Pacific Journal of Mathematics

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DANIEL GUAN

Volume 261 No. 2 February 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 261, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.261.369

TYPE I ALMOST HOMOGENEOUS MANIFOLDS OF COHOMOGENEITY ONE, III

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This paper is one of a series in which we generalize our earlier results on the equivalence of existence of Calabi extremal metrics to the geodesic stability for any type I compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we actually carry all the earlier results to the type I cases. As requested by earlier referees of this series of papers, in this third part, we shall first give an updated description of the geodesic principles and the classification of compact almost homogeneous Kähler manifolds of cohomogeneity one. Then, we shall give a proof of the equivalence of the geodesic stability and the negativity of the integral in the first part. Finally, we shall address the relation of our result to Ross–Thomas version of Donaldson's K-stability. One should easily see that their result is a partial generalization of our integral condition in the first part. And we shall give some further comments on the Fano manifolds with the Ricci classes. In Theorem 14, we give a result of Nadel type. We define the strict slope stability. In our case, it is stronger than Ross–Thomas slope stability. We strengthen two Ross–Thomas results in Theorems 15 and 16. The similar proofs of the results other than the existence for the type II cases are more complicated and will be done elsewhere.

1. Introduction

This paper is one of a series of papers in which we finished the project of studying the existence (or not) of extremal metrics in any Kähler class on any compact almost homogeneous manifolds of cohomogeneity one.

In [Guan 2011a; 2011b] we proved that for the type I compact almost homogeneous Kähler manifolds of cohomogeneity one, the existence of Calabi extremal metrics is the same as the negativity of a topological integral. We also proved in [Guan 2011b] that for any two Kähler metrics in the Mabuchi moduli space of Kähler metrics there is a smooth geodesic connecting them. That is, the geodesic principle I is true for these manifolds.

Supported by DMS-0103282.
MSC2010: 14J45, 34B18, 53C10, 53C25, 53C55.
Keywords: Kähler–Einstein, stability, cohomogeneity one.

As in [Guan 2003], the major tool is from [Guan 1999]. Although the problem of existence of the extremal metrics can be reduced to an ordinary differential equation for our manifolds, the problem of the existence of the geodesics has two variables. Thanks to the Legendre transformation, we can carry it out for the type I manifolds. But for a general type II manifolds, this method does not work any more. And we need a new method, which will be carried out in [Guan $\geq$ 2013a].

Even for the Kähler–Einstein equation, our method in [Guan 2011a] is different from [Guan and Chen 2000]. We used a semisimple method in [Guan 2011a]. One notices that our exponential map there is not the one for the geodesics. No geodesic in that situation could have infinite length. It was well known for many years that there were many nonsmooth solutions for even a real homogeneous Monge–Ampère equations. In [Chen and Tian 2008] Professor Chen gave an example which looks like a nonsmooth solution for the one-dimensional toric case, that is, $\mathbb{C}P^1$. He also mentioned it earlier to me in 1999 at Princeton. Mabuchi also mentioned it to me in Pisa, Italy in 2004. However, we already solved the smoothness question for the toric manifolds in [Guan 1999]. In this simple case, the method of X. X. Chen should also produce the smooth solution; see [Guan and Phong 2012]. The content of this note was presented in the AMS meeting in Pomona California May 2008. Recently, L. Lempert and L. Vivas claimed (also mentioned by the referee) that they found a counterexample to our geodesic principle I on the torus. However, their examples are not very explicit and not published yet. We are not able to check their examples in this paper. As we know, there is no much equivariant geometry on the torus. The geodesic problem was trivial on the torus. However, see also [Feng 2012]. We checked that all the geodesic principles hold on compact cohomogeneity-one Kähler manifolds. We conjecture that the geodesic principles hold for all the spherical manifolds. We take them as working principles in our research. *For our safety, we just require that everything is analytic.* For example, for any analytic initial value in the tangent space of the equivariant Mabuchi moduli space at a given metric, there is a geodesic ray. That is, *the geodesic principle I is not really needed for the geodesic stability.* In [Guan and Chen 2000], some possible obstructions emerged that I eventually treated in [Guan 2002], which led to the strict slope stability. After a long run, we are able to overcome all the difficulties. To solve the extremal metrics cases, we have to deal with a fourth-order ordinary differential equation, which in our cases is fortunately reduced to a second-order nonlinear equation and is successfully treated.

All the solutions we find in the cohomogeneity-one cases are not explicit except those in [Guan 1995a; Guan 2007].

In this paper, we shall prove that the negativity of the integral is actually the same as the geodesic stability.

A classification which we refer to in this paper can be found in [Guan 2003, Section 12].

Here we shall describe our updated *geodesic stability principles*. We conclude these principles by following the cumulation of other people's observations and the evidence from our examples. See [Guan 2003]. We do not assume that these principles are due to us completely, in particular the first principle.

Motivated by the Donaldson's functional in the vector bundle case, Mabuchi [1986] defined a functional on the Mabuchi moduli space of the Kähler metrics (see also a conjecture therein). It was later modified independently by several people to fit the situation of Calabi extremal metrics (see [Guan 1999; Guan and Chen 2000], etc.) on the equivariant Mabuchi moduli space of Kähler metrics, which we call the modified Mabuchi functional.

