $\omega_{\mathcal{F}}$ (alternatively, in local coordinates about the base point $T_b f_{z_1/z_2} \cap T_b F$ is a complex hyperplane of $T_0\mathbb{C}^n$, and therefore it is symplectic with respect to $\omega_{\mathcal{F}}$ because the symplectic form has type $(1, 1)$ at the origin). Thus, the blow down map identifies $\tilde{f}_z$ and W_z as 2-calibrated foliations.

Once we have described the kernel of $\tilde{w}_{\tilde{\mathcal{F}}}$, it is easy to see that the horizontal lift of $\gamma \subset \mathbb{CP}^1$ starting at $(b, \gamma(0))$ is exactly (b, γ).

It is clear that parallel transport defines a Poisson equivalence. It is obviously an equivalence of 2-calibrated foliations because (the induced) coorientations are preserved, and the 2-calibrations are restriction of the same closed 2-form on M. □

Let $c \in \Delta$ be a critical point and let us apply Theorem 31 to construct Morse coordinates for f centered at c. By restricting Morse coordinates to the leaf F containing c, we obtain Morse coordinates for the restriction of f to F. Since the restriction of $\omega_{\mathcal{F}}$ to F is mapped to a symplectic form of type $(1, 1)$ at the origin, leafwise parallel transport near c corresponds to parallel transport in $\mathbb{C}^n\backslash\{0\}$ near 0 for the function h with respect to a symplectic form of type $(1, 1)$ at the origin.

Let us consider the following system of neighborhoods of the critical point $0 \in \mathbb{C}^n$ [Seidel 2003, Section 1.2]: we fix the standard symplectic form $\Omega_{\mathbb{R}^{2n}}$ and define Σ_z, $z \in \mathbb{C}$, to be the Lagrangian sphere of points in h_z whose parallel transport over the radial segment converges to the origin. For some $r_0 > 0$ we fix the parametrization

$$\varphi_{r_0} : (T_{r_0}(\lambda), \Omega_{\mathbb{R}^{2n}}) \to (T(\lambda), d\alpha_{\mathrm{can}}).$$

For any $z \in \mathbb{C}$ small enough we define $T_z(\lambda)\backslash\Sigma_z$ by radial parallel transport to the origin and then to r_0. Of course, $T_z(\lambda)$ denotes the union of $T_z(\lambda)\backslash\Sigma_z$ and Σ_z.

Then

$$\mathcal{T}(\lambda, r) = \bigcup_{z \in \bar{D}(r)} T_z(\lambda), \quad \lambda, r > 0, \tag{25}$$

is a system of neighborhoods of the origin. We also have the corresponding annular subsets $\mathcal{A}(\lambda, \lambda', r, r')$.

Lemma 33. *Let Ω_u, $u \in K$, be a compact family of symplectic forms defined on a neighborhood W of $0 \in \mathbb{C}^n$ which make the fibers h_z symplectic submanifolds. Let us fix any $\lambda, r > 0$, $\lambda' \in (0, \lambda)$ and $r' \in (0, r)$. Then there exists $\delta > 0$ such that for any curve $\gamma \subset \bar{D}(r')\backslash\{0\}$ having the C^1-norm of $\gamma - \gamma(0)$ bounded by δ, the horizontal lift $\tilde{\gamma}_u$ starting at any $p \in \mathcal{T}(\lambda', r')$ is contained in $\mathcal{T}(\lambda, r)$, for all $u \in K$.*

Proof. Let $\mathcal{C}$ denote the topological space of (piecewise embedded or constant) curves contained in $\bar{D}(r)$ relative to the C^1-topology. Let us consider the subset

$$E = \{(\gamma, p, u, v) \subset \mathcal{C} \times \mathcal{A}(\lambda', \tilde{\lambda}, r, r') \times K \times [0, 1] \mid \gamma(0) = h(p)\},$$

and let us define the continuous map

$$G : E \to W,$$
$$(\gamma, p, u, v) \mapsto \tilde{\gamma}_u(v),$$

by sending a tuple to the evaluation for time v of the horizontal lift of γ with respect to Ω_u starting at p. The map is not everywhere defined since horizontal lifts may leave W or converge to the critical point, which is exactly what we want to control. However, inside E we have the subset $\mathcal{A}(\lambda', \tilde{\lambda}, r, r') \times K \times [0, 1]$ corresponding to constant curves. The restriction of G to this subset is the first projection. By continuity, an open neighborhood of $\mathcal{A}(\lambda', \tilde{\lambda}, r, r')$ inside E is sent into $\mathcal{T}(\lambda, r)$. Because on E we have the topology induced by the product topology, we conclude the existence of δ such that curves with $||\gamma - \gamma(0)||_{C^1} < \delta$, $\gamma(0) \in \bar{D}(r')$, have horizontal lift starting at points in $A_{\gamma(0)}(\lambda', \tilde{\lambda})$ contained in $\mathcal{T}(\lambda, r)$. If in addition such a small curve does not contain $0 \in \mathbb{C}$, the connectivity argument already used a couple of times implies that horizontal lifts starting at points in $T_{\gamma(0)}(\lambda')$ remain inside $\mathcal{T}(\lambda, r)$, and this proves the lemma. □

Remark 34. Let $\sigma \in \mathbb{C}$ and consider the constant perturbation of the complex quadratic form $h+\sigma$. Note that in the definition of $\mathcal{T}(\lambda, r) \subset \mathbb{C}^n$ for h given in (25), if we replace radial segments joining a point z to the origin by segments joining z to σ, we get exactly the same subset $\mathcal{T}(\lambda, r)$. Now assume that the parameter $u \in K$ in Lemma 33 describes not just the variation of symplectic forms, but a perturbation of h by a constant $\sigma(u)$. Then Lemma 33 holds replacing in the statement h by $h+\sigma$ and the disk of radius r' by the disk of radius r' centered at $\sigma(u)$.

5.2. ***Connected components of $W_z \cap F$ and leafwise parallel transport.*** Let us fix $z_0 \in \mathbb{CP}^1$ a regular value for f. Let $\gamma \subset \mathbb{CP}^1$ be a loop based at z_0 with empty intersection with $f(\Delta_F)$. Lemma 32 implies that parallel transport over γ defines a diffeomorphism on $W_{z_0} \cap F$ (actually a symplectomorphism). Therefore the loop acts on connected components of $W_{z_0} \cap F$ and the action descends to $\pi_1(\mathbb{CP}^1 \backslash f(\Delta_F), z_0)$.

Proposition 35. *The action of $\pi_1(\mathbb{CP}^1 \backslash f(\Delta_F), z_0)$ on connected components of $W_{z_0} \cap F$ is trivial.*

Proof. Let γ be a loop based at z_0 and not intersecting $f(\Delta_F)$. Consider H_s a homotopy connecting $H_0 = \gamma$ with the constant path z_0. We can assume without loss of generality that H misses a point of $\mathbb{CP}^1$, and therefore compose with an affine coordinate chart and work in $\mathbb{C}$.

We can assume as well that the curves γ_s in the homotopy coincide in the complement of an interval $[a, b] \subset [0, 1]$, and the C^1-norm of $\gamma_{|[a,b]} - \gamma(a)$ (rescaled to have domain $[0, 1]$) is bounded by any given $\delta > 0$: by breaking the domain of H into n^2 squares of side $1/n$, we can write H as composition of n^2 homotopies with the above property. It is possible that the starting curve γ_0 of each of the n^2 homotopies does intersect $f(\Delta_F)$, but intersections can be removed after a perturbation with does not affect the behavior we demand on the curves γ_s.

We are going to control how the lifts of the curves in the homotopy behave near Δ_F using Morse coordinates, and away from Δ_F using a compactness argument.

Let $c \in \Delta$ and let us construct Morse coordinates $z_1, \ldots, z_n, t$ as in Definition 30. We say that the restriction of the coordinates to each plaque in their domain are Morse coordinates for the restriction f to the plaque. In Morse coordinates for a given plaque the restriction of f transforms into $h + \sigma(t)$ and $\omega_{\mathcal{F}}$ transforms into a symplectic form of making the fibers $(h + \sigma(t))_z$ symplectic manifolds (this is because we can construct Morse coordinates centered at any point in Δ, and in Morse coordinates on the plaque containing c the perturbation $\sigma(t)$ can be taken to be trivial and $\omega_{\mathcal{F}}$ becomes a symplectic form of type $(1, 1)$ at the origin).

Let us cover Δ with a finite number of Morse coordinates and let us consider their associated 1-parameter families of Morse coordinates on their plaques. Let us take $\lambda, r > 0$ such that $\mathcal{T}(\lambda, r)$ as defined in (25) is contained in the image of Morse coordinates for each of the plaques. Let us also pick $\lambda' \in (0, \lambda)$ and $r' \in (0, r)$, and denote by U the points in $\tilde{M}$ whose image under at least one of the sets of Morse coordinates on its plaque is contained in $\mathcal{T}(\lambda', r')$. Note that U is a neighborhood of Δ.

For each of our Morse coordinates, its 1-parameter family of Morse coordinates on plaques fulfills the hypothesis of Lemma 33, or rather Remark 34 (we assume that the parameter space is a compact interval, and that these compact intervals cover Δ). Let δ_1 be a C^1-bound provided by Remark 34 and valid for the finite number of 1-parameter families.

Let $V \subset V'$ be open neighborhoods of Δ in $\tilde{M}$ such that $V \subset \bar{V} \subset V' \subset U$. Because $\tilde{M}\setminus V'$ is compact, there exists $\delta_2 > 0$ such that for any $p \in \tilde{M}\setminus V'$ and any curve $\gamma \subset \mathbb{CP}^1$ starting at $\tilde{f}(p)$ and such that the C^1-norm of $\gamma - \gamma(0)$ is bounded by δ_2, the horizontal lift $\tilde{\gamma}$ starting at p is contained in $\tilde{M}\setminus V$.

Let δ be the minimum of δ_1 and δ_2, and let us assume that for each γ_s in our homotopy H the C^1-norm of $\gamma_{s\,|[a,b]} - \gamma_s(a)$ is smaller than δ. Let γ_0 and γ_1 be the starting and ending curve of the homotopy and let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be their respective horizontal lifts starting at $p \in W_{z_0} \cap F$. We claim that $\tilde{\gamma}_0(1)$ and $\tilde{\gamma}_1(1)$ can be connected by a path in $W_{z_0} \cap F$, which suffices to prove the proposition.

Recall that γ_s, $s \in [0, 1]$, is independent of s in the complement of $[a, b] \subset [0, 1]$.

Let us suppose that $\tilde{\gamma}_0(a) \in \tilde{M}\setminus V'$. Because of the C^1-bound on $\gamma_{s\,|[a,b]} - \gamma_s(a)$, $s \in [0, 1]$, the horizontal lifts of $\gamma_{s\,|[a,b]}$ starting at $\tilde{\gamma}_0(a)$ are defined for all $s \in [a, b]$ and belong to $\tilde{M}\setminus V$. In particular $\tilde{\gamma}_s(b)$ is a curve in the fiber $\tilde{f}_{\gamma_0(b)}$. Since the curves $\gamma_{s\,|[b,1]}$ are all equal and avoid $f(\Delta_F)$, we can construct the horizontal lift starting at all points in the path $\tilde{\gamma}_s(b)$. What we just proved is that the homotopy H_s has a well-defined lift starting at p, and therefore $\tilde{\gamma}_s(1)$ connects $\tilde{\gamma}_0(1)$ to $\tilde{\gamma}_1(1)$.

If $\tilde{\gamma}_0(a) \in V$ then it also belongs to U. If we compose with one of the fixed Morse coordinates on the plaque u_0 containing $\tilde{\gamma}_0(a)$, the point $\tilde{\gamma}_0(a)$ is sent to $q \in \mathcal{T}(\lambda', r')$. The curves $\gamma_{0\,|[a,b]}$ and $\gamma_{1\,|[a,b]}$ meet the hypothesis of Lemma 33 (Remark 34), and therefore their horizontal lifts starting at q are contained in $\mathcal{T}(\lambda, r)$. In particular the images q_0 and q_1 of $\tilde{\gamma}_0(b)$ and $\tilde{\gamma}_1(b)$, respectively, belong to $(h + \sigma(u_0))_{\gamma_0(b)}$. All regular fibers of $h + \sigma(u_0)$ in $\mathcal{T}(\lambda, r)$ are diffeomorphic to $T(\lambda)$ and therefore they are connected. Let ζ be a path in $(h + \sigma(u_0))_{\gamma_0(b)}$ connecting q_0 to q_1. Let us also denote by ζ its image in the plaque u_0 by the Morse chart, which belongs to $\tilde{f}_{\gamma_0(b)}$. Then the ending points of the lifts of $\gamma_{0\,|[b,1]}$ starting at $\zeta(v)$, $v \in [0, 1]$, connect $\tilde{\gamma}_0(1)$ to $\tilde{\gamma}_1(1)$. $\square$

Proposition 35 is the key result to "spread" a connected component of $W_{z_0} \cap F$ onto F. Before, we need to show that $W_{z_0} \cap F$ is always nonempty. For that it suffices to prove that $\tilde{f}(F)$ contains some regular value z of $\tilde{f}$, because in that case we can use parallel transport over a curve joining z to z_0 and avoiding singular values of $f(\Delta_F)$ to find points in $W_{z_0} \cap F$: because $\tilde{M}$ is compact the regular values of $\tilde{f}$ (which are the regular values of f) are an open dense subset. The subset $\tilde{f}(F) \subset \mathbb{CP}^1$ has not empty interior and therefore it contains regular values.

Theorem 36. *Let $(M^{2n+1}, \mathcal{F}, \omega)$, $n > 1$, be a 2-calibrated foliation and (f, B, Δ) be a Lefschetz pencil structure as in Definition 30. Then any regular fiber W of the pencil intersects every leaf of $\mathcal{F}$ in a unique connected component.*

Proof. Let z_0 be a regular value and let F be a leaf. We let C be a nonempty connected component of $W_{z_0} \cap F$ (it always exists since $W_{z_0} \cap F$ is nonempty). Let us define Γ_C to be the set of horizontal curves starting at C and whose projection $\tilde{f} \circ \zeta$ is either an embedded curve or constant. We define

$$F_C := \{p \in F \backslash \Delta_F \mid \text{there exists } \zeta \in \Gamma_C,\ \zeta(1) = p\}.$$

By construction F_C is nonempty, connected and contains C. We want to show that it is open.

Let $p \in F_C$ such that the horizontal curve ζ connects $x \in C$ with p. Let us suppose that the curve $\tilde{f} \circ \zeta$ is embedded (it is not constant). Then we can find a 1-parameter family of embedded curves γ_s, $s \in (-\epsilon, \epsilon)$, defined for time $v \in [0, 1+\epsilon]$, and such that the restriction of γ_0 to $[0, 1]$ is $\tilde{f} \circ \zeta$. Because ζ is contained in $\tilde{M} \backslash \Delta$, a compactness argument implies that there exists A an open neighborhood of x inside C and $\epsilon' > 0$, such that the horizontal lift of $\gamma_{s|[0,1+\epsilon']}$ starting at any point in A exists for all $s \in (-\epsilon', \epsilon')$. It is clear that for ϵ' small enough

$$U_p = \left\{ y \in F \;\middle|\; y = \tilde{\gamma}_s(v),\ \tilde{\gamma}_s(0) \in A,\ v \in (1-\epsilon', 1+\epsilon'),\ s \in (-\epsilon', \epsilon') \right\}$$

is a neighborhood of p in F_C.

If ζ is constant we make the previous construction for a family of radial curves starting at z_0, and the open neighborhood is obtained considering horizontal lifts for time $v \in [0, \epsilon')$ starting at a neighborhood A of p inside C (we would be "spreading" the open subset A).

We claim that F_C does not contain a connected component of $W_{z_0} \cap F$ different from C. Suppose the contrary. Then we would have a loop γ with a horizontal lift connecting two different connected components of $W_{z_0} \cap B$. Since after a small perturbation we can assume without loss of generality that γ does not intersect Δ_F, this would contradict Proposition 35.

Because it is clear that any point in $F \backslash \Delta_F$ can be connected to $W_{z_0} \cap F$ by a horizontal curve lifting an embedded curve, we conclude that connected components of $W_{z_0} \cap F$ are in bijection with connected components of F, and this proves that $W_{z_0} \cap F$ is connected. □

Proof of Theorem 2. Let $(M, \mathcal{F}, \omega)$ be a 2-calibrated foliation. If it is not integral, compactness of M implies that we can slightly modify ω into ω' so that a suitable multiple $k\omega'$ defines an integral homology class. Theorem 31 implies the existence of a Lefschetz pencil (f, B, Δ).

Therefore by Theorem 36 any regular fiber $(W, \mathcal{F}_W, k\omega'_W)$ intersects every leaf in a connected component. If the dimension of W is bigger than 3, we apply the same construction to $(W, \mathcal{F}_W, k\omega'_W)$. By induction we end up with a 3-dimensional manifold with a taut foliation $(W^3, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F}, \omega)$, whose intersection with every leaf of $\mathcal{F}$ is connected. $\square$

Proof of Theorem 4. Let $l : W \hookrightarrow M$ be a submanifold as in Theorem 2. Because for all $F \in \mathcal{F}$ the intersection $W \cap F$ is connected, the map l descends to a bijection of leaf spaces

$$\tilde{l} : W/\mathcal{F}_W \to M/\mathcal{F}.$$

Open sets of $W/\mathcal{F}_W$ and $M/\mathcal{F}$ are, respectively, in one to one correspondence with saturated open sets of W and M.

Let V be an saturated open set of $(M, \mathcal{F})$. By definition $W \cap V$ is an open set of W which is clearly saturated (even without the assumption of $\tilde{l}$ being a bijection) and this shows that $\tilde{l}$ is continuous.

