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BOUND STATES OF ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATIONS WITH COMPACTLY SUPPORTED POTENTIALS

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We study the existence and concentration of bound states to N-dimensional nonlinear Schrödinger equation $-\varepsilon^2\Delta u_\varepsilon + V(x)u_\varepsilon = K(x)f(u_\varepsilon)$, where $N \geq 3$, $\varepsilon > 0$ is sufficiently small, and the function $f(s)$ is nonnegative and asymptotically linear at infinity. More concretely, when $f(s) \sim O(s)$ as $s \to +\infty$, the potential function $V(x)$ lies in $C_0^1(\mathbb{R}^N)$ with $V(x) \geq 0$ and $V(x) \not\equiv 0$, and $K(x) \geq 0$ is permitted to be unbounded under some other necessary restrictions, we can show that a positive $H^1(\mathbb{R}^N)$-solution $u_\varepsilon(x)$ exists and concentrates around the local maximum point of the corresponding ground energy function.

1. Introduction and statements of main results

This paper deals with the problem on the existence and concentration of bound states for the nonlinear Schrödinger equation

$$\begin{cases} -\varepsilon^2\triangle u_\varepsilon + V(x)u_\varepsilon = K(x)f(u_\varepsilon), & x \in \mathbb{R}^N, \\ u_\varepsilon \in H^1(\mathbb{R}^N), \quad u_\varepsilon(x) > 0, \end{cases} \tag{1-1}$$

where $N \geq 3$, $\varepsilon > 0$ is small, $K(x) \geq 0$, $V(x) \geq 0$ with $V(x) \not\equiv 0$, $f(s) \geq 0$ and $f(s) \sim O(s)$ as $s \to +\infty$, which is asymptotically linear. Such a solution u_ε is called as a bound state for $u_\varepsilon \in H^1(\mathbb{R}^N)$ and $u_\varepsilon(x) > 0$.

Consider in particular the superlinear problem given by the equation

$$\begin{cases} -\varepsilon^2\triangle u_\varepsilon + V(x)u_\varepsilon = K(x)|u_\varepsilon|^{p-1}u_\varepsilon, & x \in \mathbb{R}^N, \\ u_\varepsilon \in H^1(\mathbb{R}^N), \quad u_\varepsilon > 0, \end{cases} \tag{1-2}$$

for $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$. Under various assumptions on the potential function $V(x) \geq C_0 > 0$ for large $|x|$ or $\lim_{|x|\to\infty} V(x) = 0$ or even $V(x)$ is compactly

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supported with $V(x) \geq 0$ and $V(x) \not\equiv 0$, the existence of H^1-positive solutions has been established, and the concentration properties of u_ε can be obtained at a global or local minimum point of the ground energy function $G(\xi) \equiv V^\theta(\xi) K^{-2/(p-2)}(\xi)$ with $\theta = \frac{p}{p-2} - \frac{N}{2}$ (one can see [Ambrosetti et al. 2005; Ambrosetti and Malchiodi 2007; Ambrosetti and Wang 2005; Berestycki and Lions 1983; Bonheure and Van Schaftingen 2008; Byeon and Wang 2006; Dávila et al. 2007; del Pino and Felmer 1996; del Pino et al. 2007; Fei and Yin 2010; Gui 1996; Rabinowitz 1992; Wang and Zeng 1997; Yin and Zhang 2009]).

For the asymptotically linear problem (1-1) with $\varepsilon = 1$, there are many papers on the existence of solution in recent years. For examples, in the case of $V(x) \geq C_0 > 0$ for large $|x|$, one can see [Costa and Tehrani 2001; Jeanjean and Tanaka 2002; Liu et al. 2006; Liu and Wang 2004; Stuart and Zhou 1999]; in the special case that $V(x)$ vanishes at infinity like $a/(1+|x|^\sigma) \leq V(x) \leq A$ (the constants $\sigma \in (0,2)$, $a > 0$ and $A > 0$) and some other restrictions, the authors in [Liu et al. 2008] established the existence of bound states.

We now consider the following interesting problems indicated in [Ambrosetti and Malchiodi 2007]: if the potential function $V(x)$ decays faster than $1/(1+|x|^\sigma)$ with $\sigma \in (0,2)$ at infinity or is compactly supported with $V(x) \geq 0$ and $V(x) \not\equiv 0$, does the bound state of (1-1) still exist? If it exists, what is the concentration profile of $u_\varepsilon(x)$ as $\varepsilon \to 0$? In this paper, we will treat these two problems. We only focus on the case that $V(x)$ is compactly supported, since the other cases of $V(x) = O(1/(1+|x|^\sigma))$ with $\sigma \in \mathbb{R}$ can be treated analogously and even more simply.

To proceed, we define the *ground energy function* $G(\xi)$. The constant coefficient asymptotically linear equation is as follows:

$$
\begin{cases} -\triangle u(x) + V(\xi) u(x) = K(\xi) f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u(x) > 0, \end{cases} \tag{1-3}
$$

where $V(\xi), K(\xi) > 0$ with $\xi \in \bar{\Lambda}$, and the meaning of Λ is given in assumption (H_4) below.

The associated Euler functional is defined as

$$
I^\xi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{V(\xi)}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - K(\xi) \int_{\mathbb{R}^N} F(u) \, dx, \tag{1-4}
$$

where $F(u) = \int_0^u f(x,\tau)\, d\tau$.

In the terminology in [Wang and Zeng 1997], the function $G(\xi) = \inf_{u \in \mathcal{M}^\xi} I^\xi(u)$ is the *ground energy function* of (1-3) and $\omega(x)$ is a *ground state* of the functional

I^ξ if $G(\xi) = I^\xi(\omega)$, where $\mathcal{M}^\xi$ is the Nehari manifold, defined as

$$\mathcal{M}^\xi = \Big\{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \int_{\mathbb{R}^N} |\nabla u|^2\,dx + V(\xi)\int_{\mathbb{R}^N} |u|^2\,dx = K(\xi)\int_{\mathbb{R}^N} f(u)u\,dx\Big\}. \tag{1-5}$$

Under certain assumptions, we will solve the constant coefficient asymptotically linear problem (1-3) and prove that the ground state exists and $G(\xi)$ is a continuous function in $\bar{\Lambda}$ in Section 3 below. The assumptions are as follows:

(H_1) $V(x) \in C_0^1(\mathbb{R}^N)$, $V(x) \geq 0$; $K(x) \in C^1(\mathbb{R}^N)$, $K(x) \geq 0$.

(H_2) $f \in C(\mathbb{R}, \mathbb{R}^+) \cap C^{1,\gamma}_{\text{loc}}(\mathbb{R})$ with some constant γ satisfying $0 < \gamma \leq 1$; $f(s) = 0$ for $s \leq 0$; $f(s) = O(s^\alpha)$ with some $\alpha > 1$ near $s = 0$.

(H_3) $f(s)/s$ is a nondecreasing function for $s > 0$ and

$$\frac{f(s)}{s} \to l \in (0, +\infty) \quad \text{as } s \to +\infty. \tag{1-6}$$

(H_4) There exists a smooth bounded domain Λ of $\mathbb{R}^N$ such that $V(x) > 0$, $K(x) > 0$ on $\bar{\Lambda}$ and

$$\mu^* \equiv \max_{\xi \in \bar{\Lambda}} \frac{V(\xi)}{K(\xi)} < l, \tag{1-7}$$

$$0 < c_0 \equiv \inf_{\xi \in \Lambda} G(\xi) < \inf_{\xi \in \partial\Lambda} G(\xi). \tag{1-8}$$

(H_5) Let $N \geq 5$. There exist some constants $k > 0$ and $\beta < (\alpha - 1)(N - 2) - 2$ such that

$$0 \leq K(x) \leq k(1 + |x|)^\beta \quad \text{in } \mathbb{R}^N. \tag{1-9}$$

Our main results in this paper can be stated as follows:

Theorem 1.1 (existence and concentration). *Let assumptions (H_1)–(H_5) hold.*

(i) *Equation (1-1) has at least one bound state u_ε provided that ε is small.*

(ii) *u_ε has exactly one maximum point $x_\varepsilon \in \Lambda$, which satisfies*

$$C_1 \leq u_\varepsilon(x_\varepsilon) \leq C_2 \tag{1-10}$$

and

$$\operatorname{dist}(x_\varepsilon, M) \to 0 \quad \text{as } \varepsilon \to 0, \tag{1-11}$$

where C_1, C_2 are positive constants independent of ε, and the set M is defined by $M = \{x \in \Lambda : G(x) = c_0\}$. Moreover, if M only contains a single point x_0, then u_ε is a single peak solution; more precisely,

$$u_\varepsilon(x) = v\Big(\frac{x - x_\varepsilon}{\varepsilon}\Big) + w_\varepsilon(x), \tag{1-12}$$

where $w_\varepsilon(x) \to 0$ *in* $C^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *as* $\varepsilon \to 0$ *and* $v \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ *is the positive solution of the equation*

$$-\Delta v + V(x_0)v = K(x_0) f(v), \quad x \in \mathbb{R}^N. \tag{1-13}$$

Remark 1.1. In the assumption (H_5), $N \geq 5$ can not be removed to obtain $u_\varepsilon \in L^2(\mathbb{R}^N)$ in Theorem 1.1 since this is also necessary even for the N-dimensional linear Laplacian equation. For more details, one can see Remark 1.2 of [Yin and Zhang 2009]. On the other hand, if we do not require $u_\varepsilon \in L^2(\mathbb{R}^N)$ in Theorem 1.1, for example, only $u_\varepsilon \in L^q(\mathbb{R}^N)$ is permitted for some $q > 1$, then Theorem 1.1 still holds for all $N \geq 2$ by our proof procedure since $N \geq 5$ is only used in (4-52) of Section 4 to derive $u_\varepsilon \in L^2(\mathbb{R}^N)$ through the whole paper.

Remark 1.2. In the assumption (H_2), due to $f \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R})$, $f(s) = 0$ for $s \leq 0$ and $f(s) = O(s^\alpha)$ near $s = 0$ with $\alpha > 1$, then we actually have $0 < \gamma \leq \min\{1, \alpha - 1\}$.

Remark 1.3. With respect to the assumption (1-7) in (H_4), if $V(x) \sim l^*/(1+|x|^{\beta_1})$ with $\beta_1 > 0$ and $K(x) \sim 1/(1+|x|^{\beta_2})$ with $0 < \beta_2 < \beta_1$ or $V(x) \sim l^* e^{-|x|^{\beta_1}}$ with $\beta_1 > 0$ and $K(x) \sim e^{-|x|^{\beta_2}}$ with $0 < \beta_2 < \beta_1$, then for $0 < l^* < l$, we have $\mu^* \leq l^* < l$, namely, (1-7) holds true. However, assumption (1-7) does not satisfy the condition (K_1) in [Liu et al. 2008], to the effect that $\sup\bigl\{\frac{f(s)}{s} : s > 0\bigr\} < \inf\bigl\{\frac{V(x)}{K(x)} : |x| \geq R_0\bigr\}$ for some $R_0 > 0$, which seems to be crucial to the proof there. On the other hand, the main assumptions (K_1) and (1.8) in Theorem 1.1 of [Liu et al. 2008] are rather restricted. If we use instead of (K_1) the more natural assumption $\sup\bigl\{\frac{f(s)}{s} : s > 0\bigr\} < \inf\bigl\{\frac{V(x)}{K(x)} : x \in \mathbb{R}^n\bigr\}$, one can easily derive $l < \inf\bigl\{\frac{V(x)}{K(x)} : x \in \mathbb{R}^n\bigr\}$ and

$$\begin{aligned}\mu^* &= \inf f \int_{\mathbb{R}^N} \bigl(|\nabla u|^2 + V(x)u^2\bigr)\,dx \int_{\mathbb{R}^N} K(x)u^2\,dx \\ &\geq \inf \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + lK(x)u^2)\,dx}{\int_{\mathbb{R}^N} K(x)u^2\,dx} \geq l,\end{aligned}$$

which yields an obvious contradiction between the main assumption $l > \mu^*$ of (1.8) and (K_1) in Theorem 1.1 of [Liu et al. 2008].

Remark 1.4. The function $K(x)$ in (1-1) can be permitted to be unbounded if $\alpha > \frac{N}{N-2}$ in view of the assumption (1-9). Moreover, as in Remark 1.2 of [Yin and Zhang 2009], we can illustrate that the restriction on $\beta < (\alpha-1)(N-2)-2$ in (1-9) is optimal in order to obtain the existence of H^1-positive solution to (1-1).

Remark 1.5. The assumption in (H_3) that $f(s)/s$ is a nondecreasing function for $s > 0$ can be removed by more careful analysis than that employed in this paper. This will be done in a forthcoming paper.