Principle I. *For any two Kähler metrics in a given Kähler class*, *there is a unique* (*smooth*) *geodesic in the Mabuchi moduli space of Kähler metrics connecting them.*

This principle has been tested for toric bundles in [Guan 1999]. We also found that the same method applies to Kähler metrics on type-I and type-III compact almost homogeneous Kähler manifolds of cohomogeneity one in [Guan 2003; 2011b]; see also [Guan 2007]. It seems to us that there is not any complete geodesic except the ones induced by the holomorphic vector fields. X. X. Chen [2000] proved the existence of an unique $C^{1,1}$ solution in general.

We shall concentrate on the maximal geodesic rays. It turns out that the majority of the maximal geodesic rays are of finite length (this is different from holomorphic vector bundle theory on vector bundles; cf. [Kobayashi 1987, p. 197] and also the picture shown in [Semmes 1992, p. 544]). The maximal geodesic rays with infinite length are very special with some *strong convex* property, which we call "effective" maximal geodesic rays. The direction of the effective geodesic rays at each metric might form a *convex cone* $\mathscr{C}$.

Principle II. *The limit metrics of the maximal geodesics are concentrations*:

A. *Finite ray*: *cone concentration — partial concentration.*

B. *Infinite ray*: *blow up caused by some subvarieties outside a compact set — complete concentration outside the compact set*, *the metric on this compact set does not change.*

We call the limit of the ratio of the modified Mabuchi functional the ***generalized Futaki invariants*** *of the maximal geodesic rays. The generalized Futaki invariant is positive infinite for finite rays*, *that is*, *the only interesting generalized Futaki invariants come from the effective maximal geodesic rays.*

The second principle is based on our work on toric manifolds and cohomogeneity-one manifolds; see [Guan 2003; 2007] for examples.

For all the examples we consider in this paper, the Mabuchi equivariant moduli space is flat (see [Guan 1999]); this is similar to the vector bundle case and is not true in general (see [Mabuchi 1987]). For two maximal geodesic rays, the generalized Futaki invariants might be the same if there is a curve connecting the beginning points such that there is a parallel vector field along this curve which connects the two tangent vectors at these two points. This observation makes the definition of the generalized Futaki invariants independent of the initial Kähler metrics.

The generalized Futaki invariants define a function of the effective geodesic cone which is probably a linear function $F_{M,\omega}$, which is continuous on a certain given Banach space. Therefore, F can be defined on the closure $\bar{\mathscr{C}}$ of the effective cone $\mathscr{C}$ in the Banach space. We call $F|_{\bar{\mathscr{C}}}$ the *generalized Futaki invariant functional* or simply the *generalized Futaki invariant.* There is a seminorm $\|\cdot\|_*$, which is locally equivalent to the given norm except on some subvarieties and is zero on the functions induced by the holomorphic vector fields.

Principle III. *There is a unique extremal metric in a given Kähler class up to the automorphism group if and only if the Kähler class is geodesic stable*, *that is*, *with positive generalized Futaki invariant which is bounded below by the given seminorm.*

(Note: in many of our papers, this is called the fourth principle and the next principle is called the third, reflecting the order in which they were formulated.)

In general, the Mabuchi moduli space might not be flat. We might have some way to relate the Futaki invariants for two infinite maximal geodesic rays starting from different points. Let $\gamma_i(t)$, $i = 1, 2$ be two maximal geodesic rays. We say that they have the same infinite points if

$$d(\gamma_1, \gamma_2) = \sup_{t\in[0,+\infty)} d(\gamma_1(t), \gamma_2(t))$$

is finite. Then we have (see also [Guan 2007, Remark 4]):

Principle IV. *The Futaki invariants of two maximal geodesic rays with the same infinite point are the same.*

In the last section, we shall see that our stability in this case is the same as a version of the slope stability which is stronger than that in [Ross and Thomas 2006].

2. Preliminaries

Here we summarize some known results about the compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we only consider manifolds with a Kähler structure. For earlier results one might check [Ahiezer 1983; Huckleberry and Snow 1982].

We call a compact complex manifold an almost homogeneous manifold if its complex automorphism group has an open orbit. We say that a manifold is of cohomogeneity one if the maximal compact subgroup has a (real) hypersurface orbit. In [Guan and Chen 2000; Guan 2003], we reduced the compact complex almost homogeneous manifolds of cohomogeneity one into three types of manifolds.

We denote the manifold by M and let G be a complex subgroup of its automorphism group which has an open orbit on M.

Let us assume first that M is simply connected. Let the open orbit be G/H, K be the maximal connected compact subgroup of G, L be the generic isotropic subgroup of K, that is, K/L be a generic K-orbit. We have [Guan and Chen 2000, Theorem 1]:

Proposition 1. *If G is not semisimple, then M is a completion of a $\mathbb{C}^*$-bundle over a projective rational homogeneous space.*

If a compact almost homogeneous Kähler manifold is a completion of a $\mathbb{C}^*$-bundle over a product of a torus and a projective rational homogeneous space, we call it a *manifold of type III.* We dealt with this kind of manifold in our dissertation [Guan 1995a; 1995b]. There always exists an extremal metric in any Kähler class. In [Guan 2007], we generalized this existence result to a family of metrics connecting the extremal metric of [Guan 1995a] and the generalized quasi-Einstein metric of [Guan 1995b]; we called this family the *extremal-soliton metrics*. The existence of the extremal-soliton is the same as geodesic stability with respect to a generalized Mabuchi functional.