Now let V be an open saturated set of $(W, \mathcal{F}_W)$. We want to show that its saturation in $(M, \mathcal{F})$, denoted by $\overline{V}^{\mathcal{F}}$, is open, to conclude that $\tilde{l}$ is open.

If V is a saturated set and $x \in V$, then x is an interior point if and only if for some T_x a local manifold through x transverse to the foliation, x is an interior point of $T_x \cap V$. Hence, every $x \in V$ is an interior point of $\overline{V}^{\mathcal{F}}$. By using the holonomy, if a point in a leaf is interior, the whole leaf is made of interior points. Since every leaf of $\overline{V}^{\mathcal{F}}$ intersects V, $\overline{V}^{\mathcal{F}}$ is open, and this proves the theorem. $\square$

5.3. ***Regular fibers and Lagrangian surgery.*** Let W be a regular fiber of a Lefschetz pencil structure for $(M, \mathcal{F}, \omega)$. Theorems 36 and 31 describe the topology of $W/\mathcal{F}_W$ and part of the homology and homotopy of W in terms of the corresponding data for $(M, \mathcal{F})$. We want to understand how different regular fibers of the pencil are related as 2-calibrated foliations.

Let z and z' be regular values of the pencil belonging to the same connected component of $\mathbb{CP}^1 \backslash f(\Delta)$, and let γ be a curve in that connected component connecting z to z'. Then Lemma 32 implies that $\rho_\gamma : W_z \to W_{z'}$ is an equivalence of 2-calibrated foliations.

We notice that any two arbitrary regular values z and z' can always be joined by a curve γ transverse to $f(\Delta)$.

Theorem 37. *Let $z, z' \in \mathbb{CP}^1$ be two regular values. Let γ be an embedded curve joining z and z' and transverse to $f(\Delta)$. Then $f^{-1}(\gamma)$ is a cobordism between W_z*

and $W_{z'}$ which amounts to adding one n-handle for each point $x \in \Delta$ such that $f(x) \subset \gamma$. More precisely, if $n > 2$ and there is only one critical point $c \in f^{-1}(\gamma)$, then there exists $L \subset W_z \backslash B$ a framed Lagrangian sphere such that $W_{z'}$ is the result of performing generalized Dehn surgery on W_z along L. The framed sphere is the points in W_z that converge to c under parallel transport over γ.

Proof. Let $w \in \gamma$ and $c \in \Delta$ with $f(c) = w$. Let us take Morse coordinates around c and an affine chart on $\mathbb{CP}^1$. Let us assume for simplicity that the curve γ in the affine chart coincides with a segment of the real axis. For $r > 0$ small enough, we want to construct a Poisson equivalence $\phi : W_r \to W_{-r}$.

To that end, consider the cobordism $Z = \tilde{f}^{-1}(x_0(-r, r))$, which is a manifold with boundary because $\tilde{f}$ is transverse to γ ($\operatorname{Im} \sigma'(0) \neq 0$). The attaching of the handle in this elementary cobordism occurs in a neighborhood of c, or equivalently in a neighborhood of 0 in the Morse chart, which is where we work from now on.

We are going to arrange the current setting so that it becomes analogous to the one in Theorem 26.

The pullback of f to the t-leaf of $\mathbb{C}^n \times \mathbb{R}$ is $h + \sigma(t)$. After reparametrization of the coordinate t, we may assume without loss of generality that $\sigma(t) = (a(t), t)$.

The tangent space of Z at $0 \in \mathbb{C}^n \times \mathbb{R}$ is the hyperplane $t = 0$. Therefore the projection $Z \to \mathbb{C}^n$ is a local diffeomorphism with image an open neighborhood V of $0 \in \mathbb{C}^n$.

We define ϕ away from a neighborhood $V' \subset V$ of $0 \in \mathbb{C}^n$ as follows:

$$\phi := \rho_{x_0(-r,r)} : W_r \to W_{-r}.$$

We claim that it is possible to extend ϕ to an equivalence of 2-calibrated foliations repeating the proof of Theorem 26 with two minor modifications.

Let us define $\sigma_r(t) := (r, 0) + \sigma(t)$, $r \neq 0$, $t \in [-\epsilon, \epsilon]$. Hence the images of W_r and W_{-r} on V' are exactly $h^{-1}(\sigma_r(t))$ and $h^{-1}(\sigma_{-r}(t))$, respectively. Recall that Morse coordinates on the t-plaque send $\omega_{\mathcal{F}}$ to a symplectic form Ω_t, which makes the fibers of $h + \sigma(t)$ symplectic. Then it follows that the morphism $\rho_{x_0(-r,r)} : W_r \to W_-r$ at a point $p \in V' \cap W_r$ in the t-leaf corresponds to parallel transport $\rho_{t, x_t(r+a(t), -r+a(t))}$ (with respect to Ω_t).

The first modification we need to introduce is composing all curves used in the proof of Theorem 26 and defined in a neighborhood of $0 \in \mathbb{C}$ with the diffeomorphism $(x, y) \mapsto (x + a(y), y)$.

The second difference is that, from the very beginning, our parallel transport here is with respect to a family of symplectic forms Ω_t, and with Ω_0 of type $(1, 1)$ at the origin. This situation is not quite new since in the proof of Theorem 26 we already needed to interpolate symplectic forms (although at a later stage).

Hence we conclude that for r small enough the fiber W_{-r} is equivalent to Lagrangian surgery (and hence by Theorem 26 generalized Dehn surgery) along a framed Lagrangian sphere L; the Lagrangian sphere is the points in W_r which parallel transport over $x_0(r, 0)$ sends to the critical point c. □

Remark 38. Theorem 37 is rather natural in view of the results for contact manifolds in [Presas 2002].

5.4. ***Further directions.*** In this paper we have shown that 2-calibrated foliations are a wide enough class of codimension-one foliations and, not surprisingly, techniques from symplectic geometry are well suited to their study. We would like to finish by discussing a couple of questions that we were not able to answer.

Theorem 4 shows that our embedded 3-dimensional taut foliations capture the leaf space of $\mathcal{F}$. What it would be interesting to know is whether they capture the full transverse geometry, that is, the holonomy groupoid.

A remarkable property of 3-dimensional taut foliations is that transverse loops are never nullhomotopic. The proof of this fact uses that the universal cover of the 3-manifold is $\mathbb{R}^3$, a property which does not extend to manifolds supporting a 2-calibrated foliation. We know no examples of 2-calibrated foliations on simply connected manifolds: in [Ibort and Martínez Torres 2003] it was shown that the normal connected sum could be used to construct 5-dimensional simply connected regular Poisson manifolds with codimension-one leaves, but those methods cannot be used to construct 2-calibrated foliations since the conditions in Theorem 10 are not fulfilled. It has been recently shown that Lawson's foliation on S^5 is the symplectic foliation of a Poisson structure [Mitsumatsu 2011]. However, this Poisson structure does not admit a 2-calibration because Lawson's foliation is not taut (the compact leaf would make any transverse loop nontrivial in homology).

We conjecture that any transverse loop in a 2-calibrated foliation is not nullhomotopic.

Appendix: Legendrian surgery, open book decompositions and generalized Dehn surgery

Let (M, ξ) be an exact contact manifold and let α be a contact 1-form defining $\xi = \ker \alpha$. Recall that an open book decomposition for M is given by a pair (K, θ) such that

- K is a codimension-2 submanifold with trivial normal bundle, referred to as the binding,
- $\theta : M \backslash K \to S^1$ is a fibration that in a trivialization $D^2 \times K$ of a neighborhood of K is the angular coordinate.

Let F denote the closure of any fiber of θ. The first return map associated to a suitable lift of $\partial/\partial\theta$ to $M \backslash K$ defines a diffeomorphism of F supported away from a neighborhood of the boundary $\partial F = K$. Up to diffeomorphism M can be recovered out of F and the first return map.

The following discussion is mostly taken from [Giroux and Mohsen 2003]; alternatively, a less detailed account can be found in [Giroux 2002].

Definition 39. The contact structure ξ is supported by an open book decomposition (K, θ) if for a choice of contact form α defining ξ we have:

- α restricts to a contact form on K.
- $d\alpha$ restricts on each fiber of θ to an exact symplectic structure.
- The orientation of K as the boundary of each symplectic leaf matches the natural orientation induced by the contact form.

The form α is said to be adapted to the open book decomposition (K, θ).

In what follows we are going to discuss contact structures and cosymplectic foliations on a given manifold. Since we have been using the notion of Reeb vector field for cosymplectic foliations, we refer to contact Reeb vector fields when discussing contact structures.

Given a contact form α adapted to (K, θ), it is possible to scale it away from K to a contact 1-form α' such that the flow along its contact Reeb vector field defines a compactly supported first return map $\varphi \in \mathrm{Symp}(\mathrm{int}\, F, d\alpha')$ [Giroux and Mohsen 2003].

The isotopy class of (M, ξ) is totally determined by any open book decomposition supporting it [Giroux 2002; Giroux and Mohsen 2003]. More precisely, the relevant structure in the open book decomposition is the completion of the structure of exact symplectic manifold convex at infinity of the exact symplectic fiber $(\mathrm{int}\, F, d\alpha)$ (or $(\mathrm{int}\, F, d\alpha')$), together with the first return symplectomorphism supported inside $\mathrm{int}\, F$.

The previous characterization becomes very important in light of the following theorem:

Theorem 40 [Giroux 2002; Giroux and Mohsen 2003]. *For every exact contact manifold (M, ξ) and any contact form defining α, there exists an open book decomposition (K, θ) supporting ξ such that α is adapted to it.*

Let α be a contact form on M adapted to the open book decomposition (K, θ) and let L be a parametrized Legendrian sphere which is contained in a fiber of θ, and hence it becomes Lagrangian for the symplectic structure $d\alpha$ on the fiber.

Observe that away from the binding K, the open book decomposition defines a 2-calibrated foliation $(M \backslash K, \mathcal{F}_\theta, d\alpha)$, with $\mathcal{F}_\theta = \ker d\theta$, which is a symplectic mapping torus associated to the symplectomorphism φ supported in $\mathrm{int}\, F$. Generalized Dehn surgery along L produces a new symplectic mapping torus with return map $\varphi \circ \tau$, where τ is a generalized Dehn twist along L. Because the symplectic leaf is the same and the return map is still compactly supported, the symplectic mapping torus is in fact the open book decomposition of a unique contact manifold (up to isotopy). In [Giroux and Mohsen 2003] it has been announced that this contact manifold is (M^L, α^L), the result of performing Legendrian surgery along L [Weinstein 1991]. (This is the same result involving plumbing along a Lagrangian disk announced in [Giroux 2002, p. 411].)

The ideas developed relating Lagrangian surgery and generalized Dehn surgery allow us to give a very natural proof of this result. The key step is the following theorem.

Theorem 41. *Let $L \subset (M, \alpha)$ be a parametrized Legendrian sphere in a contact manifold and let (M^L, α^L) be the contact manifold obtained by Legendrian surgery along L. Suppose that α is adapted to the open book (K, θ) and that L is contained in a fiber of θ. Then given V any small enough neighborhood of L with empty intersection with the binding K, there exists an isotopy $\Psi_s : M \to M$, $s \in [0, 1]$, starting at the identity with the following properties*:

- *Ψ_s is supported inside V and tangent to the identity at L.*
- *$(M \backslash K, \mathcal{F}_{\theta_s}, d\alpha)$, with $\mathcal{F}_{\theta_s} := \Psi_{s*}\mathcal{F}_\theta$, is a 2-calibrated foliation and thus an open book decomposition $(K, \Psi_{s*}\theta)$ of M to which the contact form α is adapted.*
- *Let $(M^L \backslash K, \mathcal{F}_{\theta_1^L}, d\alpha^L)$ be the result of performing generalized Dehn surgery on $(M \backslash K, \mathcal{F}_{\theta_1}, d\alpha)$ along the parametrized Lagrangian sphere L. Then $(M^L \backslash K, \mathcal{F}_{\theta_1^L}, d\alpha^L)$ is an open book decomposition (K, θ_1^L) for M^L and the contact form $\alpha^L \in \Omega^1(M^L)$ is adapted to (K, θ_1^L).*

Proof. We are going to recall Weinstein's definition of Legendrian surgery using symplectic cobordisms and a Liouville vector field transverse to the boundary. Actually, we will modify the original choices to make them compatible with our setup for Lagrangian surgery, or by Theorem 26 with the setup for generalized Dehn surgery.

Recall that a boundary component of a symplectic manifold (Z, Ω) (of dimension bigger than 2) endowed with a Liouville vector field Y is said to be convex if Y is outward pointing and concave if Y is inward pointing.

We consider $(M \times [-1, 1], d(e^v\alpha))$, which is a subset of the symplectization of (M, α). The tuple $(M \times [-1, 1], d(e^v\alpha), \partial/\partial v, M \times \{0\}, L \times \{0\})$ is an isotropic setup in the language of Weinstein; see the "Neighborhoods of isotropic submanifolds" section of [Weinstein 1991]. Note that $\{1\} \times M$ and $\{-1\} \times M$ are convex and concave boundary components, respectively (beware that the notion of Liouville vector field we use is opposite to Weinstein's, since we require the flow of the vector field to expand the symplectic form exponentially).

The second isotropic setup is the one of the $(n+1)$-handle to be attached, which is the one described in the "Standard handle" section of [Weinstein 1991], up to the following change. Unlike Weinstein, we are going to glue the convex end of $(M\times[-1,1], d(e^v\alpha), \partial/\partial v, M\times\{0\}, L\times\{0\})$ to the concave end of the symplectic $(n+1)$-handle; the reason is that in our definition of Lagrangian surgery, we glued the symplectic $(n+1)$-handle along the hypersurface $H_{2,r}$ where the symplectic vector field points inward. For this reason we also define a different Liouville vector field in the $(n+1)$-handle. We use the notation introduced in Section 4.1.

The symplectic form is the standard one $\Omega_{\mathbb{R}^{2n+2}}$. We consider the function

$$q=\sum_{i=1}^{n+1} x_i^2-2y_i^2,$$

whose negative gradient with respect to the Euclidean metric,

$$E=-2x^1\frac{\partial}{\partial x^1}+4y^1\frac{\partial}{\partial y^1}-\cdots-2x^{n+1}\frac{\partial}{\partial x^{n+1}}+4y^{n+1}\frac{\partial}{\partial y^{n+1}},$$

is a Liouville vector field.

For each $r>0$ we consider the fiber q_r, which contains the Lagrangian sphere Σ_r described in Lemma 18 using Y_2 the Hamiltonian vector field of $-\operatorname{Re} h$ with respect to $\Omega_{\mathbb{R}^{2n+2}}$. Notice that $dq(Y_2)<0$ and therefore Y_2 is transverse to the level hypersurfaces q_r. Since Y_2 and E coincide at Σ_r, it follows that the sphere Σ_r is also Legendrian with respect to the contact form $\alpha_E := i_E\Omega_{\mathbb{R}^{2n+2}}$ on q_r. Moreover, at points of $\Sigma_r\subset q_r$ the contact distribution and the cosymplectic distribution coincide.

Let $V_r(\epsilon)$ be a tubular neighborhood of radius $\epsilon>0$ of Σ_r inside q_r with respect to the Euclidean metric. We claim that for any $\epsilon'>0$, $\epsilon>\epsilon'$, we have $f_r\in C^\infty(V_r(\epsilon)\backslash\Sigma_r,\mathbb{R}^+)$ a cut-off function with compact support and with the following two properties:

- $\Phi_1^{f_rY_2}(V_r(\epsilon')\backslash\Sigma_r)\subset q_{-2r}$ (note that q_{-2r} contains the Lagrangian sphere Σ_{-r}).
- $\Phi_1^{f_rY_2}(V_r(\epsilon))$ is transverse to E.

Assuming the claim, we define the hypersurface

$$H_r^L := \Phi_1^{f_rY_2}(V_r(\epsilon)\backslash\Sigma_r)\cup\Sigma_{-r}.$$

By assumption the Liouville vector field E is transverse to H_r^L, and thus the hypersurface inherits an exact contact structure α_E by restricting $i_E\Omega_{\mathbb{R}^{2n+2}}$.

The second isotropic setup is the following: the symplectic $(n+1)$-handle is the compact region bounded by H_r^L and $V_r(\epsilon)$ endowed with the standard symplectic form, the Liouville vector field is E, the hypersurface is $V_r(\epsilon)$, which is concave, and the parametrized Legendrian sphere is Σ_r.

The symplectic morphism ψ that gives rise to the symplectic elementary cobordism ([Weinstein 1991, Proposition 4.2], whose replacement for Lagrangian surgery is Lemma 16), sends $(V_r(\epsilon), \Sigma_r, \alpha_E)$ to $(\nu(L), L, \alpha)$, and therefore we can consider $(V_r(\epsilon), \Sigma_r, \alpha_E)$ as a subset of (M, α). Then $M^L := H_r^L \cup (M \backslash V_r(\epsilon))$ carries and obvious contact form α^L which extends $(M \backslash V_r(\epsilon), \alpha)$.

The data for Legendrian surgery has been chosen to be compatible with Lagrangian surgery: both H_r^L and $V_r(\epsilon)$ are transverse to Y_2 and therefore they inherit 2-calibrated foliations $(H_r^L, \mathcal{F}_r^L, \omega_r^L)$ and $(V_r(\epsilon), \mathcal{F}_r, d\alpha)$. Theorem 26 easily implies that $(H_r^L, \mathcal{F}_r^L, \omega_r^L)$ is the result of generalized Dehn surgery along $\Sigma_r \subset (V_r(\epsilon), \mathcal{F}_r, d\alpha)$.