Next let's make some comments on the proof of Theorem 1.1. First, we modify the nonlinear term $K(x)f(u_\varepsilon)$ of (1-1) outside Λ to $g_\varepsilon(x, u_\varepsilon)$, as in [Yin and Zhang

2009], with the expression

$$g_\varepsilon(x,u)=\min\bigl\{K(x)f(u),\ \varepsilon^3/(1+|x|^{\theta_0})u^+,\ \varepsilon/(1+|x|^N)\bigr\}$$

for $x\in\mathbb{R}^N$ and $u\in\mathbb{R}$, for a positive constant θ_0 to be chosen suitably. Then we study the modified equation

$$-\varepsilon^2\Delta u_\varepsilon+V(x)u_\varepsilon=\chi_\Lambda(x)K(x)f(u_\varepsilon)+(1-\chi_\Lambda(x))g_\varepsilon(x,u_\varepsilon) \tag{1-14}$$

instead of $-\varepsilon^2\triangle u_\varepsilon+V(x)u_\varepsilon=K(x)f(u_\varepsilon)$ in (1-1). It can be shown that the corresponding Euler functional I_ε of the modified equation is well-defined and has a mountain pass geometry in the weighted Sobolev space

$$E_\varepsilon\equiv\biggl\{u\in\mathfrak{D}^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u|^2+V(x)|u|^2)\,dx<\infty\biggr\},$$

with $\mathfrak{D}^{1,2}(\mathbb{R}^N)=\{u\in L^{2N/(N-2)}(\mathbb{R}^N):\nabla u\in L^2(\mathbb{R}^N)\}$. Motivated by techniques in Chapter IV of [Ekeland 1990] or [Jeanjean and Tanaka 2002], we can use a variant of the mountain pass theorem to find a so-called Cerami sequence, and further show by contradiction that such a Cerami sequence is bounded and prove the existence of a positive solution u_ε to the modified equation.

In order to show such a solution u_ε is just the solution of the original problem (1-1), we require to derive the decay property of solution u_ε and further show $g_\varepsilon(x,u_\varepsilon)=K(x)f(u_\varepsilon)$ outside the domain Λ. To this end, we establish a compactness estimate of integral type to prove that u_ε is small away from their extreme points (see Lemma 4.6 below). Based on such an integral estimate together with the Harnack inequality, we obtain the pointwise decay property of u_ε at infinity and then complete the proof of Theorem 1.1.

Here we point out that some phenomena arising from the asymptotically linear case are quite different from those in superlinear cases, since the exponent $p>1$ of $f(u)\sim u^p$ plays a crucial role in showing the concentration-compactness of u_ε and deriving the decay property of u_ε at infinity. (Especially important is the property $F(s)\equiv\int_0^s f(\tau)\,d\tau\le k_0f(s)s$, with a positive constant $k_0<\frac{1}{2}$ and $s>0$ in superlinear cases; one can see details in [Yin and Zhang 2009; Fei and Yin 2010] and the illustrations before Lemma 4.3 in this paper.) This means that some methods used in [Yin and Zhang 2009] cannot be employed directly here.

Our paper is organized as follows. In Section 2, we replace the nonlinearity $K(x)f(u_\varepsilon)$ outside Λ by a suitably truncated function $g_\varepsilon(x,u_\varepsilon)$ and give a detailed analysis of the modified equation (1-14), so that the existence of nontrivial positive solution u_ε can be established. In Section 3, we give some preliminary results regarding the properties of the nonlinear Schrödinger equation $-\Delta u+V(\xi)u=K(\xi)f(u)$. In Section 4, we derive an integral decay estimate and use the Harnack inequality to derive the pointwise decay estimate of u_ε at infinity, inspired by

Lemma 17 of [Ambrosetti et al. 2005] and Lemmas 4.3 and 4.4 of [Yin and Zhang 2009]. From these, together with some involved analysis, we can complete the proof of Theorem 1.1.

We will use the following notations:

B_r denotes the ball centered at the origin with the radius r.

For a set $A \subset \mathbb{R}^N$, we put $A^\varepsilon = \{\varepsilon^{-1}x : x \in A\}$.

2. Existence of critical points for a modified nonlinear equation

We define a class of weighted Sobolev spaces as follows:

$$E_\varepsilon : \left\{ u \in \mathfrak{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2)\, dx < \infty \right\}$$

with $\mathfrak{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-2)}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$.

The norm of the space E_ε is denoted by

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V(x)|u|^2)\, dx \right)^{1/2} \quad \text{for } u \in E_\varepsilon.$$

Towards proving Theorem 1.1, it is necessary to modify (1-1) and further discuss the existence of solution to the modified equation.

To this end, we define a function $g_\varepsilon(x, \xi)$ by

$$g_\varepsilon(x, \xi) = \min\left\{ K(x) f(\xi),\ \frac{\varepsilon^3}{1+|x|^{\theta_0}} \xi^+,\ \frac{\varepsilon}{1+|x|^N} \right\}, \quad x \in \mathbb{R}^N,\ \xi \in \mathbb{R},$$

where $\xi^+ = \max\{\xi, 0\}$, and $\theta_0 > 2$ will be suitably chosen in (4-51).

Set

$$h_\varepsilon(x, \xi) = \chi_\Lambda(x) K(x) f(\xi) + (1 - \chi_\Lambda(x)) g_\varepsilon(x, \xi),$$

where $\chi_\Lambda(x)$ represents the characteristic function of the set Λ.

We now consider the modified nonlinear equation

$$-\varepsilon^2 \Delta u + V(x) u = h_\varepsilon(x, u), \quad x \in \mathbb{R}^N. \tag{2-1}$$

The functional corresponding to (2-1) is

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|_\varepsilon^2 - \int_\Lambda K(x) F(u)\, dx - \int_{\mathbb{R}^N \setminus \Lambda} G_\varepsilon(x, u)\, dx, \tag{2-2}$$

where $F(s) = \int_0^s f(\tau)\, d\tau$ and $G_\varepsilon(x, s) = \int_0^s g_\varepsilon(x, \tau)\, d\tau$.

By (H_2) and (H_3), for any $\delta > 0$, there exists $C_\delta > 0$ such that $f(s) \le \delta s + C_\delta |s|^{2^*-1}$

and further

$$\int_{\Lambda} K(x)F(u)\,dx \le C\delta\|u\|_{\varepsilon}^{2} + C\varepsilon^{-2^*}\|u\|_{\varepsilon}^{2^*}. \tag{2-3}$$

On the other hand, a direct computation yields for $u \in E_\varepsilon$

$$\int_{\mathbb{R}^N\setminus\Lambda} G_\varepsilon(x,u)\,dx \le \int_{\mathbb{R}^N\setminus\Lambda} g_\varepsilon(x,u)u\,dx \le C\varepsilon\|u\|_{\varepsilon}^{2}. \tag{2-4}$$

It follows from (2-3) and (2-4) that $I_\varepsilon(u)$ is well-defined on E_ε. That I_ε lies in $C^1(E_\varepsilon,\mathbb{R})$ is obvious.

Next we show that I_ε has a mountain pass geometry. Given small $\varepsilon > 0$, by (2-3) and (2-4), there are two small numbers δ and $r > 0$ such that

$$I_\varepsilon(u) \ge \tfrac{1}{2}\|u\|_{\varepsilon}^{2} - C\delta\|u\|_{\varepsilon}^{2} - C\varepsilon^{-2^*}\|u\|_{\varepsilon}^{2^*} - C\varepsilon\|u\|_{\varepsilon}^{2} \ge \tfrac{1}{4}\|u\|_{\varepsilon}^{2} \quad \text{for } \|u\|_\varepsilon \le r. \tag{2-5}$$

We now claim that

$$\inf_{\psi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla\psi|^2\,dx}{\int_{\mathbb{R}^N}\psi^2\,dx} = 0. \tag{2-6}$$

Indeed, if $\psi_0(x) \ne 0 \in H^1(\mathbb{R}^N)$, then for any fixed $\lambda\in\mathbb{R}$, one has $\psi_0(\lambda x)\in H^1(\mathbb{R}^N)$. A direct computation yields that

$$\int_{\mathbb{R}^N} |\nabla\big(\psi_0(\lambda x)\big)|^2\,dx = \lambda^{2-N}\int_{\mathbb{R}^N}|\nabla\psi_0(x)|^2\,dx$$

and

$$\int_{\mathbb{R}^N}|\psi_0(\lambda x)|^2\,dx = \lambda^{-N}\int_{\mathbb{R}^N}|\psi_0(x)|^2\,dx.$$

Therefore, we arrive at

$$\frac{\int_{\mathbb{R}^N}|\nabla\big(\psi_0(\lambda x)\big)|^2\,dx}{\int_{\mathbb{R}^N}|\psi_0(\lambda x)|^2\,dx} = \lambda^2\frac{\int_{\mathbb{R}^N}|\nabla\psi_0(x)|^2\,dx}{\int_{\mathbb{R}^N}|\psi_0(x)|^2\,dx} \to 0 \quad \text{as } \lambda\to 0, \tag{2-7}$$

proving (2-6).

From (2-6), we obtain for any fixed $\xi\in\Lambda$,

$$\inf_{\psi\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla\psi|^2 + V(\xi)|\psi|^2)\,dx}{\int_{\mathbb{R}^N}K(\xi)\psi^2\,dx} = \frac{V(\xi)}{K(\xi)}. \tag{2-8}$$

This, together with (1-7), yields that for fixed $\xi\in\Lambda$ there exists a function $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\frac{\int_{\mathbb{R}^N}(|\nabla\varphi|^2 + V(\xi)|\varphi|^2)\,dx}{\int_{\mathbb{R}^N}K(\xi)\varphi^2\,dx} < l. \tag{2-9}$$

Choose $R > 0$ such that $B_R(\xi)\subset\Lambda$. We define a smooth cut-off function

$\eta:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $\eta(t)=1$ if $0\le t\le \frac{R}{4}$, $\eta(t)=0$ if $t\ge\frac{R}{2}$ and $|\eta'(t)|\le\frac{8}{R}$.

Set

$$\varphi_\varepsilon(x)=\eta(|x-\xi|)\varphi\Big(\frac{x-\xi}{\varepsilon}\Big)\in C_0^\infty(\Lambda).$$

Then

$$\begin{aligned}(2\text{-}10)\quad & I_\varepsilon(t\varphi_\varepsilon)\\ &=\varepsilon^N\Big(\frac{t^2}{2}\int_{\mathbb{R}^N}\big(|\nabla\varphi|^2+V(\xi)|\varphi|^2\big)\,dx-\frac{1}{2}\int_{\mathbb{R}^N}K(\xi)F(t\varphi)\,dx+o_\varepsilon(1)\Big);\end{aligned}$$

here and below the notation $o_\varepsilon(1)$ stands for a quantity which satisfies $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$.

Thus we have, for $\varepsilon\le 1$,

$$(2\text{-}11)\quad \liminf_{t\to+\infty}\frac{I_\varepsilon(t\varphi_\varepsilon)}{t^2}\le\frac{1}{2}\int_{\mathbb{R}^2}\big(|\nabla\varphi|^2+V(\xi)|\varphi|^2\big)\,dx-\frac{l}{2}\int_{\mathbb{R}^N}K(\xi)\varphi^2\,dx<0.$$

Consequently, there exists some $t_0>0$ such that $I_\varepsilon(t_0\varphi_\varepsilon)<0$. This, together with (2-5), means that I_ε has a mountain pass geometry. Let

$$c_\varepsilon=\inf_{\gamma\in\Gamma_\varepsilon}\max_{0\le t\le 1}I_\varepsilon(\gamma(t)),$$

where $\Gamma_\varepsilon=\{\gamma\in C([0,1],E_\varepsilon):\gamma(0)=0,\ I_\varepsilon(\gamma(1))<0\}$. By the mountain pass theorem in Chapter IV of [Ekeland 1990], as in [Liu et al. 2008], one has the following lemma.

Lemma 2.1. *Under the assumptions* (H_1)–(H_4), *for small* $\varepsilon>0$, *there exists a sequence* $\{u_n\}\subset E_\varepsilon$ *such that* $I_\varepsilon(u_n)\to c_\varepsilon$ *and* $\|I'_\varepsilon(u_n)\|_{E'_\varepsilon}(1+\|u_n\|_\varepsilon)\to 0$ *as* $n\to\infty$, *where* E'_ε *and* $\|I'_\varepsilon(u_n)\|_{E'_\varepsilon}$ *denote by the dual space of* E_ε *and the norm of* $I'_\varepsilon(u_n)$ *in* E'_ε.

Such a sequence is called a Cerami sequence. Next we will prove the sequence $\{u_n\}$ is bounded in E_ε. We reason by contradiction: we assume up to a subsequence that $\|u_n\|_\varepsilon\to+\infty$ as $n\to+\infty$, and derive a contradiction in Lemmas 2.2 and 2.3.