More recently in [Guan 2012], we even generalized the extremal-solitons to the generalized extremal solitons, which also include Nakagawa's [2011] generalized Kähler–Ricci solitons as a special case. We proved the existence of both generalized extremal solitons and the generalized Kähler–Ricci solitons on these manifolds. In a forthcoming paper [Guan ≥ 2013b], we proved the existence of the so called m-extremal metrics on these manifolds.

In general, if M is a compact almost homogeneous Kähler manifold and O is the open orbit, then $D = M - O$ is a proper closed submanifold. Moreover, D has at most two components. We call each component of D an end. If D has two components or one component, we say M is an almost homogeneous manifold with two ends or one end, respectively. We have [Huckleberry and Snow 1982, Theorem 3.2]:

Proposition 2. *If M is a compact almost homogeneous Kähler manifold with two ends, then M is a manifold of type III.*

Therefore, we only need to deal with the case with one end. In [Guan and Chen 2000], we treated the first example, that is, the blowup of the diagonal of the product of two copies of $\mathbb{C}P^n$. We treated another series in [Guan 2003]. We treated many more of them in [Guan 2009; 2011b; 2011c], etc. Again, in the case of M being simply connected, we only need to take care of the case in which G is semisimple.

If G is semisimple and M has two G orbits, one open and one closed, and moreover if the closed orbit is a complex hypersurface, there are two possibilities. Let $\mathscr{K}$, $\mathscr{L}$ be the Lie algebras of K, L. Then the centralizer of $\mathscr{L}$ in $\mathscr{K}$ is a direct sum of the center of $\mathscr{L}$ and a Lie subalgebra $\mathscr{A}$ with $\mathscr{A}$ being either one-dimensional or a 3-dimensional Lie algebra $\mathrm{su}(2)$. If $\mathscr{A}$ is one-dimensional, we call M a *manifold of type I.* If $\mathscr{A}$ is $\mathrm{su}(2)$, we call M a *manifold of type II.*

In general, if the closed orbit has a higher codimension, we can always blow up the closed orbit to obtain a manifold $\tilde{M}$ with a hypersurface end. We call the manifold M a *manifold of type I or II* if $\tilde{M}$ is of type I or II, respectively.

There is a special case of the type II manifolds. If the open orbit is a $\mathbb{C}^k$-bundle over a projective rational homogeneous manifold, we call M an *affine type manifold* (not to be confused with the closed complex submanifolds of $\mathbb{C}^m$).

Then we have (see [Guan 2003, Section 12]):

Proposition 3. *Any compact almost homogeneous Kähler manifold M of cohomogeneity one is an* $\mathrm{Aut}_0(M)$ *equivariant fibration over a product of a rational projective homogeneous manifold Q and a complex torus T with a fiber F. Therefore, M can be regarded as a fiber bundle over T with a simply connected fiber M_1. One of following holds*:

(i) *M is a manifold of type III.*

(ii) *M_1 is of type II but not affine.*

(iii) *M_1 is affine.*

(iv) *M_1 is of type I.*

We say that M is a *manifold of type I, or type II, affine,* if M_1 is, respectively, a manifold of type I or type II, affine.

We actually can also obtain the structure of an M_1-bundle over T from [Huckleberry and Snow 1982]. We only need to understand the bundle structure for the open orbit. By [ibid., Corollary 4.4] we have that the bundle structure is a product unless, when we apply Proposition 3 to $\tilde{M}$, $F = Q^k$. In the latter case, there is an unbranched double covering $\bar{M}$ of M such that the bundle structure of $\bar{M}$ is a product.

Proposition 4. *The M_1-bundle over T is a product except in the case where the open orbit is an F_0-bundle over $Q \times T$ such that F_0 is in the second*, *sixth and eighth cases in* [Ahiezer 1983, p. 67]. *In the latter cases*, *the M_1-bundle has an unbranched double covering which is a product of M_1 and T.*

In [Guan 2011a; 2011b], we dealt with the type I cases.

One updated remark is that since we are dealing with the Kähler metrics it is more convenient to separate the type II case into two cases in [Guan 2009] and [Guan 2011c]. We call the cases in [Guan 2009] (and the papers between [ibid.] and [Guan 2003]) the *type IV* cases. They are the affine cases such that the group

$\pi(G_F)$, the restriction of the subgroup G_F of G fixing a given fiber F, is not of type A. Therefore, one might also call them the *non-type-A type II* cases. All of them are Fano.

One might call the rest (in [Guan 2011c]) of the type II cases the new type II cases (or simply the type II cases). They are those type II cases such that $\pi(G_F)$ is of type A. Therefore, one might also call them the *type A type II* cases.

This note is a continuation of the first part and the second part of this paper [Guan 2011a; 2011b]. We shall retain all the notation from those papers here.

3. The complex structures of the type I almost homogeneous manifolds

In this section, we shall deal with the complex structure of the type I almost homogeneous manifolds. We retain the notation in [Guan 2011a; 2011b]. Let us recall some basic notation of the Lie algebras.

Let G be the complex Lie group action and S be the connected complex Lie subgroup acting on a given fiber. According to [Guan 2003, p. 283, Theorem 12.1(ii)], a compact complex almost homogeneous manifold of cohomogeneity one is type I if and only if the fiber F is one of (1) the second and third case with $n \geq 3$, (2) the fourth case, (3) the eight and ninth cases, (4) the fifth case in [Ahiezer 1983, p. 67].

The fiber F in (4) has $S = \pi(G_F) = F_4$, so $G = F_4 = S$, that is, $M = F$ is homogeneous. Therefore, every Kähler class of M has a metric with constant scalar curvature. So, we do not need to do anything with (4).