On $V_r(\epsilon)$ we have two structures of 2-calibrated foliation, $(\mathcal{F}_r, d\alpha)$ and $(\mathcal{F}, d\alpha)$. The reason is that ψ preserves contact forms and hence contact Reeb vector fields, but it does not preserve the 1-forms defining the cosymplectic foliations (or their associated Reeb vector fields). However, at Σ_r the Liouville and Hamiltonian vector fields coincide, and this implies that at points in L the contact distribution is tangent to $\mathcal{F}_r$. In particular the contact Reeb vector field for α is transverse to $\mathcal{F}_r$ near L. It is also transverse to $\mathcal{F}$ because α is adapted to the open book. Therefore we can use the trajectories of the contact Reeb vector field to construct an isotopy Ψ_s tangent to the identity at L and supported inside V in a small neighborhood of Σ_r contained in $V_r(\epsilon)$.

The claim about the existence of the function f_r is easily proved when $n = 1$ by inspecting the trajectories of E and Y_2. The general case can be reduced to the previous one: each point $(x_1, y_1, \dots, x_{n+1}, y_{n+1})$ in $\mathbb{C}^{n+1}$ and away from the union of stable and unstable manifolds (these are the same for both Morse functions $\operatorname{Re} h$ and q) determines $[x_1 : \cdots : x_{n+1}], [y_1 : \cdots : y_{n+1}]$, a point in $\mathbb{RP}^n \times \mathbb{RP}^n$, which gives rise to a line in $\mathbb{R}^{n+1}$ and one in $i\mathbb{R}^{n+1}$. These lines span a plane in $\mathbb{C}^{n+1} = \mathbb{R} \oplus i\mathbb{R}^{n+1}$. Each plane in the family is preserved by the flow of E and Y_2; moreover, the flows restrict on the planes to the flows of the 1-dimensional case. From this observation the claim follows easily. $\square$

Theorem 41 provides an isotopy Ψ_s supported away from K so that α is adapted to the 1-parameter family of open book decompositions $(K, \Psi_s\theta)$. Therefore we can identify the symplectic fiber F and symplectic monodromy $\varphi \in \operatorname{Symp}(\operatorname{int} F, d\alpha)$ of (K, θ) with those of $(K, \Psi_1\theta)$ (again following the contact Reeb flow). Hence the third point in Theorem 41 asserts that (M^L, α^L) is adapted to an open book decomposition with the same symplectic leaf $(F, d\alpha)$ and monodromy $\varphi \circ \tau \in \operatorname{Symp}(\operatorname{int} F, d\alpha)$, which is exactly what we wanted to prove.

Remark 42. If we attach the convex end of the symplectic handle to the concave end of the symplectization, we get the contact manifold (M^{L^-}, α^{L^-}). α^{L^-} is adapted to an open book decomposition whose monodromy is $\varphi \circ \tau^{-1}$.

Proposition 6.1 of [Durfee and Kauffman 1975] implies that in dimensions 5 and 13 the manifolds M^L and M^{L^-} are diffeomorphic. In [van Koert and Niederkrüger 2005, Section 3], it is shown that there are instances (coming from Brieskorn manifolds) in which (M^L, α^L) and (M^{L^-}, α^{L^-}) are not contactomorphic, and hence the authors can deduce that τ^2 is not isotopic to the identity in $\mathrm{Symp}^{\mathrm{comp}}(T^*S^6, d\alpha_{\mathrm{can}})$, a result already proved by Seidel for $n = 2$ [1999]; similar results are also drawn for powers of the Dehn twists known to be isotopic to the identity in $\mathrm{Diff}^{\mathrm{comp}}(T(\lambda))$, for all n even.

Remark 43. For any contact form α on M representing the given contact structure ξ and L a Legendrian submanifold, Giroux and Mohsen [2003] announce the existence of relative open book decompositions, meaning that α is adapted to the open book decomposition and L is contained in a fiber.

The interested reader familiar with approximately holomorphic geometry [Donaldson 1996] and its version for contact manifolds [Ibort et al. 2000; Presas 2002] can write a proof along the following lines: the open book decomposition is the result of pulling back the canonical open book decomposition of $\mathbb{C}$ by an approximately holomorphic function. To make sure the binding does not contain L, we use reference sections supported near L which achieve the value 1 when restricted to L; they come from an explicit formula once we identify a tubular neighborhood of L with a tubular neighborhood of the zero section of the first jet bundle with its canonical contact structure $(\mathcal{J}^1 L, \alpha_{\mathrm{can}})$. It is necessary to further add perturbations whose restrictions to L attain real values: they are such that its restrictions to $T^*L \times \{0\} \subset \mathcal{J}^1 L$ are small real multiples of reference sections equivariant with respect to the involution on $(\mathcal{J}^1, \alpha_{\mathrm{can}})$ that reverses the sign of the fiber and conjugation on $\mathbb{C}$ (this construction is analogous to the content of the remark after Lemma 3 in [Auroux et al. 2001]).

Therefore we conclude that Lagrangian surgery includes Legendrian surgery, for we can bypass the latter by choosing appropriate compatible open book decompositions and then performing Lagrangian surgery. According to Theorem 26 we can even claim that generalized Dehn surgery contains Legendrian surgery, and forget about the cobordisms.

Actually, generalized Dehn surgeries for different open book decompositions supporting the contact structure give the same contact manifold because there is a contact surgery behind. Now consider (L, χ) where L is a Legendrian submanifold of (M, α) and $\chi \in \mathrm{Symp}^{\mathrm{comp}}(T^*L, d\alpha_{\mathrm{can}})$. Let us take any open book decomposition relative to L and such that α is adapted to it, and consider the new manifold M^L associated to the open book decomposition with symplectic monodromy $\varphi \circ \chi$. It is clear that the diffeomorphism type of the manifold does not depend on the open book decomposition, but it is not clear whether in general the contact structure depends on the choice of open book decomposition. In either case, it would be an interesting situation because it would give either a new contact surgery — possibly a Legendrian surgery based on a block different from a symplectic handle — or different contact structures.

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THE TRACE OF FROBENIUS OF ELLIPTIC CURVES AND THE p-ADIC GAMMA FUNCTION

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We define a function in terms of quotients of the p-adic gamma function which generalizes earlier work of the author on extending hypergeometric functions over finite fields to the p-adic setting. We prove, for primes $p > 3$, that the trace of Frobenius of any elliptic curve over $\mathbb{F}_p$, whose j-invariant does not equal 0 or 1728, is just a special value of this function. This generalizes results of Fuselier and Lennon which evaluate the trace of Frobenius in terms of hypergeometric functions over $\mathbb{F}_p$ when $p \equiv 1 \pmod{12}$.

1. Introduction and statement of results

Let $\mathbb{F}_p$ denote the finite field with p, a prime, elements. Consider $E/\mathbb{Q}$ an elliptic curve with an integral model of discriminant $\Delta(E)$. We denote E_p the reduction of E modulo p. We note that E_p is nonsingular, and hence an elliptic curve over $\mathbb{F}_p$, if and only if $p \nmid \Delta(E)$, in which case we say p is a prime of good reduction. Regardless, we define

$$a_p(E) := p + 1 - \#E_p(\mathbb{F}_p). \tag{1-1}$$

If p is not a prime of good reduction we know $a_p(E) = 0, \pm 1$ depending on the nature of the singularity. If p is a prime of good reduction, we refer to $a_p(E)$ as the *trace of Frobenius* as it can be interpreted as the trace of the Frobenius endomorphism of $E/\mathbb{F}_p$. For a given elliptic curve $E/\mathbb{Q}$, these a_p are important quantities. Recall the Hasse–Weil L-function of E (viewed as function of a complex variable s) is defined by

$$L(E, s) := \prod_{p \mid \Delta} \frac{1}{1 - a_p(E) p^{-s}} \prod_{p \nmid \Delta} \frac{1}{1 - a_p(E) p^{-s} + p^{1-2s}}.$$

This Euler product converges for $\mathrm{Re}(s) > \frac{3}{2}$ and has analytic continuation to the whole complex plane. The Birch and Swinnerton-Dyer conjecture concerns the behavior of $L(E, s)$ at $s = 1$.

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The main result of this paper relates the trace of Frobenius to a special value of a function which we define in terms of quotients of the p-adic gamma function. Let $\Gamma_p(\cdot)$ denote Morita's p-adic gamma function and let ω denote the Teichmüller character of $\mathbb{F}_p$ with $\overline{\omega}$ denoting its character inverse. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ the fractional part of x, i.e., $x - \lfloor x \rfloor$.

Definition 1.1. Let p be an odd prime and let $t \in \mathbb{F}_p$. For $n \in \mathbb{Z}^+$ and $1 \le i \le n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define

$$
{}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \,\middle|\, t\right]_p := \frac{-1}{p-1} \sum_{j=0}^{p-2} (-1)^{jn}\, \overline{\omega}^j(t) \\
\times \prod_{i=1}^{n} \frac{\Gamma_p\big(\langle a_i - \frac{j}{p-1}\rangle\big)}{\Gamma_p\big(\langle a_i\rangle\big)} \frac{\Gamma_p\big(\langle -b_i + \frac{j}{p-1}\rangle\big)}{\Gamma_p\big(\langle -b_i\rangle\big)} (-p)^{-\left\lfloor \langle a_i\rangle - \frac{j}{p-1}\right\rfloor - \left\lfloor \langle -b_i\rangle + \frac{j}{p-1}\right\rfloor}.
$$

Throughout the paper we will refer to this function as ${}_nG_n[\cdots]$. The value of ${}_nG_n[\cdots]$ depends only on the fractional part of the a and b parameters. Therefore, we can assume $0 \le a_i, b_i < 1$.

This function has some very nice properties. It generalizes the function defined by the author in [McCarthy 2012a], which exhibits relationships to Fourier coefficients of modular forms. This earlier function has only one line of parameters and corresponds to ${}_nG_n[\cdots]$ when all the bottom line parameters are integral and $t = 1$. The earlier function also extended, to the p-adic setting, hypergeometric functions over finite fields with trivial bottom line parameters. In Section 3 we will see that ${}_nG_n[\cdots]$ extends hypergeometric functions over finite fields in their full generality, to the p-adic setting. By definition, results involving hypergeometric functions over finite fields will often be restricted to primes in certain congruence classes; see for example [Evans 2010; Fuselier 2010; Lennon 2011; Mortenson 2005; Vega 2011]. The motivation for developing ${}_nG_n[\cdots]$ is that it can often allow these results to be extended to a wider class of primes [McCarthy 2012a; 2012b], as we exhibit in our main result below. We will discuss these properties in more detail in Section 3.

We now state our main result, which relates the trace of Frobenius of an elliptic curve over $\mathbb{F}_p$ to a special value of ${}_nG_n[\cdots]$. We first note that if $p > 3$ then any elliptic curve over $\mathbb{F}_p$ is isomorphic to an elliptic curve of the form

$$E : y^2 = x^3 + ax + b,$$

that is, short Weierstrass form, and that the trace of Frobenius of isomorphic curves are equal. Let $j(E)$ denote the j-invariant of the elliptic curve E. Let $\phi_p(\cdot)$ be the Legendre symbol modulo p. We will often omit the subscript p when it is clear from the context.

Theorem 1.2. *Let $p > 3$ be prime. Consider an elliptic curve $E/\mathbb{F}_p$ of the form $E : y^2 = x^3 + ax + b$ with $j(E) \neq 0, 1728$. Then*

$$a_p(E) = \phi(b) \cdot p \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \,\middle|\, -\frac{27b^2}{4a^3}\right]_p. \tag{1-2}$$

Independent of Theorem 1.2, we will see later from Proposition 3.1 that the right-hand side of (1-2) is p-integral.

Theorem 1.2 generalizes Theorem 1.2 of [Fuselier 2010] and Theorem 2.1 of [Lennon 2011], which evaluate the trace of Frobenius in terms of hypergeometric functions over $\mathbb{F}_p$ when $p \equiv 1 \pmod{12}$. The results in the latter paper are in fact over $\mathbb{F}_q$, for $q \equiv 1 \pmod{12}$ a prime power, and hence allow calculation of a_p up to sign when $p \not\equiv 1 \pmod{12}$ via the relation $a_p^2 = a_{p^2} + 2p$. Theorem 1.2 however gives a direct evaluation of a_p for all primes $p > 3$ and resolves this sign issue.

One of the nice features of the main result in [Lennon 2011] is that it is independent of the Weierstrass model of the elliptic curve. Recall an elliptic curve over a field $\mathbb{K}$ in Weierstrass form is given by

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{1-3}$$

with $a_1, a_2, \dots, a_6 \in \mathbb{K}$. We can define the quantities

$$b_2 := {a_1}^2 + 4a_2, \quad b_4 := 2a_4 + a_1a_3, \quad b_6 := {a_3}^2 + 4a_6,$$
$$b_8 := {a_1}^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2{a_3}^2 - {a_4}^2, \quad c_4 := b_2^2 - 24b_4,$$
$$c_6 := -b_2^3 + 36b_2b_4 - 216b_6,$$

in the standard way. These can then be used to calculate $\Delta(E) = (c_4^3 - c_6^2)/1728$ and $j(E) = c_4^3/\Delta(E)$. An admissible change of variables, $x = u^2x' + r$ and $y = u^3y' + su^2x' + t$ with $u, r, s, t, \in \mathbb{K}$ and $u \neq 0$ in (1-3) will result in an isomorphic curve also given in Weierstrass form, and any two isomorphic curves over $\mathbb{K}$ are related by such an admissible change of variables. Two curves related by an admissible change of variables will have the same j-invariant but their discriminants will differ by a factor of a twelfth-power, namely u^{12}, and their respective c_i quantities will differ by a factor of u^i. This allows the main result [ibid.], which is stated in terms of $j(E)$ and $\Delta(E)$, to be expressed independently of the Weierstrass model of the elliptic curve. We can do something similar with Theorem 1.2.

Corollary 1.3. *Let $p > 3$ be prime. Consider an elliptic curve $E/\mathbb{F}_p$ in Weierstrass form with $j(E) \neq 0, 1728$. Then*

$$a_p(E) = \phi(-6 \cdot c_6) \cdot p \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \,\middle|\, 1 - \frac{1728}{j(E)}\right]_p.$$

Please refer to [Knapp 1992; Silverman 2009] for a detailed account of any of the properties of elliptic curves mentioned in the above discussion. The rest of this paper is organized as follows. In Section 2 we recall some basic properties of multiplicative characters, Gauss sums and the p-adic gamma function. We discuss some properties of ${}_nG_n[\cdots]$ in Section 3, including its relationship to hypergeometric functions over finite fields. The proofs of our main results are contained in Section 4. Finally, we make some closing remarks in Section 5.

2. Preliminaries

Let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$.

2A. ***Multiplicative characters and Gauss sums.*** Let $\widehat{\mathbb{F}}_p^*$ denote the group of multiplicative characters of $\mathbb{F}_p^*$. We extend the domain of $\chi \in \widehat{\mathbb{F}}_p^*$ to $\mathbb{F}_p$, by defining $\chi(0) := 0$ (including the trivial character ε) and denote by $\overline{\chi}$ the inverse of χ. We recall the following orthogonal relations. For $\chi \in \widehat{\mathbb{F}}_p^*$ we have

$$\sum_{x \in \mathbb{F}_p} \chi(x) = \begin{cases} p-1 & \text{if } \chi = \varepsilon, \\ 0 & \text{if } \chi \neq \varepsilon, \end{cases} \tag{2-1}$$

and, for $x \in \mathbb{F}_p$ we have

$$\sum_{\chi \in \widehat{\mathbb{F}}_p^*} \chi(x) = \begin{cases} p-1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1. \end{cases} \tag{2-2}$$

We now introduce some properties of Gauss sums. For further details see [Berndt et al. 1998], noting that we have adjusted results to take into account $\varepsilon(0) = 0$.

Let ζ_p be a fixed primitive p-th root of unity in $\overline{\mathbb{Q}}_p$. We define the additive character $\theta : \mathbb{F}_p \to \mathbb{Q}_p(\zeta_p)$ by $\theta(x) := \zeta_p^x$. It is easy to see that

$$\theta(a+b) = \theta(a)\theta(b), \tag{2-3}$$

$$\sum_{x \in \mathbb{F}_p} \theta(x) = 0. \tag{2-4}$$

We note that $\mathbb{Q}_p$ contains all $(p-1)$-th roots of unity and in fact they are all in $\mathbb{Z}_p^*$. Thus we can consider multiplicative characters of $\mathbb{F}_p^*$ to be maps $\chi : \mathbb{F}_p^* \to \mathbb{Z}_p^*$. Recall then that for $\chi \in \widehat{\mathbb{F}}_p^*$, the Gauss sum $g(\chi)$ is defined by $g(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x)\theta(x)$. It easily follows from (2-2) that we can express the additive character as a sum of Gauss sums. Specifically, for $x \in \mathbb{F}_p^*$ we have

$$\theta(x) = \frac{1}{p-1} \sum_{\chi \in \widehat{\mathbb{F}}_p^*} g(\overline{\chi})\, \chi(x). \tag{2-5}$$

The following important result gives a simple expression for the product of two Gauss sums. For $\chi \in \widehat{\mathbb{F}}_p^*$ we have

$$g(\chi)g(\overline{\chi}) = \begin{cases} \chi(-1)p & \text{if } \chi \neq \varepsilon, \\ 1 & \text{if } \chi = \varepsilon. \end{cases} \tag{2-6}$$

Another important product formula for Gauss sums is the Hasse–Davenport formula.