So assume $\|u_n\|_\varepsilon\to\infty$ and set $\omega_n=u_n/\|u_n\|_\varepsilon$. By the boundedness of $\{\omega_n\}$ in E_ε there exists $\omega\in E_\varepsilon$ satisfying, after passing to a subsequence if necessary,

$$(2\text{-}12)\qquad \omega_n\rightharpoonup\omega\quad\text{weakly in } E_\varepsilon,$$

$$(2\text{-}13)\qquad \omega_n\to\omega\quad\text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^N)\text{ with } 2\le t<\frac{2N}{N-2},$$

$$(2\text{-}14)\qquad \omega_n\to\omega\quad\text{almost everywhere in } \mathbb{R}^N.$$

Lemma 2.2. *Under the assumptions* (H_1)–(H_3), *if* $\|u_n\|_\varepsilon\to+\infty$, *then* $\omega(x)\ge 0$ *with* $\omega(x)\not\equiv 0$ *and* ω *solves the following equation weakly in* E_ε:

$$(2\text{-}15)\qquad -\varepsilon^2\Delta u+V(x)u=\chi_\Lambda(x)lK(x)u.$$

Proof. Since it follows from Lemma 2.1 that $I_\varepsilon'(u_n)u_n^- = o_n(1)$, then $\|u_n^-\|_\varepsilon = o_n(1)$ holds true. This means $\|\omega_n^-\|_\varepsilon = o_n(1)$; hence $\omega^- = 0$ and $\omega \geq 0$.

On the other hand, by Lemma 2.1 and (2-4), we have

$$o_n(1) = \frac{I_\varepsilon'(u_n)u_n}{\|u_n\|_\varepsilon^2} = 1 - \int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n^2\,dx - \int_{\mathbb{R}^N\setminus\Lambda}\frac{g_\varepsilon(x,u_n)u_n}{\|u_n\|_\varepsilon^2}dx$$
$$\geq 1 - \int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n^2\,dx - C\varepsilon;$$

here and below $o_n(1)$ denotes a quantity that vanishes as $n \to \infty$.

From this, for small ε and large n we obtain

$$C\int_\Lambda \omega_n^2\,dx \geq \int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n^2\,dx \geq 1 - o_n(1) - C\varepsilon \geq \tfrac{1}{2}. \tag{2-16}$$

Combining (2-13) with (2-16) yields $\int_\Lambda \omega^2\,dx \geq C$, which obviously leads to $\omega \not\equiv 0$.

Next we prove that ω satisfies (2-15).

In fact, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have $\dfrac{I_\varepsilon'(u_n)\phi}{\|u_n\|_\varepsilon} = o_n(1)$, which is equivalent to

$$\int_{\mathbb{R}^N}(\varepsilon^2\nabla\omega_n\nabla\phi + V(x)\omega_n\phi)\,dx = \int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n\phi\,dx - \int_{\mathbb{R}^N\setminus\Lambda}\frac{g_\varepsilon(x,u_n)}{\|u_n\|_\varepsilon}\phi\,dx + o_n(1). \tag{2-17}$$

Due to (2-12) and (2-17), there holds

$$\int_{\mathbb{R}^N}(\varepsilon^2\nabla\omega\nabla\phi + V(x)\omega\phi)\,dx = \lim_{n\to\infty}\left(\int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n\phi\,dx - \int_{\mathbb{R}^N\setminus\Lambda}\frac{g_\varepsilon(x,u_n)}{\|u_n\|_\varepsilon}\phi\,dx\right). \tag{2-18}$$

Noting that

$$\int_\Lambda\left(K(x)\frac{f(u_n)}{u_n}\omega_n\right)^2dx \leq C\int_\Lambda V(x)\omega_n^2\,dx \leq C$$

and

$$K(x)\frac{f(u_n)}{u_n}\omega_n \to lK(x)\omega \quad \text{almost everywhere in } \Lambda,$$

we get

$$\lim_{n\to\infty}\int_\Lambda K(x)\frac{f(u_n)}{u_n}\omega_n\phi\,dx = \int_\Lambda lK(x)\omega\phi\,dx. \tag{2-19}$$

In addition, one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Lambda}\frac{g(x,u_n)}{\|u_n\|_\varepsilon}\phi\,dx=0. \tag{2-20}$$

Substituting (2-19) and (2-20) into (2-18) yields the conclusion of Lemma 2.2. □

Lemma 2.3. *Under the assumptions* (H_1)–(H_4), *Equation* (2-15) *has no nontrivial solution* $\omega(x)$ *with* $\omega(x)\geq 0$.

Proof. By (1-7), along the proof line of (2-9), there exists $v_\varepsilon\in C_0^\infty(\Lambda)$ such that

$$\frac{\int_\Lambda(\varepsilon^2|\nabla v_\varepsilon|^2+V(x)|v_\varepsilon|^2)\,dx}{\int_\Lambda K(x)v_\varepsilon^2\,dx}<l.$$

Let Λ_0 be a set satisfying $\operatorname{supp} v_\varepsilon\subsetneq\Lambda_0\subsetneq\Lambda$ and

$$\mu_0=\inf_{\varphi\in C_0^\infty(\Lambda_0)}\frac{\int_{\Lambda_0}(\varepsilon^2|\nabla\varphi|^2+V(x)|\varphi|^2)\,dx}{\int_{\Lambda_0}K(x)\varphi^2\,dx};$$

then $\mu_0<l$.

Due to the compactness of the embedding $H_0^1(\Lambda_0)\hookrightarrow L^2(\Lambda_0)$, a direct argument then shows there exists a nontrivial nonnegative function $v_0\in H_0^1(\Lambda_0)$ such that

$$-\varepsilon^2\triangle v_0+V(x)v_0=\mu_0K(x)v_0,\quad x\in\Lambda_0. \tag{2-21}$$

In addition, by the strong maximum principle [Gilbarg and Trudinger 1983, Lemma 3.4 and Theorem 3.5], one has

$$v_0>0,\quad x\in\Lambda_0,\qquad \frac{\partial v_0}{\partial\nu}<0,\quad x\in\partial\Lambda_0.$$

Moreover, we can assert that if $\omega\geq 0$ is a nontrivial solution of (2-15), then $\omega\not\equiv 0$ in Λ for small ε. Indeed, if $\omega\equiv 0$ in Λ, we get $\|\omega\|_\varepsilon^2=0$ by (2-15), which yields a contradiction since ω is nontrivial.

Hence, we can choose the domain Λ_0 so that $\int_{\Lambda_0}K(x)v_0\omega\,dx>0$. In this case, we have

$$\begin{aligned}\mu_0\int_{\Lambda_0}K(x)v_0\omega\,dx&=\int_{\Lambda_0}(-\varepsilon^2\triangle v_0+V(x)v_0)\omega\,dx\\&=l\int_{\Lambda_0}K(x)v_0\omega\,dx-\int_{\partial\Lambda_0}\varepsilon^2\frac{\partial v_0}{\partial\nu}\omega\,d\sigma\geq l\int_{\Lambda_0}K(x)v_0\omega\,dx.\end{aligned}$$

This means $\mu_0\geq l$, which contradicts with $\mu_0<l$. Hence we complete the proof of Lemma 2.3. □

Combining Lemma 2.2 with Lemma 2.3, we immediately obtain the announced result:

Lemma 2.4. *Under the assumptions* (H_1)–(H_4), *the sequence* $\{u_n\}$ *in Lemma 2.1 is bounded in* E_ε.

Next we state the main result in this section.

Lemma 2.5. *Under the assumptions* (H_1)–(H_4), *for small* $\varepsilon > 0$, *the modified functional* I_ε *of* (2-1) *has a nontrivial critical point* $u_\varepsilon \in E_\varepsilon$ *with the level* $I_\varepsilon(u_\varepsilon) = c_\varepsilon$.

Proof. The boundedness of $\{u_n\}$ in E_ε implies that there exists $u_\varepsilon \in E_\varepsilon$ satisfying, after passing to a subsequence if necessary,

$$u_n \rightharpoonup u_\varepsilon \quad \text{weakly in } E_\varepsilon, \tag{2-22}$$

$$u_n \to u_\varepsilon \quad \text{strongly in } L^t_{\mathrm{loc}}(R^N) \text{ with } 2 \le t < \frac{2N}{N-2}. \tag{2-23}$$

Next we show $\|u_n\|_\varepsilon \to \|u_\varepsilon\|_\varepsilon$ as $n \to \infty$, which together with (2-22) leads to the strong convergence of $\{u_n\}$ in E_ε.

In fact, by $I'_\varepsilon(u_n)u_\varepsilon \to 0$ and (2-22), we arrive at

$$o_n(1) = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u_n \cdot \nabla u_\varepsilon + V(x) u_n u_\varepsilon)\, dx \\ - \int_\Lambda K(x) f(u_n) u_\varepsilon \, dx - \int_{\mathbb{R}^N \setminus \Lambda} g_\varepsilon(x, u_n) u_\varepsilon \, dx, \tag{2-24}$$

which implies

$$\|u_\varepsilon\|_\varepsilon^2 - \int_\Lambda K(x) f(u_n) u_\varepsilon \, dx - \int_{\mathbb{R}^N \setminus \Lambda} g_\varepsilon(x, u_n) u_\varepsilon \, dx = o_n(1). \tag{2-25}$$

In addition, we have

$$\|u_n\|_\varepsilon^2 - \int_\Lambda K(x) f(u_n) u_n \, dx - \int_{\mathbb{R}^N \setminus \Lambda} g_\varepsilon(x, u_n) u_n \, dx = I'_\varepsilon(u_n) u_n = o_n(1). \tag{2-26}$$

On the other hand, by use of (2-23), we find

$$\lim_{n\to\infty} \int_\Lambda K(x) f(u_n) u_n \, dx = \lim_{n\to\infty} \int_\Lambda K(x) f(u_n) u_\varepsilon \, dx, \tag{2-27}$$

and for any fixed large $R > 0$ (without loss of generality, $\Lambda \subset B_R$ is assumed),

$$\lim_{n\to\infty} \int_{B_R \setminus \Lambda} g_\varepsilon(x, u_n) u_n \, dx = \lim_{n\to\infty} \int_{B_R \setminus \Lambda} g_\varepsilon(x, u_n) u_\varepsilon \, dx. \tag{2-28}$$

Thus, in order to obtain $\|u_n\|_\varepsilon \to \|u_0\|_\varepsilon$, it follows from (2-25)–(2-28) that we only need to prove the following statement:

For any given $\delta > 0$, there exists $R > 0$ such that for all n

$$\left| \int_{\mathbb{R}^N \setminus B_R} g_\varepsilon(x, u_n) u_\varepsilon \, dx \right| < \delta, \qquad \left| \int_{\mathbb{R}^N \setminus B_R} g_\varepsilon(x, u_n) u_n \, dx \right| < \delta. \tag{2-29}$$

It is only enough to check the first inequality in (2-29) since the second one is similar. By direct computations, we have

$$\left|\int_{\mathbb{R}^N\setminus B_R} g_\varepsilon(x,u_n)u_\varepsilon\,dx\right| \le \frac{C\varepsilon}{R^{(\theta_0-2)/2}}\|u_n\|_\varepsilon\|u_\varepsilon\|_\varepsilon \to 0 \quad \text{as } R\to\infty.$$

The last estimate follows from the choice of $\theta_0>2$ and the boundedness of $\{u_n\}$. Thus we have shown that $u_n\to u_\varepsilon$ in E_ε, which completes the proof of Lemma 2.5. □

Remark 2.1. Since $h_\varepsilon(x,\xi)$ is Lipschitzian continuous in ξ for fixed x, it follows from second order elliptic regularity theory that u_ε is a classical solution of (2-1). Furthermore, $u_\varepsilon>0$.

3. Solving a related constant coefficient problem

In this section, toward the proof of Theorem 1.1 in Section 4, we study the asymptotically linear problem (1-3) with constant coefficients. Some conclusions and techniques in this section are very similar to those in Section 2, but we give the argument anyway, for the reader's convenience.

We consider the functional $I^\xi(u)$ defined in (1-4) for $u\in E\equiv H_r^1(\mathbb{R}^N)=\{u\in H^1(\mathbb{R}^N): u(x)=u(|x|)\}$. Set

$$\|u\|_\xi=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(\xi)|u|^2)\,dx\right)^{1/2},$$

which is a norm equivalent to the $H^1(\mathbb{R}^N)$ norm. We now verify that I^ξ has a mountain pass geometry. Similar to the proof of (2-5), there are two small numbers $\delta, r>0$ such that

$$I^\xi(u)\ge \tfrac12\|u\|_\xi^2 - C\delta\|u\|_\xi^2 - C\|u\|_\xi^{2^*}\ge \tfrac14\|u\|_\xi^2 \quad \text{for } \|u\|_\xi\le r. \tag{3-1}$$

In addition, by (2-9), there exists a function $\varphi\in H^1(\mathbb{R}^N)\setminus\{0\}$ such that

$$\frac{\int_{\mathbb{R}^N}(|\nabla\varphi|^2+V(\xi)|\varphi|^2)\,dx}{\int_{\mathbb{R}^N}K(\xi)\varphi^2\,dx}<l. \tag{3-2}$$

Let φ^* be the symmetrization of φ (see [Berestycki and Lions 1983, Appendix A.III]). Then $\varphi^*(x)=\varphi^*(|x|)$ is a nonnegative function. Moreover, for any continuous function $H(s)$ such that $H(\varphi(x))$ is integrable in $\mathbb{R}^N$ there holds

$$\int_{\mathbb{R}^N} H(\varphi^*)\,dx=\int_{\mathbb{R}^N} H(\varphi)\,dx \tag{3-3}$$

and

$$\int_{\mathbb{R}^N}|\nabla\varphi^*|^2dx\le\int_{\mathbb{R}^N}|\nabla\varphi|^2dx. \tag{3-4}$$

By (3-2)–(3-4), we have

$$\frac{\int_{\mathbb{R}^N}(|\nabla\varphi^*|^2+V(\xi)|\varphi^*|^2)\,dx}{\int_{\mathbb{R}^N}K(\xi)|\varphi^*|^2\,dx}<l; \tag{3-5}$$

by the same argument as in (2-11) we can derive

$$\liminf_{t\to+\infty}\frac{I^{\xi}(t\varphi^*)}{t^2}<0. \tag{3-6}$$

Thus there exists $t_0>0$ such that $I^{\xi}(t_0\varphi^*)<0$, showing that I^{ξ} has a mountain pass geometry. Define the mountain level

$$c_1=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1}I^{\xi}(\gamma(t)), \tag{3-7}$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\ I^{\xi}(\gamma(1))<0\}$.