In [Guan 2011a], we look at three special possible fiber cases [Ahiezer 1983, p. 67] first:

(1) $F = F(\mathrm{OP}_n)$: The third case in [Ahiezer 1983, p. 67] with $n \geq 3$. We have $F = \mathbb{C}P^n$ and
$$S = \pi(G_F) = \mathrm{SO}(n, \mathbb{C}),$$
regarding $\mathbb{C}P^n$ as a completion of $\mathbb{C}^n$. The corresponding compact rank-one symmetric space is the real n-dimensional real projective space. It has an equivariant branched double covering Q^n of the second case. We denote the latter case by $F(\mathrm{OQ}_n)$.

(2) $F = F(\mathrm{Gr}_k)$: The fourth case with a standard $S = \mathrm{Sp}(k, \mathbb{C})$-action on the manifold $F = \mathrm{Gr}(2k, 2)$. The corresponding compact rank one symmetric space is the quaternionic projective space.

(3) $F = F(\mathrm{Sp}_7^p)$: The ninth case with an $S = \mathrm{Spin}(7, \mathbb{C})$-action on $F = \mathbb{C}P^7$. This is the restriction of (1) with $n + 1 = 8$ to the complex Lie subgroup $\mathrm{Spin}(7, \mathbb{C})$. It has an equivariant branched double covering Q^7 of the eighth case. In [Guan

2011a], we also denote the latter case by $F(\mathrm{Sp}_7^q)$ and denote both of them by $F(\mathrm{Sp}_7)$ whenever there is no confusion.

In [ibid.], we defined a certain basis of the Lie algebra α, F_α and G_α for positive roots α. And, we considered a fixed point p_0 and its orbit p_s generated by a semisimple element $-iH$ in the Lie algebra. Let T be the tangent vector of p_s and p_∞ be the limit point in the closed orbit.

In the case (1), we obtained:

Proposition 5. *For $F(\mathrm{OP}_n)$ and $F(\mathrm{OQ}_n)$, along p_s we have*
$$J(F_{e_1+e_i} \pm F_{e_1-e_i}) = -(\tanh s)^{\mp 1}(G_{e_1+e_i} \pm G_{e_1-e_i})$$
(*and $JF_{e_1} = -(\tanh s)G_{e_1}$*). *We also have that $F_{e_i \pm e_k} = G_{e_i \pm e_k} = 0$ (and $F_{e_i} = G_{e_i} = 0$) for $i > 1$. In particular, at p_∞, $JF_\alpha = -G_\alpha$ for $\alpha \neq e_i \pm e_k$ (and e_i), $1 < i < k$.*

In the case of (2), we obtained:

Proposition 6. *For $F(\mathrm{Gr}_k)$, we have*
$$JF_{\alpha_1} = -(\tanh 2s)G_{\alpha_1},$$
$$J(F_{2e_1} \pm F_{2e_2}) = -(\tanh 2s)^{\mp 1}(G_{2e_1} \mp G_{2e_2}),$$
$$J(F_{e_1-e_k} \pm G_{e_2-e_k}) = -(\tanh s)^{\mp 1}(G_{e_1-e_k} \pm F_{e_2-e_k}),$$
$$J(F_{e_1+e_k} \pm G_{e_2+e_k}) = -(\tanh s)^{\mp 1}(G_{e_1+e_k} \pm F_{e_2+e_k}).$$

$F_\alpha = G_\alpha = 0$ for $\alpha = e_1 + e_2, e_i - e_k, 2e_i, e_i + e_k$ with $i > 2$.

At p_∞, we have $F_\alpha = G_\alpha = 0$ if $\alpha = e_1 + e_2, 2e_i, e_i \pm e_k, i > 2$, and $JF_\alpha = G_\alpha$ if $\alpha = 2e_2, e_2 \pm e_k$. Otherwise $JF_\alpha = -G_\alpha$.

Before we consider the isolated case (3), we can look at the general cases in which $G \neq S = \pi(G_F) \subset \mathrm{Aut}(F)$, where G_F is the subgroup that acts on the fiber F and $\pi : G_F \to \mathrm{Aut}(F)$ is the induced map from G_F to $\mathrm{Aut}(F)$. As in [Ahiezer 1983], G is semisimple, $U_G = H$ is the 1-subgroup. There is a parabolic subgroup $P = SS_1R$ with S, S_1 semisimple and R solvable such that $U_G = US_1R$ where $U = H \cap S$ is a 1-subgroup of S. The manifold is a fibration over G/P with the completion of $P/U_G = S/U$ as the isotropic open orbit of the almost homogeneous fiber. In this case, the root system of S is a subsystem of the root system of G. In the Lie algebra of G, we also have some other F_α, G_α outside $\mathcal{S}$. Let K be a maximal connected compact Lie subgroup of G and L be the isotropic subgroup of K at a generic orbit. Let $\mathcal{K}$, $\mathcal{L}$ be the corresponding Lie algebras. The tangent space of G/U_G along p_s is decomposed into irreducible $\mathcal{L}$-representations. These F_α, G_α are in the complement representation of the Lie algebra $\mathcal{S}$ of S. As it is in the tangent space of G/P, $JF_\alpha = -G_\alpha (\mathrm{mod}\, \mathcal{S})$. Therefore, we have $JF_\alpha = -G_\alpha$ for any α which is not in the root system of S.