Theorem 2.1 (Hasse, Davenport [Berndt et al. 1998, Theorem 11.3.5]). *Let χ be a character of order m of $\mathbb{F}_p^*$ for some positive integer m. For a character ψ of $\mathbb{F}_p^*$ we have*

$$\prod_{i=0}^{m-1} g(\chi^i\psi) = g(\psi^m)\psi^{-m}(m)\prod_{i=1}^{m-1} g(\chi^i).$$

We now recall a formula for counting zeros of polynomials in affine space using the additive character. If $f(x_1, x_2, \ldots, x_n) \in \mathbb{F}_p[x_1, x_2, \ldots, x_n]$, then the number of points, N_p, in $\mathbb{A}^n(\mathbb{F}_p)$ satisfying $f(x_1, x_2, \ldots, x_n) = 0$ is given by

$$pN_p = p^n + \sum_{y\in\mathbb{F}_p^*}\ \sum_{x_1,x_2,\ldots,x_n\in\mathbb{F}_p} \theta(y\, f(x_1, x_2, \ldots, x_n))\,. \tag{2-7}$$

2B. ***p-adic preliminaries.*** We define the Teichmüller character to be the primitive character $\omega : \mathbb{F}_p \to \mathbb{Z}_p^*$ satisfying $\omega(x) \equiv x \pmod{p}$ for all $x \in \{0, 1, \ldots, p-1\}$. We now recall the p-adic gamma function. For further details, see [Koblitz 1980]. Let p be an odd prime. For $n \in \mathbb{Z}^+$ we define the p-adic gamma function as

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<j<n \\ p\nmid j}} j,$$

and extend to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{n\to x} \Gamma_p(n)$$

for $x \neq 0$, where n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how n approaches x, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^*$. We now state a product formula for the p-adic gamma function. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x = r/(p-1)$ with $0 \leq r \leq p-1$ then

$$\prod_{h=0}^{m-1} \Gamma_p\Big(\frac{x+h}{m}\Big) = \omega\big(m^{(1-x)(1-p)}\big)\, \Gamma_p(x) \prod_{h=1}^{m-1} \Gamma_p\Big(\frac{h}{m}\Big). \tag{2-8}$$

We note also that

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}, \tag{2-9}$$

where $x_0 \in \{1, 2, \dots, p\}$ satisfies $x_0 \equiv x \pmod p$. The Gross–Koblitz formula [1979] allows us to relate Gauss sums and the p-adic gamma function. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Then we have the following result.

Theorem 2.2 [Gross and Koblitz 1979]. *For $j \in \mathbb{Z}$,*

$$g(\overline{\omega}^j) = -\pi^{(p-1)\langle \frac{j}{p-1} \rangle} \Gamma_p\Big(\Big\langle \frac{j}{p-1} \Big\rangle\Big).$$

3. Properties of ${}_nG_n[\cdots]$.

As both $\Gamma_p(\cdot)$ and $\omega(\cdot)$ are in $\mathbb{Z}_p^*$, we see immediately from its definition that ${}_nG_n[\cdots]_p \in p^\delta \mathbb{Z}_p$ for some $\delta \in \mathbb{Z}$. We describe δ explicitly in the following proposition. We first define

$$\langle b_i \rangle^* := 1 - \langle -b_i \rangle = \begin{cases} \langle b_i \rangle & \text{if } b_i \notin \mathbb{Z}, \\ 1 & \text{if } b_i \in \mathbb{Z}. \end{cases}$$

Proposition 3.1. *Let p be an odd prime and let $t \in \mathbb{F}_p$. Let $n \in \mathbb{Z}^+$, $1 \le i \le n$ and $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. For $j \in \mathbb{Z}$ we define*

$$f(j) := \#\big\{ a_i \;\big|\; \langle a_i \rangle < \tfrac{j}{p-1}, 1 \le i \le n \big\} - \#\big\{ b_i \;\big|\; \langle b_i \rangle^* \le \tfrac{j}{p-1}, 1 \le i \le n \big\}.$$

Then

$${}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \;\Big|\; t \right]_p \in p^\delta \mathbb{Z}_p,$$

where $\delta = \mathrm{Min}\{ f(j) \mid 0 \le j \le p-2 \}$.

Proof. As $\Gamma_p(\cdot), \omega(\cdot)$ and $\frac{1}{p-1}$ are all in $\mathbb{Z}_p^*$, the result follows from noting that

$$\big\lfloor \langle a_i \rangle - \tfrac{j}{p-1} \big\rfloor = \begin{cases} -1 & \text{if } \langle a_i \rangle < j/(p-1), \\ 0 & \text{if } \langle a_i \rangle \ge j/(p-1), \end{cases}$$

and

$$\big\lfloor \langle -b_i \rangle + \tfrac{j}{p-1} \big\rfloor = \begin{cases} 1 & \text{if } \langle b_i \rangle^* \le j/(p-1), \\ 0 & \text{if } \langle b_i \rangle^* > j/(p-1). \end{cases}$$

□

We note that ${}_nG_n[\cdots]$ generalizes the function defined in [McCarthy 2012a]. This earlier function has only one line of parameters and corresponds to ${}_nG_n[\cdots]$ when all the bottom line parameters are integral and $t = 1$. Therefore the results from [McCarthy 2012a; 2012b] can be restated using ${}_nG_n[\cdots]$. The motivation for developing ${}_nG_n[\cdots]$ and its predecessor was to allow results involving hypergeometric functions over finite fields, which are often restricted to primes in certain congruence classes, to be extended to a wider class of primes. While the function defined in [McCarthy 2012a] extended to the p-adic setting, hypergeometric functions over finite fields with trivial bottom line parameters, we now show, in Lemma 3.3, that ${}_nG_n[\cdots]$ extends hypergeometric functions over finite fields in their full generality.

Hypergeometric functions over finite fields were originally defined by Greene [1987], who first established these functions as analogues of classical hypergeometric functions. Functions of this type were also introduced by Katz [1990] about the same time. In the present article we use a normalized version of these functions defined in [McCarthy 2012c], which is more suitable for our purposes. The reader is directed to [ibid., §2] for the precise connections among these three classes of functions.

Definition 3.2 [McCarthy 2012c, Definition 1.4]. For $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ in $\widehat{\mathbb{F}}_p^*$ and x in $\mathbb{F}_p$, define

$$
\tag{3-1} {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \,\middle|\, x\right)_p := \frac{1}{p-1} \sum_{\chi \in \widehat{\mathbb{F}}_p^*} \prod_{i=0}^{n} \frac{g(A_i\chi)}{g(A_i)} \prod_{j=1}^{n} \frac{g(\overline{B_j\chi})}{g(\overline{B_j})} g(\overline{\chi})\chi(-1)^{n+1}\chi(x).
$$

Many of the results concerning hypergeometric functions over finite fields that we quote from other articles were originally stated using Greene's function. If this is the case, note then that we have reformulated them in terms ${}_{n+1}F_n(\cdots)$ as defined above.

We have the following relationship between ${}_nG_n[\cdots]$ and ${}_{n+1}F_n(\cdots)$.

Lemma 3.3. *For a fixed odd prime p, let $A_i, B_k \in \widehat{\mathbb{F}}_p^*$ be given by $\overline{\omega}^{a_i(p-1)}$ and $\overline{\omega}^{b_k(p-1)}$ respectively, where ω is the Teichmüller character. Then*

$$
{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \,\middle|\, t\right)_p = {}_{n+1}G_{n+1}\left[\begin{matrix} a_0, & a_1, & \ldots, & a_n \\ 0, & b_1, & \ldots, & b_n \end{matrix} \,\middle|\, t^{-1}\right]_p.
$$

Proof. Starting from the definition of ${}_{n+1}F_n(\cdots)$, we convert the right-hand side of (3-1) to an expression involving the p-adic gamma function and Teichmüller character. We note $\widehat{\mathbb{F}}_p^*$ can be given by $\{\omega^j \mid 0 \le j \le p-2\}$. Then, straightforward applications of the Gross–Koblitz formula (Theorem 2.2) with $\chi = \omega^j$ yield

$$
g(\overline{\chi}) = -\pi^j \Gamma_p\left(\tfrac{j}{p-1}\right),
$$
$$
\frac{g(A_i\chi)}{g(A_i)} = \pi^{-j-(p-1)\left(\left\lfloor a_i - \frac{j}{p-1}\right\rfloor - \lfloor a_i \rfloor\right)} \frac{\Gamma_p\left(\langle a_i - \frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\langle a_i \rangle\right)},
$$
$$
\frac{g(\overline{B_k\chi})}{g(\overline{B_k})} = \pi^{j-(p-1)\left(\left\lfloor -b_k + \frac{j}{p-1}\right\rfloor - \lfloor -b_k \rfloor\right)} \frac{\Gamma_p\left(\langle -b_k + \frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\langle -b_k \rangle\right)},
$$

where π is as defined in Section 2B. Substituting these expressions into (3-1) and tidying up yields the result. $\square$

We note that if $\chi \in \widehat{\mathbb{F}}_p^*$ is a character of order d and is given by $\overline{\omega}^{x(p-1)}$ then $x = m/d \in \mathbb{Q}$ and $p \equiv 1 \pmod{d}$. Therefore, given a hypergeometric function over $\mathbb{F}_p$ whose arguments are characters of prescribed order, the function will only be defined for primes p in certain congruence classes. By Lemma 3.3, for primes in these congruence classes, the finite field hypergeometric function will be related to an appropriate ${}_nG_n[\cdots]$ function. However this corresponding ${}_nG_n[\cdots]$ will be defined at all primes not dividing the orders of the particular characters appearing in the finite field hypergeometric function. This opens the possibility of extending results involving hypergeometric functions over finite fields to all but finitely many primes.

For example, we have the following result from [McCarthy 2012b], which relates a special value of the hypergeometric function over finite fields to a p-th Fourier coefficient of a certain modular form. Let

$$f(z) := f_1(z) + 5f_2(z) + 20f_3(z) + 25f_4(z) + 25f_5(z) = \sum_{n=1}^{\infty} c(n)q^n, \tag{3-2}$$

where $f_i(z) := \eta^{5-i}(z)\eta^4(5z)\eta^{i-1}(25z)$, $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function and $q := e^{2\pi i z}$. Then f is a cusp form of weight four on the congruence subgroup $\Gamma_0(25)$.

Theorem 3.4 [McCarthy 2012b, Corollary 1.6]. *If $p \equiv 1 \pmod 5$ is prime, $\chi_5 \in \widehat{\mathbb{F}}_p^*$ is a character of order 5 and $c(p)$ is as defined in (3-2), then*

$${}_4F_3\left(\begin{matrix}\chi_5, & \chi_5^2, & \chi_5^3, & \chi_5^4 \\ & \varepsilon, & \varepsilon, & \varepsilon\end{matrix} \,\middle|\, 1\right)_p - p = c(p).$$

This result can be extended to almost all primes using ${}_nG_n[\cdots]$, as follows.

Theorem 3.5 [McCarthy 2012b, Theorem 1.4]. *If $p \neq 5$ is an odd prime and $c(p)$ is as defined in (3-2), then*

$${}_4G_4\left[\begin{matrix}\frac{1}{5}, & \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5} \\ 0, & 0, & 0, & 0\end{matrix} \,\middle|\, 1\right]_p - \left(\tfrac{5}{p}\right)p = c(p),$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol modulo p.

Results in [Mortenson 2005] establish congruences modulo p^2 between the classical hypergeometric series and the hypergeometric function over $\mathbb{F}_p$, for primes p in certain congruence classes. In [McCarthy 2012a] we extend these results to primes in additional congruence classes and, in some cases to modulo p^3, using the predecessor to ${}_nG_n[\cdots]$.

The main purpose of this paper is to extend to almost all primes the results in [Lennon 2011], which relate the trace of Frobenius a_p to a special value of a hypergeometric function over $\mathbb{F}_p$ when $p \equiv 1 \pmod{12}$. In addition to their formal statement, the results in [ibid.] appear in various forms throughout that paper, all of which are related by known transformations for hypergeometric function over finite fields. We recall one such version of [ibid., Theorem 2.1].

Theorem 3.6 [Lennon 2011, §2.2]. *Let $p \equiv 1 \pmod{12}$ be prime and let $\psi \in \widehat{\mathbb{F}}_p^*$ be a character of order* 12. *Consider an elliptic curve $E/\mathbb{F}_p$ of the form*

$$E : y^2 = x^3 + ax + b$$

with $j(E) \neq 0, 1728$. Then

$$a_p(E) = \psi^3\Big(-\frac{a^3}{27}\Big) \cdot {}_2F_1\left(\begin{matrix}\psi, & \psi^5 \\ & \varepsilon\end{matrix} \,\middle|\, \frac{4a^3+27b^2}{4a^3}\right)_p.$$

Theorem 3.6 generalizes [Fuselier 2010, Theorem 1.2] and other results from Fuselier's thesis [2007] that provide similar results for various families of elliptic curves. In attempting to extend Theorem 3.6 beyond $p \equiv 1 \pmod{12}$, one might consider using

$${}_2G_2\left[\begin{matrix}\frac{1}{12}, & \frac{5}{12} \\ 0, & 0\end{matrix} \,\middle|\, \frac{4a^3}{4a^3+27b^2}\right]_p,$$

as suggested by Lemma 3.3. However this leads to poor results when $p \not\equiv 1 \pmod{12}$. Results where ${}_nG_n[\cdots]$ extend those involving ${}_{n+1}F_n(\cdots)$ seem to work best when the arguments of ${}_nG_n[\cdots]$ appear in sets such that for each denominator all possible relatively prime numerators are represented. This is reflected in Theorem 1.2.

Hypergeometric functions over finite fields have been applied to many areas but most interestingly perhaps has been their relationships to modular forms [Ahlgren and Ono 2000; Evans 2010; Fuselier 2010; Frechette et al. 2004; McCarthy 2012b; Mortenson 2005; Ono 1998; Papanikolas 2006] and their use in evaluating the number of points over $\mathbb{F}_p$ on certain algebraic varieties [Ahlgren and Ono 2000; Fuselier 2010; McCarthy 2012b; Vega 2011]. Lemma 3.3 allows these results to be expressed in terms of ${}_nG_n[\cdots]$ also. Many of these cited results are based on ${}_{n+1}F_n(\cdots)$ with arguments which are characters of order at most 2 and hold for all odd primes. However there is much scope for developing results where the characters involved have higher orders, in which case these functions will be defined for primes in certain congruence classes and ${}_nG_n[\cdots]$ allows the possibility to extend these results to a wider class of primes.

4. Proofs of Theorem 1.2 and Corollary 1.3

We first prove a preliminary result which we will require later for the proof of our main result.

Lemma 4.1. *Let p be prime. For $0 \leq j \leq p-2$ and $t \in \mathbb{Z}^+$ with $p \nmid t$, we have*

$$\Gamma_p\big(\big\langle\tfrac{tj}{p-1}\big\rangle\big)\omega(t^{tj}) \prod_{h=1}^{t-1} \Gamma_p\big(\tfrac{h}{t}\big) = \prod_{h=0}^{t-1} \Gamma_p\big(\big\langle\tfrac{h}{t} + \tfrac{j}{p-1}\big\rangle\big) \tag{4-1}$$

and

$$\Gamma_p\big(\big\langle\tfrac{-tj}{p-1}\big\rangle\big)\omega(t^{-tj}) \prod_{h=1}^{t-1} \Gamma_p\big(\tfrac{h}{t}\big) = \prod_{h=0}^{t-1} \Gamma_p\big(\big\langle\tfrac{1+h}{t} - \tfrac{j}{p-1}\big\rangle\big). \tag{4-2}$$

Proof. Fix $0 \leq j \leq p-2$ and let $k \in \mathbb{Z}_{\geq 0}$ be defined such that

$$k\big(\tfrac{p-1}{t}\big) \leq j < (k+1)\big(\tfrac{p-1}{t}\big). \tag{4-3}$$

Letting $m = t$ and $x = (tj/(p-1)) - k$ in (2-8) yields

$$\prod_{h=0}^{t-1} \Gamma_p\big(\tfrac{j}{p-1} + \tfrac{h-k}{t}\big) = \omega\Big(t^{\left(1-\frac{tj}{p-1}+k\right)(1-p)}\Big)\Gamma_p\big(\tfrac{tj}{p-1} - k\big) \prod_{h=1}^{t-1} \Gamma_p\big(\tfrac{h}{t}\big). \tag{4-4}$$

We note that $0 \leq k < t$. Using (4-3) we see that if $0 \leq h < t$ then

$$0 \leq \tfrac{h-k}{t} + \tfrac{j}{p-1} < 1.$$

Therefore, if $1 \leq k < t$ then

$$\begin{aligned}
\prod_{h=0}^{t-1} \Gamma_p\big(\tfrac{h-k}{t} + \tfrac{j}{p-1}\big) &= \prod_{h=0}^{t-1} \Gamma_p\big(\big\langle\tfrac{h-k}{t} + \tfrac{j}{p-1}\big\rangle\big) \\
&= \prod_{h=0}^{k-1} \Gamma_p\big(\big\langle\tfrac{t+h-k}{t} + \tfrac{j}{p-1}\big\rangle\big) \prod_{h=k}^{t-1} \Gamma_p\big(\big\langle\tfrac{h-k}{t} + \tfrac{j}{p-1}\big\rangle\big) \\
&= \prod_{h=t-k}^{t-1} \Gamma_p\big(\big\langle\tfrac{h}{t} + \tfrac{j}{p-1}\big\rangle\big) \prod_{h=0}^{t-k-1} \Gamma_p\big(\big\langle\tfrac{h}{t} + \tfrac{j}{p-1}\big\rangle\big) \\
&= \prod_{h=0}^{t-1} \Gamma_p\big(\big\langle\tfrac{h}{t} + \tfrac{j}{p-1}\big\rangle\big).
\end{aligned} \tag{4-5}$$

The result in (4-5) also holds when $k = 0$. Substituting (4-5) into (4-4) and noting that

$$\Gamma_p\big(\big\langle\tfrac{tj}{p-1}\big\rangle\big) = \Gamma_p\big(\tfrac{tj}{p-1} - k\big),$$

by (4-3), yields (4-1).

