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GEOGRAPHY OF SIMPLY CONNECTED NONSPIN SYMPLECTIC 4-MANIFOLDS WITH POSITIVE SIGNATURE

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We construct new families of closed simply connected nonspin irreducible symplectic 4-manifolds with positive signature that are interesting with respect to the geography problem.

1. Introduction

Given a closed smooth 4-manifold M, let $e(M)$ and $\sigma(M)$ denote the Euler characteristic and the signature of M, respectively. We define

$$\chi_h(M) = \frac{e(M)+\sigma(M)}{4} \quad \text{and} \quad c_1^2(M) = 2e(M)+3\sigma(M).$$

Note that $e(M)$ and $\sigma(M)$ are in turn completely determined by $\chi_h(M)$ and $c_1^2(M)$, that is,

$$e(M) = 12\chi_h(M) - c_1^2(M) \quad \text{and} \quad \sigma(M) = c_1^2(M) - 8\chi_h(M).$$

When M is a complex surface, $\chi_h(M)$ is the holomorphic Euler characteristic of M while $c_1^2(M)$ is the square of the first Chern class of M. The classical "geography problem" in algebraic geometry, originally posed by Persson [1981], asks which ordered pairs of positive integers can be realized as the pair $(\chi_h(M), c_1^2(M))$ for some minimal complex surface M of general type. The related "botany problem", which is a lot more difficult, asks for the classification of all minimal complex surfaces with a given pair of invariants (χ_h, c_1^2).

The symplectic geography problem, first posed in [McCarthy and Wolfson 1994], asks which ordered pairs of integers can be realized as $(\chi_h(M), c_1^2(M))$ for some minimal symplectic 4-manifold M. There has been steady progress on the symplectic geography problem in recent years and the problem has been completely solved for simply connected minimal symplectic 4-manifolds with negative signature

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(cf. [Akhmedov et al. 2010a; Akhmedov and Park 2010a; Park and Szabó 2000]). The symplectic botany problem, that is, the classification problem for minimal symplectic 4-manifolds with a given pair of invariants (χ_h, c_1^2), seems to be an intractable problem at the moment. However, we now know that most ordered pairs are realized by infinitely many pairwise nondiffeomorphic simply connected minimal symplectic 4-manifolds; see [Gompf and Stipsicz 1999].

In this paper, we will focus our attention on the symplectic geography problem for simply connected minimal symplectic 4-manifolds with *nonnegative* signature. Unlike the negative signature case, the existing literature [Akhmedov and Park 2008; 2010b; Akhmedov et al. 2010b; Li and Stipsicz 2002; Niepel 2005; Park 2002; 2003; Stipsicz 1998; 1999] is far from capturing all possible (χ_h, c_1^2) coordinates, even if we allow nontrivial fundamental groups. The main goal of this paper is to summarize the current state of our knowledge when the simply connected symplectic 4-manifolds are required to be nonspin, or equivalently, are required to have odd intersection form. By Freedman's classification theorem [1982] for simply connected topological 4-manifolds, our problem is then equivalent to finding a minimal symplectic 4-manifold M with signature σ that is homeomorphic to $k\mathbb{CP}^2 \# (k-\sigma)\overline{\mathbb{CP}}^2$, where k is any odd positive integer and σ is any integer satisfying $0 \le \sigma \le k$. Here, $\mathbb{CP}^2$ is the complex projective plane, $\overline{\mathbb{CP}}^2$ is the underlying smooth 4-manifold $\mathbb{CP}^2$ equipped with the opposite orientation, and $k\mathbb{CP}^2 \# (k-\sigma)\overline{\mathbb{CP}}^2$ is the connected sum of k copies of $\mathbb{CP}^2$ and $k-\sigma$ copies of $\overline{\mathbb{CP}}^2$. Note that a simply connected symplectic 4-manifold M has odd $b_2^+(M)$, and hence our integer k must be odd.

A closed 4-manifold with signature σ corresponds to a point (χ_h, c_1^2) on the line $c_1^2 = 8\chi_h + \sigma$. For technical reasons, it will be convenient to fix the signature and deal with each of these lines separately. It is now well-known (see [Akhmedov and Park 2008; Park 2003]) that for each signature $\sigma \ge 0$, there exists a constant $\lambda(\sigma)$ depending only on σ such that any point (χ_h, c_1^2) on the line $c_1^2 = 8\chi_h + \sigma$ satisfying $\chi_h \ge \lambda(\sigma)$ is realized by at least one simply connected nonspin minimal symplectic 4-manifold and infinitely many simply connected nonspin irreducible nonsymplectic 4-manifolds (Definition 13 in Section 6). In other words, $k\mathbb{CP}^2 \# (k-\sigma)\overline{\mathbb{CP}}^2$ is homeomorphic to at least one minimal symplectic 4-manifold and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, provided that k is odd and $k \ge 2\lambda(\sigma)-1$ for some constant $\lambda(\sigma)$ that depends only on the signature σ.

The main result of this paper is the explicit formulation of the smallest values of $\lambda(\sigma)$ that are currently known to the authors. In [Akhmedov and Park 2008], small $\lambda(\sigma)$ values are given when $0 \le \sigma \le 4$, and these values are listed in Table 1. In this paper, we will concentrate on the cases when $\sigma \ge 5$ (see Table 2 in Section 6). For example, when $0 \le \sigma \le 100$, we realize more than 20,000 new (χ_h, c_1^2) points that were not covered by the results in [Akhmedov and Park 2008; Park 2003].

σ	0	1	2	3	4
$\lambda(\sigma) \leq$	25	25	24	27	26

Table 1. Results from [Akhmedov and Park 2008].

If a 4-manifold M is simply connected, then $2\chi_h(M) - 1 = b_2^+(M) \geq \sigma(M)$. Thus we obtain an *a priori* lower bound $\chi_h \geq \lceil(\sigma + 1)/2\rceil$, where

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$$

is the ceiling function. It is tempting to conjecture that our *a posteriori* lower bound for χ_h can eventually be improved down to $\lambda(\sigma) = \lceil(\sigma + 1)/2\rceil$, which will result in the complete solution of the geography problem for simply connected nonspin minimal symplectic 4-manifolds.

Our paper is organized as follows. In Section 2, we present a branched covering construction of Lefschetz fibrations with positive signature, which is a generalization of Stipsicz's constructions [1998; 1999]. In Section 3, we show how to glue together semifree cyclic group actions on closed 2-manifolds, and then we use these actions to construct new examples of Lefschetz fibrations with positive signature. In Section 4, we show how to obtain simply connected 4-manifolds from nonsimply connected Lefschetz fibrations by performing generalized fiber sums with certain 4-manifolds that were constructed in [Akhmedov and Park 2010a]. In Section 5, we implement the strategies from previous sections to construct new families of simply connected irreducible 4-manifolds with positive signature. In Section 6, we compute the lower bounds $\lambda(\sigma)$ for many small values of σ.

2. Branched covering construction

Let Σ_g be a closed 2-dimensional manifold of genus $g > 0$. Let $\zeta : \Sigma_g \to \Sigma_g$ be an orientation-preserving self-diffeomorphism of Σ_g with q fixed points $\{y_1, \ldots, y_q\}$. Assume that

$$\zeta^p = \underbrace{\zeta \circ \cdots \circ \zeta}_{p} = \mathrm{id},$$

for some positive integer $p \geq 2$, and that ζ generates a semifree $\mathbb{Z}/p$ action on Σ_g. If $\zeta_* : H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z})$ is the induced homomorphism on the first homology group, then we also assume that

$$\zeta_*^{p-1} + \zeta_*^{p-2} + \cdots + \zeta_* + \mathrm{id} = 0 \tag{1}$$

on $H_1(\Sigma_g; \mathbb{Z})$, which is equivalent to 1 not being an eigenvalue of ζ_*. See Examples 3 and 5 below for some concrete examples of ζ.

We will consider $\Sigma_g \times \Sigma_g$ as a symplectic 4-manifold equipped with a product symplectic form $\tilde{\omega} = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$, where ω is a symplectic volume form on Σ_g and $\mathrm{pr}_j : \Sigma_g \times \Sigma_g \to \Sigma_g$ $(j = 1, 2)$ is the projection map onto the j-th factor. For each $i = 1, \ldots, p$, let

$$\Gamma_i = \mathrm{graph}(\zeta^i) = \{(x, \zeta^i(x)) \mid x \in \Sigma_g\} \subset \Sigma_g \times \Sigma_g.$$

Note that Γ_p is equal to the diagonal $\{(x, x) \mid x \in \Sigma_g\}$. The graphs $\Gamma_1, \ldots, \Gamma_p$ are symplectic submanifolds of $\Sigma_g \times \Sigma_g$ with respect to $\tilde{\omega}$ (see Lemma 2.1 in [Akhmedov and Park 2008]), and the graphs intersect at q points

$$\{(y_j, y_j) \mid j = 1, \ldots, q\}.$$

If we symplectically blow up $\Sigma_g \times \Sigma_g$ at these q intersection points, then the proper transform B of the union $\Gamma_1 \cup \cdots \cup \Gamma_p$ consists of p disjoint genus g symplectic submanifolds of $(\Sigma_g \times \Sigma_g) \# q\overline{\mathbb{CP}}^2$.

Let $\{\gamma_k \mid k = 1, \ldots, 2g\}$ be a basis for $H_1(\Sigma_g; \mathbb{Z})$ and let $\{\gamma^\ell \mid \ell = 1, \ldots, 2g\}$ be the dual basis under the intersection product so that $\gamma_k \cdot \gamma^\ell = \delta_k^\ell$. If we introduce the notation

$$[\Delta] = [\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}'\} \times \Sigma_g],$$

then the homology class of Γ_i is given by

$$[\Gamma_i] = [\Delta] - \sum_{k=1}^{2g} \gamma^k \times \zeta_*^i(\gamma_k).$$

Using (1), we can express the homology class of B as

$$[B] = p\left([\Delta] - \sum_{j=1}^{q} [E_j]\right),$$

where $E_1, \ldots, E_q$ are the exceptional spheres of the blowups. We also note that

$$c_1((\Sigma_g \times \Sigma_g) \# q\overline{\mathbb{CP}}^2) = \mathrm{PD}\left((2 - 2g)[\Delta] - \sum_{j=1}^{q} [E_j]\right),$$

where PD denotes the Poincaré duality isomorphism.