If S is B_2, G can be B_n, C_n, F_4. If S is B_3, G can be B_n, F_4. If S is C_3, G can be C_n, F_4. If S is B_n with $n > 3$, G can only be B_{m+n}. If S is C_n with $n > 3$, then G can be C_{n+m}. The case of a B_2-action that has an isotropic group of $\mathrm{SO}(4, \mathbb{C})$ generated by roots $\pm e_1 \pm e_2$ is exactly the same as the case of an $\mathrm{Sp}(2, \mathbb{C})$-action, which has an isotropic subgroup of $\mathrm{Sp}(1, \mathbb{C}) \times \mathrm{Sp}(1, \mathbb{C})$ generated by $\pm 2e_1, \pm 2e_2$.

We have a few more possibilities. If $S = D_k$, $k > 3$, G can only be D_n, $n > 3$ or E_n, $n > k$. If $S = D_3$, that is an A_3, G can be A_n, $n > 2$, B_n, $n > 3$, C_n $n > 3$, D_n $n > 2$ and E_n. If $S = D_2$, G can be any simple group or product of simple groups other than G_2.

We then treated the isolated case (3) of the $\mathrm{Spin}(7, \mathbb{C})$-action on $\mathbb{C}P^7$ in [Guan 2011a]. This case is the restriction of the case (1) with an $G = S = \mathrm{SO}(8, \mathbb{C})$-action to the $\mathrm{Spin}(7, \mathbb{C})$-action induced by the spinor representation.

We obtained:

Proposition 7. *For $F(\mathrm{Sp}_7)$, we have*

$$J(\sqrt{2}F_{h_i} \pm F_{h_j+h_k}) = -\Big(\tanh \frac{\sqrt{3}}{2}s\Big)^{\mp 1}\big(\sqrt{2}G_{h_i} \pm G_{h_j+h_k}\big),$$

$$JH = -T,$$

$$F_{e_i-e_j} = G_{e_i-e_j} = 0 \quad \textit{for } 0 < i < j < 4.$$

At p_∞, $JF_{h_i} = -G_{h_i}$, $JF_{h_j+h_k} = -G_{h_j+h_k}$, $F_{h_i-h_k} = G_{h_i-h_k} = 0$.

However, in this case $S = B_3$, G can only be B_n or F_4.

4. The Kähler structures

In [Guan 2011a], we examined the Kähler structure for the $S = \mathrm{SO}(n, \mathbb{C})$-actions and obtained that for any possible G and $S = \mathrm{SO}(n, \mathbb{C})$ we always have a Kähler metric: $\omega([X, Y]) = (aH + I, [X, Y])$ with the I in the C center of $\mathscr{L}$ and a a nonpositive function of s.

See [Guan 2011a, Section 3].

Therefore, we have the volume formula

$$V = -Ma'a^{2(n-1)}\prod_1^r(a_i - a)\prod_1^s(b_j + a)$$

$$\Big(\text{or} \quad V = Ma'a^{2n-1}(\tanh s)\prod_1^r(a_i - a)\prod_1^s(b_j + a)\Big),$$

with some positive numbers a_i and b_j.

Then in [Guan 2011a], we dealt with the Kähler metrics with $\mathrm{Sp}(k,\mathbb{C})$ and $\mathrm{Spin}(7,\mathbb{C})$-actions. We have the volume form

$$V = Ma'a^{4k-5}(\tanh 2s)\prod_1^r (a_i - a)\prod_1^s (b_j + a)$$

for the $\mathrm{Sp}(k,\mathbb{C})$-actions.

For the $S = \mathrm{Spin}(7,\mathbb{C})$-action, we obtained the volume form

$$V = -Ma'a^6\prod_{i=1}^r (a_i - a)\prod_{j=1}^s (b_j + a).$$

We also observe that a_i and b_j come in pairs, and $b_{j(i)} = a_i$.

Altogether, we have:

Proposition 8. *For the type I case the volume is*

$$V = -Ma'a^{2m}\prod (a_i^2 - a^2)$$

for the cases $S = D_k$ or $\mathrm{Spin}(7,\mathbb{C})$ and

$$V = Ma'a^{2m+1}(\tanh bs)\prod (a_i^2 - a^2)$$

for the cases $S = B_k$ (or C_k) with $b = 1$ (or 2), where M and a_i are positive numbers, m are nonnegative integers. We also have that $2m+1$ (or $2m+2$) are the dimensions of the fiber. Moreover, the vectors in Propositions 5, 6 and 7 are orthogonal to each other.

Let $h = \log V$. In [Guan 2011a, Section 5, Theorem 2] we obtained:

Proposition 9. *If the fiber with the S-action is of type I of complex dimension n, then the function a for the Ricci form ρ is*

$$a_\rho = \tfrac{1}{2}\biggl(\biggl(\log\biggl(a'a^{n-1}\prod_1^r (a_i^2 - a^2)\biggr)\biggr)' - 2\sum_1^{n-1} N_i \coth 2N_i s\biggr).$$

Moreover, the N_i are (1) 1 for $S = \mathrm{SO}(n+1,\mathbb{C})$ and (2) 1 except three of them being 2 for S of type C_k, (3) $\sqrt{3}/2$ for the case $S = \mathrm{Spin}(7,\mathbb{C})$. Other coefficients come from the Ricci curvature of G/P, which is $-(q_{G/P},[X,Y])$ with $q_{G/P} = \sum_{\alpha\in\Delta^+-\Delta_P} H_\alpha$ with the standard inner product.