We use a similar argument to prove (4-2). The result is trivial for $j = 0$. Fix $0 < j \leq p-2$ and let $k \in \mathbb{Z}^+$ be defined such that

$$(k-1)\left(\tfrac{p-1}{t}\right) < j \leq k\left(\tfrac{p-1}{t}\right). \tag{4-6}$$

Letting $m = t$ and $x = k - tj/(p-1)$ in (2-8) yields

$$\prod_{h=0}^{t-1} \Gamma_p\left(\tfrac{k+h}{t} - \tfrac{tj}{p-1}\right) = \omega\left(t^{\left(1-k+\frac{tj}{p-1}\right)(1-p)}\right)\Gamma_p\left(k - \tfrac{tj}{p-1}\right)\prod_{h=1}^{t-1}\Gamma_p\left(\tfrac{h}{t}\right). \tag{4-7}$$

We note that $1 \leq k \leq t$. Using (4-6) we see that if $0 \leq h < t$ then

$$0 \leq \tfrac{k+h}{t} - \tfrac{j}{p-1} < 1.$$

Therefore, if $1 < k \leq t$ then

$$\begin{aligned}\prod_{h=0}^{t-1} \Gamma_p\left(\tfrac{k+h}{t} - \tfrac{j}{p-1}\right) &= \prod_{h=0}^{t-1} \Gamma_p\left(\left\langle\tfrac{k+h}{t} - \tfrac{j}{p-1}\right\rangle\right)\\ &= \prod_{h=0}^{t-k} \Gamma_p\left(\left\langle\tfrac{k+h}{t} - \tfrac{j}{p-1}\right\rangle\right)\prod_{h=t-k+1}^{t-1} \Gamma_p\left(\left\langle\tfrac{k+h-t}{t} - \tfrac{j}{p-1}\right\rangle\right)\\ &= \prod_{h=k-1}^{t-1} \Gamma_p\left(\left\langle\tfrac{1+h}{t} - \tfrac{j}{p-1}\right\rangle\right)\prod_{h=0}^{k-2} \Gamma_p\left(\left\langle\tfrac{1+h}{t} - \tfrac{j}{p-1}\right\rangle\right)\\ &= \prod_{h=0}^{t-1} \Gamma_p\left(\left\langle\tfrac{1+h}{t} - \tfrac{j}{p-1}\right\rangle\right).\end{aligned} \tag{4-8}$$

The result in (4-8) also holds when $k = 1$. Now (4-2) follows by substituting (4-8) into (4-7) and noting that, by (4-6),

$$\Gamma_p\left(\left\langle\tfrac{-tj}{p-1}\right\rangle\right) = \Gamma_p\left(\tfrac{-tj}{p-1} + k\right). \qquad \square$$

Proof of Theorem 1.2. We note that $a \neq 0, b \neq 0$ and $-27b^2/(4a^3) \neq 1$ as $j(E) \neq 0, 1728$. Initially the proof proceeds along similar lines to the proofs of [Fuselier 2010, Theorem 1.2; Lennon 2011, Theorem 2.1] by using (2-7) to evaluate $\#E(\mathbb{F}_p)$. However we then transfer to the p-adic setting using the Gross–Koblitz formula (Theorem 2.2) and use properties of the p-adic gamma function, including

Lemma 4.1, to prove the desired result. By (2-7) we have

$$
\begin{aligned}
\text{(4-9)}\quad & p(\#E(\mathbb{F}_p)-1)\\
&= p^2 + \sum_{y\in\mathbb{F}_p^*}\sum_{x_1,x_2\in\mathbb{F}_p} \theta(y(x_1^3+ax_1+b-x_2^2))\\
&= p^2 + \sum_{y\in\mathbb{F}_p^*}\theta(yb) + \sum_{y,x_2\in\mathbb{F}_p^*}\theta(yb-yx_2^2) + \sum_{y,x_1\in\mathbb{F}_p^*}\theta(yx_1^3+ayx_1+yb)\\
&\qquad + \sum_{y,x_1,x_2\in\mathbb{F}_p^*}\theta(yx_1^3+ayx_1+by-yx_2^2)).
\end{aligned}
$$

We now examine each sum of (4-9) in turn and will refer to them as S_1 to S_4, respectively. Using (2-4) we see that

$$S_1 = \sum_{y\in\mathbb{F}_p^*}\theta(yb) = -1.$$

We use (2-3) and (2-5) to expand the remaining terms as expressions in Gauss sums. This exercise has also been carried out in the proof of [Lennon 2011, Theorem 2.1] so we only give a brief account here. Let T be a fixed generator for the group of characters of $\mathbb{F}_p^*$. Then

$$
\begin{aligned}
S_2 &= \sum_{y,x_2\in\mathbb{F}_p^*}\theta(yb-yx_2^2)\\
&= \frac{1}{(p-1)^2}\sum_{r,s=0}^{p-2} g(T^{-r})g(T^{-s})T^r(b)\,T^s(-1)\sum_{x_2\in\mathbb{F}_p^*}T^{2s}(x_2)\sum_{y\in\mathbb{F}_p^*}T^{r+s}(y).
\end{aligned}
$$

We now apply (2-1) to the last summation on the right, which yields $(p-1)$ if $r=-s$ and zero otherwise. So

$$S_2 = \frac{1}{(p-1)}\sum_{s=0}^{p-2} g(T^s)g(T^{-s})T^{-s}(b)T^s(-1)\sum_{x_2\in\mathbb{F}_p^*}T^{2s}(x_2).$$

Again we apply (2-1) to the last summation on the right, which yields $(p-1)$ if $s=0$ or $s=(p-1)/2$, and zero otherwise. Thus, and using (2-6), we get that

$$S_2 = g(\varepsilon)\,g(\varepsilon) + \ g(\phi)\,g(\phi)\,\phi(-b) = 1 + p\,\phi(b).$$

Similarly,

$$S_3 = \sum_{y,x_1 \in \mathbb{F}_p^*} \theta(yx_1^3 + ayx_1 + yb)$$
$$= \frac{1}{(p-1)^3} \sum_{r,s,t=0}^{p-2} g(T^{-r})g(T^{-s})g(T^{-t})T^s(a)T^t(b) \cdot \sum_{x_1 \in \mathbb{F}_p^*} T^{3r+s}(x_1) \sum_{y \in \mathbb{F}_p^*} T^{r+s+t}(y),$$

$$S_4 = \sum_{y,x_1,x_2 \in \mathbb{F}_p^*} \theta(yx_1^3 + ayx_1 + by - yx_2^2))$$
$$= \frac{1}{(p-1)^4} \sum_{j,r,s,t=0}^{p-2} g(T^{-j})g(T^{-r})g(T^{-s})g(T^{-t})T^r(a)T^s(b)T^t(-1) \cdot \sum_{x_1 \in \mathbb{F}_p^*} T^{3j+r}(x_1) \sum_{y \in \mathbb{F}_p^*} T^{j+r+s+t}(y) \sum_{x_2 \in \mathbb{F}_p^*} T^{2t}(x_2).$$

We now apply (2-1) to the last summation on the right of S_4, which yields $p-1$ if $t=0$ or $t=(p-1)/2$ and zero otherwise. In the case $t=0$ we find that

$$S_{4,t=0} = -S_3.$$

When $t=(p-1)/2$ we get, after applying (2-1) twice more,

$$S_{4,t=\frac{p-1}{2}} = \frac{\phi(-b)}{(p-1)} \sum_{j=0}^{p-2} g(T^{-j})g(T^{\frac{p-1}{2}-2j})g(T^{3j})g(T^{\frac{p-1}{2}})T^{-3j}(a)T^{2j}(b).$$

Combining (1-1), (4-9) and the evaluations of S_1, S_2, S_3 and S_4 we find that

$$\text{(4-10)} \quad a_p(E) = -\frac{\phi(b)\,p}{(p-1)} - \frac{\phi(-b)}{p(p-1)} \sum_{j=1}^{p-2} g(T^{-j})g(T^{\frac{p-1}{2}-2j})g(T^{3j})g(T^{\frac{p-1}{2}})T^j\big(\tfrac{b^2}{a^3}\big).$$

We know from Theorem 2.1 with $\chi = \phi = T^{\frac{p-1}{2}}$ and $\psi = T^{-2j}$ that

$$\text{(4-11)} \qquad g(T^{\frac{p-1}{2}-2j}) = \frac{g(T^{-4j})g(T^{\frac{p-1}{2}})T^{4j}(2)}{g(T^{-2j})}.$$

Accounting for (4-11) in (4-10) and applying (2-6) with $\chi = \phi = T^{\frac{p-1}{2}}$ gives us

$$a_p(E) = \frac{-\phi(b)\, p}{(p-1)}\left[1 + \frac{1}{p}\sum_{j=1}^{p-2} \frac{g(T^{-j})g(T^{3j})g(T^{-4j})}{g(T^{-2j})} T^{j}\left(\tfrac{16b^2}{a^3}\right)\right]. \tag{4-12}$$

We now take T to be the inverse of the Teichmüller character, that is, $T = \overline{\omega}$, and use the Gross–Koblitz formula (Theorem 2.2) to convert (4-12) to an expression involving the p-adic gamma function. This yields

$$\begin{aligned} a_p(E) = \frac{-\phi(b)\, p}{(p-1)}\Bigg[1 - \sum_{j=1}^{p-2} (-p)^{\left(\lfloor \frac{-2j}{p-1}\rfloor - \lfloor \frac{-j}{p-1}\rfloor - \lfloor \frac{3j}{p-1}\rfloor - \lfloor \frac{-4j}{p-1}\rfloor - 1\right)} \\ \cdot \frac{\Gamma_p\left(\langle \frac{-j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{3j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{-4j}{p-1}\rangle\right)}{\Gamma_p\left(\langle \frac{-2j}{p-1}\rangle\right)} \overline{\omega}^{j}\left(\tfrac{16b^2}{a^3}\right)\Bigg]. \end{aligned} \tag{4-13}$$

Next we use Lemma 4.1 to transform the components of (4-13) which involve the p-adic gamma function. After some tidying up we then get

$$\begin{aligned} a_p(E) = \frac{-\phi(b)\, p}{(p-1)}\Bigg[1 - \sum_{j=1}^{p-2} (-p)^{\left(\lfloor \frac{-2j}{p-1}\rfloor - \lfloor \frac{-j}{p-1}\rfloor - \lfloor \frac{3j}{p-1}\rfloor - \lfloor \frac{-4j}{p-1}\rfloor - 1\right)} \Gamma_p\left(1 - \tfrac{j}{p-1}\right) \\ \cdot \Gamma_p\left(\tfrac{j}{p-1}\right) \frac{\Gamma_p\left(\langle \frac{1}{4} - \frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{3}{4} - \frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{1}{3} + \frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{2}{3} + \frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)} \overline{\omega}^{j}\left(\tfrac{27b^2}{4a^3}\right)\Bigg]. \end{aligned}$$

We note for $0 \leq j \leq p-2$,

$$\left\lfloor \tfrac{-4j}{p-1} \right\rfloor - \left\lfloor \tfrac{-2j}{p-1} \right\rfloor = \left\lfloor \tfrac{1}{4} - \tfrac{j}{p-1} \right\rfloor + \left\lfloor \tfrac{3}{4} - \tfrac{j}{p-1} \right\rfloor,$$

and when $1 \leq j \leq p-2$,

$$\left\lfloor \tfrac{-j}{p-1} \right\rfloor + \left\lfloor \tfrac{3j}{p-1} \right\rfloor + 1 = \left\lfloor \tfrac{1}{3} + \tfrac{j}{p-1} \right\rfloor + \left\lfloor \tfrac{2}{3} + \tfrac{j}{p-1} \right\rfloor.$$

Also, by (2-9) we have, for $0 \leq j \leq p-1$,

$$\Gamma_p\left(1 - \tfrac{j}{p-1}\right)\Gamma_p\left(\tfrac{j}{p-1}\right) = (-1)^{p-j} = (-1)^p\, \overline{\omega}^{j}(-1).$$

Therefore

$$
\begin{aligned}
a_p(E) = \frac{-\phi(b)\,p}{(p-1)}\Bigg[\sum_{j=0}^{p-2}(-p)^{\left(-\lfloor \frac{1}{4}-\frac{j}{p-1}\rfloor-\lfloor \frac{3}{4}-\frac{j}{p-1}\rfloor-\lfloor \frac{1}{3}+\frac{j}{p-1}\rfloor-\lfloor \frac{2}{3}+\frac{j}{p-1}\rfloor\right)}
&\cdot\frac{\Gamma_p\left(\langle \frac{1}{4}-\frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle \frac{3}{4}-\frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)} \\
&\cdot\frac{\Gamma_p\left(\langle -\frac{2}{3}+\frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle -\frac{1}{3}+\frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\langle -\frac{2}{3}\rangle\right)\Gamma_p\left(\langle -\frac{1}{3}\rangle\right)}\overline{\omega}^j\left(-\frac{27b^2}{4a^3}\right)\Bigg] \\
= \phi(b)\cdot p\cdot {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\;\Big|\; -\frac{27b^2}{4a^3}\right]_p. \qquad \square
\end{aligned}
$$

Remark 4.2. Using (2-7) to evaluate the number of points on certain algebraic varieties over finite fields is by no means new. However, the author first observed the technique in the work of Fuselier [2007; 2010] where it was used to relate these evaluations to hypergeometric functions over finite fields. These methods were subsequently used by Lennon [2011] in generalizing Fuselier's work and, as we've seen, also form part of our proof of Theorem 1.2.

Proof of Corollary 1.3. As noted in the introduction, when $p > 3$, any elliptic curve $E/\mathbb{F}_p$ is isomorphic to an elliptic curve of the form $E' : y^2 = x^3 + ax + b$. Therefore $a_p(E) = a_p(E')$ and Theorem 1.2 can be used to evaluate $a_p(E)$. We also note that

$$j(E) = j(E') = \frac{1728\cdot 4a^3}{4a^3+27b^2},$$

and so

$$1-\frac{1728}{j(E)} = -\frac{27b^2}{4a^3}.$$

As E and E' are related by an admissible change of variables, this implies $c_6(E) = c_6(E')\cdot u^6$ for some $u \in \mathbb{F}_p^*$. Now $c_6(E') = -27\cdot 32\cdot b$ so $\phi(b) = \phi(-6\cdot c_6(E))$ as required. $\square$

5. Concluding remarks

5A. ***The $p = 3$ case.*** Theorem 1.2 considers elliptic curves over $\mathbb{F}_p$ for primes $p > 3$. While ${}_nG_n[\cdots]_p$ is not defined for $p = 2$, it is defined for $p = 3$ once the parameters are 3-adic integers. As the parameters of the ${}_2G_2[\cdots]_p$ in Theorem 1.2 are not all 3-adic integers it is clear that the result cannot be extended to $p = 3$ using the same function. However we can say something about the $p = 3$ case. Any elliptic curve over $\mathbb{F}_3$, whose j-invariant is nonzero, is isomorphic to a curve of the form $E : y^2 = x^3 + ax^2 + b$ with both a and b nonzero [Silverman 2009, Apppendix A]. It is an easy exercise to evaluate $a_3(E)$ and to show

$$a_3(E) = \phi(a) \cdot {}_2G_2\left[\begin{matrix} 0, & 0 \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, -\frac{a}{b}\right]_3.$$

This relationship is somewhat contrived however and direct calculation of $a_3(E)$ is much more straightforward.

5B. ***Transformation properties of*** ${}_n G_n[\cdots]_p$. As mentioned in Section 3, hypergeometric functions over finite fields were originally defined by Greene [1987] as analogues of classical hypergeometric functions. His motivation was to develop the area of character sums and their evaluations through parallels with the classical functions, and, in particular, with their transformation properties. His endeavor was largely successful and analogues of various classical transformations were found [ibid.]. Some others were recently provided by the author in [McCarthy 2012c]. These transformations for hypergeometric functions over finite fields can obviously be rewritten in terms of ${}_nG_n[\cdots]_p$ via Lemma 3.3 and these results will hold for all p where the original characters existed over $\mathbb{F}_p$. It is an interesting question to consider if these transformations can then be extended to almost all p and become transformations for ${}_nG_n[\cdots]_p$ in full generality. This is something yet to be considered and may be the subject of forthcoming work.

5C. ***q-version of*** ${}_n G_n[\cdots]_p$. As discussed in Section 3, ${}_nG_n[\cdots]_p$ extends hypergeometric functions over finite fields, as defined in Definition 3.2, to the p-adic setting. Definition 3.2 can easily be extended to $\mathbb{F}_q$ where q is a prime power and indeed, this is how it was originally defined in [McCarthy 2012c, Definition 1.4]. In a similar manner to the proof of Lemma 3.3, we can then use the Gross–Koblitz formula (not as quoted in Theorem 2.2 but its $\mathbb{F}_q$-version) to transform the hypergeometric function over $\mathbb{F}_q$ to an expression involving products of the p-adic gamma function. Generalizing the resulting expression yields the following q-version of ${}_nG_n[\cdots]_p$. We now let ω denote the Teichmüller character of $\mathbb{F}_q$.

Definition 5.1. Let $q = p^r$, for p an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define

$$\begin{aligned}{}_nG_n\left[\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{matrix} \,\middle|\, t\right]_q := \frac{-1}{q-1} \sum_{j=0}^{q-2} (-1)^{jn}\, \overline{\omega}^j(t) \\ \times \prod_{i=1}^{n} \prod_{k=0}^{r-1} \frac{\Gamma_p\big(\langle (a_i - \frac{j}{q-1}) p^k \rangle\big)}{\Gamma_p\big(\langle a_i p^k \rangle\big)} \frac{\Gamma_p\big(\langle (-b_i + \frac{j}{q-1}) p^k \rangle\big)}{\Gamma_p\big(\langle -b_i p^k \rangle\big)} \\ (-p)^{-\lfloor \langle a_i p^k \rangle - \frac{jp^k}{q-1} \rfloor - \lfloor \langle -b_i p^k \rangle + \frac{jp^k}{q-1} \rfloor}.\end{aligned}$$

When $q = p$ in Definition 5.1 we recover ${}_nG_n[\cdots]_p$ as per Definition 1.1. We believe ${}_nG_n[\cdots]_q$ could be used to generalize results involving hypergeometric functions over $\mathbb{F}_q$ which are restricted to q in certain congruence classes (e.g., those in [Lennon 2011]). However we do not examine this here for the following reason. The main purpose of this paper is to demonstrate that ${}_nG_n[\cdots]_p$ can be used to extend results involving hypergeometric functions over $\mathbb{F}_p$, which are limited to primes in certain congruence classes, and thus avoid the need to work over $\mathbb{F}_q$.