Since $[B]$ is divisible by p, we may take the cyclic p-fold branched cover of $(\Sigma_g \times \Sigma_g) \# q\overline{\mathbb{CP}}^2$ that is branched along B. We will denote this branched covering by $\beta : X^{\zeta}_{g,p,q} \to (\Sigma_g \times \Sigma_g) \# q\overline{\mathbb{CP}}^2$. The total space $X^{\zeta}_{g,p,q}$ inherits a symplectic structure from $(\Sigma_g \times \Sigma_g)\,\#\,q\overline{\mathbb{CP}^2}$, and we have

$$\begin{aligned} c_1(X^{\zeta}_{g,p,q}) &= \beta^*\Big(c_1((\Sigma_g \times \Sigma_g)\,\#\,q\overline{\mathbb{CP}^2}) - \frac{p-1}{p}\mathrm{PD}[B]\Big) \\ &= \beta^*\mathrm{PD}\Big((3-2g-p)[\Delta] + (p-2)\sum_{j=1}^{q}[E_j]\Big). \end{aligned}$$

The characteristic numbers of $X^{\zeta}_{g,p,q}$ can be computed as follows.

$$\begin{aligned} e(X^{\zeta}_{g,p,q}) &= pe((\Sigma_g \times \Sigma_g)\,\#\,q\overline{\mathbb{CP}^2}) - p(p-1)e(\Sigma_g) \\ &= p((2-2g)^2+q) - p(p-1)(2-2g) \\ &= p(4g^2+2gp-10g-2p+q+6), \\ c_1^2(X^{\zeta}_{g,p,q}) &= p\Big((3-2g-p)[\Delta] + (p-2)\sum_{j=1}^{q}[E_j]\Big)^2 \\ &= p(2(3-2g-p)^2 - q(p-2)^2) \\ &= p(-p^2q+8g^2+2p^2+8gp+4pq-24g-12p-4q+18), \\ \sigma(X^{\zeta}_{g,p,q}) &= \tfrac{1}{3}\big(c_1^2(X^{\zeta}_{g,p,q}) - 2e(X^{\zeta}_{g,p,q})\big) \\ &= \tfrac{1}{3}p(-p^2q+2p^2+4gp+4pq-4g-8p-6q+6), \\ \chi_h(X^{\zeta}_{g,p,q}) &= \tfrac{1}{4}\big(e(X^{\zeta}_{g,p,q}) + \sigma(X^{\zeta}_{g,p,q})\big) \\ &= \tfrac{1}{12}p(-p^2q+12g^2+2p^2+10gp+4pq-34g-14p-3q+24). \end{aligned}$$

Let $\epsilon : (\Sigma_g \times \Sigma_g)\#q\overline{\mathbb{CP}^2} \to \Sigma_g \times \Sigma_g$ be the blowdown map. Then the composition of maps

$$X^{\zeta}_{g,p,q} \xrightarrow{\beta} (\Sigma_g \times \Sigma_g)\,\#\,q\overline{\mathbb{CP}^2} \xrightarrow{\epsilon} \Sigma_g \times \Sigma_g \xrightarrow{\mathrm{pr}_1} \Sigma_g \tag{2}$$

gives a fibration of $X^{\zeta}_{g,p,q}$ over Σ_g. A regular fiber of this fibration is a cyclic p-fold branched cover of Σ_g that is branched over p points. Thus a regular fiber is a closed surface of genus equal to

$$\tfrac{1}{2}(p^2+2gp-3p+2). \tag{3}$$

The proper transform of each graph Γ_i $(i = 1, \ldots, p)$ gives rise to a section of (2) whose image is a genus g surface S_i in $X^{\zeta}_{g,p,q}$ with self-intersection equal to

$$\begin{aligned} [S_i]^2 &= \big\langle c_1(X^{\zeta}_{g,p,q}), [S_i]\big\rangle - e(\Sigma_g) \\ &= 2g-2+\frac{1}{p}\Big((3-2g-p)[\Delta] + (p-2)\sum_{j=1}^{q}[E_j]\Big)\cdot[B] \\ &= pq-2g-2p-2q+4. \end{aligned}$$

Lemma 1. *Let* $f : X^{\zeta}_{g,p,q} \to \Sigma_g$ *denote the composition of maps in* (2). *Then* f *is a relatively minimal Lefschetz fibration with* pq *critical points. Moreover, each critical point of* f *corresponds to a nonseparating vanishing cycle.*

Proof. Clearly the only singular fibers of f are $\{f^{-1}(y_j) \mid j = 1, \dots, q\}$. We will prove that each $f^{-1}(y_j)$ contains exactly p Lefschetz critical points. To describe each $f^{-1}(y_j)$ explicitly, we will view $X^{\zeta}_{g,p,q}$ as the minimal desingularization of another branched cover that we will define below.

Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_p$. Since $[\Gamma] = p[\Delta] \in H_2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ is divisible by p, we may take the cyclic p-fold branched cover of $\Sigma_g \times \Sigma_g$ that is branched along Γ. We will denote this branched covering by $\widehat{\beta} : \widehat{X}^{\zeta}_{g,p,q} \to \Sigma_g \times \Sigma_g$. The total space $\widehat{X}^{\zeta}_{g,p,q}$ has q singular points, $\{\widehat{\beta}^{-1}(y_j, y_j) \mid j = 1, \dots, q\}$, each of which can be locally modeled by

$$\{(x, y, z) \in \mathbb{C}^3 \mid z^p = x^p + y^p\}. \tag{4}$$

In these local coordinates, the singular point $\widehat{\beta}^{-1}(y_j, y_j)$ corresponds to $(0, 0, 0)$, and a neighborhood of the singular point corresponds to the cyclic p-fold cover of the (x, y)-plane that is branched over p complex lines that intersect transversely at $(0, 0)$.

Next let $\widehat{f} : \widehat{X}^{\zeta}_{g,p,q} \to \Sigma_g$ denote the singular fibration given by the composition

$$\widehat{X}^{\zeta}_{g,p,q} \xrightarrow{\widehat{\beta}} \Sigma_g \times \Sigma_g \xrightarrow{\mathrm{pr}_1} \Sigma_g.$$

A regular fiber of $\widehat{f}$ is again a closed surface of genus equal to (3). There are exactly q singular fibers $\{\widehat{f}^{-1}(y_j) \mid j = 1, \dots, q\}$. For each $j = 1, \dots, q$, note that $\widehat{f}^{-1}(y_j) \setminus \{\widehat{\beta}^{-1}(y_j, y_j)\}$ is a smooth and connected surface since it is the unbranched cyclic p-fold cover of the once punctured surface $(\{y_j\} \times \Sigma_g) \setminus \{(y_j, y_j)\}$ coming from a surjective homomorphism

$$\pi_1((\{y_j\} \times \Sigma_g) \setminus \{(y_j, y_j)\}) \cong F_{2g} \longrightarrow \mathbb{Z}/p \subset S_p, \tag{5}$$

where F_{2g} is the free group with $2g$ generators and S_p is the symmetric group on p symbols. Since $\mathbb{Z}/p$ is abelian, (5) can be factored as the composition

$$\pi_1((\{y_j\} \times \Sigma_g) \setminus \{(y_j, y_j)\}) \longrightarrow \pi_1(\Sigma_g) \longrightarrow \mathbb{Z}/p.$$

Thus the cover $\widehat{f}^{-1}(y_j) \setminus \{\widehat{\beta}^{-1}(y_j, y_j)\} \to (\{y_j\} \times \Sigma_g) \setminus \{(y_j, y_j)\}$ can be viewed as a restriction of the unbranched cyclic p-fold cover of the closed surface Σ_g. In other words, $\widehat{f}^{-1}(y_j) \setminus \{\widehat{\beta}^{-1}(y_j, y_j)\}$ can be embedded into the unbranched cyclic p-fold cover of Σ_g. This implies that $\widehat{f}^{-1}(y_j) \setminus \{\widehat{\beta}^{-1}(y_j, y_j)\}$ is diffeomorphic to a surface of genus $gp - p + 1$ having p punctures, and $\widehat{f}^{-1}(y_j)$ is a connected surface that is smooth away from the point $\widehat{\beta}^{-1}(y_j, y_j)$, which is a multiple point of order p.

Now recall from [Gompf and Stipsicz 1999; Némethi 1999] that $X^{\zeta}_{g,p,q}$ is the minimal desingularization of $\widehat{X}^{\zeta}_{g,p,q}$. The standard algorithm for resolution of singularities (see [Némethi 1999, Example 1.20(h)]) replaces each singular point $\widehat{\beta}^{-1}(y_j, y_j)$ of $\widehat{X}^{\zeta}_{g,p,q}$ having local model (4) with a closed surface of genus $\frac{1}{2}(p^2-3p+2)$ and self-intersection $-p$. This surface is just $\beta^{-1}(E_j)$, which is a cyclic p-fold branched cover of the exceptional sphere E_j branched over p points. It follows that each singular fiber $f^{-1}(y_j)$ is the union of two closed surfaces that intersect each other transversely at p distinct points. One of the surfaces is $\beta^{-1}(E_j)$, and the other is a genus $gp-p+1$ surface of self-intersection $-p$, which is the smooth completion of $\widehat{f}^{-1}(y_j)\setminus\{\widehat{\beta}^{-1}(y_j, y_j)\}$. The p transverse intersection points between the two surfaces are exactly the p Lefschetz critical points of f that get mapped to y_j. Finally, comparing the sum of genera with (3), we observe that each union of the two surfaces is obtained by replacing the annular neighborhoods of p nonseparating circles in a regular fiber with p pairs of transversely intersecting disks. This implies that all the vanishing cycles are nonseparating. □

Remark 2. We can verify the number of critical points of f by computing the difference

$$e(X^{\zeta}_{g,p,q}) - e(\text{regular fiber})\cdot e(\text{base}) = pq.$$

We can split the singular fibers of f so that each new singular fiber contains only one critical point (cf. [Harris and Morrison 1998; Takamura 2004]) but we do not need to do so for our applications below.