Then we calculated the scalar curvature in [Guan 2011a, Section 6, Theorem 3]. We write

$$V = -Ma'\tilde{Q}(a) = -Ma'(-a)^{n-1}Q_1(a)g(s),$$

with $g(s) = 1$ for $S = D_k$ or $\mathrm{Spin}(7,\mathbb{C})$ and $g(s) = \tanh bs$ for $S = B_k$ or C_k. We write $Q(a) = (-a)^{n-1}Q_1(a)$ and obtained $\rho\wedge\omega^{N-1} = M((-a_\rho Q(a))' + p_0 a')$.

Proposition 10. *The scalar curvature is*

$$R = \frac{2(-a_\rho Q)' + pa'}{-a' Q}.$$

Moreover, $p(a) = (-a)^{n-1} p_1(a)$ with $p_1(a)$ a polynomial of a and is a positive linear sum of Q_1 and product of $\deg Q_1 - 1$ *factors of Q_1. The contribution of each constant factor k_j (that is, the vector F_α such that the corresponding metrics $\omega(F_\alpha, JF_\alpha) = k_j$ is a constant along p_s) is $2k_{\rho,j}/k_j$ for the Q_1 factor. The contribution of each $a_i \pm a$ is $2a_{\rho,i} Q_1/q_i$.*

Therefore, we have

$$R_0 = \frac{\int_0^{-l} [(2u_\rho Q)_x + p]\, dx}{\int_0^{-l} Q\, dx} = \frac{2u_\rho(-l)Q(-l) + \int_0^{-l} p\, dx}{\int_0^{-l} Q\, dx},$$

where we let $u = -a$ and $l = \lim_{s \to +\infty} a$. We also obtained in [Guan 2011a] that $a_\rho(0) = 0$.

5. Geodesic stability and existence of the Calabi extremal metrics

In [Guan 2011b, Section 2], for any metric we obtained a function $\Gamma(s)$ such that $-4a = 4u = \Gamma'$ and the geodesic equation is $\ddot{\Gamma}\Gamma'' = (\dot{\Gamma}')^2$, where $'$ is the derivative with respect to s, the parameter from the manifold, and $\dot{}$ is the derivative with respect to t, the parameter for the geodesic. We obtain the smooth geodesics and so the uniqueness. Therefore, we might regard $U = 4u$ as g in [Guan 2011a].

We also have

$$4u_s(+\infty) = \Gamma_{ss}(+\infty) = 0$$

since u is increasing and bounded by $-l$ (see the end of last section).

We shall apply the method in [Guan 2003] to prove the second and third *geodesic stability principles* for all the type I Kähler almost homogeneous manifolds of cohomogeneity one.

The proof is parallel to what we have in [ibid.] but even simpler (with our advanced notation).

Letting H be the Legendre transformation of Γ as in [ibid.], a path Γ_t represents a geodesic in the Mabuchi moduli space of the equivariant Kähler metrics in a given Kähler class is a geodesic if and only if H_t is linear on t. We denote $h = \dot{H}$.

Recall that R is the scalar curvature, HR its average, Q the volume function appeared right before Proposition 10. Applying the scalar curvature formula in Proposition 10, we have that with a positive constant C the derivative of Mabuchi functional is:

$$
\begin{aligned}
-\int_M \dot\Gamma(R-HR)\omega^{2n}
&= -C\int_0^{-l}\dot\Gamma(s,t)\Big(2u_\rho Q+\int(p-R_0Q)\,du\Big)_x\,dx\\
&= C\int_0^{-l}\dot H(x,t)\Big(2u_\rho Q-\int(R_0Q-p)\,du\Big)_x\,dx\\
&= C\Big(2h(-l)u_\rho(-l)Q(-l)-2h(0)u_\rho(0)Q(0)-R_0h(-l)\int_0^{-l}Q\,dx\\
&\quad + R_0\int_0^{-l}h'\Big(\int_0^x Q\,du\Big)dx+h(-l)\int_0^{-l}p\,dx-\int_0^{-l}h'\Big(\int_0^x p\,du\Big)dx\\
&\qquad -2\sum_1^{n-1}\int_0^{-l}N_i\coth(2N_is)h'Q\,dx+\int_0^{-l}h'(\log(Qu_s))_sQ\,dx\Big)\\
&= C\Big(R_0\int_0^{-l}h'\Big(\int_0^x Q\,du\Big)dx-\int_0^{-l}h'\Big(\int_0^x p\,du\Big)dx\\
&\qquad -2\sum_1^{n-1}N_i\int_0^{-l}\coth(2N_is)h'Q\,dx+\int_0^{-l}h'(Qu_s)_x\,dx\Big)\\
&= C\Big(R_0\int_0^{-l}h'\Big(\int_0^x Q\,du\Big)dx-\int_0^{-l}h'\Big(\int_0^x p\,du\Big)dx\\
&\qquad -2\sum_1^{n-1}N_i\int_0^{-l}\coth(2N_is)h'Q\,dx-\int_0^{-l}Qu_sh''\,dx\Big)\\
&= C\Big(R_0\int_0^{-l}h'\Big(\int_0^x Q\,du\Big)dx-\int_0^{-l}h'\Big(\int_0^x p\,du\Big)dx\\
&\qquad -2\sum_1^{n-1}N_i\int_0^{-l}\coth(2N_iH_x)h'Q\,dx-\int_0^{-l}Q(H_{xx})^{-1}h''\,dx\Big).
\end{aligned}
$$

The change of sign in the second equality comes from $\dot\Gamma(s,t)=-\dot H(x,t)$ for the Legendre transformation as in [Guan 2003].