The next two lemmas are established analogously to Lemma 2.1 and Lemma 2.4, respectively.

Lemma 3.1. *There exists a sequence $\{u_n\}\subset E$ such that $I^{\xi}(u_n)\to c_1$ and*

$$\|(I^{\xi})'(u_n)\|_{H^{-1}}(1+\|u_n\|_{\xi})\to 0\quad \text{as } n\to\infty.$$

Lemma 3.2. *The sequence $\{u_n\}$ given in Lemma 3.1 is bounded in E.*

Based on Lemma 3.2, we have:

Lemma 3.3. *The functional I^{ξ} has a positive critical point $\omega\in H_r^1(\mathbb{R}^N)$ with the level $I^{\xi}(\omega)=c_1$. That is, ω is a radially symmetric solution to the problem (1-3).*

Proof. It follows from the boundedness of $\{u_n\}$ in Lemma 3.2 that there exists $\omega\in E$ satisfying, after passing to a subsequence if necessary,

$$u_n\rightharpoonup\omega\quad\text{weakly in } E, \tag{3-8}$$

$$u_n\to\omega\quad\text{strongly in } L^t_{\mathrm{loc}}(R^N)\text{ with } 2\le t<\frac{2N}{N-2}. \tag{3-9}$$

As in Lemma 2.5, we only need to show $\|u_n\|_{\xi}\to\|\omega\|_{\xi}$ as $n\to\infty$, which together with (3-8) leads to the strong convergence of $\{u_n\}$ in E.

Since $(I^{\xi})'(u_n)\omega\to 0$ and using (3-8), we arrive at

$$o_n(1)=\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\omega+V(\xi)u_n\omega)\,dx-\int_{\mathbb{R}^N}K(\xi)f(u_n)\omega\,dx.$$

This implies

$$\|\omega\|_{\xi}^2-\int_{\mathbb{R}^N}K(\xi)f(u_n)\omega\,dx=o_n(1). \tag{3-10}$$

In addition, we have

$$\|u_n\|_\xi^2 - \int_{\mathbb{R}^N} K(\xi) f(u_n)u_n \, dx = o_n(1). \tag{3-11}$$

On the other hand, it follows from (3-9) and the Hölder inequality that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - \omega) \, dx \right| &\le C \int_{\mathbb{R}^N} |u_n| \, |u_n - \omega| \, dx \\ &\le C \|u_n\|_{L^2} \|u_n - \omega\|_{L^2} = o_n(1). \end{aligned} \tag{3-12}$$

Hence, collecting (3-10)–(3-12) yields $\|u_n\|_\xi \to \|\omega\|_\xi$ as $n \to \infty$ and $I^\xi(\omega) = c_1$. Moreover, ω is a nontrivial critical point of I^ξ to E. By the principle of symmetric criticality (see [Willem 1996, Theorem 1.28]), ω is also a nontrivial critical point of I^ξ to $H^1(\mathbb{R}^N)$. In addition, $\omega > 0$ can be shown as in Remark 2.1. Therefore, Lemma 3.3 is proved. □

Next we assert that the radial function $\omega(x) = \omega(V(\xi), K(\xi); x)$ found in Lemma 3.3 is a ground state of the functional I^ξ, that is,

$$G(\xi) = I^\xi(\omega). \tag{3-13}$$

Obviously, $G(\xi) \le I^\xi(\omega)$ since $\omega \in \mathcal{M}^\xi$, ω being defined in (1-5). What is left is to show $I^\xi(\omega) \le G(\xi)$ in order to get (3-13).

For any $u \in \mathcal{M}^\xi$, let u^* be the symmetrization of u. Then $u^* \in H^1(\mathbb{R}^N)$ and $u^* \ge 0$. Consider the function

$$J(t) = I^\xi(tu^*) = \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\nabla u^*|^2 + V(\xi)|u^*|^2 \right) dx - K(\xi) \int_{\mathbb{R}^N} F(tu^*) \, dx. \tag{3-14}$$

A direct computation yields

$$\begin{aligned} \lim_{t\to\infty} \frac{J(t)}{t^2} &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u^*|^2 + V(\xi)|u^*|^2 \right) dx - \frac{lK(\xi)}{2} \int_{\mathbb{R}^N} |u^*|^2 \, dx \\ &\le \frac{K(\xi)}{2} \int_{\mathbb{R}^N} \left(\frac{f(u^*)}{u^*} - l \right) |u^*|^2 \, dx. \end{aligned} \tag{3-15}$$

In addition, by the Strauss inequality [Willem 1996, Lemma 4.5], we have $u^*(x) \to 0$ as $|x| \to +\infty$. On the other hand, it follows from $\lim_{s\to 0^+} f(s)/s = 0$ that there exists $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$ such that

$$\left(\frac{f(u^*(x))}{u^*(x)} - l \right) |u^*(x)|^2 < 0 \tag{3-16}$$

for $x \in \Omega$. If $x \in R^N \setminus \Omega$, the left-hand side of (3-16) is nonnegative, by (H_3). Thus, we have

$$\int_{\mathbb{R}^N} \left(\frac{f(u^*)}{u^*} - l \right) |u^*|^2 \, dx < 0.$$

This, together with (3-15), yields that there exists $t_0 = t_0(u^*) > 0$ such that $I^\xi(t_0u^*) < 0$. Define $\gamma(t) = tt_0u^*$; then $\gamma(t) \in \Gamma$. By the definition of c_1, we see that

$$I^\xi(\omega) = c_1 \le \max_{0\le t\le 1} I^\xi(tt_0u^*) \le \max_{0\le t\le 1} I^\xi(tt_0u) \le \max_{t\ge 0} I^\xi(tu) = I^\xi(u).$$

Since u is arbitrary, we have $I^\xi(\omega) \le G(\xi)$ and (3-13) is shown.

Remark 3.1. By the Gidas–Ni–Nirenberg result [Fei and Yin 2010, Theorem 2 and following remark], 0 is the unique maximum point of $\omega(x)$ in $\mathbb{R}^N$. This motivates us to establish a similar result in Lemma 4.5 in Section 4 below.

Finally, we show that the ground energy function $G(\xi)$ is continuous for $\xi \in \bar{\Lambda}$. Here we point out that the continuity of $G(\xi)$ corresponding to the superlinear case of $f(u)$ in (1-3) has been proved in [Wang and Zeng 1997].

Lemma 3.4. *$G(\xi)$ is continuous with respect to $\xi \in \bar{\Lambda}$.*

Proof. Consider a sequence $\{\xi_j\} \subseteq \bar{\Lambda}$ such that $\xi_j \to \xi_0 \in \bar{\Lambda}$ as $j \to +\infty$. Then $V(\xi_j) \to V(\xi_0)$, $K(\xi_j) \to K(\xi_0)$ as $j \to \infty$. Set

$$I_j(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{V(\xi_j)}{2}\int_{\mathbb{R}^N} |u|^2\,dx - K(\xi_j)\int_{\mathbb{R}^N} F(u)\,dx,$$

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{V(\xi_0)}{2}\int_{\mathbb{R}^N} |u|^2\,dx - K(\xi_0)\int_{\mathbb{R}^N} F(u)\,dx,$$

and

$$\Gamma_j = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I_j(\gamma(1)) < 0\},$$

$$\Gamma_0 = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ I_0(\gamma(1)) < 0\}.$$

From (3-7) and (3-13), we have

$$G(\xi_j) = \inf_{\gamma\in\Gamma_j}\max_{0\le t\le 1} I_j(\gamma(t)) \quad\text{and}\quad G(\xi_0) = \inf_{\gamma\in\Gamma_0}\max_{0\le t\le 1} I_0(\gamma(t)).$$

The proof of the continuity of $G(\xi)$ now proceeds in two steps.

Step 1: $\limsup_{j\to\infty} G(\xi_j) \le G(\xi_0)$.

For any fixed path $\gamma(t)$ satisfying $\gamma(0) = 0$ and $I_0(\gamma(1)) < 0$, we have $I_j(\gamma(1)) < 0$ for large j and

$$\limsup_{j\to\infty} G(\xi_j) \le \limsup_{j\to\infty} \max_{0\le t\le 1} I_j(\gamma(t)) = \max_{0\le t\le 1} I_0(\gamma(t)).$$

Since the path $\gamma(t)$ is arbitrary, this yields

$$\limsup_{j\to\infty} G(\xi_j) \le G(\xi_0). \tag{3-17}$$

Step 2: $\liminf_{j\to\infty} G(\xi_j) \geq G(\xi_0)$.

We split this step into four parts.

Let $\omega_j(x) \in H_r^1(\mathbb{R}^N)$ satisfy $G(\xi_j) = I_j(\omega_j(x))$ (the existence of $\omega_j(x)$ has been shown in Lemma 3.3).

Part 1. $\int_{\mathbb{R}^N} |\nabla \omega_j|^2\,dx$ is uniformly bounded with respect to j.

According to Pohozaev identity [Willem 1996, Appendix], we have

$$\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla \omega_j|^2\,dx = -\frac{V(\xi_j)}{2}\int_{\mathbb{R}^N} |\omega_j|^2\,dx + K(\xi_j)\int_{\mathbb{R}^N} F(\omega_j)\,dx.$$

This implies

$$G(\xi_j) = I_j(\omega_j) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla \omega_j|^2\,dx. \tag{3-18}$$

It follows from (3-17) and (3-18) that there is a positive constant C such that

$$\int_{\mathbb{R}^N} |\nabla \omega_j|^2\,dx \leq C \quad \text{for any } j. \tag{3-19}$$

Part 2. $\int_{\mathbb{R}^N} \omega_j^2\,dx$ has a uniform upper bound independent of j.

Note that up to a subsequence, there exists a radial symmetric function $\omega(x)$ such that, as $j \to \infty$,

$$\omega_j \rightharpoonup \omega, \quad \text{weakly in } \mathfrak{D}^{1,2}(\mathbb{R}^N), \tag{3-20}$$

$$\omega_j \to \omega, \quad \text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^N), \quad 1 \leq t < \frac{2N}{N-2}, \tag{3-21}$$

$$\omega_j \to \omega, \quad \text{almost everywhere in } \mathbb{R}^N. \tag{3-22}$$

By the Strauss inequality [Berestycki and Lions 1983, Lemma A.III, p. 340] for the radial function in $\mathfrak{D}^{1,2}(\mathbb{R}^N)$, we have

$$|\omega_j(x)|^2 \leq C(N)|x|^{2-N}\int_{\mathbb{R}^N} |\nabla \omega_j(x)|^2\,dx, \quad \text{for all } |x| \geq 1, \tag{3-23}$$

where the positive constant $C(N)$ only depends on N.

Since $f(s)/s \to 0$ as $s \to 0$ by the assumption (H_2), we get from (3-23) and the fact that $N \geq 5$ that

$$\frac{f(\omega_j(x))}{\omega_j(x)} \to 0 \quad \text{as } |x| \to \infty \text{ uniformly with respect to } j.$$

This implies that there exists a large number $R > 0$ such that

$$\int_{|x|\geq R}\left(V(\xi_j) - K(\xi_j)\frac{f(\omega_j)}{\omega_j}\right)|\omega_j|^2\,dx \geq C\int_{|x|\geq R} |\omega_j|^2\,dx, \tag{3-24}$$

where $C > 0$ is independent of R and j.

It follows from (3-24) and the partial differential equation satisfied by ω_j that for large R,

$$C\int_{|x|\geq R}|\omega_j|^2\,dx \leq \int_{|x|\geq R}\left(V(\xi_j)-K(\xi_j)\frac{f(\omega_j)}{\omega_j}\right)|\omega_j|^2\,dx \tag{3-25}$$
$$\leq C\int_{|x|\leq R}|\omega_j|^2\,dx \to C\int_{|x|\leq R}|\omega|^2\,dx \quad \text{as } j\to\infty.$$

Combining (3-24) with (3-25) yields that $\int_{\mathbb{R}^N}|\omega_j|^2\,dx$ has a uniform supper bound with respect to j. Thus $\omega\in L^2(\mathbb{R}^N)$ and further $\omega\in H^1(\mathbb{R}^N)$. Moreover, ω is a solution of the equation

$$-\Delta\omega(x)+V(\xi_0)\omega(x)=K(\xi_0)f(\omega),\quad x\in\mathbb{R}^N. \tag{3-26}$$

Part 3. $\int_{\mathbb{R}^N}|\omega_j|^2\,dx$ has a uniform positive lower bound with respect to j.