Given a positive integer u, let $\eta_u : \Sigma_k \to \Sigma_g$ be a u-fold unbranched covering of Σ_g, where $k = u(g-1)+1$. We pull back the branched covering

$$X^{\zeta}_{g,p,q} \xrightarrow{\beta} (\Sigma_g\times\Sigma_g)\,\#\,q\overline{\mathbb{CP}}^2 \xrightarrow{\epsilon} \Sigma_g\times\Sigma_g$$

by the product map $\eta_{u_1}\times\eta_{u_2} : \Sigma_{k_1}\times\Sigma_{k_2}\to\Sigma_g\times\Sigma_g$, where u_i is a positive integer and $k_i = u_i(g-1)+1$ for each $i = 1, 2$. The total space of this pullback is a new symplectic 4-manifold $X^{\zeta}_{g,p,q}(u_1, u_2)$, which is a p-fold branched cover of $\Sigma_{k_1}\times\Sigma_{k_2}$ and a u_1u_2-fold unbranched cover of $X^{\zeta}_{g,p,q}$. The composition

$$f_{u_1,u_2} : X^{\zeta}_{g,p,q}(u_1, u_2) \longrightarrow \Sigma_{k_1}\times\Sigma_{k_2} \xrightarrow{\mathrm{pr}_1} \Sigma_{k_1}$$

gives a new relatively minimal Lefschetz fibration, where $X^{\zeta}_{g,p,q}(1, 1) = X^{\zeta}_{g,p,q}$ and $f_{1,1} = f$. A regular fiber of f_{u_1,u_2} is a u_2-fold unbranched cover of the fiber of f (or equivalently a p-fold branched cover of Σ_{k_2} branched along u_2p points) and hence has genus equal to

$$1+\frac{u_2}{2}(p^2+2gp-3p).$$

A section of f gives rise to a section of f_{u_1,u_2} whose image is a genus k_1 surface of self-intersection equal to

$$u_1(pq-2g-2p-2q+4).$$

Since $X^{\zeta}_{g,p,q}(u_1,u_2)$ is a u_1u_2-fold unbranched cover of $X^{\zeta}_{g,p,q}$, we have

$$e(X^{\zeta}_{g,p,q}(u_1,u_2)) = u_1u_2\cdot e(X^{\zeta}_{g,p,q}), \qquad \sigma(X^{\zeta}_{g,p,q}(u_1,u_2)) = u_1u_2\cdot\sigma(X^{\zeta}_{g,p,q}),$$
$$\chi_h(X^{\zeta}_{g,p,q}(u_1,u_2)) = u_1u_2\cdot\chi_h(X^{\zeta}_{g,p,q}), \quad c_1^2(X^{\zeta}_{g,p,q}(u_1,u_2)) = u_1u_2\cdot c_1^2(X^{\zeta}_{g,p,q}).$$

Example 3. Recall from Section 2 of [Akhmedov and Park 2008] that there exists a semifree $\mathbb{Z}/(g+1)$ action on Σ_g with 4 fixed points satisfying (1). Applying the above machinery, we obtain a family of symplectic 4-manifolds $X^g_{u_1,u_2} = X^{\zeta}_{g,g+1,4}(u_1,u_2)$, where g, u_1 and u_2 are positive integers, satisfying

$$\begin{aligned} e(X^g_{u_1,u_2}) &= 2u_1u_2(g+1)(3g^2-5g+4),\\ \sigma(X^g_{u_1,u_2}) &= \tfrac{2}{3}u_1u_2(g+1)(g^2+2g-6),\\ \chi_h(X^g_{u_1,u_2}) &= \tfrac{1}{6}u_1u_2(g+1)(10g^2-13g+6),\\ c_1^2(X^g_{u_1,u_2}) &= 2u_1u_2(g+1)(7g^2-8g+2). \end{aligned}$$

For each triple of positive integers g, u_1, u_2, there exists a relatively minimal Lefschetz fibration $f_{u_1,u_2} : X^g_{u_1,u_2} \to \Sigma_{k_1}$ such that the genus of a regular fiber is equal to $1+\frac{1}{2}u_2(g+1)(3g-2)$ and there is a section whose image is a surface of genus $k_1 = u_1(g-1)+1$ and self-intersection $-2u_1$.

Remark 4. The 4-manifolds X_g, $X_g(n)$ and $\tilde{X}_g(n^2)$ in [Akhmedov and Park 2008] are equal to $X^g_{1,1}$, $X^g_{n,1}$ and $X^g_{n,n}$, respectively.

3. Gluing self-diffeomorphisms of surfaces

In light of the machinery in Section 2, it will be desirable to find lots of semifree $\mathbb{Z}/p$ actions on closed surfaces. One way to produce such actions is to glue together semifree $\mathbb{Z}/p$ actions on surfaces of low genera as we explain below.

Let $v \geq 2$ be an integer. For each $i = 1, \ldots, v$, let $\alpha_i : \Sigma_{g_i} \to \Sigma_{g_i}$ be an orientation-preserving self-diffeomorphism of a closed surface of genus g_i with q_i fixed points $\{y_{i,1}, \ldots, y_{i,q_i}\}$. Assume that each α_i generates a semifree $\mathbb{Z}/p$ action on Σ_{g_i}. For each $j = 1, \ldots, q_i$, let $\rho_{i,j}$ be the rotational number of α_i at the fixed point $y_{i,j}$ so that α_i induces rotation by angle $2\pi\rho_{i,j}/p$ in the tangent space at $y_{i,j}$. The rotational numbers are well-defined mod p and are relatively prime to p. They satisfy (see [Nielsen 1937])

$$\sum_{j=1}^{q_i} \frac{1}{\rho_{i,j}} \equiv 0 \pmod{p},$$

where $1/\rho_{i,j}$ denotes the multiplicative inverse of $\rho_{i,j}$ in $(\mathbb{Z}/p)^\times$. We can reverse the signs of $\rho_{i,1}, \dots, \rho_{i,q_i}$ simultaneously by reversing the orientation of Σ_{g_i}.

Now choose a single fixed point of α_i for $i = 1, v$, and choose two fixed points of α_i for $i = 2, \dots, v-1$. Without loss of generality, we may choose $y_{1,2}$, $y_{v,1}$ and $y_{i,1}$, $y_{i,2}$ for $i = 2, \dots, v-1$. We remove small $\mathbb{Z}/p$-equivariant neighborhoods of these chosen fixed points and then glue the boundary circle at $y_{i,2}$ to the boundary circle at $y_{i+1,1}$ for $i = 1, \dots, v-1$. Such gluing of one-holed and two-holed surfaces results in a closed surface of genus $g = \sum_{i=1}^{v} g_i$. If $\rho_{i,2} = -\rho_{i+1,1}$ for all $i = 1, \dots, v-1$, that is, the rotational numbers are negatives of each other at the gluing points, then the restrictions of α_i's to the punctured surfaces can also be glued together to form an orientation-preserving self-diffeomorphism $\zeta : \Sigma_g \to \Sigma_g$ with q fixed points, where

$$q = -2(v-1) + \sum_{i=1}^{v} q_i .$$

We will say that ζ is an *equivariant sum* of $\alpha_1, \dots, \alpha_v$, and write $\zeta = \alpha_1 \# \cdots \# \alpha_v$. In case when $\alpha_1 = \cdots = \alpha_v$, we will write $\zeta = v\alpha_1$ for short.

Example 5. For each odd integer $p \geq 3$, there exists a semifree $\mathbb{Z}/p$ action on $\Sigma_{(p-1)/2}$ as follows. Consider $\Sigma_{(p-1)/2}$ as the quotient of a regular $2p$-gon by identifying the opposite sides. The rotation of the $2p$-gon by angle $2\pi/p$ gives an orientation-preserving self-diffeomorphism $\tau_p : \Sigma_{(p-1)/2} \to \Sigma_{(p-1)/2}$ with 3 fixed points. The fixed points of τ_p are the center of the $2p$-gon and the 2 points coming from the vertices. The center of the $2p$-gon has rotational number 1, and the other 2 fixed points both have rotational number -2.

We can find a basis of $H_1(\Sigma_{(p-1)/2}; \mathbb{Z})$ such that the induced homomorphism $(\tau_p)_* : H_1(\Sigma_{(p-1)/2}; \mathbb{Z}) \to H_1(\Sigma_{(p-1)/2}; \mathbb{Z})$ is represented by the $(p-1) \times (p-1)$ matrix

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & -1 \\ \hline & & & -1 \\ & I_{p-2} & & \vdots \\ & & & -1 \end{array}\right], \tag{6}$$

where I_{p-2} is the identity $(p-2) \times (p-2)$ matrix. It is easy to check that this matrix satisfies (1).

For each positive integer v, let $\zeta = v\tau_p$ be the equivariant sum of v copies of τ_p. (We glue along fixed points with rotational number -2, and we alternate the orientations of the punctured $\Sigma_{(p-1)/2}$'s so that the rotational numbers are $+2$ and -2 at each gluing.) Then $\zeta : \Sigma_{v(p-1)/2} \to \Sigma_{v(p-1)/2}$ generates a semifree $\mathbb{Z}/p$ action on $\Sigma_{v(p-1)/2}$ with $v+2$ fixed points. The induced homomorphism

$\zeta_* : H_1(\Sigma_{v(p-1)/2}; \mathbb{Z}) \to H_1(\Sigma_{v(p-1)/2}; \mathbb{Z})$ satisfies (1) since it can be represented by a block diagonal matrix each of whose blocks is conjugate to (6).

From the branched covering construction in Section 2, we obtain a family of symplectic 4-manifolds $W^{p,v}_{u_1,u_2} = X^{v\tau_p}_{v(p-1)/2,\,p,\,v+2}(u_1,u_2)$, where $p \geq 3$ is an odd integer and v, u_1, u_2 are positive integers, satisfying

$$\begin{aligned}
e(W^{p,v}_{u_1,u_2}) &= pu_1u_2[(v^2+v)p^2 - 2(v^2+3v+1)p + v^2+6v+8],\\
\sigma(W^{p,v}_{u_1,u_2}) &= \tfrac{1}{3}pu_1u_2(vp^2-4v-6),\\
\chi_h(W^{p,v}_{u_1,u_2}) &= \tfrac{1}{12}pu_1u_2[(3v^2+4v)p^2 - 6(v^2+3v+1)p + 3v^2+14v+18],\\
c_1^2(W^{p,v}_{u_1,u_2}) &= pu_1u_2[(2v^2+3v)p^2 - 4(v^2+3v+1)p + 2v^2+8v+10].
\end{aligned}$$

Moreover, for each quadruple of positive integers p, v, u_1, u_2 with odd $p \geq 3$, we have a relatively minimal Lefschetz fibration $f_{u_1,u_2} : W^{p,v}_{u_1,u_2} \to \Sigma_{k_1}$ such that the genus of a regular fiber is equal to $1 + \frac{1}{2}pu_2[(v+1)p - v - 3]$ and there is a section whose image is a surface of genus $k_1 = 1 + u_1[-1 + v(p-1)/2]$ and self-intersection $-u_1v$.

Note that $c_1^2(W^{p,v}_{u_1,u_2}) \leq 9\chi_h(W^{p,v}_{u_1,u_2})$, with equality if and only if $p=5$ and $v=1$. If we view the quotient $c_1^2(W^{p,v}_{u_1,u_2})/\chi_h(W^{p,v}_{u_1,u_2})$ as a function of p and v, then its gradient vector is

$$\begin{bmatrix} -\dfrac{24\big((v^3+3v^2+v)p^2 - (5v^3+16v^2+14v)p + 4v^3+18v^2+22v+6\big)}{\big((3v^2+4v)p^2 - 6(v^2+3v+1)p + 3v^2+14v+18\big)^2} \\ -\dfrac{12\big((p^2-4)(p-1)^2v^2 - 12(p-1)^2v + 2p^3 - 14p^2 + 28p - 4\big)}{\big((3v^2+4v)p^2 - 6(v^2+3v+1)p + 3v^2+14v+18\big)^2} \end{bmatrix}$$

When $p \geq 7$ and $v \geq 1$, both components of this gradient vector are negative and hence $c_1^2(W^{p,v}_{u_1,u_2})/\chi_h(W^{p,v}_{u_1,u_2})$ is decreasing as p and v increase. We observe that $\lim_{v\to\infty} c_1^2(W^{p,v}_{u_1,u_2})/\chi_h(W^{p,v}_{u_1,u_2}) = 8$, and

$$\lim_{p\to\infty} \frac{c_1^2(W^{p,v}_{u_1,u_2})}{\chi_h(W^{p,v}_{u_1,u_2})} = \frac{12(2v+3)}{3v+4} \leq \frac{60}{7},$$

where the rational function $12(2v+3)/(3v+4)$ is decreasing for $v \geq 1$. Therefore most $W^{p,v}_{u_1,u_2}$'s lie well below the Bogomolov–Miyaoka–Yau (BMY) line, $c_1^2 = 9\chi_h$.