If h'' is negative somewhere, then the geodesic is finite and the limit is a cone metric. The point $-l$ cannot be a singular point. At the singular points h'' is negative. Therefore, the last term of the right hand side is positive infinite. The second term from the right hand side is finite if 0 is not a singular point and positive if 0 is a singular point since in that case $h''(0)<0$ and $h'(0)=s(0)-s_0(0)=0$, $h'<0$ near 0.

If h'' is nonnegative, then the geodesic ray is infinite and h' is increasing. s becomes infinite at each point with $h' > 0$, so $\coth(2N_i s)$ is 1 at such points. It is not difficult to see that $(H_{xx})^{-1}$ is zero whenever h'' is not zero. The limit of the derivative is:

Theorem 11. *For type I compact Kähler almost homogeneous manifolds of cohomogeneity one, the generalized Futaki invariant of a maximal geodesic ray with a convex function h is*

$$C\left(\int_0^{-l} h'\left(\int_0^x (R_0 Q - p)\, du - 2\sum_1^{n-1} N_i Q\right) dx\right)$$

with a constant $C > 0$.

According to [Guan 2011a, (14)], this is proportional to the negative of

$$\int_0^{-l} h' g_l \, dx.$$

We notice that all the generalized Futaki invariants of the maximal geodesic rays do not depend on the initial metrics and they are positive if there is an extremal metric.

Moreover, if there is a Kähler metric with a constant scalar curvature, then at the corresponding H_0 we have that the slopes of Mabuchi functionals are zeros. Therefore, for any h,

$$\int_0^{-l}\left[h'\left[\int_0^x (R_0 Q - p)\, du - 2\sum_1^{n-1} N_i Q \coth(2N_i H_{0,x})\right] - Q(H_{0,xx})^{-1} h''\right] dx = 0.$$

In general, the slope of the Mabuchi functional is

$$C\int_0^{-l} Q\left(2\sum_1^{n-1} N_i(\coth(2N_i H_{0,x}) - \coth(2N_i H_x))h' + ((H_{0,xx})^{-1} - (H_{xx})^{-1})h''\right) dx$$

$$= C\int_0^{-l} Q\left(4\sum_1^{n-1} N_i \frac{e^{2N_i H_{0,x}}(e^{2N_i t h'} - 1)}{(e^{2N_i(H_{0,x}+th')} - 1)(e^{2N_i H_{0,x}} - 1)} h' + \left((H_{0,xx})^{-1} - (H_{xx})^{-1}\right)h''\right) dx.$$

It turns into

$$C\int_0^{-l} Q\left(\sum_1^{n-1} \frac{4N_i}{e^{2N_i H_{0,x}} - 1} h' + H_{0,xx}^{-1} h''\right) dx.$$

Therefore, using this formula as a hint, we can define

$$\|h\|_*^{2,1} = \int_0^{-l} Q\left(\sum_1^{n-1} \frac{4N_i}{e^{2N_i H_{0,x}} - 1}|h'| + H_{0,xx}^{-1}|h''|\right) dx$$

to be the norm of $W_*^{2,1}$. A calculation shows that this is related to

$$\int_0^{-l} |\Delta_0 h|\, Q\, dx \quad \text{and also} \quad \int_0^{-l} \sup\{|\partial^2 h(v)|_0/|v|_0\}\, dV,$$

with dV the volume element. The generalized Futaki functional is positive on the closure of the effective cone in $W_*^{2,1}$.

The generalized Futaki functional is positive if and only if it is positive for

$$h' = \begin{cases} 1 & \text{if } x > x_0, \\ 0 & \text{if } x \le x_0, \end{cases}$$

with $x_0 \in [0, -l)$. These functions h' correspond to functions of h in $W_*^{2,1}$ which are the extremal rays of the effective cone. As we see in the sentence right after Theorem 11, this is the same as the partial integral

$$\int_{x_0}^{-l} g_l\, du = \int_{x_0^2}^{l^2} f_l\, dx < 0$$

for the g_l, f_l in [Guan 2011a]. This is the same as the necessary and sufficient condition in [ibid.] (see (7) and (16) there) for the existence of the Kähler metrics with constant scalar curvatures.

Therefore, we obtain:

Theorem 12. *For type I Kähler compact almost homogeneous manifolds of cohomogeneity one*, *there is a unique extremal metric in a Kähler class on the manifold up to the automorphism group if and only if the Kähler class is geodesically stable.*

The same method works for some of Kähler classes on type II compact Kähler almost homogeneous manifolds of cohomogeneity one. But in general, we will use a different method. Theorem 12 and a result similar to Theorem 11 are true for general compact almost homogeneous manifolds of cohomogeneity one. But it will take us some more time to publish the related results and proofs. We also expect that Theorem 12 is true for any Kähler class on any compact Kähler manifold.

Theorem 11 also gives another proof for the stability (the necessary condition) in [ibid.]. However, the integral itself and its partial integrals *do not occur directly* as generalized Futaki invariants of any (smooth) geodesic.

A generalization of our argument is essential to prove the necessary condition for the type II cases (and the type IV case in [Guan 2009]). However, since we have not seen any example with a zero value of the integral for the Ricci classes, for all the known cases so far in [Guan 2009], etc., the corresponding result in the next section is enough for the necessary part for the Kähler–Einstein case.