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(DN)-(Ω)-TYPE CONDITIONS FOR FRÉCHET OPERATOR SPACES

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We introduce (DN)-(Ω)-type conditions for Fréchet operator spaces. We investigate which quantizations carry over the above conditions from the underlying Fréchet space onto the operator space structure. This holds in particular for the minimal and maximal quantizations in case of a Fréchet space and — additionally — for the row, column and Pisier quantizations in case of a Fréchet–Hilbert space. We also reformulate these conditions in the language of matrix polars.

1. Introduction

The aim of this paper is to continue building a satisfactory theory for Fréchet operator spaces. The first motivation comes from the work of Effros and Webster [1997] and Effros and Winkler [1997] who started to build such a theory. The setting in both of these articles is very general — they define the operator analogues of arbitrary locally convex spaces. Another paper dealing with local analogues of operator spaces is [Beien and Dierolf 2001]. Motivated by the preface to the book of Effros and Ruan [2000] we restrict ourselves to the class of Fréchet spaces. Moreover the structure theory of Fréchet spaces is highly developed. One of the aspects of this structure theory are the so-called (DN)-(Ω) type conditions which play a very important role in several problems. They appear in the splitting theory of short exact sequences; see [Meise and Vogt 1997, Chapter 30; Poppenberg and Vogt 1995]. They play a role in characterizing when $L(X, Y) = LB(X, Y)$ that is, when every linear and continuous operator between Fréchet spaces is bounded in the sense that it maps some zero neighborhood into a bounded set; see [Meise and Vogt 1997, Chapter 29; Vogt 1983]. These conditions appear also in the lately defined concept of tameness; see [Dubinsky and Vogt 1989; Piszczek 2009]. Both boundedness and tameness are strongly connected with the longstanding open problem of Pełczyński of whether every complemented subspace of a nuclear Fréchet space with a basis

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has a basis itself. So far all known tame Fréchet spaces bring a positive answer to Pełczyński's question. In this paper we will try to build a theory that will enable us to follow the above described point of view.

Section 2 recalls the basic and necessary definitions of the objects we deal with together with the definitions of our conditions. In Section 3 we investigate which quantizations satisfy the operator (DN)-(Ω)-type conditions whenever the underlying Fréchet space possesses any of these properties. The main result is contained in Theorem 14. Recall that many natural Fréchet spaces are nuclear and by [Effros and Webster 1997, Theorem 7.4] such a space has only one quantization (up to a complete isomorphism). Therefore if some quantization of a nuclear space carries over our conditions then any other does, and so it does not seem to be interesting to consider various quantizations for such spaces. However there do exist Fréchet spaces that are not nuclear but seem to be important (see [Taskinen 1991]). Therefore we believe the content of Section 3 is useful. Section 4 shows our conditions from another point of view. We are able to rewrite (oDN) and $(o\Omega)$ in the language of matrix polars.

For unexplained details we refer the reader to [Meise and Vogt 1997] in case of the structure theory of Fréchet spaces and to [Effros and Ruan 2000] and [Pisier 2003] in case of the operator space theory.

2. Preliminaries

Recall that a Fréchet space X is a locally convex space that is metrizable and complete. The topology of such a space can always be given by a nondecreasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms and in this case $X = \mathrm{proj}_k X_k$, where $(X_k, \|\cdot\|_k)$ are local Banach spaces and $\iota_k^{k+1}\colon X_{k+1} \to X_k$ are the linking maps. The closed unit ball in the k-th seminorm in the space X will be usually denoted by U_k and its polar by U_k°, i.e., $U_k^\circ = \{x' \in X'\colon\ |x'x| \leqslant 1\ \forall\, x \in U_k\}$. The closed unit ball in the k-th local Banach space X_k will be denoted by B_{X_k}. Following [Effros and Webster 1997] we define a *Fréchet operator space* to be the projective limit of a sequence of operator spaces with the linking maps being completely bounded. To indicate this we will sometimes write $X = \text{m-proj}_k X_k$. Usually it will be clear from the context what kind of projective limit we deal with, therefore we will omit the symbol m-. This means that the Fréchet space $M_n(X)$ of $n \times n$ matrices with entries in X is given by $M_n(X) = \mathrm{proj}_k M_n(X_k)$ and the linking maps are just

$$\left(\iota_k^{k+1}\right)_n\colon\ M_n(X_{k+1}) \to M_n(X_k), \quad \left(\iota_k^{k+1}\right)_n\big((x_{ij})_{i,j=1}^n\big) := \left(\iota_k^{k+1} x_{ij}\right)_{i,j=1}^n.$$

By $M_n(X')$ we mean the linear space of all completely bounded maps $\phi\colon X \to M_n$. Using [Effros and Ruan 2000, Lemma 4.1.1] we see that $M_n(X') = T_n(X)'$ linearly and this isomorphism allows us to endow $M_n(X')$ with the (DF)-topology (recall that here $T_n(X) = \mathrm{proj}_k T_n(X_k)$ is a Fréchet space therefore its dual is a (DF)-space).

We can also quantize Fréchet spaces. If $X = \mathrm{proj}_k X_k$ is a Fréchet space and $Q \colon \mathfrak{B} \to \mathfrak{O}$ is a strict quantization from the category of Banach spaces into the category of operator spaces then by definition

$$Q(X) := \text{m-proj}_k \mathfrak{Q}(X_k).$$

For convenience we will write

$$\min X = \mathrm{proj}_k \min X_k, \quad \max X = \mathrm{proj}_k \max X_k,$$

and in case of Fréchet–Hilbert spaces

$$H_c = \mathrm{proj}_k (H_k)_c, \quad H_r = \mathrm{proj}_k (H_k)_r, \quad OH = \mathrm{proj}_k OH_k.$$

Let us recall that by [Effros and Webster 1997, Theorem 7.4], all the quantizations for nuclear Fréchet spaces are equal (up to a complete isomorphism).

Examples. 1. The space $C(\mathbb{R}) = \mathrm{proj}_k C([-k, k])$ of continuous functions on the real line is a Fréchet space that carries an operator space structure. In $M_n(C(\mathbb{R}))$ we define seminorms

$$\|(f_{ij})\|_k := \sup\{\|(f_{ij}(x))\|_{M_n} : x \in [-k, k]\}.$$

In a similar fashion we can introduce an operator space structure on the spaces $C^\infty(K)$, $C^\infty(\Omega)$ for arbitrary subsets K compact and Ω open of $\mathbb{R}^d$.

2. In order to give an example of a Fréchet operator space arising in quantum physics, let

$$s = \left\{ x = (x_j)_{j \in \mathbb{N}} : \|x\|_k^2 := \sum_{j=1}^{+\infty} |x_j|^2 j^{2k} < +\infty \ \forall k \in \mathbb{N} \right\},$$

be the (nuclear) Fréchet space of rapidly decreasing sequences, with the topology given by the sequence of norms $(\|\cdot\|_k)_{k \in \mathbb{N}}$: in short,

$$s = \mathrm{proj}_k \ell_2((j^k)_j).$$

Following [Dubin and Hennings 1990] we call $s \tilde{\otimes}_\pi s$ the space of physical states and we endow it with the Fréchet operator space structure

$$\mathcal{T}_{op} = s_r \hat{\otimes}_{op} s_c,$$

where $\hat{\otimes}_{op}$ stands for the operator projective tensor product.

3. A moment's reflection shows that the above space $\mathcal{T}_{op}$ is in fact $L(s', s)$ with a suitable operator space structure. We can generalize this by introducing such a

structure on $L(X', Y)$ for arbitrary Fréchet spaces X and Y. Recall that $L(X', Y)$ is a Fréchet space with a sequence $(|||\cdot|||_k)_k$ of seminorms defined by

$$|||T|||_k := \sup\{\|Tf\|_k : f \in U_k^\circ\},$$

where $(U_k)_k$ is a zero neighborhood basis in X and $(\|\cdot\|_k)_k$ defines the topology of Y. Then the linear isomorphism $M_n(L(X', Y)) = L(\ell_p^n(X)', \ell_p^n(Y))$ for arbitrary $1 \leqslant p \leqslant +\infty$ provides $L(X', Y)$ with an operator space structure.

Let us now define the operator analogues of the conditions (DN) and (Ω).

Definition 1. (i) We will say that a Fréchet operator space X satisfies the property (oDN) if there exists a seminorm p such that for any other seminorm q and arbitrary number $\tau \in (0, 1)$ there exist another seminorm r and a constant $C > 0$ such that the inequality

$$\|(x_{ij})\|_q \leqslant C\big(\|(x_{ij})\|_p\big)^{1-\tau}\big(\|(x_{ij})\|_r\big)^\tau \tag{2-1}$$

holds for every matrix $(x_{ij}) \in M_n(X)$ of arbitrary size $n \in \mathbb{N}$.

(ii) We will say that a Fréchet operator space X satisfies the property $(o\Omega)$ if for every seminorm p there exists another seminorm q such that for any other seminorm r there exist a number $\theta \in (0, 1)$ and a constant $C > 0$ such that the inequality

$$\|(\phi_{ij})\|_q^* \leqslant C\big(\|(\phi_{ij})\|_p^*\big)^\theta\big(\|(\phi_{ij})\|_r^*\big)^{1-\theta} \tag{2-2}$$

holds for every matrix $(\phi_{ij}) \in M_n(X')$ of arbitrary size $n \in \mathbb{N}$.

Remarks. 1. If one of the above conditions holds for a Fréchet operator space X then we write (respectively) $X \in (oDN)$, $X \in (o\Omega)$.

2. If $X \in (oDN)$ then the seminorm p is in fact a norm and so all the seminorms become norms.

3. In the above definition the symbol $\|(\phi_{ij})\|_k^*$ stands for the cb-norm of a map $(\phi_{ij})\colon X \to M_n$. We stress that — in general — it is finite for all but finitely many k.

4. If we restrict the above definitions to $n = 1$ then we get the classical (DN) and (Ω) conditions of Vogt; see [Meise and Vogt 1997, page 367].

5. There are other versions of these conditions: if we change the quantifiers in (1) to "$\ldots \exists\, r \in \mathbb{N}, \tau \in (0, 1) \ldots$" then we get the condition $(o\underline{DN})$. If we change in (2) the quantifiers to "$\forall\, p \in \mathbb{N}, \theta \in (0, 1) \ldots$" then we get the condition $(o\overline{\Omega})$ and the change to "$\ldots \forall\, r \in \mathbb{N}, \theta \in (0, 1) \ldots$" leads to the condition $(o\overline{\overline{\Omega}})$. We have obvious implications

$$(o\overline{\overline{\Omega}}) \Rightarrow (o\overline{\Omega}) \Rightarrow (o\Omega), \qquad (oDN) \Rightarrow (o\underline{DN}).$$

6. It is not difficult to show (see [Meise and Vogt 1997, Lemma 29.10]) that (oDN) is satisfied whenever (2-1) holds with $\tau = \frac{1}{2}$.

7. Recall that by [Tomiyama 1983, Lemma 1.1] (compare also [Paulsen 2002, page 41]) we have for all operator spaces

$$\|(a_{ij})\| \leqslant \Bigl(\sum_{i,j=1}^{n} \|a_{ij}\|^2\Bigr)^{1/2} \leqslant n\|(a_{ij})\|.$$

Therefore if X has (DN) as a Fréchet space then all the Fréchet spaces $M_n(X)$ satisfy this property with some constant $C_n = C_{p,q,r}(n)$. The point is that these constants be uniformly bounded (with respect to the matrix size n). The same can be observed for (Ω).

8. Both conditions are invariants in the category of Fréchet operator spaces.

Proposition 2. *The Fréchet operator space $\mathcal{T}_{op} = s_r \hat{\otimes}_{op} s_c$ of physical states satisfies both properties (DN) and (Ω).*

Proof. By [Effros and Webster 1997, Theorem 7.5] and the commutativity of $\hat{\otimes}_{op}$ we have the complete isomorphism

$$\mathcal{T}_{op} = s_c \check{\otimes}_{op} s_r = \operatorname{proj}_k \bigl(\ell_2(j^k)_c \check{\otimes}_{op} \ell_2(j^k)_r\bigr),$$

where $\check{\otimes}_{op}$ stands for the operator injective tensor product. Applying [Effros and Ruan 2000, 9.3.1 and 9.3.4] we get the complete isometry

$$\ell_2(j^k)_c \check{\otimes}_{op} \ell_2(j^k)_r \cong \mathcal{K}\bigl(\ell_2(j^k)', \ell_2(j^k)\bigr),$$

where $\mathcal{K}$ stands for the compact operators. Therefore

$$\begin{aligned} M_n(s_c \check{\otimes}_{op} s_r) &= \operatorname{proj}_k \mathcal{K}\bigl(\ell_2^n(\ell_2(j^k))', \ell_2^n(\ell_2(j^k))\bigr) \\ &= \operatorname{proj}_k \bigl(\ell_2^n(\ell_2(j^k)) \widetilde{\otimes}_\varepsilon \ell_2^n(\ell_2(j^k))\bigr) = \ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s). \end{aligned}$$

By [Meise and Vogt 1997, Lemma 29.2] the space s satisfies (DN) and it is easy to see that $\ell_2^n(s)$ satisfies this condition with exactly the same constant C in (2-1). Applying [Piszczek 2010, Theorem 4] we observe that $\ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s)$ satisfies (DN) with constant C independent of n therefore $\mathcal{T}_{op} \in (oDN)$. In order to show the other property let us recall that by [Meise and Vogt 1997, Lemma 29.11] $s \in (\Omega)$ therefore $\ell_2^n(s) \in (\Omega)$ (with unchanged constants). By [Piszczek 2010, Theorem 5] $\ell_2^n(s) \widetilde{\otimes}_\varepsilon \ell_2^n(s)$ satisfies (Ω) with constants C independent of the matrix size. This shows that $\mathcal{T}_{op} \in (o\Omega)$. □

The (DN)-(Ω)-type conditions have equivalent forms which are often used in proofs. Since we will be using these equivalent forms extensively in the sequel we state them below for convenience of the reader.

Theorem 3 [Vogt 1977; Vogt and Wagner 1980]. *Let X be a Fréchet space.*

(1) *X satisfies the property (DN) if and only if there exists a seminorm p such that for any other seminorm q there exist another seminorm r and a constant $C > 0$ such that the inclusion*

$$U_q^\circ \subset sU_p^\circ + \frac{C}{s} U_r^\circ$$

is satisfied for all numbers $s > 0$.

(2) *X satisfies the property (Ω) if and only if for every seminorm p there exists another seminorm q such that for any other seminorm r there exist a number $\gamma > 0$ and a constant $C > 0$ such that the inclusion*

$$U_q \subset sU_p + \frac{C}{s^\gamma} U_r$$

is satisfied for all numbers $s > 0$.

3. Hereditary properties of quantizations

Now we will try to answer the following question: Suppose X has one of the properties (DN) or (Ω). Which quantizations of X automatically satisfy the operator analogues of these conditions? Let us indicate that the proofs are exactly the same for all versions of (DN)-type conditions as well as those of (Ω) type. Therefore we will always give proofs precisely for (DN) and (Ω). Moreover the results are formulated in such a way that only sufficiency will require an argument. First we focus on the condition (DN). We start this a little bit technical section with the following result.

Proposition 4. *Let X be a Fréchet space. Then X satisfies (DN) if and only if* $\min X$ *satisfies (oDN).*

Proof. Recall that for arbitrary $n \in \mathbb{N}$ we have in $M_n(\min X)$ the seminorms

$$\|(x_{ij})\|_k = \sup\left\{\|\xi(x_{ij})\|_{M_n} : \xi \in U_k^\circ\right\},$$

where U_k° is the polar of the zero neighborhood U_k. Choosing all the parameters according to (2-1) and assuming $X \in (DN)$ we obtain by Theorem 3 a chain of inequalities

$$\begin{aligned}\|(x_{ij})\|_q &\leqslant \sup\left\{\left\|\left(s\xi(x_{ij}) + Cs^{-1}\eta(x_{ij})\right)\right\|_{M_n} : \xi \in U_p^\circ, \eta \in U_r^\circ\right\}\\ &\leqslant s\|(x_{ij})\|_p + Cs^{-1}\|(x_{ij})\|_r.\end{aligned}$$

Taking the infimum over positive s we get

$$\|(x_{ij})\|_q^2 \leqslant 4C\|(x_{ij})\|_p\|(x_{ij})\|_r,$$

and since the constant C is independent of the matrix size (that is, C does not depend on n) we obtain the condition (oDN). □

In order to prove the analogous result for the max quantization we will need two lemmata.