Remark 6. According to Section 4.5 of [Luo 2000], there is a unique $\mathbb{Z}/3$ action on Σ_g with $g+2$ fixed points. It follows that $W^{3,2}_{u_1,u_2}$ is exactly equal to $X^2_{u_1,u_2}$ in Example 3. More generally, for each odd integer $p \geq 5$, we conjecture that $W^{p,2}_{u_1,u_2}$ is diffeomorphic to $X^{p-1}_{u_1,u_2}$ in Example 3. We also conjecture that the 4-manifolds Z_g, $Z_g(n)$ and $\tilde{Z}_g(n^2)$ in Section 3 of [Akhmedov and Park 2008] are diffeomorphic to $W^{2g+1,1}_{1,1}$, $W^{2g+1,1}_{n,1}$ and $W^{2g+1,1}_{n,n}$, respectively. In particular, we conjecture that

$W^{5,1}_{1,1}$, $W^{5,1}_{n,1}$ and $W^{5,1}_{n,n}$, lying on the BMY line $c_1^2 = 9\chi_h$, are diffeomorphic to complex surfaces $H = H(1)$, $H(n)$ and $H(n^2)$ in [Chen 1991; Stipsicz 1998; 1999], respectively.

4. Generalized fiber sums

Let Σ_b denote a closed Riemann surface of genus $b > 0$. Suppose $f : X \to \Sigma_b$ is a Lefschetz fibration with generic fiber F diffeomorphic to a closed Riemann surface Σ_a with genus $a > 0$. Assume that f is a relatively minimal Lefschetz fibration (i.e., no fiber contains a sphere of self-intersection -1) so that X is a minimal symplectic 4-manifold (Theorem 1.4 of [Stipsicz 2000]). Also assume that f has a section whose image S in X has self-intersection d. From Theorem 10.2.18 in [Gompf and Stipsicz 1999], X can be equipped with a symplectic structure such that both F and S are symplectic submanifolds. From Proposition 8.1.9 in [Gompf and Stipsicz 1999], we have an exact sequence

$$\pi_1(F) \longrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1(\Sigma_b) \longrightarrow 1. \tag{7}$$

Let $t > 0$ be an integer. By symplectically resolving the intersection points, we can find a symplectic genus $ta + b$ surface $\Sigma \subset X$ representing the homology class $t[F] + [S] \in H_2(X; \mathbb{Z})$ with self-intersection $2t + d$. By taking t large enough, we can assume that $2t + d \geq 0$. Let $\widetilde{X} = X \# (2t + d)\overline{\mathbb{CP}}^2$, where each of the $2t + d$ symplectic blowups take place at points on $\Sigma \subset X$. The proper transform $\widetilde{\Sigma} \subset \widetilde{X}$ is a symplectic submanifold with genus $ta + b$ and self-intersection 0. Note that we have

$$e(\widetilde{X}) = e(X) + 2t + d,$$
$$\sigma(\widetilde{X}) = \sigma(X) - 2t - d.$$

Lemma 7. *Let $\tilde{i} : \widetilde{\Sigma}^{\|} \hookrightarrow \widetilde{X} \setminus \nu\widetilde{\Sigma}$ be the inclusion map of a parallel copy of $\widetilde{\Sigma}$ into the complement of a tubular neighborhood $\nu\widetilde{\Sigma}$ in $\widetilde{X} = X \# (2t + d)\overline{\mathbb{CP}}^2$. Then we have*

$$\frac{\pi_1(\widetilde{X} \setminus \nu\widetilde{\Sigma})}{\left\langle \tilde{i}_*(\pi_1(\widetilde{\Sigma}^{\|}))\right\rangle} = 1, \tag{8}$$

where $\langle \tilde{i}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle$ is the normal subgroup of $\pi_1(\widetilde{X} \setminus \nu\widetilde{\Sigma})$ generated by the image $\tilde{i}_*(\pi_1(\widetilde{\Sigma}^{\|}))$.

Proof. Let $i : \Sigma^{\|} \hookrightarrow X \setminus \nu\Sigma$ be the inclusion map of a parallel copy of Σ. From exact sequence (7), we deduce that $\pi_1(X)/\langle i_*(\pi_1(\Sigma^{\|}))\rangle = 1$. Since the blowups do not effect the fundamental groups, we conclude that $\pi_1(\widetilde{X})/\langle \tilde{i}_*(\pi_1(\widetilde{\Sigma}^{\|}))\rangle = 1$. If

$2t+d>0$, then any meridian $\mu(\widetilde{\Sigma})$ of $\widetilde{\Sigma}$ in $\pi_1(\widetilde{X}\setminus\nu\widetilde{\Sigma})$ bounds a disk that comes from a punctured exceptional sphere. Hence $\pi_1(\widetilde{X}\setminus\nu\widetilde{\Sigma})=\pi_1(\widetilde{X})$ and (8) follows from our last conclusion.

If $2t+d=0$, then $\widetilde{X}=X$, $\widetilde{\Sigma}=\Sigma$, $\widetilde{\Sigma}^{\parallel}=\Sigma^{\parallel}$, and $\tilde{i}=i$. Any meridian $\mu(\Sigma)$ in $\pi_1(X\setminus\nu\Sigma)$ is conjugate to a meridian of S. Since $[F]\cdot[S]=1$, $\mu(\Sigma)$ is in the normal subgroup generated by the generators of $\pi_1(F)$, which in turn lies in $\langle i_*(\pi_1(\Sigma^{\parallel}))\rangle$. This implies that $\pi_1(X\setminus\nu\Sigma)/\langle i_*(\pi_1(\Sigma^{\parallel}))\rangle=\pi_1(X)/\langle i_*(\pi_1(\Sigma^{\parallel}))\rangle=1$. □

For each pair of integers $m\geq 1$ and $n\geq 2$, let $Y_n(m)$ denote the irreducible 4-manifold constructed in Section 2 of [Akhmedov and Park 2010a] that has the same cohomology ring as the connected sum $(2n-3)(S^2\times S^2)$. Recall that $Y_n(m)$ is obtained by performing $2n+4$ surgeries along Lagrangian tori in the product 4-manifold $\Sigma_2\times\Sigma_n$. Thus $Y_n(m)$ contains a pair of submanifolds $\Sigma_2=\Sigma_2\times\{\mathrm{pt}\}$ and $\Sigma_n=\{\mathrm{pt}'\}\times\Sigma_n$, both of self-intersection 0. When $m=1$, $Y_n(1)$ is a minimal symplectic 4-manifold. Moreover, Σ_2 and Σ_n are symplectic submanifolds of $Y_n(1)$. When $n\geq 3$, there exist $2n-4$ pairs of geometrically dual Lagrangian tori which, together with Σ_2 and Σ_n, form a basis for $H_2(Y_n(1);\mathbb{Z})\cong\mathbb{Z}^{4n-6}$.

Theorem 8. *Let $f:X\to\Sigma_b$ be a relatively minimal Lefschetz fibration as above having at least one nonseparating vanishing cycle. Suppose that $n=ta+b\geq 2$. For a suitable choice of the gluing diffeomorphism $\varphi:\partial(\nu\widetilde{\Sigma})\to\partial(\nu\Sigma_n)$, the generalized fiber sum*

$$P_n^m(X)=\widetilde{X}\,\#_\varphi Y_n(m)=(\widetilde{X}\setminus\nu\widetilde{\Sigma})\cup_\varphi(Y_n(m)\setminus\nu\Sigma_n) \tag{9}$$

along $\widetilde{\Sigma}$ and Σ_n is simply connected, and satisfies

$$\begin{aligned}
e(P_n^m(X))&=e(X)+d+(8a+2)t+8b-8,\\
\sigma(P_n^m(X))&=\sigma(X)-2t-d,\\
\chi_h(P_n^m(X))&=\chi_h(X)+2at+2b-2,\\
c_1^2(P_n^m(X))&=c_1^2(X)-d+(16a-2)t+16b-16,\\
b_2^+(P_n^m(X))&=b_2^+(X)-b_1(X)+4at+4b-4\geq 3,\\
b_2^-(P_n^m(X))&=b_2^-(X)-b_1(X)+d+(4a+2)t+4b-4.
\end{aligned}$$

If $\sigma(P_n^m(X))$ is not divisible by 16 or if $2t+d>0$, then $P_n^m(X)$ is nonspin and the set $\{P_n^m(X)\mid m\geq 1\}$ contains infinitely many homeomorphic but pairwise nondiffeomorphic irreducible 4-manifolds. When $m=1$, $P_n^1(X)$ is symplectic and irreducible. If $n=ta+b\geq 3$, then $P_n^1(X)$ contains disjoint symplectic tori T_1 and T_2 of self-intersection 0 satisfying $\pi_1(P_n^1(X)\setminus(T_1\cup T_2))=1$.

Proof. Recall from [Akhmedov and Park 2010a] that $e(Y_n(m)) = 4n - 4$ and $\sigma(Y_n(m)) = 0$ since torus surgeries change neither e nor σ. Hence we have

$$\begin{aligned} e(P_n^m(X)) &= e(\widetilde{X}) + e(Y_n(m)) - 2e(\Sigma_n) \\ &= e(X) + 2t + d + 4n - 4 - 2(2-2n) \\ &= e(X) + 2t + d + 8n - 8 \\ &= e(X) + 2t + d + 8ta + 8b - 8, \end{aligned}$$

$$\sigma(P_n^m(X)) = \sigma(\widetilde{X}) + \sigma(Y_n(m)) = \sigma(X) - 2t - d.$$

The other characteristic numbers can be computed from the formulas $\chi_h = \frac{1}{4}(e+\sigma)$, $c_1^2 = 2e + 3\sigma$, $b_2^+ = b_1 - 1 + \frac{1}{2}(e+\sigma)$, and $b_2^- = b_1 - 1 + \frac{1}{2}(e-\sigma)$.