We now show that $\int_{\mathbb{R}^N}|\omega_j|^2\,dx$ has a uniform positive lower bound with respect to j. If so, this assertion together with (3-21) and (3-25) will yield

$$\omega\not\equiv 0. \tag{3-27}$$

Note that $V(\xi_0)/K(\xi_0)<l$ and $V(\xi_j)\to V(\xi_0)$, $K(\xi_j)\to K(\xi_0)$ as $j\to\infty$. Thus we can choose a fixed small number $\eta>0$ satisfying

$$\frac{V(\xi_0)-\eta}{K(\xi_0)+\eta}<l, \tag{3-28}$$

and, for large j,

$$V(\xi_j)>V(\xi_0)-\eta,\quad K(\xi_j)<K(\xi_0)+\eta. \tag{3-29}$$

Let m_0 be the ground energy of the functional

$$H^1(\mathbb{R}^N)\ni u\mapsto\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{V(\xi_0)-\eta}{2}\int_{\mathbb{R}^N}|u|^2\,dx-(K(\xi_0)+\eta)\int_{\mathbb{R}^N}F(u)\,dx$$

in the Nehari manifold $\mathcal{M}^\eta$, which is defined as

$$\mathcal{M}^\eta=\Big\{u\in H^1(\mathbb{R}^N)\setminus\{0\}:$$
$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx+(V(\xi_0)-\eta)\int_{\mathbb{R}^N}|u|^2\,dx=(K(\xi_0)+\eta)\int_{\mathbb{R}^N}f(u)u\,dx\Big\}.$$

By (3-28) and the similar proof on Lemma 3.3, one can show that m_0 is achieved and is positive (in the arguments of Lemma 3.3, we have used the condition $V(\xi)/K(\xi)<l$ parallel to (3-28)).

Consider the function

$$g_j(t) = \int_{\mathbb{R}^N} |\nabla(t\omega_j)|^2\,dx+(V(\xi_0)-\eta)\int_{\mathbb{R}^N} |t\omega_j|^2\,dx-(K(\xi_0)+\eta)\int_{\mathbb{R}^N} f(t\omega_j)t\omega_j\,dx.$$

Recalling that $\lim_{s\to 0} F(s)/s^2 = \lim_{s\to 0} f(s)/(2s) = 0$, we get $g_j(t) > 0$ for $0 < t \ll 1$. In addition, by (3-29) we get $g_j(1) < I'_j(\omega_j)\omega_j = 0$. Therefore there exists a $t_j \in (0, 1)$ such that $g_j(t_j\omega_j) = 0$, that is,

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t_j\omega_j)|^2\,dx + \frac{V(\xi_0)-\eta}{2}\int_{\mathbb{R}^N} |t_j\omega_j|^2\,dx - (K(\xi_0)+\eta)\int_{\mathbb{R}^N} F(t_j\omega_j)\,dx \ge m_0. \tag{3-30}$$

Set

$$h_j(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(t\omega_j)|^2\,dx + \frac{V(\xi_j)}{2}\int_{\mathbb{R}^N} |t\omega_j|^2\,dx - K(\xi_j)\int_{\mathbb{R}^N} F(t\omega_j)\,dx.$$

It follows from a direct computation and the assumption (H_3) that, for $t \in (0, 1]$,

$$\begin{aligned} h'_j(t) &= t\int_{\mathbb{R}^N} |\nabla\omega_j|^2\,dx+tV(\xi_j)\int_{\mathbb{R}^N}|\omega_j|^2\,dx-K(\xi_j)\int_{\mathbb{R}^N} f(t\omega_j)\omega_j\,dx \\ &\ge 0. \end{aligned} \tag{3-31}$$

Combining (3-29), (3-30), and (3-31), we obtain, for large j,

$$I_j(\omega_j) \ge m_0.$$

Together with (3-18), this yields, for large j,

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla\omega_j|^2\,dx = I_j(\omega_j) \ge m_0. \tag{3-32}$$

In addition, since

$$\left(\frac{F(s)}{s^2}\right)' = \frac{f(s)s-2F(s)}{s^3} \ge 0 \quad \text{and} \quad \lim_{s\to+\infty}\frac{F(s)}{s^2} = \lim_{s\to+\infty}\frac{f(s)}{2s} = \frac{l}{2},$$

we have

$$0 \le \frac{F(s)}{s^2} \le \frac{l}{2}, \qquad s \ne 0. \tag{3-33}$$

Therefore, by (3-32), (3-33), and the Pohozaev identity we find that

$$0 < C \le \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla\omega_j|^2\,dx \le C\int_{\mathbb{R}^N} \omega_j^2\,dx, \tag{3-34}$$

where C is a generic positive constant independent of j, that is, $\int_{\mathbb{R}^N} |\omega_j|^2\,dx$ have a uniform positive lower bound with respect to j.

Part 4. $\lim_{j\to\infty}\int_{\mathbb{R}^N} F(\omega_j)\,dx = \int_{\mathbb{R}^N} F(\omega)\,dx$.

In order to show

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} F(\omega_j)\,dx = \int_{\mathbb{R}^N} F(\omega)\,dx, \tag{3-35}$$

then by (3-21) we only need to prove:

For any given $\delta > 0$, there exists $R > 0$ such that, for large j,

$$\left|\int_{\mathbb{R}^N\setminus B_R} F(\omega_j)\,dx\right| < \delta. \tag{3-36}$$

In fact, if we set η_R to be a smooth cut-off function such that $\eta_R = 0$ for $|x| \le \frac{R}{2}$, $\eta_R = 1$ for $|x| \ge R$ and $|\nabla\eta| \le \frac{4}{R}$, then multiplying by $\eta_R\omega_j$ the equation

$$-\Delta\omega_j + V(\xi_j)\omega_j = K(\xi_j)f(\omega_j), \quad x \in \mathbb{R}^N,$$

yields, for large R and j,

$$C\int_{|x|\ge R}\left(|\nabla\omega_j|^2 + |\omega_j|^2\right)dx \le \frac{C}{R} \to 0 \quad \text{as } R \to +\infty,$$

which means that (3-36) and further (3-35) hold.

Finally, we show $\liminf_{j\to\infty} G(\xi_j) \ge G(\xi_0)$. In view of (3-35), (3-26)–(3-27) and the fact that $G(\xi_0)$ is the ground energy of the functional I_0, we have

$$\begin{aligned}\liminf_{j\to\infty} G(\xi_j) &\\ = \liminf_{j\to\infty}&\left\{\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla\omega_j|^2 + V(\xi_j)|\omega_j|^2\right)dx - K(\xi_j)\int_{\mathbb{R}^N} F(\omega_j)\,dx\right\} \ge G(\xi_0).\end{aligned} \tag{3-37}$$

Thus the continuity of $G(\xi)$ is derived from (3-17) and (3-37), that is, Lemma 3.4 is proved. □

4. The proof of Theorem 1.1

At first, we intend to obtain an upper bound estimate of the critical value c_ε corresponding to the functional $I_\varepsilon(u)$ defined in Section 2, which will play a crucial role in establishing the concentration and decay estimates of solution u_ε to Equation (2-1). From the decay estimates of u_ε we can show $g_\varepsilon(x, u_\varepsilon) \equiv K(x)f(u_\varepsilon)$ in $\mathbb{R}^N \setminus \Lambda$ and subsequently complete the proof of Theorem 1.1.

Lemma 4.1. *Under the hypotheses* (H_1)–(H_4), *and with* c_0 *as in* (H_4), *we have, for small* $\varepsilon > 0$,

$$c_\varepsilon \le (c_0 + o_\varepsilon(1))\varepsilon^N. \tag{4-1}$$

Proof. For $\xi \in \Lambda$, choose $R > 0$ such that $B_R(\xi) \subset \Lambda$. Define a smooth cut-off function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\eta(t) = 1$ if $0 \le t \le \frac{R}{4}$, $\eta(t) = 0$ if $t \ge \frac{R}{2}$ and $|\eta'(t)| \le \frac{8}{R}$. Set

$$w_\varepsilon(x) = \eta(|x-\xi|)\omega\Big(\frac{x-\xi}{\varepsilon}\Big),$$

where $\omega(x) = \omega(V(\xi), K(\xi); x)$ is the solution of (1-3).

Noting that w_ε is compactly supported in Λ, one can get $G_\varepsilon(x, tw_\varepsilon) = 0$ for all $t \ge 0$ and $x \in \Lambda$, where $G_\varepsilon(x, u)$ is the function defined in (2-2). Then as in the argument in (2-11), there exists a sufficiently large $T > 0$ such that $I_\varepsilon(Tw_\varepsilon) < 0$. This implies that the path $\gamma_\varepsilon(t) = \{tTw_\varepsilon : t \in [0, 1]\}$ is an element of Γ_ε satisfying $c_\varepsilon \le \max_{0\le t\le 1} I_\varepsilon(\gamma_\varepsilon(t))$. Also, similar to the proof of (2-10), we infer that $I_\varepsilon(tTw_\varepsilon) = \varepsilon^N(I^\xi(tTw) + o_\varepsilon(1))$. Hence

$$\max_{0\le t\le 1} I_\varepsilon(\gamma_\varepsilon(t)) = \max_{0\le t\le 1} I_\varepsilon(tTw_\varepsilon) = \varepsilon^N(\max_{0\le t\le 1} I^\xi(tTw) + o_\varepsilon(1)) = \varepsilon^N(G(\xi) + o_\varepsilon(1)).$$

Since ξ is arbitrary and the smallness of ε is independent of the choice of ξ, then Lemma 4.1 is proved. □

The next result illustrates that the maximum of u_ε on $\overline{\Lambda}$ has a uniform positive lower bound.

Lemma 4.2. *Let x_ε be the maximum point of u_ε on $\overline{\Lambda}$, then there exists a positive constant C independent of ε such that*

$$u_\varepsilon(x_\varepsilon) \ge C. \tag{4-2}$$

Proof. By (H_2) and (H_3), for any $\delta > 0$, there exists $C_\delta > 0$ such that $f(s) \le \delta s + C_\delta |s|^2$. From $I'_\varepsilon(u_\varepsilon)u_\varepsilon = 0$, one has, for small δ and ε,

$$\begin{aligned}\|u_\varepsilon\|_\varepsilon^2 &= \int_\Lambda K(x) f(x, u_\varepsilon) u_\varepsilon \, dx + \int_{\mathbb{R}^N \setminus \Lambda} g_\varepsilon(x, u_\varepsilon) u_\varepsilon \, dx \\ &\le \frac{1}{2}\|u_\varepsilon\|_\varepsilon^2 + C\|u_\varepsilon\|_\varepsilon^2 \max_{\overline{\Lambda}} u_\varepsilon.\end{aligned}$$

Obviously this means that there exists a positive number C independent of ε such that $u_\varepsilon(x_\varepsilon) \ge C$ holds true due to $\|u_\varepsilon\|_\varepsilon \ne 0$, then the proof of Lemma 4.2 is completed. □

Note that since $f(s)$ is asymptotically linear, then in the general case, there is no number $\theta > 0$ such that $(2+\theta)F(s) \le f(s)s$ for any $s > 0$, here $F(s) = \int_0^s f(\tau)\, d\tau$. However, in the superlinear case, this property of $(2+\theta)F(s) \le f(s)s$ with $\theta > 0$ play a crucial role in obtaining the uniform boundedness of $\varepsilon^{-N}\|u_\varepsilon\|_\varepsilon$ from (4-1), which will be used to derive the decay estimate of u_ε at infinity and the concentration of u_ε as $\varepsilon \to 0$ (one can see the details in [Fei and Yin 2010] and some references therein). To overcome this kind of difficulty, next we will use some different ingredients (motivated by the proofs of Lemmas 2.2–2.3) to treat the uniform boundedness of $\varepsilon^{-N}\|u_\varepsilon\|_\varepsilon$.

Lemma 4.3. *There exists a positive constant C independent of small ε such that*

$$\varepsilon^{-N}\int_{\mathbb{R}^N}\bigl(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)|u_\varepsilon|^2\bigr)\,dx\le C, \tag{4-3}$$

namely,

$$\int_{\mathbb{R}^N}\bigl(|\nabla v_\varepsilon|^2+V(\varepsilon x+x_\varepsilon)|v_\varepsilon|^2\bigr)\,dx\le C, \tag{4-4}$$

where $v_\varepsilon(x)=u_\varepsilon(\varepsilon x+x_\varepsilon)$ and the meaning of x_ε is given in Lemma 4.2.