Lemma 5. *Let X, E be locally convex spaces. Suppose $U, V \subset X$ and $B \subset E$ are absolutely convex subsets. Then $(U \cap V) \otimes B = U \otimes B \cap V \otimes B$.*

Proof. Since the inclusion $\subset$ is obvious we take an element ϕ with representations $\phi = x \otimes b \in U \otimes B$ and $\phi = y \otimes c \in V \otimes B$. If $g(b) = 0$ for all functionals $g \in E'$ then $b = 0$ and so $\phi = 0 \in (U \cap V) \otimes B$. Therefore we may suppose $f(b) \neq 0$ for some functional $f \in E'$. If $f(c) = 0$ then for every functional $x' \in X'$ we have

$$(x' \otimes f)(x \otimes b) = (x' \otimes f)(y \otimes c),$$

which means $x'x = 0$ for every $x' \in X'$. This gives $x = 0$ and $\phi = 0 \in (U \cap V) \otimes B$. So let us suppose $f(c) \neq 0$. We may also assume $|f(b)/f(c)| \leqslant 1$ (otherwise we take the inverse). For every $x' \in X'$ we get $x'(x) f(b) = x'(y) f(c)$ which leads to $y = (f(b)/f(c))x$. But we deal with absolutely convex sets, therefore $y \in U$ and so $\phi \in (U \cap V) \otimes B$, which shows the other inclusion. □

Lemma 6. *Let X be a Fréchet space and E a Banach space. If X has the property (DN) or (Ω) then their projective tensor product as well as the injective one satisfy the same condition too.*

Proof. We start with the projective tensor product. By [Köthe 1979, Chapter VIII, §41, 2(4)] one basis of zero neighborhoods in $X \tilde{\otimes}_\pi E$ has the form

$$\left(\overline{\Gamma(U_k \otimes B_E)}\right)_{k \in \mathbb{N}},$$

where $(U_k)_k$ is a basis of zero neighborhoods in X and B_E is the closed unit ball in E. Let us now assume $X \in (DN)$. By Theorem 3(1) we have

$$\frac{1}{2s} U_p \cap \frac{s}{2C} U_r \subset U_q.$$

Tensoring by B_E and taking polars we obtain, with the help of Lemma 5,

$$\left(U_q \otimes B_E\right)^\circ \subset \left(\frac{1}{2s}\left(U_p \otimes B_E\right) \cap \frac{s}{2C}\left(U_r \otimes B_E\right)\right)^\circ.$$

If now U and V are arbitrary zero neighborhoods in a locally convex space Y then by the Bipolar theorem we have

$$(U \cap V)^\circ \subset \overline{\Gamma(U^\circ + V^\circ)}^{\sigma(Y',Y)} = U^\circ + V^\circ,$$

the last equality being a consequence of absolute convexity and weak* compactness of U° and V°. Adapting the above inclusion to $Y = X \tilde{\otimes}_\pi E$ and the considered zero neighborhoods we get

$$\big(U_q \otimes B_E\big)^\circ \subset 2s\big(U_p \otimes B_E\big)^\circ + \frac{2C}{s}\big(U_r \otimes B_E\big)^\circ.$$

Taking $t = 2s$, $D = 4C$ and recalling that $\overline{\Gamma(A)}^\circ = A^\circ$ for every set A we arrive at

$$\overline{\Gamma(U_q \otimes B_E)}^\circ \subset t\overline{\Gamma(U_p \otimes B_E)}^\circ + \frac{D}{t}\overline{\Gamma(U_r \otimes B_E)}^\circ.$$

Again by Theorem 3(1) we obtain the property (DN) for the space $X\tilde{\otimes}_\pi E$. Moreover the crucial constant D in the above inclusion does not depend on the Banach space E but only on the Fréchet space X. The case of the condition (Ω) is even simpler since by Theorem 3(2) we have

$$\begin{aligned}\overline{\Gamma(U_q \otimes B_E)} &\subset \overline{s\Gamma(U_p \otimes B_E) + Cs^{-\gamma}\Gamma(U_q \otimes B_E)}\\ &\subset s\overline{\Gamma(U_p \otimes B_E)} + (C+1)s^{-\gamma}\overline{\Gamma(U_r \otimes B_E)}.\end{aligned}$$

Again the constant $C+1$ depends only on the Fréchet space X.

In the case of the injective tensor product we recall that by [Köthe 1979, Chapter VIII, §44, 2(3)] one basis of zero neighborhoods in $X\tilde{\otimes}_\varepsilon E$ is of the form

$$\big((U_k^\circ \otimes B_E^\circ)^\circ\big)_{k\in\mathbb{N}}.$$

Suppose now that $X \in (\Omega)$. By Theorem 3(2) this gives

$$U_q \subset sU_p + Cs^{-\gamma}U_r,$$

whence the meaning of all the parameters follows. Using a technique similar to the one in the beginning of the proof we obtain

$$(U_q^\circ \otimes B_E^\circ)^\circ \subset \Big(\frac{1}{2s}(U_p^\circ \otimes B_E^\circ) \cap \frac{s^\gamma}{2C}(U_r^\circ \otimes B_E^\circ)\Big)^\circ.$$

By [Köthe 1969, Chapter IV, §20, 8(10)] we get

$$\begin{aligned}(U_q^\circ \otimes B_E^\circ)^\circ &\subset \overline{\Gamma\big\{2s\big(U_p^\circ \otimes B_E^\circ\big)^\circ \cup 2Cs^{-\gamma}\big(U_r^\circ \otimes B_E^\circ\big)^\circ\big\}}\\ &\subset \overline{2s\big(U_p^\circ \otimes B_E^\circ\big)^\circ + 2Cs^{-\gamma}\big(U_r^\circ \otimes B_E^\circ\big)^\circ}.\end{aligned}$$

But the sets under consideration are zero neighborhoods therefore we may drop the closure by increasing one of them which leads to

$$(U_q^\circ \otimes B_E^\circ)^\circ \subset 2s(U_p^\circ \otimes B_E^\circ)^\circ + 3Cs^{-\gamma}(U_r^\circ \otimes B_E^\circ)^\circ.$$

Now taking $t = 2s$, $D = 3 \cdot 2^\gamma C$ and applying Theorem 3(2) we arrive at the property (Ω) in the injective tensor product. To show that the property (DN) also passes onto $X\tilde{\otimes}_\varepsilon E$ we start with $u=\sum_{j=1}^m x_j\otimes a_j\in X\otimes_\varepsilon E$. By [Köthe 1979, Chapter VIII, §44, 2(5)] its seminorms are calculated as

$$\|u\|_k=\sup\left\{\sum_{j=1}^m |f(x_j)g(a_j)| : f\in U_k^\circ,\ g\in B_{E'}\right\}.$$

If X satisfies (DN) then for $f\in U_q^\circ$ we obtain by Theorem 3(1) functionals $f_1\in U_p^\circ$ and $f_2\in U_r^\circ$ with $f=sf_1+Cs^{-1}f_2$. Consequently,

$$\sum_{j=1}^m |f(x_j)g(a_j)|\leqslant s\sum_{j=1}^m |f_1(x_j)g(a_j)|+Cs^{-1}\sum_{j=1}^m |f_2(x_j)g(a_j)|.$$

Taking the supremum over all such f, f_1, f_2 we get

$$\|u\|_q\leqslant s\|u\|_p+Cs^{-1}\|u\|_r,$$

and taking the infimum over all $s>0$ we arrive at our condition. Finally it is easy to observe that the above property passes onto the completion, therefore $X\tilde{\otimes}_\varepsilon E$ satisfies (DN) and the constant C does not depend on the Banach space E. □

Proposition 7. *Let X be a Fréchet space. Then X satisfies (DN) if and only if* $\max X$ *satisfies (oDN).*

Proof. Recall that by [Effros and Ruan 2000, 3.3] for $(x_{ij})\in M_n(\max(X))$ we have

$$\|(x_{ij})\|_k=\sup\left\|\left(f_{uv}(x_{ij})\right)\right\|_{M_{nm}},$$

where the supremum runs over all $(f_{uv})\in L(X,M_m)$ with $\|(f_{uv})\|_{L(X_k,M_m)}\leqslant 1$ and all $m\in\mathbb{N}$. We have $L(X,M_m)=(X\otimes_\pi M_m)'$ by [Köthe 1979, Chapter VIII, §41, 3(3)] (since M_m is finite-dimensional we may drop the tensor product completion). Moreover, by [Meise and Vogt 1997, Remark 24.5(b)], $B_{L(X_k,M_m)}=(U_k\otimes B_{M_m})^\circ$. If X satisfies the property (DN) then by Lemma 6 we get

$$B_{L(X_q,M_m)}\subset sB_{L(X_p,M_m)}+Cs^{-1}B_{L(X_r,M_m)},$$

where all the parameters are chosen according to Theorem 3(1). Choosing (f_{uv}) in $L(X,M_m)$ with $\|(f_{uv})\|_{L(X_q,M_m)}\leqslant 1$ we obtain

$$\left\|\left(f_{uv}(x_{ij})\right)\right\|_{M_{nm}}=\left\|\left((sg_{uv}+Cs^{-1}h_{uv})(x_{ij})\right)\right\|_{M_{nm}}$$

for some $(g_{uv})\in B_{L(X_p,M_m)}$, $(h_{uv})\in B_{L(X_r,M_m)}$. Now taking the supremum over all such (f_{uv}), (g_{uv}), (h_{uv}) and all natural m we obtain

$$\|(x_{ij})\|_q\leqslant s\|(x_{ij})\|_p+Cs^{-1}\|(x_{ij})\|_r.$$

Finally, taking the infimum over all $s>0$ we arrive at

$$\|(x_{ij})\|_q^2\leqslant 4C\|(x_{ij})\|_p\|(x_{ij})\|_r,$$

and since the constant C is independent of the matrix size we obtain the condition (oDN). □

We now move to row and column quantizations of Fréchet–Hilbert spaces.

Proposition 8. *Let H be a Fréchet–Hilbert space.*

(1) *$H \in (DN)$ if and only if $H_r \in (oDN)$.*

(2) *$H \in (DN)$ of and only if $H_c \in (oDN)$.*

Proof. We start with the row quantization. Recall that by [Pisier 2003, page 22] the seminorms in $M_n(H_r)$ are given by the following formula: If $\phi_{ij} \in H$ for $i, j = 1, \dots, n$, then

$$\|(\phi_{ij})\|_k = \sup\left\{\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2\right)^{1/2} : (x_j)_{j=1}^n \in B_{\ell_2^n(H_k)'}\right\}.$$

If H satisfies the property (DN) then there exists p such that for all q we can find r and $C > 0$ with

$$\|h\|_q^2 \leqslant C\|h\|_p\|h\|_r, \quad \forall\, h \in H.$$

Using the Cauchy–Schwarz inequality we get

$$\|(h_j)\|_q^2 \leqslant C\|(h_j)\|_p\|(h_j)\|_r, \quad \forall\,(h_j) \in \ell_2^n(H)$$

with the same constant C. By Theorem 3(1) this gives

$$B_{\ell_2^n(H_q)'} \subset s B_{\ell_2^n(H_p)'} + Cs^{-1} B_{\ell_2^n(H_r)'}$$

for all positive s and a constant C independent of n. For arbitrary $(x_j)_{j=1}^n \in B_{\ell_2^n(H_q)'}$ the above inclusion allows us to find $(y_j)_{j=1}^n \in B_{\ell_2^n(H_p)'}$ and $(z_j)_{j=1}^n \in B_{\ell_2^n(H_r)'}$ with

$$\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2 = \sum_{i=1}^n \left|\sum_{j=1}^n \langle s y_j + Cs^{-1} z_j, \phi_{ij}\rangle\right|^2.$$

Applying once again the Cauchy–Schwarz inequality we arrive at

$$\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle x_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}} \leqslant s\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle y_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}} + Cs^{-1}\left(\sum_{i=1}^n \left|\sum_{j=1}^n \langle z_j, \phi_{ij}\rangle\right|^2\right)^{\frac{1}{2}}.$$

Taking the supremum over all such (x_j), (y_j), (z_j) we obtain

$$\|(\phi_{ij})\|_q \leqslant s\|(\phi_{ij})\|_p + Cs^{-1}\|(\phi_{ij})\|_r.$$

Finally taking the infimum over all $s > 0$ leads to the condition (oDN). Moving to the column quantization we recall that by [Pisier 2003, page 22] the seminorms in

$M_n(H_c)$ are given by the following formula: If $\phi_{ij} \in H$ for $i, j = 1, \dots, n$, then

$$\|(\phi_{ij})\|_k = \sup\left\{\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_k^2\right)^{1/2} : (\xi_j)_{j=1}^n \in B_{\ell_2^n}\right\}.$$

If $H \in (DN)$ then by the Cauchy–Schwarz inequality we have for arbitrary $(\xi_j)_{j=1}^n$ in $B_{\ell_2^n}$ that

$$\begin{aligned}\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_q^2 &\leqslant C \sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_p \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_r \\ &\leqslant C\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_p^2\right)^{1/2}\left(\sum_{i=1}^n \left\|\sum_{j=1}^n \xi_j \phi_{ij}\right\|_r^2\right)^{1/2}.\end{aligned}$$

This leads to

$$\|(\phi_{ij})\|_q^2 \leqslant C\|(\phi_{ij})\|_p\|(\phi_{ij})\|_r$$

with the constant C independent of the matrix size, therefore we conclude that the column quantization also carries over the property (DN). □

Proposition 9. *Let H be a Fréchet–Hilbert space.*

(1) *$H \in (DN)$ if and only if $OH \in (oDN)$.*

(2) *$H \in (\Omega)$ if and only if $OH \in (o\Omega)$.*

Proof. Recall that if K is a Hilbert space then by [Effros and Ruan 2000, Proposition 3.5.2] the norm in $M_n(OK)$ is given by

$$\|\phi\| = \|\langle\!\langle \phi, \phi\rangle\!\rangle\|^{1/2} := \left\|\left(\left(\langle \phi_{ij}, \phi_{kl}\rangle\right)_{k,l=1}^n\right)_{i,j=1}^n\right\|^{1/2},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in K. With the above notation the scalar matrix $\langle\!\langle \phi, \phi\rangle\!\rangle$ need not be positive in M_{n^2}, therefore (for the reasons that will become apparent shortly) we quickly describe how to change it isometrically into a positive one. Suppose $A = (A_{i,j})$ is in $M_n(M_n)$ and each $A_{i,j} = (a_{i,j,k,l}) \in M_n$. We reorder the first row of A in the following way: the first row of $A_{1,1}$ remains untouched, the first row of $A_{1,2}$ exchanges with the second row of $A_{1,1}$ and in general the first row of $A_{1,j}$ exchanges with the j-th row of $A_{1,1}$. Next the second row of $A_{1,2}$ remains untouched and the second row of $A_{1,j} (j \geqslant 3)$ exchanges with the j-th row of $A_{1,2}$. We continue until the first row of A is completely reordered and apply the same procedure to any other row of A. Such a reordering (call it ρ) is an isometry and $\rho(\langle\!\langle \phi, \phi\rangle\!\rangle)$ is positive in M_{n^2}. Indeed, if $\xi = (\xi^i)_i \in \ell_2^n(\ell_2^n)$ and each $\xi^i = (\xi^i_j)_j \in \ell_2^n$ then

$$\big\langle \rho(\langle\!\langle \phi, \phi \rangle\!\rangle)\xi, \xi \big\rangle = \Big\langle \sum_{i,j=1}^{n} \overline{\xi}^i_j \phi_{ij}, \sum_{k,l=1}^{n} \overline{\xi}^k_l \phi_{kl} \Big\rangle = \Big\| \sum_{i,j=1}^{n} \overline{\xi}^i_j \phi_{ij} \Big\|^2, \tag{3-1}$$

and the last quantity is nonnegative. Suppose now H satisfies the condition (DN) and let p, q, r, C have the same meaning as in (2-1). Take n in $\mathbb{N}$ and $x = (x_{ij})$ in $M_n(OH)$. By (3-1) and positivity of $\rho(\langle\!\langle x, x \rangle\!\rangle)$ we get

$$\begin{aligned}
\|x\|_q^2 &= \|\rho(\langle\!\langle x, x \rangle\!\rangle_q)\|^2 \\
&= \sup\Big\{ \Big\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \Big\|_q^2 : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \Big\} \\
&\leqslant C \sup\Big\{ \Big\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \Big\|_p : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \Big\} \sup\Big\{ \Big\| \sum_{i,j=1}^{n} \overline{\xi}^i_j x_{ij} \Big\|_r : \|\xi\|_{\ell_2^{n^2}} \leqslant 1 \Big\} \\
&= C\|x\|_p \|x\|_r .
\end{aligned}$$

Since the constant C does not depend on the matrix size n, we get the condition (oDN).

In order to prove the other equivalence recall first that for every functional $\phi \in H'$ we get a sequence of functionals $(\phi_k)_{k \geqslant k_0}$ acting on the local Hilbert steps which satisfy

$$\phi_k \circ \iota_k = \phi \quad (k \geqslant k_0),$$

where ι_ks are the canonical projections. We also have $\|\phi\|_k^* = \|\phi_k\|_{H_k'}$. If now $\phi = (\phi_{ij}) \in M_n((OH)')$ then we may find matrices $\phi_k = (\phi_{k,i,j}) \in M_n((OH_k)')$ of functionals such that

$$\|\phi\|_k^* = \|\phi_k\|_{(OH_k)'}. \tag{3-2}$$

By the selfduality of Pisier's quantization we have $\|\phi_k\|^*_{(OH_k)'} = \|\phi_k\|_{OH_k}$. Suppose now $H \in (\Omega)$ and let p, q, r, θ, C have the same meaning as in (2-2). We take $n \in \mathbb{N}$, $\phi = (\phi_{ij}) \in M_n(H')$ and apply exactly the same reasoning as above to obtain the inequality

$$\|\phi_q\|_{OH_q} \leqslant C \|\phi_p\|^{\theta}_{OH_p} \|\phi_r\|^{1-\theta}_{OH_r},$$

where the constant C does not depend on the matrix size n. Applying (3-2) we get the condition $(o\Omega)$. □

Now we will investigate the minimal and maximal quantizations in view of the condition (Ω). Here the Blecher duality will play an important role.