To compute $\pi_1(P_n^m(X))$, we first choose a standard presentation

$$\pi_1(\Sigma_n) = \Big\langle c_1, d_1, \dots, c_n, d_n \;\Big|\; \prod_{j=1}^{n} [c_j, d_j] = 1 \Big\rangle.$$

From the presentation of $\pi_1(Y_n(m))$ in [Akhmedov and Park 2010a], we know that $\pi_1(Y_n(m))/\langle z \rangle = 1$, where $\langle z \rangle$ is the normal subgroup generated by the image z of any one of the four generators c_1, d_1, c_2, d_2 of $\pi_1(\Sigma_n)$ under the inclusion induced homomorphism $\pi_1(\Sigma_n) \to \pi_1(Y_n(m))$. We also know that any meridian of Σ_n is conjugate to the image of $[a_1, b_1][a_2, b_2]$ in $\pi_1(Y_n(m) \setminus \nu\Sigma_n)$, where a_i, b_i $(i = 1, 2)$ are the images of standard generators of $\pi_1(\Sigma_2 \times \{\mathrm{pt}\})$. All relations of $\pi_1(Y_n(m))$ listed in [Akhmedov and Park 2010a], except $[a_1, b_1][a_2, b_2] = 1$, continue to hold in $\pi_1(Y_n(m) \setminus \nu\Sigma_n)$ since these relations come from torus surgeries that occur away from $\nu\Sigma_n$. Since $z = 1$ still implies $a_i = b_i = 1$ $(i = 1, 2)$ in $\pi_1(Y_n(m) \setminus \nu\Sigma_n)$, we deduce that $\pi_1(Y_n(m) \setminus \nu\Sigma_n)/\langle z \rangle = 1$.

When forming the generalized fiber sum $P_n^m(X)$, we choose the gluing diffeomorphism φ such that the induced homomorphism φ_* maps the element of $\pi_1(\widetilde{\Sigma}^{\parallel})$ represented by a nonseparating vanishing cycle of the Lefschetz fibration X to z, viewed as an element of $\pi_1(\Sigma_n^{\parallel})$. Thus $z = 1$ in $\pi_1(P_n^m(X))$, which then implies that the inclusion induced homomorphism

$$\pi_1(Y_n(m) \setminus \nu\Sigma_n) \longrightarrow \pi_1(P_n^m(X)) \tag{10}$$

is trivial. Note that the inclusion induced homomorphism $\pi_1(\widetilde{\Sigma}^{\parallel}) \to \pi_1(P_n^m(X))$ is also trivial since it can be factored through homomorphism (10) after $\widetilde{\Sigma}^{\parallel}$ is identified with $\Sigma_n^{\parallel}$ via φ. It follows from Lemma 7 that the inclusion induced homomorphism $\pi_1(\widetilde{X} \setminus \nu\widetilde{\Sigma}) \to \pi_1(P_n^m(X))$ is trivial as well. By the Seifert–van Kampen theorem, we conclude that $\pi_1(P_n^m(X)) = 1$.

If $2t + d > 0$, then $P_n^m(X)$ contains a genus 2 surface of self-intersection -1 that is the internal sum of a punctured exceptional sphere in $\widetilde{X} \setminus \nu\widetilde{\Sigma}$ and a punctured

Σ_2 in $Y_n(m) \setminus \nu\Sigma_n$. In this case, the intersection form of $P_n^m(X)$ is odd and $P_n^m(X)$ is nonspin. Also recall that the signature of a spin 4-manifold is divisible by 16 according to Rohlin's theorem [1952].

Note that $e(P_n^m(X))$ and $\sigma(P_n^m(X))$ are independent of m. If $\sigma(P_n^m(X))$ is not divisible by 16 or if $2t+d>0$, then for fixed n, the set $\{P_n^m(X) \mid m \geq 1\}$ consists of homeomorphic simply connected nonspin 4-manifolds by Freedman's classification theorem (cf. [Freedman 1982]).

Since $Y_n(1)$ is symplectic, the corresponding fiber sum $P_n^1(X)$ is symplectic as well (cf. [Gompf 1995; McCarthy and Wolfson 1994]). Since $(\widetilde{X}, \widetilde{\Sigma})$ is a relatively minimal pair (i.e., every sphere of self-intersection -1 intersects $\widetilde{\Sigma}$) by Corollary 3 in [Li 1999], $P_n^1(X)$ is minimal by Usher's theorem [2006]. Recall from [Hamilton and Kotschick 2006; Kotschick 1997] that a simply connected minimal symplectic 4-manifold is irreducible, and thus $P_n^1(X)$ is irreducible.

Any Lefschetz fibration X with fiber genus a and base genus b satisfies $b_1(X) \leq 2a+2b$. Since X has at least one nonseparating vanishing cycle, we have $b_1(X) < 2a+2b \leq 2at+2b$. Thus we deduce that $b_2^+(P_n^m(X)) > b_2^+(X) \geq 1$. Since $P_n^1(X)$ is symplectic and simply connected, $b_2^+(P_n^1(X)) = b_2^+(P_n^m(X))$ is odd. It follows that $b_2^+(P_n^m(X)) \geq 3$ and the Seiberg–Witten invariant of $P_n^m(X)$ is well defined.

Let Y_0 denote the symplectic 4-manifold that is obtained by performing the same torus surgeries on $\Sigma_2 \times \Sigma_n$ as for $Y_n(m)$, except $(a_1'' \times d_2', d_2', +m)$ surgery (cf. [Akhmedov and Park 2010a]). Let $P_0 = \widetilde{X} \#_\varphi Y_0$ be the generalized fiber sum of $\widetilde{X}$ and Y_0 along $\widetilde{\Sigma}$ and Σ_n using the same gluing diffeomorphism φ that was used in the construction of $P_n^m(X)$. Note that P_0 is symplectic and minimal for the same reasons as $P_n^1(X)$. We have $b_2(P_0) = b_2(P_n^m(X)) + 2$, and there is an orthogonal decomposition $H^2(P_0; \mathbb{Z}) = H \oplus H^\perp$, where H is the 2-dimensional hyperbolic summand generated by the Poincaré duals of $[a_1 \times d_2]$ and $[b_1 \times c_2]$. Using the adjunction inequality, we can easily see that every Seiberg–Witten basic class of P_0 lies in $H^\perp$.

Since $P_n^m(X)$ can be obtained from $P_n^1(X)$ by performing a $1/(m-1)$ surgery on a null-homologous torus, we can apply the product formula in [Morgan et al. 1997] as in [Akhmedov et al. 2008; Fintushel et al. 2007; Szabó 1998] and deduce that there exist surjective homomorphisms

$$\xi_m : H^\perp \longrightarrow H^2(P_n^m(X); \mathbb{Z})$$

that preserve the cup product pairing and satisfy

$$\mathrm{SW}_{P_n^m(X)}(\xi_m(L_0)) = \mathrm{SW}_{P_n^1(X)}(\xi_1(L_0)) + (m-1)\,\mathrm{SW}_{P_0}(L_0), \tag{11}$$

for every characteristic element $L_0 \in H^\perp \subset H^2(P_0; \mathbb{Z})$. We note that the right side of (11) contains only one SW_{P_0} term for the reasons given in the proof of Corollary 2 in [Fintushel et al. 2007]. By a theorem of Taubes [1994], we have

$\mathrm{SW}_{P_0}(c_1(P_0)) = \pm 1$. By setting $L_0 = c_1(P_0)$ in (11) and observing that there are infinitely many values for the Seiberg–Witten invariants of $P_n^m(X)$, we conclude that $\{P_n^m(X) \mid m \geq 1\}$ contains infinitely many pairwise nondiffeomorphic 4-manifolds.

Next we prove that $P_n^m(X)$ is irreducible for all m large enough, or more specifically when $\mathrm{SW}_{P_n^m(X)}(\xi_m(c_1(P_0))) \neq 0$. We will argue the same way as in the proof of Theorem 5.4 in [Kotschick 1997]. Suppose $P_n^m(X) = M \# N$ is a connected sum of two smooth 4-manifolds M and N. Both M and N are simply connected since $P_n^m(X)$ is. If $b_2^+(M)$ and $b_2^+(N)$ are both positive, then the Seiberg–Witten invariant of $P_n^m(X)$ is trivial (cf. [Witten 1994]), a contradiction. Without loss of generality, assume $b_2^+(N) = 0$. If $b_2(N) = 0$, then the simply connected 4-manifold N must be homeomorphic to S^4 by Freedman's theorem in [Freedman 1982]. Thus it remains to rule out the case when $b_2(N) = b_2^-(N) > 0$. In this case, the intersection form of N is a nontrivial negative definite form, so by Donaldson's theorem in [Donaldson 1983], it is equivalent to the standard diagonal form. Let $e_1, \dots, e_{b_2(N)}$ be a basis for $H^2(N; \mathbb{Z})$ such that $e_i^2 = -1$ for each $i = 1, \dots, b_2(N)$, and $e_i \cdot e_j = 0$ when $i \neq j$. Using the neck pinching argument as in [Donaldson 1996; Kotschick 1997], we deduce that M has nontrivial Seiberg–Witten invariant. Moreover, if L is any Seiberg–Witten basic class of M, then the cohomology classes

$$L + \sum_{i=1}^{b_2(N)} a_i e_i, \tag{12}$$

where $a_i = \pm 1$ for each $i = 1, \dots, b_2(N)$, are all Seiberg–Witten basic classes of $P_n^m(X) = M \# N$. Furthermore, every Seiberg–Witten basic class of $P_n^m(X)$ can be written as (12).

Let $L_m = \xi_m(c_1(P_0))$ be a Seiberg–Witten basic class of $P_n^m(X)$. By changing any basis element e_i to $-e_i$ if necessary, we can assume that $L_m = L - e_1 - \cdots - e_{b_2(N)}$ for some L. Thus $L_m + 2e_1 = L + e_1 - e_2 - \cdots - e_{b_2(N)}$ is also a Seiberg–Witten basic class of $P_n^m(X)$. By the adjunction inequality, we can assume that $\xi_1(c_1(P_0)) = c_1(P_n^1(X))$. It now follows from (11) that there exists $\bar{e}_1 \in \xi_m^{-1}(e_1) \subset H^\perp$ such that $c_1(P_n^1(X)) + 2\xi_1(\bar{e}_1)$ or $c_1(P_0) + 2\bar{e}_1$ is a Seiberg–Witten basic class of $P_n^1(X)$ or P_0, respectively. By a theorem of Taubes [1996], we can then deduce that the Poincaré dual of $\xi_1(\bar{e}_1)$ or $\bar{e}_1$ is represented by an embedded symplectic sphere of self-intersection -1 in $P_n^1(X)$ or P_0, respectively (cf. Remark 10.1.16(b) in [Gompf and Stipsicz 1999]). This implies that $P_n^1(X)$ or P_0 is not minimal, a contradiction.

Finally, if $n \geq 3$, then $Y_n(1)$ contains $2n-4$ pairs of geometrically dual Lagrangian tori that are all disjoint from Σ_n. The images of these $4n-8$ tori in the fiber sum $P_n^1(X)$ are again Lagrangian submanifolds (cf. [Gompf 1995]). Let T_1 and T_2 be two of these $4n-8$ Lagrangian tori in $P_n^1(X)$ that are not geometrically dual to each other. By perturbing the symplectic form on $P_n^1(X)$, we can turn both T_1 and T_2 into symplectic submanifolds of $P_n^1(X)$ (cf. [Gompf 1995, Lemma 1.6]).