Proof. For convenience we will use the notation $\|v_\varepsilon\|$ with

$$\|v_\varepsilon\|=\left(\int_{\mathbb{R}^N}\bigl(|\nabla v_\varepsilon|^2+V(\varepsilon x+x_\varepsilon)|v_\varepsilon|^2\bigr)\,dx\right)^{1/2}.$$

If (4-4) does not hold, there exists a sequence of functions $v_n(x)\equiv u_{\varepsilon_n}(\varepsilon_n x+x_n)$ such that $\|v_n\|\to+\infty$ as $n\to\infty$ and $v_n(x)$ satisfies

$$\begin{aligned}-\Delta v_n+V(\varepsilon_n x+x_n)v_n&\\ =\chi_{\Omega_n}(x)K(\varepsilon_n x+x_n)f(v_n)&+(1-\chi_{\Omega_n}(x))g_{\varepsilon_n}(\varepsilon_n x+x_n,v_n),\end{aligned} \tag{4-5}$$

where $\Omega_n\equiv\varepsilon_n^{-1}(\Lambda-x_n)$ and $x_n\equiv x_{\varepsilon_n}\in\overline{\Lambda}$.

Set $\omega_n=v_n/\|v_n\|$, then $\|\omega_n\|=1$ and $\omega_n(x)$ satisfies

$$\begin{aligned}-\Delta\omega_n+V(\varepsilon_n x+x_n)\omega_n&\\ =\chi_{\Omega_n}(x)K(\varepsilon_n x+x_n)\frac{f(v_n)}{v_n}\omega_n&+(1-\chi_{\Omega_n}(x))\frac{g_{\varepsilon_n}(\varepsilon_n x+x_n,v_n)}{\|v_n\|}.\end{aligned} \tag{4-6}$$

We rewrite (4-6) as

$$-\Delta\omega_n=a_n(x)\omega_n, \tag{4-7}$$

where

$$a_n(x)=-V(\varepsilon_n x+x_n)+\chi_{\Omega_n}(x)K(\varepsilon_n x+x_n)\frac{f(v_n)}{v_n}+(1-\chi_{\Omega_n}(x))\frac{g_{\varepsilon_n}(\varepsilon_n x+x_n,v_n)}{v_n}.$$

For any fixed and bounded smooth domain $\Omega\subset\mathbb{R}^N$ and fixed $\alpha\in(0,1)$, due to $\|a_n(x)\|_{L^\infty(\Omega)}\le C(\Omega)$, it follows from $\|\omega_n\|=1$ and the elliptic equation (4-7) that $\|\omega_n\|_{C^{1,\alpha}(\bar\Omega)}\le C(\Omega,\alpha)$, where the positive constants $C(\Omega)$ and $C(\Omega,\alpha)$ depend on Ω and Ω,α respectively. Therefore, for fixed $\beta\in(0,\alpha)$, there exists a subsequence still denoted by $\{\omega_n\}$ and a function ω such that $\omega_n\to\omega$ in $C^{1,\beta}(\bar\Omega)$.

In particular, for a series of closed ball sequences $B_k(0)$, $k=1,2,\dots$, then there exists a subsequence $\{\omega_{1n}\}$ and a function ω_1 such that $\omega_{1n}\to\omega_1$ in $C^{1,\beta}(B_1(0))$, and there exists a subsequence $\{\omega_{(k+1)n}\}\subseteq\{\omega_{kn}\}$ and a function ω_{k+1} such that

$\omega_{(k+1)n} \to \omega_{k+1}$ in $C^{1,\beta}(B_{k+1}(0))$ as $n \to \infty$ for $k \geq 1$. By the diagonal process, one knows that there exists a subsequence still denoted by $\{\omega_n\}$ and a function ω such that $\omega_n \to \omega$ in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$. Of course, $\lim_{n\to\infty} \omega_n(x) = \omega(x)$ holds for $x \in \mathbb{R}^N$.

Let $x_n \to x_0 \in \overline{\Lambda}$. We consider two cases.

Case I: $\lim_{n\to\infty} \operatorname{dist}(x_n, \partial\Lambda)/\varepsilon_n = +\infty$.

In this case, by taking a subsequence, we can assume $x_n \in \Lambda$. Hence $0 \in \Omega_n$ and $\lim_{n\to\infty} \operatorname{dist}(0, \partial\Omega_n) = \lim_{n\to\infty} \operatorname{dist}(x_n, \partial\Lambda)/\varepsilon_n = +\infty$, which leads to $\lim_{n\to\infty} \Omega_n = \mathbb{R}^N$.

For any fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds $\operatorname{supp}\varphi \subseteq \Omega_n$ for lager n. Multiplying φ on two hand sides of (4-6) and integrating by parts yield, for large n,

$$\int [\nabla\omega_n \nabla\varphi + V(\varepsilon_n x + x_n)\omega_n\varphi]dx = \int K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega_n\varphi\, dx. \tag{4-8}$$

Note that

$$\lim_{n\to\infty} \int [\nabla\omega_n \nabla\varphi + V(\varepsilon_n x + x_n)\omega_n\varphi]dx = \int [\nabla\omega\nabla\varphi + V(x_0)\omega\varphi]dx. \tag{4-9}$$

Next we show that

$$\lim_{n\to\infty} \int K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega_n\varphi\, dx = \int K(x_0)l\omega\varphi\, dx. \tag{4-10}$$

Define the set $A = \{x \in \mathbb{R}^N : \lim_{n\to\infty} v_n(x) = +\infty\}$ and let $A^c = \mathbb{R}^N \setminus A$. If $x \in A$, then $\lim_{n\to\infty} f(v_n(x))/v_n(x) = l$. If $x \in A^c$, since $\lim_{n\to\infty} \|v_n\| = +\infty$, we have $\omega(x) = \lim_{n\to\infty} \omega_n(x) = \liminf_{n\to\infty} v_n(x)/\|v_n\| = 0$.

On the other hand, since $K(\varepsilon_n x + x_n)$ is uniformly bounded for $x \in \operatorname{supp}\varphi$ with respect to n and $f(s)/s$ is also bounded, we have

$$\lim_{n\to\infty} \int K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega_n\varphi\, dx = \lim_{n\to\infty} \int K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega\varphi\, dx. \tag{4-11}$$

Therefore,

$$\lim_{n\to\infty} \int_{\operatorname{supp}\varphi\cap A} K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega\varphi\, dx = \int_{\operatorname{supp}\varphi\cap A} K(x_0)l\omega\varphi\, dx. \tag{4-12}$$

In addition, obviously,

$$\lim_{n\to\infty} \int_{\operatorname{supp}\varphi\cap A^c} K(\varepsilon_n x + x_n)\frac{f(v_n)}{v_n}\omega\varphi\, dx = 0 = \int_{\operatorname{supp}\varphi\cap A^c} K(x_0)l\omega\varphi\, dx. \tag{4-13}$$

Collecting (4-11)–(4-13) yields (4-10).

From (4-8)–(4-10), we arrive at

$$\int_{\mathbb{R}^N} \nabla\omega\nabla\varphi + V(x_0)\omega\varphi = \int_{\mathbb{R}^N} K(x_0)l\omega\varphi, \tag{4-14}$$

which means that ω solves

$$-\Delta\omega + V(x_0)\omega = K(x_0)l\omega. \tag{4-15}$$

Case II: $\liminf_{n\to\infty} \operatorname{dist}(x_n, \partial\Lambda)/\varepsilon_n \le C$.

In this case, we can show that $x_0 \in \partial\Lambda$. Thus, up to a rotation, we can obtain $\lim_{n\to\infty} \Omega_n = \{x \in \mathbb{R}^N : x_1 < 0\}$. Similarly to Case I, we conclude that the function $\omega(x)$ satisfies

$$-\Delta\omega + V(x_0)\omega = K(x_0)l\omega\chi_{\{x_1<0\}}(x). \tag{4-16}$$

In Case I or Case II, for any fixed bounded domain $M \subset \mathbb{R}^N$ or $M \subset \{x \in \mathbb{R}^N : x_1 < 0\}$ we have

$$\begin{aligned}\int_M \big[|\nabla\omega|^2 + V(x_0)\omega^2\big]dx &= \lim_{n\to\infty} \int_M \big[|\nabla\omega_n|^2 + V(\varepsilon_n x + x_n)\omega_n^2\big]dx \\ &\le \int_{\mathbb{R}^N} \big[|\nabla\omega_n|^2 + V(\varepsilon_n x + x_n)\omega_n^2\big]dx = 1;\end{aligned}$$

then

$$\int_{\mathbb{R}^N} \big[|\nabla\omega|^2 + V(x_0)\omega^2\big]dx \le 1, \tag{4-17}$$

which means $\omega \in H^1(\mathbb{R}^N)$ due to $V(x_0) > 0$.

It follows the equations (4-15)–(4-16), together with (4-17), the fact that $\omega \ge 0$, regularity theory and the strong maximum principle for second-order elliptic equations, that we can get $\omega(x) \in C^{2,\gamma}(\mathbb{R}^N)$ in Case I and $\omega(x) \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0, 1)$ in Case II, and $\omega(x) > 0$ with $\omega(x) \to 0$ as $|x| \to \infty$. However, this is contradictory with the conclusion of Lemma 2.3. Thus (4-15) and (4-16) have no nontrivial nonnegative solutions. Lemma 4.3 is proved. □

Next we assert that the maximum point of u_ε on $\bar{\Lambda}$ must lie in the interior of Λ.

Lemma 4.4. $\lim_{\varepsilon\to 0} \max_{\partial\Lambda} u_\varepsilon = 0$.

Proof. To prove this, we argue by contradiction assuming that there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that for each n,

$$\max_{\partial\Lambda} u_{\varepsilon_n} \ge C > 0. \tag{4-18}$$

Let $x_n \in \partial\Lambda$ such that $u_{\varepsilon_n}(x_n) = \max_{\partial\Lambda} u_{\varepsilon_n}$ and $x_n \to x_0 \in \partial\Lambda$ as $n \to \infty$. Define $v_n(x) = u_{\varepsilon_n}(\varepsilon_n x + x_n)$, then $v_n(0) \ge C$ and $v_n(x)$ satisfies

$$\begin{aligned}&-\Delta v_n + V(\varepsilon_n x + x_n)v_n \\ &\qquad = \chi_{\Omega_n}(x)K(\varepsilon_n x + x_n)f(v_n) + (1 - \chi_{\Omega_n}(x))g_{\varepsilon_n}(\varepsilon_n x + x_n, v_n),\end{aligned} \tag{4-19}$$

where $\Omega_n \equiv \varepsilon_n^{-1}(\Lambda - x_n)$.

By (4-4), there holds

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \le C,$$

which deduces that for large n, for any fixed $R > 0$, there exists a positive constant $C(R)$ depending on R such that

$$\int_{B_R(0)} \bigl(|\nabla v_n|^2 + v_n^2\bigr)\,dx \le C(R).$$

In terms of this and (4-19), as in the proof of Lemma 4.3, there exists some nonnegative function $v(x)$ such that $v_n \to v(x)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $v(x)$ satisfies

$$-\Delta v + V(x_0)v = K(x_0)\chi_{\{x_1<0\}} f(v), \quad x = (x_1, x') \in \mathbb{R}^N. \tag{4-20}$$

Note that $v_n(0) \ge C$, then $v(0) \ge C$ and further $v(x) > 0$ in $\mathbb{R}^N$ by the maximum principle and Equation (4-20).

On the other hand, acting the test function $\partial_{x_1} v$ on (4-20) yields

$$\int_{\mathbb{R}^{N-1}} F(v(0, x'))\,dx' = 0,$$

which leads to $v(0, x') = 0$. However, this is impossible due to $v(x) > 0$ in $\mathbb{R}^N$. Thus Lemma 4.4 is proved. □

Lemma 4.5. *For small ε, u_ε possesses at most one maximum point x_ε on $\overline{\Lambda}$ and $G(x_\varepsilon) \to c_0$ as $\varepsilon \to 0$.*

Proof. First, we prove $G(x_\varepsilon) \to c_0$ as $\varepsilon \to 0$.

If not, we have $\limsup_{\varepsilon\to 0} G(x_\varepsilon) > c_0$. Let $x_{\varepsilon_j} \to x_0 \in \overline{\Lambda}$; then $\lim_{j\to\infty} G(x_{\varepsilon_j}) = \limsup_{\varepsilon\to 0} G(x_\varepsilon) > c_0$, which means $G(x_0) > c_0$.

Set $v_j(x) = u_{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j})$. Then v_j solves

$$\begin{aligned} -\Delta v_j + V(\varepsilon_j x + x_{\varepsilon_j})v_j \\ = \chi_{\Omega_j}(x) K(\varepsilon_j x + x_{\varepsilon_j}) f(v_j) + (1 - \chi_{\Omega_j}(x)) g_{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j}, v_j). \end{aligned} \tag{4-21}$$

As before, we can show that v_j converges in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for $\alpha \in (0, 1)$ to some function v_0 that satisfies

$$-\Delta v_0 + V(x_0)v_0 = K(x_0) f(v_0), \quad x \in \mathbb{R}^N \tag{4-22}$$

or

$$-\Delta v_0 + V(x_0)v_0 = K(x_0)\chi_{\{x_1<0\}} f(v_0), \quad x = (x_1, x') \in \mathbb{R}^N. \tag{4-23}$$

The case of (4-23) can be excluded by the same argument as in Lemma 4.4, so we focus on the case of (4-22).