Proposition 10. *Let X be a Fréchet space. Then X satisfies* (Ω) *if and only if* $\max X$ *satisfies* $(o\Omega)$.

Proof. Recall that for an arbitrary Banach space X we have by [Blecher 1992, Corollary 2.8] that $(\max X)' = \min X'$ completely isometrically. Therefore if X is a Fréchet space then for $(\phi_{ij}) \in M_n((\max X)')$ we have

$$\|(\phi_{ij})\|_k = \sup\left\{\|x''(\phi_{ij})\|_{M_n} : x'' \in B_{X_k''}\right\}.$$

Taking together the Separation theorem [Köthe 1969, Chapter IV, §20, 7(1)] and the Bipolar theorem [Köthe 1969, Chapter IV, §20, 8(5)] it is enough to take in the above supremum vectors $x \in B_{X_k}$. By the density of U_k in B_{X_k} we may restrict ourselves to vectors $x \in U_k$. If X satisfies (Ω) then by Theorem 3(2) we get for arbitrary p a number q such that for all r there exist positive C and γ with

$$U_q \subset sU_p + \frac{C}{s^\gamma} U_r$$

for all $s > 0$. Repeating the proof of Proposition 4 we obtain the condition $(o\Omega)$. □

In order to prove the analogous result for the minimal quantization we will need a lemma. It seems to be known to specialists but for the sake of convenience we will state and prove it.

Proposition 11. *Let X be a Fréchet space.*

(1) $X \in (DN)$ *if and only if* $X'' \in (DN)$.

(2) $X \in (\Omega)$ *if and only if* $X'' \in (\Omega)$.

Proof. Suppose X satisfies the condition (DN). By Theorem 3(1) we find p such that for all q there exist r and $C > 0$ with

$$U_q^\circ \subset sU_p^\circ + Cs^{-1}U_r^\circ$$

for all $s > 0$. For arbitrary $x'' \in X''$ we have by [Meise and Vogt 1997, Proposition 25.9] that

$$\begin{aligned}\|x''\|_q &= \sup\left\{|x''x'| : x' \in U_q^\circ\right\}\\ &\leqslant s\sup\left\{|x''y'| : y' \in U_p^\circ\right\} + Cs^{-1}\sup\left\{|x''z'| : z' \in U_r^\circ\right\}\\ &= s\|x''\|_p + Cs^{-1}\|x''\|_r.\end{aligned}$$

taking the infimum over positive s we get the property (DN) for the bidual. Since by [Meise and Vogt 1997, Corollary 25.10] every Fréchet space is a topological subspace of its bidual, the converse of (1) follows. Suppose now that X satisfies the condition (Ω). By Theorem 3(2) we get

$$U_q \subset sU_p + Cs^{-\gamma}U_r.$$

Taking polars twice (each of which in the consecutive dual) and applying [Köthe 1969, Chapter IV, §20, 8(9)] we obtain

$$U_q^{\circ\circ} \subset (2sU_p + 2Cs^{-\gamma}U_r)^{\circ\circ}.$$

By the Separation theorem [Köthe 1969, Chapter IV, §20, 7(1)] $U_k^{\circ\circ} = \overline{U_k}^{\sigma(X'',X')}$ and these sets constitute a basis of zero neighborhoods in the bidual, therefore

$$\overline{U_q}^{\sigma(X'',X')} \subset \overline{2sU_p + 2Cs^{-\gamma}U_r}^{\sigma(X'',X')} \subset 2s\overline{U_p}^{\sigma(X'',X')} + 2Cs^{-\gamma}\overline{U_r}^{\sigma(X'',X')}.$$

Taking $t = 2s$ and $D = 2^{\gamma+1}C$ we arrive at the (Ω) property in the bidual. The converse of (2) is valid by the Separation theorem which implies that for every functional ϕ acting on X we have $\|\phi\|_{k,X'}^* = \|\phi\|_{k,X'''}^*$. □

Proposition 12. *Let X be a Fréchet space. Then X satisfies (Ω) if and only if* $\min X$ *satisfies* $(o\Omega)$.

Proof. Recall that for an arbitrary Banach space X we have by [Blecher 1992, Corollary 2.8] that $(\min X)' = \max X'$ completely isometrically. Therefore if X is a Fréchet space then for $(\phi_{ij}) \in M_n((\min X)')$ we have

$$\|(\phi_{ij})\|_k^* = \sup \big\|\big(f_{uv}(\phi_{ij})\big)\big\|_{M_{nm}},$$

where the supremum runs over all $(f_{uv}) \in L(X', M_m)$ with $\|(f_{uv})\|_{L(X_k', M_m)} \leqslant 1$ and all $m \in \mathbb{N}$. We have $L(X', M_m) = X'' \otimes_\varepsilon M_m$ by [Köthe 1979, Chapter VIII, §44, 2(6)] (since M_m is finite-dimensional we may drop the tensor product completion). Moreover $B_{L(X_k', M_m)} = (V_k^\circ \otimes B_{M_m'})^\circ$, where $V_k = \overline{U_k}^{\sigma(X'',X')}$ and $(U_k)_k$ is a basis of zero neighborhoods in X. By Lemma 6 and Proposition 11(2) we observe that for every p there exists q such that for all r we can find positive C and γ with

$$B_{L(X_q', M_m)} \subset sB_{L(X_p', M_m)} + Cs^{-\gamma}B_{L(X_r', M_m)}.$$

Choosing $\|(f_{uv})\|_{L(X_q', M_m)} \leqslant 1$ we obtain

$$\big\|\big(f_{uv}(\phi_{ij})\big)\big\|_{M_{nm}} = \big\|\big((sg_{uv} + Cs^{-1}h_{uv})(\phi_{ij})\big)\big\|_{M_{nm}}$$

for some $(g_{uv}) \in B_{L(X_p', M_m)}$, $(h_{uv}) \in B_{L(X_r', M_m)}$. Now taking the supremum over all such (f_{uv}), (g_{uv}), (h_{uv}) and all natural m we obtain

$$\|(\phi_{ij})\|_q^* \leqslant s\|(\phi_{ij})\|_p^* + Cs^{-1}\|(\phi_{ij})\|_r^*.$$

Finally, taking the infimum over all $s > 0$ we arrive at

$$\|(\phi_{ij})\|_q \leqslant D\big(\|(\phi_{ij})\|_p^*\big)^{1-\theta}\big(\|(\phi_{ij})\|_r^*\big)^\theta,$$

where $\theta = 1/(\gamma + 1)$ and $D = (C\gamma)^{1/(\gamma+1)}(1 + \gamma^{-\gamma})$. Since the constant D is independent of the matrix size of (ϕ_{ij}) we obtain the condition $(o\Omega)$. □

By the duality of row and column Hilbert spaces (see [Effros and Ruan 2000, page 59]) we get the following result. Its proof is analogous to that of Proposition 8, therefore we omit it.

Proposition 13. *Let H be a Fréchet–Hilbert space. Then $H \in (\Omega)$ if and only if $H_c \in (o\Omega)$ if and only if $H_r \in (o\Omega)$.*

We now put together all the previously obtained results.

Theorem 14. *Let X be an arbitrary Fréchet space. Then the minimal and maximal quantizations carry over both properties (DN) as well as (Ω) onto their operator space structures. If H is an arbitrary Fréchet–Hilbert space then the above remains valid for the additional row, column and Pisier quantizations.*

4. The conditions of type (oDN)–$(o\Omega)$ in the language of polars

In this section we will prove an analogous version of Theorem 3 for Fréchet operator spaces. In order to do that we will slightly change the notation. So far we have worked with a sequence $(M_n(X))_n$ of spaces where the n-th space denoted $n \times n$ matrices with entries in X. Now we prefer to have one space of infinite matrices. This will enable us to provide an operator space with a suitable notation of weak topologies and polars. Suppose that X is an operator space so that we have a sequence $(M_n(X), \|\cdot\|_n)$ of Banach spaces with $(\|\cdot\|_n)_n$ satisfying Ruan's axioms (see [Effros and Ruan 2000, page 20]). Let us denote by $I(X)$ the linear space of infinite matrices with entries in X and identify $M_n(X)$ with a subspace of $I(X)$ of the form

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : A \in M_n(X) \right\}.$$

This subspace can be naturally endowed with a norm that makes it isometric to $M_n(X)$. Therefore we will still denote it by $M_n(X)$. The above identification allows us to consider $M_n(X)$ isometrically embedded into $M_{n+1}(X)$. Therefore $\bigcup_n M_n(X)$ has a structure of a normed space and its completion will be denoted by $K(X)$. The norm of $x \in K(X)$ is given by

$$\|x\| = \lim_n \|x^n\|,$$

where x^ns are the truncations of x to $M_n(X)$. Following [Effros and Ruan 2000, Chapter 10] we will also use the notation

$$T(X) := \{w = \alpha x \beta : \alpha, \beta \in HS(\ell_2),\ x \in K(X)\}$$

and endow this space with a norm defined by

$$|||w||| := \inf \|\alpha\|_2 \|x\| \|\beta\|_2,$$

where the infimum runs over all such decompositions. Additionally we write

$$M(X) = \{x \in I(X) : \|x\| < +\infty\}.$$

As a simple example let us note that $K(\mathbb{C}) = \mathcal{K}(\ell_2)$, $T(\mathbb{C}) = \mathcal{N}(\ell_2)$, $M(\mathbb{C}) = \mathcal{B}(\ell_2)$. Moreover by [Effros and Ruan 2000, Theorem 10.1.4], we have isometrically

$$K(X)' = T(X'), \quad T(X)' = M(X'). \tag{4-1}$$

This is in fact a complete isometry but this will be beyond our interests here. The above notation may also be introduced for an arbitrary locally convex operator space. As usual we restrict ourselves to Fréchet operator spaces. If $X = \operatorname{proj}_k X_k$ is such a space then we obtain Fréchet spaces $K(X)$, $T(X)$, $M(X)$. These spaces may be viewed as

$$K(X) = \operatorname{proj}_k K(X_k), \quad T(X) = \operatorname{proj}_k T(X_k), \quad M(X) = \operatorname{proj}_k M(X_k).$$

Equivalently we can easily observe that

$$K(X) = \Big(\bigcup_n M_n(X)\Big)^{\sim}, \quad T(X) = \Big(\bigcup_n T_n(X)\Big)^{\sim},$$
$$M(X) = \{x \in I(X) : \|x\|_k < +\infty \quad \forall k \in \mathbb{N}\},$$

where $\tilde{}$ stands for the completion. We can also define a dual Fréchet operator space to be the linear space

$$K(X') := \bigcup_k K(X_k')$$

equipped with the topology inherited from the space $B(K(X), K(\ell_2))$, as well as the space

$$M(X') := \bigcup_k M(X_k')$$

equipped with the topology inherited from the space $B(K(X), B(\ell_2))$. For the sake of correctness let us point out that if X is a Fréchet operator space then $K(X')$ is no longer a Fréchet space and that the Ruan's axioms are now fulfilled by the dual norms

$$\|\phi\|_k^* = \sup\big\{\|\langle\langle \phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} : x \in K(X),\ \|x\|_k \leqslant 1\big\}.$$

In fact $K(X')$ has the structure of a (DF)-space where the fundamental sequence of bounded sets consists of absolutely matrix convex sets (we recall this definition below). Therefore we may introduce the notion of a (DF)-operator space but we will not go into details since this concept lies beyond our interests. With the above introduced topologies we also obtain for a Fréchet operator space complete isomorphisms (4-1).

Recall that by the unitary isometry $\ell_2(\ell_2)=\ell_2$ we may always think that $\langle\langle\phi, x\rangle\rangle$ is in $\mathcal{B}(\ell_2)$. Let us now define weak matrix topologies, absolutely matrix convex sets and matrix polars. We follow the notation of [Effros and Webster 1997; Effros and Winkler 1997; Effros and Ruan 2000, Chapter 5.5] with only slight modification coming from the fact that instead of the space $\bigcup_n M_n(X)$ we consider its completion. Suppose X is a Fréchet operator space. We define on $K(X)$ the *weak matrix topology* $m\sigma(K(X), K(X'))$ to be determined by the seminorms

$$\rho_{\xi,\phi,\eta}(x) := |\langle\langle\langle\phi, x\rangle\rangle\eta, \xi\rangle|,$$

where $\phi \in K(X')$, $\xi, \eta \in \ell_2$. Analogously we define on $K(X')$ the *weak* matrix topology* $m\sigma(K(X'), K(X))$ to be determined by the seminorms

$$\rho_{\xi,x,\eta}(\phi) := |\langle\langle\langle\phi, x\rangle\rangle\eta, \xi\rangle|,$$

where $x \in K(X)$, $\xi, \eta \in \ell_2$. It is easy to notice that

$$m\sigma(K(X), K(X')) = \sigma(K(X), K(X)'),$$
$$m\sigma(K(X'), K(X)) = \sigma(T(X)', T(X))|_{K(X')}.$$

A subset $S \subset K(X)$ is called *absolutely matrix convex* if the following two conditions hold:

(1) If $x, y \in S$ then $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in S$.

(2) If $x \in S$ and α, β are contractions on ℓ_2 then $\alpha x \beta \in S$.

Since the intersection of two absolutely matrix convex sets is again absolutely matrix convex we may define $\operatorname{amc}(S)$ to be the *absolutely matrix convex hull* of S, i.e., the smallest absolutely matrix convex set containing S. It can be precisely described (see [Effros and Webster 1997, Lemma 3.2]). If $A \subset K(X)$ then its *matrix polar* $A^{\circledcirc} \subset K(X')$ is defined as

$$A^{\circledcirc} := \{\phi \in K(X') : \|\langle\langle\phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} \leqslant 1 \quad \text{for all } x \in A\}.$$

Similarly for $A \subset K(X')$ we define

$$A^{\circledcirc} := \{x \in K(X) : \|\langle\langle\phi, x\rangle\rangle\|_{\mathcal{B}(\ell_2)} \leqslant 1 \quad \text{for all } \phi \in A\}.$$

As in the classical case we have the Bipolar theorem. The original proof is for $\bigcup_n M_n(X)$ while we work with its completion but the argument is analogous.

Theorem 15 [Effros and Webster 1997]. *Let X be a Fréchet operator space.*

(1) *If $A \subset K(X)$ then* $A^{\circledcirc\circledcirc} = \overline{\operatorname{amc}(A)}^{m\sigma(K(X), K(X'))}$.

(2) *If $A \subset K(X')$ then* $A^{\circledcirc\circledcirc} = \overline{\operatorname{amc}(A)}^{m\sigma(K(X'), K(X))}$.

We are now ready to reformulate our (oDN)-$(o\Omega)$ conditions in the spirit of Theorem 3. The proofs are analogous to the ones in [Vogt 1977, Lemma 1.4] and [Vogt and Wagner 1980, Lemma 2.1].

Theorem 16. *Let X be a Fréchet operator space and let $(U_k)_{k\in\mathbb{N}}$ be a basis of zero neighborhoods in $K(X)$.*

(1) *X satisfies the property (oDN) if and only if*

$$\exists\, p\in\mathbb{N}\ \forall q\in\mathbb{N}\ \exists\, r\in\mathbb{N}, C>0\ \forall s>0\colon\quad U_q^{\circledcirc}\subset sU_p^{\circledcirc}+\frac{C}{s}U_r^{\circledcirc},$$

(2) *X satisfies the property $(o\Omega)$ if and only if*

$$\forall p\in\mathbb{N}\ \exists\, q\in\mathbb{N}\ \forall r\in\mathbb{N}\ \exists\,\gamma>0, C>0\ \forall s>0\colon\quad U_q\subset sU_p+\frac{C}{s^\gamma}U_r.$$

We end this section by operator space versions of Lemma 6 and Proposition 11. Let us first note that if X and Y are operator spaces and $U\subset K(X)$, $V\subset K(Y)$ then

$$U\otimes V=\{x\otimes y : x\in U,\ y\in V\}.$$

Recalling the definitions of the operator projective $\hat{\otimes}_{op}$ and injective $\check{\otimes}_{op}$ tensor products and denoting by B_E the unit ball of E we can observe that

$$B_{K(X\hat{\otimes}_{op}Y)}=\overline{\operatorname{amc}(B_{K(X)}\otimes B_{K(Y)})},\quad B_{K(X\check{\otimes}_{op}Y)}=\big(B_{K(X)}^{\circledcirc}\otimes B_{K(Y)}^{\circledcirc}\big)^{\circledcirc}.$$

Therefore repeating the proof of Lemma 6 we obtain the following result.

Theorem 17. *Let X be a Fréchet operator space and E an operator space. If X has the property (oDN) or $(o\Omega)$ then their operator projective tensor product as well as the operator injective one satisfy the same condition too.*

Theorem 18. *Let X be a Fréchet operator space.*

(1) *$X\in(oDN)$ if and only if $X''\in(oDN)$.*

(2) *$X\in(o\Omega)$ if and only if $X''\in(o\Omega)$.*

Proof. (1) Observe that X satisfies (oDN) if and only if $K(X)$ satisfies (DN). By Proposition 11(1) $K(X)''=M(X'')\in(DN)$ and $K(X'')$ is a topological subspace of $M(X'')$, therefore it satisfies the condition (DN) and so $X''\in(oDN)$. Conversely $X''\in(oDN)$ leads to $K(X'')\in(DN)$ and by [Effros and Webster 1997, Corollary 8.2, Proposition 9.1] $K(X)$ is its topological subspace which gives $X\in(oDN)$.

(2) Observe that by [Effros and Ruan 2000, Lemma 4.1.1] $X\in(o\Omega)$ if and only if $T_n(X)\in(\Omega)$ with the constants $C_{p,q,r}(n)$ uniformly bounded with respect to n. By Proposition 11(2) this is equivalent to $T_n(X)''=T_n(X'')\in(\Omega)$ which is then equivalent to $X''\in(o\Omega)$. □

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