To show $\pi_1(P_n^1(X) \setminus (T_1 \cup T_2)) = 1$, it will be convenient to fix T_1 and T_2, say $T_1 = a_1' \times c_3''$ and $T_2 = a_2' \times d_3''$. Here, a_1', a_2', c_3'' and d_3'' are parallel copies of a_1, a_2, c_3 and d_3 as defined in [Fintushel et al. 2007]. Then $\pi_1(P_n^1(X) \setminus (T_1 \cup T_2))$ is normally generated by meridians of T_1 and T_2, which are all conjugate to the commutators $[b_1^{-1}, d_3]$ or $[b_2^{-1}, c_3]$. Note that the generators b_1, b_2, c_3 and d_3 are still trivial in $\pi_1(P_n^1(X) \setminus (T_1 \cup T_2))$ since the Luttinger surgery relations in Section 2 of [Akhmedov and Park 2010a] still hold true in $\pi_1(P_n^1(X) \setminus (T_1 \cup T_2))$. It follows that meridians of T_1 and T_2 are all trivial and hence $\pi_1(P_n^1(X) \setminus (T_1 \cup T_2)) = \pi_1(P_n^1(X)) = 1$. □

Instead of using $Y_n(m)$ summand in generalized fiber sum (9), we may use $Y_{n-2}(m) \# 2\overline{\mathbb{CP}}^2$ when $n \geq 4$. Specifically, we resolve the intersection between Σ_2 and Σ_{n-2} in $Y_{n-2}(m)$ to obtain a genus n submanifold of $Y_{n-2}(m)$ with self-intersection 2. Next we blow up two points on this submanifold to obtain a genus n submanifold Σ_n' of self-intersection 0 in $Y_{n-2}(m) \# 2\overline{\mathbb{CP}}^2$. When $m = 1$, the resolution and the blowups can be performed symplectically, and hence $(Y_{n-2}(1) \# 2\overline{\mathbb{CP}}^2, \Sigma_n')$ is a relatively minimal pair of symplectic manifolds. The advantage of using $Y_{n-2}(m) \# 2\overline{\mathbb{CP}}^2$ summand is that the resulting generalized fiber sum has slightly smaller characteristic numbers than $P_n^m(X)$.

Theorem 9. *Let $f : X \to \Sigma_b$ be a relatively minimal Lefschetz fibration as above having at least one nonseparating vanishing cycle. Suppose that $n = ta + b \geq 4$. For a suitable choice of the gluing diffeomorphism $\psi : \partial(\nu\widetilde{\Sigma}) \to \partial(\nu\Sigma_n')$, the generalized fiber sum*

$$\begin{aligned} Q_n^m(X) &= \widetilde{X} \#_\psi (Y_{n-2}(m) \# 2\overline{\mathbb{CP}}^2) \\ &= (\widetilde{X} \setminus \nu\widetilde{\Sigma}) \cup_\psi \big((Y_{n-2}(m) \# 2\overline{\mathbb{CP}}^2) \setminus \nu\Sigma_n'\big) \end{aligned}$$

along $\widetilde{\Sigma}$ and Σ_n' is simply connected, nonspin, and satisfies

$$\begin{aligned} e(Q_n^m(X)) &= e(X) + d + (8a+2)t + 8b - 14, \\ \sigma(Q_n^m(X)) &= \sigma(X) - 2t - d - 2, \\ \chi_h(Q_n^m(X)) &= \chi_h(X) + 2at + 2b - 4, \\ c_1^2(Q_n^m(X)) &= c_1^2(X) - d + (16a-2)t + 16b - 34, \\ b_2^+(Q_n^m(X)) &= b_2^+(X) - b_1(X) + 4at + 4b - 8 \geq 3, \\ b_2^-(Q_n^m(X)) &= b_2^-(X) - b_1(X) + d + (4a+2)t + 4b - 6. \end{aligned}$$

The set $\{Q_n^m(X) \mid m \geq 1\}$ contains infinitely many homeomorphic but pairwise nondiffeomorphic irreducible 4-manifolds. When $m = 1$, $Q_n^1(X)$ is symplectic and irreducible. If $n = ta + b \geq 5$, then $Q_n^1(X)$ contains disjoint symplectic tori T_1' and T_2' of self-intersection 0 satisfying $\pi_1(Q_n^1(X) \setminus (T_1' \cup T_2')) = 1$.

Proof. We compute that

$$
\begin{aligned}
e(Q_n^m(X)) &= e(\widetilde{X}) + e(Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2) - 2e(\Sigma_n') \\
&= e(X) + 2t + d + 4(n-2) - 4 + 2 - 2(2-2n) \\
&= e(X) + 2t + d + 8n - 14 \\
&= e(X) + 2t + d + 8ta + 8b - 14,
\end{aligned}
$$

$$
\sigma(Q_n^m(X)) = \sigma(\widetilde{X}) + \sigma(Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2) = \sigma(X) - 2t - d - 2.
$$

The other characteristic numbers can be computed from these as before.

Since the exceptional sphere of a blowup intersects Σ_n' once transversely, any meridian of Σ_n' is null-homotopic in the complement of a tubular neighborhood $\nu\Sigma_n'$. Hence we conclude that

$$
\pi_1\big((Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2) \setminus \nu\Sigma_n'\big) = \pi_1\big(Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2\big) = \pi_1(Y_{n-2}(m)).
$$

From [Akhmedov and Park 2010a], we know that $\pi_1(Y_{n-2}(m))/\langle z\rangle = 1$, where z is the image of any one of the generators c_1, d_1, c_2, d_2 of $\pi_1(\Sigma_{n-2})$ under the inclusion induced homomorphism.

Let $\widetilde{\Sigma}^{\|}$ and $\Sigma_n'^{\|}$ denote parallel copies of $\widetilde{\Sigma}$ and Σ_n' in the boundaries $\partial(\nu\widetilde{\Sigma})$ and $\partial(\nu\Sigma_n')$, respectively. When forming the generalized fiber sum $Q_n^m(X)$, we choose the gluing diffeomorphism ψ such that ψ_* maps the element of $\pi_1(\widetilde{\Sigma}^{\|})$ represented by a nonseparating vanishing cycle of X to z, viewed as an element of $\pi_1(\Sigma_n'^{\|})$. Thus $z = 1$ in $\pi_1(Q_n^m(X))$, which then implies that the inclusion induced homomorphism

$$
\pi_1\big((Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2) \setminus \nu\Sigma_n'\big) \longrightarrow \pi_1(Q_n^m(X)) \tag{13}
$$

is trivial. Note that the inclusion induced homomorphism $\pi_1(\widetilde{\Sigma}^{\|}) \to \pi_1(Q_n^m(X))$ is also trivial since it can be factored through homomorphism (13) after $\widetilde{\Sigma}^{\|}$ is identified with $\Sigma_n'^{\|}$. It follows from Lemma 7 that the inclusion induced homomorphism $\pi_1(\widetilde{X} \setminus \nu\widetilde{\Sigma}) \to \pi_1(Q_n^m(X))$ is trivial as well. By Seifert–van Kampen theorem, we conclude that $\pi_1(Q_n^m(X)) = 1$.

$Q_n^m(X)$ is nonspin since it contains a surface of self-intersection -1 and genus $a > 0$, namely the internal sum of the image of a punctured fiber of X in $\widetilde{X} \setminus \nu\widetilde{\Sigma}$ and a punctured exceptional sphere in $(Y_{n-2}(m)\,\#\,2\overline{\mathbb{CP}}^2) \setminus \nu\Sigma_n'$. Since $Y_{n-2}(1)\,\#\,2\overline{\mathbb{CP}}^2$ is symplectic, the corresponding fiber sum $Q_n^1(X)$ is symplectic as well. The irreducibility of $Q_n^1(X)$ and the fact that $\{Q_n^m(X) \mid m \geq 1\}$ contains infinitely many homeomorphic but pairwise nondiffeomorphic irreducible 4-manifolds can be proved exactly the same way as in the proof of Theorem 8.

Finally, if $n \geq 5$, then $Y_{n-2}(1)$ contains $2n - 8$ pairs of geometrically dual Lagrangian tori. The images of these $4n - 16$ tori in the blowup $Y_{n-2}(1)\,\#\,2\overline{\mathbb{CP}}^2$ are disjoint from Σ'_n, and hence their images in $Q^1_n(X)$ are Lagrangian submanifolds of $Q^1_n(X)$. Let T'_1 and T'_2 denote two of these $4n-16$ Lagrangian tori, say $T'_1 = a'_1 \times c''_3$ and $T'_2 = a'_2 \times d''_3$. By perturbing the symplectic form on $Q^1_n(X)$, we can turn both T'_1 and T'_2 into symplectic submanifolds of $Q^1_n(X)$. We can deduce that $\pi_1(Q^1_n(X) \setminus (T'_1 \cup T'_2)) = 1$ in exactly the same way as in the proof of Theorem 8. □

For comparison, we note that

$$\begin{aligned} e(Q^m_n(X)) &= e(P^m_n(X)) - 6, & \sigma(Q^m_n(X)) &= \sigma(P^m_n(X)) - 2, \\ \chi_h(Q^m_n(X)) &= \chi_h(P^m_n(X)) - 2, & c_1^2(Q^m_n(X)) &= c_1^2(P^m_n(X)) - 18, \\ b_2^+(Q^m_n(X)) &= b_2^+(P^m_n(X)) - 4, & b_2^-(Q^m_n(X)) &= b_2^-(P^m_n(X)) - 2. \end{aligned} \tag{14}$$

Remark 10. The irreducible symplectic 4-manifolds M and N (homeomorphic to $47\mathbb{CP}^2 \# 45\overline{\mathbb{CP}}^2$ and $51\mathbb{CP}^2 \# 47\overline{\mathbb{CP}}^2$, respectively) in Section 4 of [Akhmedov and Park 2008] are respectively equal to $Q^1_n(X)$ and $P^1_n(X)$ with $a = 7$, $b = 2$, $t = 1$, $d = -2$, $n = 9$, $e(X) = 36$, and $\sigma(X) = 4$.

5. Simply connected 4-manifolds with positive signature

We now apply Theorems 8 and 9 to Lefschetz fibrations in Sections 2 and 3 to obtain new families of simply connected irreducible 4-manifolds with positive signature.