Set

$$J_{\varepsilon_j}(v_j)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_j|^2\,dx+V(\varepsilon_j x+x_{\varepsilon_j})|v_j|^2\,dx \\ -\int_{(\Lambda-x_{\varepsilon_j})/\varepsilon_j}K(\varepsilon_j x+x_{\varepsilon_j})F(v_j)\,dx-\int_{\mathbb{R}^N\setminus(\Lambda-x_{\varepsilon_j})/\varepsilon_j}G(\varepsilon_j x+x_{\varepsilon_j},v_j)\,dx. \tag{4-24}$$

By invoking Lemma 2.2 in [del Pino and Felmer 1996] together with $2F(s)\le f(s)s$, we conclude that

$$\liminf_{j\to\infty}J_{\varepsilon_j}(v_j)\ge I^{x_0}(v_0). \tag{4-25}$$

This, together with (4-1), yields

$$c_0\ge\liminf_{j\to\infty}\varepsilon_j^{-N}I_{\varepsilon_j}(u_{\varepsilon_j})=\liminf_{j\to\infty}J_{\varepsilon_j}(v_j)\ge I^{x_0}(v_0)\ge G(x_0)>c_0,$$

which leads to a contradiction.

In addition, using the arguments in [del Pino and Felmer 1996, p. 133], we can show that u_ε possesses at most one maximum point x_ε on $\overline{\Lambda}$. We omit the details. This concludes the proof of Lemma 4.5. □

Next we establish a compactness result for u_ε which will be crucial to derive the decay of $u_\varepsilon(x)$ as $|x|\to\infty$.

Lemma 4.6. *For any $\nu>0$, there exist $\rho_0(\nu)$, $\varepsilon_0(\nu)>0$ such that for $\rho>\rho_0(\nu)$, $\varepsilon<\varepsilon_0(\nu)$, then*

$$\operatorname{dist}(x_\varepsilon,M)<\nu, \tag{4-26}$$

and

$$\varepsilon^{-N}\int_{\mathbb{R}^N\setminus B_{\varepsilon\rho}(x_\varepsilon)}\bigl(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)|u_\varepsilon|^2\bigr)\,dx<\nu, \tag{4-27}$$

where $M=\{\xi\in\Lambda:G(\xi)=c_0\}$, and the meaning of c_0 is given in (1-8).

Proof. Since the first conclusion can be directly derived from Lemma 4.5, then it suffices to prove (4-27).

As a consequence of Lemma 4.5 and the assumption on $G(x)$ in (H_4), we have $d=\inf_n\operatorname{dist}(x_n,\partial\Lambda)>0$ and $\Lambda_n=(\Lambda-x_n)/\varepsilon_n\supset B_{d/\varepsilon_n}\equiv B_{\tilde\rho_n}$.

If (4-27) does not hold, then we can assume that there exist $\nu_0>0$, $\tilde\rho_n>\rho_n\to+\infty$, $\varepsilon_n\to 0$ as $n\to\infty$ such that

$$\mathcal{T}\equiv\varepsilon_n^{-N}\int_{\mathbb{R}^N\setminus B_{\varepsilon_n\rho_n}(x_n)}\bigl(\varepsilon^2|\nabla u_n|^2+V(x)|u_n|^2\bigr)\,dx>\nu_0, \tag{4-28}$$

where $x_n\equiv x_{\varepsilon_n}$, $u_n\equiv u_{\varepsilon_n}$.

Set $v_n(x) = u_n(\varepsilon_n x + x_n)$, $V_n(x) = V(\varepsilon_n x + x_n)$ and $v_n \rightharpoonup v_0$, $x_n \to x_0 \in M$ as $n \to \infty$. Then, by (4-1) and (4-25) as $n \to \infty$,

$$\tfrac{1}{2}\mathcal{T} = \varepsilon_n^{-N}\frac{1}{2}\int_{\mathbb{R}^N}\bigl(\varepsilon_n^2|\nabla u_n|^2 + V(x)|u_n|^2\bigr)\,dx \\ - \varepsilon_n^{-N}\frac{1}{2}\int_{B_{\varepsilon_n\rho_n}(x_n)}\bigl(\varepsilon_n^2|\nabla u_n|^2 + V(x)|u_n|^2\bigr)\,dx \to 0,$$

which is contradictory with (4-28). We have completed the proof of Lemma 4.6. □

Before we treat the decay estimate of u_ε at infinity, we need to establish more integration estimates based on Lemma 4.6.

Note that by the assumptions in (H_2) and (H_3), then for any fixed $p > 1$, there exists a positive constant $C_1 = C_1(p)$ depending on p such that

$$f(s) \le \frac{1}{16}\max_{\xi\in\bar{\Lambda}}\frac{V(\xi)}{K(\xi)}s + C_1|s|^p. \tag{4-29}$$

Furthermore we have a relation between $\|u\|_\varepsilon$ and $\int_\Lambda K(x)|u|^{p+1}dx$ for any $1 < p < \frac{N+2}{N-2}$ as follows, which comes from Lemma 2.1 of [Yin and Zhang 2009].

Lemma 4.7. *Under the assumptions (H_1) and (H_4), for each $\varepsilon \in (0, 1]$, then there exists a positive constant $C_2 = C_2(p)$ depending only on p such that*

$$\int_\Lambda K(x)|u|^{p+1}dx \le C_2\varepsilon^{-N(p-1)/2}\|u\|_\varepsilon^{p+1} \quad \text{for } u \in E_\varepsilon, \tag{4-30}$$

where the domain Λ is defined in the assumption (H_4).

For later use, we introduce two fixed positive numbers $K_0 > 128$ and $c > 0$ such that $c^2 \ge 128K_0^2/(d_0^2V_1)$, where $d_0 = \operatorname{dist}(\partial\Lambda, M) > 0$ and $V_1 = \frac{1}{2}\min_{x\in\Lambda} V(x) > 0$.

Set $\nu_0 = \min\{d_0/K_0, (16C_1C_2)^{-2/(p-1)}\}$, where C_1 and C_2 are given in (4-29)–(4-30). Take $\varepsilon_1 = \min\{\varepsilon_0(\nu_0), d_0/(K_0\rho_0(\nu_0)), (\ln 2)/c\}$, where $\varepsilon_0(\nu_0)$ and $\rho_0(\nu_0)$ are given in Lemma 4.6. From now on, we always assume $\varepsilon < \varepsilon_1$ and $\nu < \nu_0$ in (4-26)–(4-27).

It follows from (4-26) that, for $\varepsilon < \varepsilon_1$ and $\nu < \nu_0$,

$$\operatorname{dist}(x_\varepsilon, \partial\Lambda) > \frac{d_0}{2} \quad \text{and} \quad \varepsilon\rho_0(\nu_0) < \frac{d_0}{K_0}. \tag{4-31}$$

Define $\Omega_{n,\varepsilon} = \mathbb{R}^N \setminus B_{R_{n,\varepsilon}}(x_\varepsilon)$ with $R_{n,\varepsilon} = e^{c\varepsilon n}$ and let $\tilde{n} > \hat{n}$ be integers such that

$$R_{\hat{n}-1,\varepsilon} < \frac{d_0}{K_0} \le R_{\hat{n},\varepsilon}, \qquad R_{\tilde{n}+2,\varepsilon} \le \frac{d_0}{2} < R_{\tilde{n}+3,\varepsilon}. \tag{4-32}$$

By the second inequality in (4-31), one gets $R_{n,\varepsilon} \geq R_{\hat{n},\varepsilon} \geq d_0/K_0 > \varepsilon\rho_0(\nu_0)$ for $n \geq \hat{n}$ and $\varepsilon < \varepsilon_1$, and this also yields

$$\Omega_{n,\varepsilon} \cap B_{\varepsilon\rho_0(\nu_0)}(x_\varepsilon) = \varnothing. \tag{4-33}$$

Let $\chi_{n,\varepsilon}(x)$ be smooth cut-off functions such that $\chi_{n,\varepsilon}(x) = 0$ in $B_{R_{n,\varepsilon}}(x_\varepsilon)$, $\chi_{n,\varepsilon}(x) = 1$ in $\Omega_{n+1,\varepsilon}$, $0 \leq \chi_{n,\varepsilon} \leq 1$ and $|\nabla\chi_{n,\varepsilon}| \leq 2/(R_{n+1,\varepsilon} - R_{n,\varepsilon})$.

Lemma 4.8. *Under assumptions* (H_1) *and* (H_2), *if* $\varepsilon < \varepsilon_1$ *and* $\hat{n} \leq n \leq \tilde{n}$, *we have*

$$\int_{\mathbb{R}^N} A_{n,\varepsilon} dx \leq \frac{1}{2}\int_{\Omega_{n,\varepsilon}} \left(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) dx, \tag{4-34}$$

where $A_{n,\varepsilon}(x) = \varepsilon^2|\nabla(\chi_{n,\varepsilon}u_\varepsilon)|^2 + V(x)(\chi_{n,\varepsilon}u_\varepsilon)^2$.

Proof. For $\varepsilon < \varepsilon_1$, it follows from a straightforward computation that

$$R_{n+1,\varepsilon} - R_{n,\varepsilon} \geq \frac{c\varepsilon R_{n+1,\varepsilon}}{2}.$$

This yields

$$\varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 \leq \frac{4\varepsilon^2}{|R_{n+1,\varepsilon} - R_{n,\varepsilon}|^2} \leq \frac{16}{c^2 R_{n+1,\varepsilon}^2}. \tag{4-35}$$

From the choice of c, for $\varepsilon < \varepsilon_1$ and $\hat{n} \leq n \leq \tilde{n}$, we arrive at

$$\frac{128}{c^2 R_{n+1,\varepsilon}^2} \leq V(x) \quad \text{for } x \in \{x : R_{n,\varepsilon} \leq |x - x_\varepsilon| < R_{n+1,\varepsilon}\}. \tag{4-36}$$

Noting that $\nabla\chi_{n,\varepsilon}$ is supported in $\{x : R_{n,\varepsilon} \leq |x - x_\varepsilon| < R_{n+1,\varepsilon}\}$, then for $\varepsilon < \varepsilon_1$ and $\hat{n} \leq n \leq \tilde{n}$, by (4-35) and (4-36), we obtain

$$\varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 \leq \tfrac{1}{8}V(x) \quad \text{in } \mathbb{R}^N. \tag{4-37}$$

Multiplying (2-1) by $\chi_{n,\varepsilon}^2 u_\varepsilon$ and integrating over $\mathbb{R}^N$ yields

$$\int_{\mathbb{R}^N} A_{n,\varepsilon} dx = I + II + III,$$

where

$$\begin{aligned}
I &= \int_{\Omega_{n,\varepsilon}} \varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 u_\varepsilon^2\, dx,\\
II &= \int_{\Lambda\cap\Omega_{n,\varepsilon}} K(x) f(u_\varepsilon)\chi_{n,\varepsilon}^2 u_\varepsilon\\
&\leq \frac{1}{16}\int_{\Lambda\cap\Omega_{n,\varepsilon}} V(x)u_\varepsilon^2 dx + C_1\int_{\Lambda\cap\Omega_{n,\varepsilon}} K(x)|u_\varepsilon|^{p+1} dx,\\
III &= \int_{(\mathbb{R}^N\setminus\Lambda)\cap\Omega_{n,\varepsilon}} g_\varepsilon(x, u_\varepsilon)\chi_{n,\varepsilon}^2 u_\varepsilon\, dx.
\end{aligned}$$

By (4-37), we have

$$|I| \le \frac{1}{8}\int_{\Omega_{n,\varepsilon}} V(x)u_\varepsilon^2\,dx. \tag{4-38}$$

Next we treat $|II|$.

Clearly, we only need to consider the case $\Lambda \cap \Omega_{n,\varepsilon} \neq \varnothing$. In this situation, there is a set $\Sigma_{n,\varepsilon}$ such that $\Sigma_{n,\varepsilon} \cap \Omega_{n,\varepsilon}$ has the uniform cone property and $\Lambda \subset \Sigma_{n,\varepsilon} \subset \Lambda_{r_0} = \{x : \operatorname{dist}(x, \Lambda) \le r_0\}$, where $r_0 > 0$ is a small constant such that $V(x) \ge V_1$ holds true for $x \in \Lambda_{2r_0}$.

By (4-30), one has

$$\int_{\Sigma_{n,\varepsilon}\cap\Omega_{n,\varepsilon}} K(x)|u_\varepsilon|^{p+1}dx \le C_2\varepsilon^{-N(p-1)/2}\biggl(\int_{\Sigma_{n,\varepsilon}\cap\Omega_{n,\varepsilon}} \bigl(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\bigr)\,dx\biggr)^{(p+1)/2}. \tag{4-39}$$

In addition, by (4-33), we arrive at $\Sigma_{n,\varepsilon} \cap \Omega_{n,\varepsilon} \subset \mathbb{R}^N \setminus B_{\varepsilon\rho_0(\nu_0)}(x_\varepsilon)$ for $\varepsilon < \varepsilon_1$ and $n \ge \hat{n}$. Thus, it follows from (4-27), (4-39) and the definition of ν_0 that

$$|II| \le \frac{1}{8}\int_{\Omega_{n,\varepsilon}} \bigl(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\bigr)\,dx. \tag{4-40}$$

Finally, we estimate $|III|$.

Similar to the proof of (2-3), for $\varepsilon < \varepsilon_1$, we have

$$|III| \le \int_{\Omega_{n,\varepsilon}} \frac{2\varepsilon^3}{1+|x|^{\theta_0}}u_\varepsilon^2\,dx \le \frac{1}{8}\int_{\Omega_{n,\varepsilon}} \bigl(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\bigr)\,dx. \tag{4-41}$$

Combining (4-38), (4-40) with (4-41) yields the conclusion of Lemma 4.8. □

From Lemma 4.8, repeating the same argument as in Lemma 3.3 of [Fei and Yin 2010] leads to the following result.

Lemma 4.9. *Under the assumptions of Lemma 4.8, for small $\varepsilon < \varepsilon_1$, one has*

$$\int_{\mathbb{R}^N} |\nabla(\chi_{\tilde{n},\varepsilon}u_\varepsilon)|^2\,dx \le C\varepsilon^{N-2}2^{-(\ln 2)/(c\varepsilon)}. \tag{4-42}$$

Next, we establish an estimate of $u_\varepsilon(x)$ for large $|x|$.

Lemma 4.10. *Under the assumptions of Lemma 4.8, for $x \in \mathbb{R}^N$ satisfying $|x - x_\varepsilon| \ge d_0/2$, where the meaning of x_ε is given in Lemma 4.2, we have*

$$u_\varepsilon(x) \le C2^{-(\ln 2)/(2c\varepsilon)}. \tag{4-43}$$

Proof. First we assert that

$$\max_{\overline{\Lambda}} u_\varepsilon \le C, \tag{4-44}$$

where $C > 0$ is independent of small ε.

In fact, for any fixed p with $1 < p < \frac{N+2}{N-2}$, it follows from (2-1) that $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ satisfies

$$-\triangle v_\varepsilon + V(\varepsilon x)v_\varepsilon = K(\varepsilon x) f(v_\varepsilon) \le \tfrac{1}{16} V(\varepsilon x)v_\varepsilon + C(p)v_\varepsilon^p \quad \text{in } B_{d_0}(\varepsilon^{-1}x_\varepsilon)), \tag{4-45}$$

where $C(p)$ is a positive constant dependent of p.

Define $a_\varepsilon(x) = \frac{15}{16} V(\varepsilon x) - C(p)v_\varepsilon^{p-1}$; then $v_\varepsilon(x)$ is a weak subsolution of the equation

$$-\triangle v_\varepsilon + a_\varepsilon(x)v_\varepsilon = 0 \quad \text{in } B_{d_0}(\varepsilon^{-1}x_\varepsilon)). \tag{4-46}$$

By (4-3), then we obtain, for $\dfrac{N}{2} < q = \dfrac{2N}{(p-1)(N-2)}$ and small ε,

$$\left(\int_{B_{d_0}(\varepsilon^{-1}x_\varepsilon)} |a_\varepsilon|^q \, dx\right)^{1/q} \le C + C(\varepsilon^{-N/2}\|u_\varepsilon\|_\varepsilon)^{2N/(q(N-2))} \le C.$$

This, together with the weak Harnack inequality (see [Gilbarg and Trudinger 1983, p. 193]), yields that there is a positive constant C depending only on the space dimension N and the $L^q(B_{d_0}(\varepsilon^{-1}x_\varepsilon))$ norm of $a_\varepsilon(x)$ such that

$$\begin{aligned}\max_{\bar\Lambda} u_\varepsilon = u_\varepsilon(x_\varepsilon) = v_\varepsilon(\varepsilon^{-1}x_\varepsilon) &\le C\left(\int_{B_{d_0}(\varepsilon^{-1}x_\varepsilon)} v_\varepsilon^2 \, dx\right)^{1/2} \\ &= C\left(\varepsilon^{-N}\int_{B_{\varepsilon d_0}(x_\varepsilon)} u_\varepsilon^2 \, dx\right)^{1/2} \le C\varepsilon^{-N/2}\|u_\varepsilon\|_\varepsilon \le C,\end{aligned}$$

namely, (4-44) is proved.

In addition, as in (4-45)–(4-46), one knows that $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ is also a weak subsolution of the equation

$$-\triangle v_\varepsilon + b_\varepsilon(x)v_\varepsilon = 0, \tag{4-47}$$

where $b_\varepsilon(x) = \frac{15}{16} V(\varepsilon x) - C(p)\chi_\varepsilon(x)v_\varepsilon^{p-1} - (1-\chi_\varepsilon(x))\varepsilon^3/(1+|\varepsilon x|^{\theta_0})$, and χ_ε is a characteristic function of $\Lambda^\varepsilon = \{\varepsilon^{-1}x : x \in \Lambda\}$. Moreover, $b_\varepsilon(x)$ has a uniform L^∞ bound independent of small ε by (4-44).

On the other hand, it is noted that for $x \in \mathbb{R}^N$ with $x \in \mathbb{R}^N \setminus B_{d_0/2}(x_\varepsilon)$, then $B_{\varepsilon c d_0}(x) \subset \Omega_{\tilde n+1,\varepsilon}$ holds true for small ε and a direct computation yields, for $2^* = 2N/(N-2)$,

$$\begin{aligned}\left(\int_{B_{cd_0}(\varepsilon^{-1}x)} |v_\varepsilon|^{2^*} dy\right)^{1/2^*} &\le C\varepsilon^{-(N-2)/2}\left(\int_{\mathbb{R}^N} |\nabla(\chi_{\tilde n,\varepsilon}u_\varepsilon)|^2(z)\, dz\right)^{1/2} \\ &\le C2^{-(\ln 2)/(2c\varepsilon)}.\end{aligned} \tag{4-48}$$

Subsequently, with the aid of Harnack inequality [Gilbarg and Trudinger 1983, Theorem 8.17] and (4-48), we arrive at

$$u_\varepsilon(x) = v_\varepsilon(\varepsilon^{-1}x) \le C\left(\int_{B_{cd_0}(\varepsilon^{-1}x)} |v_\varepsilon|^{2^*} dy\right)^{1/2^*} \le C2^{-(\ln 2)/(2c\varepsilon)}, \tag{4-49}$$

where $C > 0$ depends only on d_0, N and the uniform L^∞ bound of $b_\varepsilon(x)$.

Since the L^∞ norm of $b_\varepsilon(x)$ is uniformly bounded, the proof of Lemma 4.10 is complete. □

Remark 4.1. By Lemma 4.10, for $\theta \ge 1$, there exists an ε_0 such that for $\varepsilon < \varepsilon_0$,

$$|u_\varepsilon(x)| \le \varepsilon^\theta \quad \text{for } x \in \mathbb{R}^N \setminus B_{d_0/2}(x_\varepsilon). \tag{4-50}$$

Next, we show that the local maximum point x_ε of $u_\varepsilon(x)$ in the domain $\bar{\Lambda}$ is also a maximum point of $u_\varepsilon(x)$ in the whole space.

Lemma 4.11. *Under the assumptions of Lemma 4.8, x_ε is the maximum point of u_ε in $\mathbb{R}^N$.*

Proof. Let y_ε be the maximum point of u_ε in $\mathbb{R}^N$; then $u_\varepsilon(y_\varepsilon) = \max_{\mathbb{R}^N} u_\varepsilon \ge \max_{\bar{\Lambda}} u_\varepsilon \ge C$. According to (4-50), we have $y_\varepsilon \subset B_{d_0/2}(x_\varepsilon) \subset \Lambda$ for small ε. Hence $y_\varepsilon = x_\varepsilon$ for small ε by Lemma 4.5. Namely, the proof of Lemma 4.11 is completed. □

Proof of Theorem 1.1. It follows from the assumption (H_5) that there exist positive constants σ_0, θ_0, θ_1 and θ_2 such that

$$\beta < (\alpha - \theta_1)\sigma_0 - \theta_0 \quad \text{and} \quad 4 + 2(\alpha - \theta_1) \le (\theta_1 - 1)\theta_2, \tag{4-51}$$

where $N - \frac{9}{4} < \sigma_0 < N - 2$, $\theta_0 > 2$, $\theta_1 > 1$.

We define the comparison function

$$U(x) = \frac{1}{|x - x_\varepsilon|^{\sigma_0}} \quad \text{for } x \in \mathbb{R}^N \setminus B_{d_0/2}(x_\varepsilon).$$

It is easy to know that $Z(x) = U(x) - \varepsilon^2 u_\varepsilon(x) \ge 0$ on $\partial(B_{d_0/2}(x_\varepsilon))$ for small ε. Recalling that $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ vanishes at infinity, this is also true for $Z(x)$.

On the other hand, using the expression for $h_\varepsilon(x, u_\varepsilon)$ and noting that $\sigma_0 < N - 2$, we conclude from (4-50) that $\Delta Z = \Delta U - \varepsilon^2 \Delta u_\varepsilon \le 0$ holds for $x \in \mathbb{R}^N \setminus B_{d_0/2}(x_\varepsilon)$ and sufficiently small ε.

Thus, by the maximum principle, we deduce $u_\varepsilon \le U/\varepsilon^2$ in $x \in \mathbb{R}^N \setminus B_{d_0/2}(x_\varepsilon)$. This and the uniform boundedness of x_ε imply

$$u_\varepsilon(x) \le \frac{C}{\varepsilon^2(1 + |x|^{\sigma_0})} \quad \text{in } \mathbb{R}^N \setminus \Lambda. \tag{4-52}$$

Next we verify that u_ε actually solves Equation (1-1). Indeed, since $f(s) = O(s^\alpha)$ near $s = 0$, together with (4-50) we have, for small ε,

$$f(u_\varepsilon) \le C|u_\varepsilon|^\alpha \quad \text{in } \mathbb{R}^N \setminus \Lambda. \tag{4-53}$$

Combining (4-50)–(4-53), we have, for small ε,

$$K(x) f(u_\varepsilon) \le Ck(1+|x|^\beta)|u_\varepsilon|^\alpha \le \frac{\varepsilon^3}{1+|x|^{\theta_0}}|u_\varepsilon| \quad \text{in } \mathbb{R}^N \setminus \Lambda. \tag{4-54}$$

Choose two positive numbers θ_3 and θ_4 such that

$$\beta < (\alpha-\theta_3)\sigma_0 - N \quad \text{and} \quad 2+2(\alpha-\theta_3) \le \theta_3\theta_4. \tag{4-55}$$

Collecting (4-50), (4-52), (4-53), and (4-55) yields for small ε,

$$K(x) f(u_\varepsilon) \le Ck(1+|x|^\beta)|u_\varepsilon|^{\alpha-\theta_3}|u_\varepsilon|^{\theta_3} \le \frac{\varepsilon}{1+|x|^N} \quad \text{in } \mathbb{R}^N \setminus \Lambda. \tag{4-56}$$

Therefore, it follows from (4-54) and (4-56) that $g_\varepsilon(x, u_\varepsilon) \equiv K(x) f(u_\varepsilon)$ holds true in $\mathbb{R}^N \setminus \Lambda$ and subsequently u_ε solves the original equation (1-1). In addition, noting that $N - \frac{9}{4} < \sigma_0$, then the estimate (4-52) leads to $u_\varepsilon \in L^2(\mathbb{R}^N)$ for $N \ge 5$.

Finally, combining the conclusions in Lemma 4.2, Lemma 4.5 and Lemma 4.11, in order to finish the proof of Theorem 1.1, we only need to verify (1-12). Set $M = \{x_0\}$, due to (4-26), one has $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Let $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$, then v_ε is uniformly bounded in $H^1_{\text{loc}}(\mathbb{R}^N)$ and satisfies the equation

$$-\triangle v_\varepsilon + V(\varepsilon x + x_\varepsilon)v_\varepsilon = K(\varepsilon x + x_\varepsilon) f(v_\varepsilon), \quad x \in \mathbb{R}^N. \tag{4-57}$$

As in the arguments of Lemma 4.3 or Lemma 4.5, we can show that v_ε converges to $v \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ in $C^2_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$. With the aid of (4-43), v_ε converges to v in $L^\infty(\mathbb{R}^N)$ as $\varepsilon \to 0$. Therefore v is a solution of the equation

$$-\triangle v + V(x_0)v = K(x_0) f(v), \quad x \in \mathbb{R}^N; \tag{4-58}$$

moreover, by virtue of strong maximum principle, $v > 0$ can be derived. On the other hand, as a consequence of Theorem 2 [Gidas et al. 1981] and the subsequent remark, v is radially symmetric and decays exponentially.

Thus the proof of Theorem 1.1 is completed. □

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