Example 11. For each triple of positive integers g, u_1, u_2, recall from Example 3 that there is a Lefschetz fibration $f_{u_1,u_2} : X^g_{u_1,u_2} \to \Sigma_b$ such that the genus of a regular fiber is $a = 1 + \frac{1}{2}u_2(g+1)(3g-2)$ and there is a section whose image is a surface of genus $b = u_1(g-1)+1$ and self-intersection $d = -2u_1$. Since $2t+d \geq 0$, we require $t \geq u_1$. Let

$$n = t + \tfrac{1}{2}tu_2(g+1)(3g-2) + u_1(g-1) + 1.$$

Applying Theorem 8 to $f_{u_1,u_2} : X^g_{u_1,u_2} \to \Sigma_b$, we obtain a family of simply connected 4-manifolds $P^m_n(X^g_{u_1,u_2})$, with $m \geq 1$ and $n \geq 3$, satisfying

$$\begin{aligned} e(P^m_n(X^g_{u_1,u_2})) &= 2u_1u_2(g+1)(3g^2-5g+4) \\ &\qquad + 4tu_2(g+1)(3g-2) + 8u_1g + 10t - 10u_1, \\ \sigma(P^m_n(X^g_{u_1,u_2})) &= \tfrac{2}{3}u_1u_2(g+1)(g^2+2g-6) - 2t + 2u_1, \\ \chi_h(P^m_n(X^g_{u_1,u_2})) &= \tfrac{1}{6}u_1u_2(g+1)(10g^2-13g+6) \\ &\qquad + tu_2(g+1)(3g-2) + 2t + 2u_1(g-1), \\ c_1^2(P^m_n(X^g_{u_1,u_2})) &= 2u_1u_2(g+1)(7g^2-8g+2) \\ &\qquad + 8tu_2(g+1)(3g-2) + 16u_1g + 14t - 14u_1, \end{aligned} \tag{15}$$

$$b_2^+(P_n^m(X^g_{u_1,u_2})) = \tfrac{1}{3}u_1u_2(g+1)(10g^2-13g+6) \\ \qquad + 2tu_2(g+1)(3g-2) + 4t + 4u_1(g-1) - 1,$$

$$b_2^-(P_n^m(X^g_{u_1,u_2})) = \tfrac{1}{3}u_1u_2(g+1)(8g^2-17g+18) \\ \qquad + 2tu_2(g+1)(3g-2) + 4u_1g + 6t - 6u_1 - 1. \tag{16}$$

From Theorem 9, we obtain another family of simply connected nonspin 4-manifolds $Q_n^m(X^g_{u_1,u_2})$, with $m \geq 1$ and $n \geq 5$, whose characteristic numbers can be computed from (14) (15), and (16). Moreover, when $m = 1$, both $P_n^1(X^g_{u_1,u_2})$ and $Q_n^1(X^g_{u_1,u_2})$ are irreducible symplectic 4-manifolds and contain symplectic tori T_j and T'_j ($j = 1, 2$) of self-intersection 0 such that

$$\pi_1(P_n^1(X^g_{u_1,u_2}) \setminus (T_1 \cup T_2)) = 1 \quad \text{and} \quad \pi_1(Q_n^1(X^g_{u_1,u_2}) \setminus (T'_1 \cup T'_2)) = 1.$$

Example 12. For each quadruple of positive integers p, v, u_1, u_2 with odd $p \geq 3$, recall from Example 5 that there is a Lefschetz fibration $f_{u_1,u_2} : W^{p,v}_{u_1,u_2} \to \Sigma_b$ such that the genus of a regular fiber is $a = 1 + \frac{1}{2}pu_2[(v+1)p - v - 3]$ and there is a section whose image is a surface of genus $b = 1 + u_1[-1 + v(p-1)/2]$ and self-intersection $d = -u_1v$. Since $2t + d \geq 0$, we require

$$t \geq \lceil u_1v/2 \rceil,$$

where $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$. From Theorems 8 and 9, we obtain two families of simply connected 4-manifolds $P_n^m(W^{p,v}_{u_1,u_2})$ and $Q_n^m(W^{p,v}_{u_1,u_2})$ with $m \geq 1$ and

$$n = t + \tfrac{1}{2}tpu_2[(v+1)p - v - 3] + u_1[-1 + v(p-1)/2] + 1 \geq 5.$$

We compute that

$$e(P_n^m(W^{p,v}_{u_1,u_2})) = pu_1u_2\big[(v^2+v)p^2 - 2(v^2+3v+1)p + v^2 + 6v + 8\big] \\ \qquad + 4tu_2(v+1)p^2 + 4\big[u_1v - tu_2(v+3)\big]p + 10t - 5u_1v - 8u_1,$$

$$\sigma(P_n^m(W^{p,v}_{u_1,u_2})) = \tfrac{1}{3}pu_1u_2(vp^2 - 4v - 6) - 2t + u_1v,$$

$$\chi_h(P_n^m(W^{p,v}_{u_1,u_2})) = \tfrac{1}{12}pu_1u_2\big[(3v^2+4v)p^2 - 6(v^2+3v+1)p + 3v^2 + 14v + 18\big] \\ \qquad + tu_2(v+1)p^2 + \big[u_1v - tu_2(v+3)\big]p + 2t - u_1v - 2u_1,$$

$$c_1^2(P_n^m(W^{p,v}_{u_1,u_2})) = pu_1u_2\big[(2v^2+3v)p^2 - 4(v^2+3v+1)p + 2v^2 + 8v + 10\big] \\ \qquad + 8tu_2(v+1)p^2 + 8\big[u_1v - tu_2(v+3)\big]p + 14t - 7u_1v - 16u_1,$$

$$b_2^+(P_n^m(W^{p,v}_{u_1,u_2})) = \tfrac{1}{6}pu_1u_2\big[(3v^2+4v)p^2 - 6(v^2+3v+1)p + 3v^2 + 14v + 18\big] \\ \qquad + 2tu_2(v+1)p^2 + 2\big[u_1v - tu_2(v+3)\big]p + 4t - 2u_1v - 4u_1 - 1,$$

$$b_2^-(P_n^m(W^{p,v}_{u_1,u_2})) = \tfrac{1}{6}pu_1u_2\big[(3v^2+2v)p^2 - 6(v^2+3v+1)p + 3v^2 + 22v + 30\big] \\ \qquad + 2tu_2(v+1)p^2 + 2\big[u_1v - tu_2(v+3)\big]p + 6t - 3u_1v - 4u_1 - 1.$$

The characteristic numbers of $Q_n^m(W_{u_1,u_2}^{p,v})$ can be computed from these values via (14). When $m=1$, both $P_n^1(W_{u_1,u_2}^{p,v})$ and $Q_n^1(W_{u_1,u_2}^{p,v})$ are irreducible symplectic 4-manifolds and contain symplectic tori T_j and T_j' $(j=1,2)$ of self-intersection 0 such that $\pi_1(P_n^1(W_{u_1,u_2}^{p,v})\setminus(T_1\cup T_2))=1$ and $\pi_1(Q_n^1(W_{u_1,u_2}^{p,v})\setminus(T_1'\cup T_2'))=1$.

6. Upper bounds for the lower bound

We start this section by giving a more rigorous definition of $\lambda(\sigma)$ from the introduction.

Definition 13. Given an integer $\sigma\geq 0$, let $\lambda(\sigma)$ be the smallest positive integer with the following properties.

(i) $\lambda(\sigma)\geq\lceil(\sigma+1)/2\rceil$.

(ii) Every point (χ_h, c_1^2) on the line $c_1^2=8\chi_h+\sigma$ satisfying $\chi_h\geq\lambda(\sigma)$ is realized as $(\chi_h(M_i), c_1^2(M_i))$, where $\{M_i \mid i\in\mathbb{Z}\}$ is an infinite family of homeomorphic but pairwise nondiffeomorphic closed simply connected nonspin irreducible 4-manifolds such that M_i is symplectic for each $i\geq 0$ and M_i is nonsymplectic for each $i<0$.

As in the introduction, we make the following conjecture.

Conjecture 14. $\lambda(\sigma)=\lceil(\sigma+1)/2\rceil$ for every integer $\sigma\geq 0$.

Our goal in this section is to calculate explicit upper bounds on $\lambda(\sigma)$ for many small values of σ. First we restate a result from [Akhmedov and Park 2008] (see also [Akhmedov et al. 2010a, Theorem 23; Akhmedov and Park 2010a, Theorem 2]).

Theorem 15 [Akhmedov and Park 2008, Theorem 5.3]. *Let X be a closed symplectic 4-manifold that contains a symplectic torus T of self-intersection 0. Let νT be a tubular neighborhood of T and $\partial(\nu T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(\nu T))\to\pi_1(X\setminus\nu T)$ induced by the inclusion is trivial. Then for any pair of integers (χ, c) satisfying*

$$\chi\geq 1 \ \text{ and } \ 0\leq c\leq 8\chi, \tag{17}$$

there exists a symplectic 4-manifold Y with $\pi_1(Y)=\pi_1(X)$,

$$\chi_h(Y)=\chi_h(X)+\chi \ \text{ and } \ c_1^2(Y)=c_1^2(X)+c.$$

Moreover, if X is minimal then Y is minimal as well. If $c<8\chi$, or if $c=8\chi$ and X has an odd intersection form, then the corresponding Y has an odd indefinite intersection form. □

The next theorem gives us a means for constructing infinitely many distinct smooth structures on some topological 4-manifolds.

Theorem 16. *Let Y be a closed simply connected minimal symplectic 4-manifold with $b_2^+(Y) > 1$. Assume that Y contains a symplectic torus T of self-intersection 0 such that $\pi_1(Y \setminus T) = 1$. Then there exist an infinite family of pairwise nondiffeomorphic irreducible symplectic 4-manifolds and an infinite family of pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to Y.*

Proof. We can perform a knot surgery on Y along T using a knot $K \subset S^3$ (see [Fintushel and Stern 2009, Lecture 3]). Let Y_K denote the resulting 4-manifold. Since $\pi_1(Y \setminus T) = 1$, Y_K is homeomorphic to Y. By varying the knot K, we obtain infinitely many pairwise nondiffeomorphic 4-manifolds. If K is a fibered knot, then Y_K can be viewed as a symplectic fiber sum [Fintushel and Stern 1998], is minimal by Usher's theorem [2006], and hence is irreducible [Hamilton and Kotschick 2006; Kotschick 1997].

Given an integer $k \neq 0$, let $T(k)$ denote the k-twist knot on page 372 of [Fintushel and Stern 1998] with Alexander polynomial $kt - (2k+1) + kt^{-1}$. If $k = \pm 1$, then $T(\pm 1)$ is fibered, and thus $Y_{T(\pm 1)}$ is symplectic and irreducible. If $k \neq 0, \pm 1$, then $Y_{T(k)}$ is nonsymplectic. It only remains to prove that $Y_{T(k)}$ is irreducible when $k \neq 0, \pm 1$. We will argue the same way as in the proof of Theorem 8. The computation of the Seiberg–Witten invariant of $Y_{T(k)}$ in [Fintushel and Stern 2009] implies that there exists an isomorphism $\xi_{T(k)} : H^2(Y_{T(1)}; \mathbb{Z}) \longrightarrow H^2(Y_{T(k)}; \mathbb{Z})$ that preserves the cup product pairing and restricts to a one-to-one correspondence between the Seiberg–Witten basic classes of $Y_{T(1)}$ and $Y_{T(k)}$. Suppose that $Y_{T(k)}$ is not irreducible. Then there will be some $e_1 \in H^2(Y_{T(k)}; \mathbb{Z})$ such that $e_1^2 = -1$ and $\xi_{T(k)}(c_1(Y_{T(1)})) + 2e_1$ is a Seiberg–Witten basic class of $Y_{T(k)}$. This will imply that $c_1(Y_{T(1)}) + 2\xi_{T(k)}^{-1}(e_1)$ is a Seiberg–Witten basic class of $Y_{T(1)}$. By a result of Taubes [1996], we can then conclude that the Poincaré dual of $\xi_{T(k)}^{-1}(e_1)$ is represented by an embedded symplectic sphere of self-intersection -1 in $Y_{T(1)}$. Hence $Y_{T(1)}$ is not minimal, a contradiction. □

By combining Theorems 15 and 16, we may deduce the following.

Corollary 17. *Let X be a closed simply connected nonspin minimal symplectic 4-manifold with $b_2^+(X) > 1$ and $\sigma(X) \geq 0$. Assume that X contains disjoint symplectic tori T_1 and T_2 of self-intersection 0 such that $\pi_1(X \setminus (T_1 \cup T_2)) = 1$. Suppose σ is a fixed integer satisfying $0 \leq \sigma \leq \sigma(X)$. If $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$ and if we define*

$$\ell(\sigma) = \left\lceil \frac{\sigma(X) - \sigma}{8} - 1 \right\rceil,$$

then

$$\lambda(\sigma) \leq \chi_h(X) + \ell(\sigma) + 1.$$

In other words, if k is any odd integer satisfying $k \geq b_2^+(X) + 2\ell(\sigma) + 2$, then there exist an infinite family of pairwise nondiffeomorphic irreducible symplectic 4-manifolds and an infinite family of pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to $k\mathbb{CP}^2 \# (k-\sigma)\overline{\mathbb{CP}}^2$.

Proof. We can write $\sigma(X) - \sigma = 8\ell(\sigma) + r(\sigma)$ for integers $\ell(\sigma)$ and $r(\sigma)$ satisfying $\ell(\sigma) \geq -1$ and $1 \leq r(\sigma) \leq 8$. Since $\pi_1(X \setminus \nu T_1) = 1$, we can apply Theorem 15 to the pair, X and T_1. Let (χ, c) and Y be as in the conclusion of Theorem 15. Since $\pi_1(Y) = \pi_1(X) = 1$, we have $b_2^+(Y) = b_2^+(X) + 2\chi$ and $b_2^-(Y) = b_2^-(X) + 10\chi - c$. By Freedman's classification theorem [1982], Y must be homeomorphic to

$$(b_2^+(X) + 2\chi)\mathbb{CP}^2 \# (b_2^-(X) + 10\chi - c)\overline{\mathbb{CP}}^2.$$

By setting $c = 8\chi + \sigma - \sigma(X)$ in (17), we obtain a minimal symplectic 4-manifold Y that is homeomorphic to $k\mathbb{CP}^2 \# (k-\sigma)\overline{\mathbb{CP}}^2$, where $k = b_2^+(X) + 2\chi$. Since c is nonnegative, we must have $8\chi + \sigma - \sigma(X) = 8(\chi - \ell(\sigma)) - r(\sigma) \geq 0$, which implies that $\chi \geq \ell(\sigma) + 1$. It follows that $\chi_h(Y) \geq \chi_h(X) + \ell(\sigma) + 1$ and $k \geq b_2^+(X) + 2\ell(\sigma) + 2$.

We recall from [Akhmedov et al. 2010a; Akhmedov and Park 2008; 2010a] that for each pair of integers (χ, c) satisfying (17), there exist a minimal symplectic 4-manifold Z with $\chi_h(Z) = \chi$, $c_1^2(Z) = c$, and a symplectic torus $T'' \subset Z$ of self-intersection 0 such that Y is the generalized fiber sum of X and Z along T_1 and T''. Note that $T_2 \subset (X \setminus \nu T_1) \subset Y$ is a symplectic torus of self-intersection 0 in Y (cf. [Gompf and Stipsicz 1999, Theorem 10.2.1]). Since $\pi_1(X \setminus (\nu T_1 \cup T_2)) = 1$, we have $\pi_1(Y \setminus T_2) = 1$. We can now apply Theorem 16 to the pair, Y and T_2, and conclude that there are infinitely many distinct smooth structures on Y. $\square$

Next we show that $\lambda(\sigma)$ is subadditive in the following sense.

Corollary 18. *Let σ_1 and σ_2 be positive integers such that $\sigma_1 + \sigma_2$ is not divisible by 16. For each $j = 1, 2$, suppose that there exists a closed simply connected nonspin minimal symplectic 4-manifold N_j containing a symplectic torus $T_j \subset N_j$ of self-intersection 0 such that*

(i) $\pi_1(N_j \setminus T_j) = 1$,

(ii) $\chi_h(N_j) = \lambda(\sigma_j)$, *and* $\sigma(N_j) = \sigma_j$.

Then we have $\lambda(\sigma_1 + \sigma_2) \leq \lambda(\sigma_1) + \lambda(\sigma_2)$.

Proof. Let X be the generalized fiber sum of N_1 and N_2 along T_1 and T_2. It is easy to check that X is a closed simply connected minimal symplectic 4-manifold. Since

$$\sigma(X) = \sigma(N_1) + \sigma(N_2) = \sigma_1 + \sigma_2 \not\equiv 0 \pmod{16},$$

X is nonspin by Rohlin's theorem [1952]. Let T be a parallel copy of T_1 (and T_2) in X. From (i), there are topological disks bounding the meridians of T_1 and T_2, and these disks can be glued together to form a topological sphere that intersects T transversely once. It follows that $\pi_1(X \setminus T) = 1$ and thus we can apply Corollary 17 with $\sigma = \sigma(X)$ and conclude that

$$\lambda(\sigma_1 + \sigma_2) \leq \chi_h(X) = \chi_h(N_1) + \chi_h(N_2) = \lambda(\sigma_1) + \lambda(\sigma_2). \qquad \square$$

We now proceed to list the smallest upper bounds on $\lambda(\sigma)$ currently known to the authors. We begin by first finding parameters g, p, v, u_1, u_2 and t in Examples 11 and 12 that yield 4-manifolds with small χ_h values. By Rohlin's theorem, these 4-manifolds are nonspin if their signatures are not divisible by 16. Unfortunately, given an integer $\sigma \geq 0$, there is no clear pattern as to which family or parameters

σ	$\lambda(\sigma) \leq$	X	σ	$\lambda(\sigma) \leq$	X
0–1	25	$Q^1_9(W^{3,2}_{1,1})$	50	86	$P^1_{19}(W^{5,1}_{2,1})$
2	24	$Q^1_9(W^{3,2}_{1,1})$	51	111	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_9(W^{3,2}_{1,1})$
3	27	$P^1_9(W^{3,2}_{1,1})$	52	110	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_9(W^{3,2}_{1,1})$
4	26	$P^1_9(W^{3,2}_{1,1})$	53	113	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi P^1_9(W^{3,2}_{1,1})$
5	47	$Q^1_{15}(W^{3,2}_{1,2})$	54	112	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi P^1_9(W^{3,2}_{1,1})$
6	46	$Q^1_{15}(W^{3,2}_{1,2})$	55	133	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_{15}(W^{3,2}_{1,2})$
7	49	$P^1_{15}(W^{3,2}_{1,2})$	56	132	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_{15}(W^{3,2}_{1,2})$
8	48	$P^1_{15}(W^{3,2}_{1,2})$	57	135	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi P^1_{15}(W^{3,2}_{1,2})$
9–13	59	$Q^1_{18}(W^{5,1}_{1,1})$	58	134	$P^1_{19}(W^{5,1}_{2,1}) \# \varphi P^1_{15}(W^{3,2}_{1,2})$
14–21	58	$Q^1_{18}(W^{5,1}_{1,1})$	59–61	143	$Q^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_{18}(W^{5,1}_{1,1})$
22	57	$Q^1_{18}(W^{5,1}_{1,1})$	62–69	142	$Q^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_{18}(W^{5,1}_{1,1})$
23	60	$P^1_{18}(W^{5,1}_{1,1})$	70	141	$Q^1_{19}(W^{5,1}_{2,1}) \# \varphi Q^1_{18}(W^{5,1}_{1,1})$
24	59	$P^1_{18}(W^{5,1}_{1,1})$	71	144	$Q^1_{36}(W^{5,1}_{3,1})$
25	84	$P^1_{18}(W^{5,1}_{1,1}) \# \varphi Q^1_9(W^{3,2}_{1,1})$	72	143	$Q^1_{36}(W^{5,1}_{3,1})$
26	83	$P^1_{18}(W^{5,1}_{1,1}) \# \varphi Q^1_9(W^{3,2}_{1,1})$	73	146	$P^1_{36}(W^{5,1}_{3,1})$
27	86	$P^1_{18}(W^{5,1}_{1,1}) \# \varphi P^1_9(W^{3,2}_{1,1})$	74	145	$P^1_{36}(W^{5,1}_{3,1})$
28	85	$P^1_{18}(W^{5,1}_{1,1}) \# \varphi P^1_9(W^{3,2}_{1,1})$	75–81	167	$Q^1_{34}(W^{5,1}_{2,2})$
29–31	87	$Q^1_{19}(W^{5,1}_{2,1})$	82–89	166	$Q^1_{34}(W^{5,1}_{2,2})$
32–39	86	$Q^1_{19}(W^{5,1}_{2,1})$	90–97	165	$Q^1_{34}(W^{5,1}_{2,2})$
40–47	85	$Q^1_{19}(W^{5,1}_{2,1})$	98	164	$Q^1_{34}(W^{5,1}_{2,2})$
48	84	$Q^1_{19}(W^{5,1}_{2,1})$	99	167	$P^1_{34}(W^{5,1}_{2,2})$
49	87	$P^1_{19}(W^{5,1}_{2,1})$	100	166	$P^1_{34}(W^{5,1}_{2,2})$

Table 2. Upper bounds on $\lambda(\sigma)$.

will yield a simply connected nonspin 4-manifold X with $\sigma(X) \geq \sigma$ having the smallest $\chi_h(X) + \ell(\sigma) + 1$. Hence we had to resort to a computer search.

Table 2 on the previous page lists some of the smallest upper bounds on $\lambda(\sigma)$ that we found. For example, when $\sigma = 10$, Table 2 says that $\lambda(10) \leq 59$, that is, for each odd integer $k \geq 2 \cdot 59 - 1 = 117$, there exist an infinite family of pairwise nondiffeomorphic irreducible symplectic 4-manifolds and an infinite family of pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to $k\mathbb{CP}^2 \# (k-10)\overline{\mathbb{CP}}^2$. The third column in Table 2 lists the simply connected 4-manifold X that was used to obtain the upper bound via Corollary 17. The $\#\varphi$ symbol denotes a generalized fiber sum along the tori T_j and/or T_j'. We have compiled upper bounds on $\lambda(\sigma)$ for σ up to about 1,000,000 but we will only list a small sample here.

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