6. Geodesic stability and strict slope stability

In this section, we shall discuss our result and the strict slope stability. This is something also similar to the holomorphic vector bundle case and can be defined on any Kähler class of any compact Kähler manifold.

6.1. To make the things simpler, first we assume that the Kähler class is the anticanonic class $-K_M$, N is a smooth subvariety and $M(N)$ is the blow-up of M along N. Let E be the exceptional divisor and e be the largest number such that $-K_M - aE > 0$ on $M(N)$, regarding K_M as the pullback line bundle for any a such that $0 < a < e$,

$$m(N) = \int_0^e (-K_M - (n - \dim N)E)(-K_M - aE)^{n-1}\, da,$$

$$m = \int_0^e (-K_M - aE)^n\, da.$$

We say that M is *strictly* slope stable if for any subvariety N (not necessary smooth) *that is not a component of the fixed point set of a holomorphic vector field* we have $m(N) < m$. That is

$$\int_0^e (a - (n - \dim N))E(-K_M - aE)^{n-1}\, da < 0.$$

Notice that there is only one possible zero for $a - (n - \dim N)$, we see that if $m(N) - m < 0$ then

$$\int_0^c (a - (n - \dim N))E(-K - aE)^{n-1}\, da < 0$$

for any $0 < c < e$. That is, when N is smooth, our stability is stronger than Ross–Thomas's *slope stability* in [Ross and Thomas 2006], which only requires the inequality for rational c with $0 < c < e$, while our inequality is true for any c with $0 < c \leq e$. If N is not smooth, we do not know whether the slope stability in [Ross and Thomas 2006] implies these inequalities or not.

A smooth N destabilizes M only if $-K_M - (n - \dim N)E$ is ample, therefore, $-K(E)$ is ample on E if E is smooth, and is kind of ample even if E is singular. When N is smooth, we see that E is Fano. By [Futaki 1987], we see that N is Fano also. This is quite similar to the calculation in [Guan 2003; 2011a].

Actually, when $F = \mathbb{C}P^k$ or $\mathrm{Gr}(2k, 2)$, we have $D(F) = 2$ by [Guan 2011b, Section 3, Theorem 15]. Therefore, for the closed orbit N, $e = -2^{-1} l_\rho$ and the codimension can only be 1; see [ibid., Section 3]. If $y = -l_\rho - 2a$, the integral above is

$$\int_0^{-l_\rho}(-2^{-1}(y+l_\rho)-1)E(\omega+2^{-1}(y+l_\rho)E)^{n-1}2^{-1}\,dy = C\int_0^{-K(F)}(-K(F)-D(F)-y)Q\,dy,$$

with a positive number C. That is exactly the same condition as in Theorem 15 just cited.

When $F=Q^k$, $D(F)=1$. Therefore, $e=-l_\rho$. Let

$$y=-l_\rho-a=-K(F)+m-1-a,$$

with $m=n-\dim N$. The integral above is

$$\int_0^{-l_\rho}(-l_\rho-y-m)E(\omega+(y+l_\rho)E)^{n-1}\,dy = C\int_{-K(F)+m-1}(-K(F)-D(F)-y)Q\,dy,$$

with $C>0$. Again, that is exactly Theorem 15 in [ibid.].

6.2. In general, for any given Kähler class ω we let

$$m_c(N)=\int_0^c(-K_M-(n-\dim N)E)(\omega-aE)^{n-1}\,da,$$
$$m_c=\int_0^c(\omega-aE)^n\,da,$$

with $0<c\le e$ and e the largest number such that $\omega-aE>0$ for $0<a<e$. We let $\mu_c(N)=m_c(N)/m_c$. If $N=M$, we let $m(M)=(-K_M)\omega^{n-1}$ and $\mu=m(M)/\omega^n$. Then the strict slope stability says that $\mu_c(N)-\mu<0$ for all $0<c\le e$. Similar obstructions appeared in [Guan 2003]. At that time I was not able to understand the general meaning of this obstruction and related it to the Ding–Tian generalized Futaki invariant forcibly. But it was clear in [Guan 2003] it was not the Ding–Tian generalized Futaki invariant. I also talked on this at Pisa, Italy in 2004. Ross and Thomas [2006] partially generalized this obstruction but without the *strict* part for a smooth N, that is, they assume that $0<c<e$. Also, they assume that c is rational, which makes their slope stability much weaker. For a nonsmooth subvariety N, I am not sure that their stability implies these inequalities or not. For our case, our strict slope stability is equivalent to the existence. But the Ross–Thomas slope stability is only a necessary condition. Therefore, *a Kähler class with the integral equal to zero when $c=e$ or c is irrational would give a counterexample for the equivalence between the Ross–Thomas slope stability or Donaldson K-stability and the existence.* See also [Guan 2003; 2007].

It is very easy to check that if K_M is the Kähler class and we replace $-K_M - aE$ by $K_M - aE$, and let

$$m_c(N) = \int_0^c (-K_M - (n - \dim N)E)(K_M - aE)^{n-1}\, da,$$

the strict slope stability means that $m_c(N) + m_c < 0$ holds automatically. Moreover, if $K_M = 0$, for any Kähler metric ω we replace $-K_M - aE$ by $\omega - aE$ and let

$$m_c(N) = \int_0^c (-n + \dim N)E(\omega - aE)^{n-1}\, da,$$

the strict slope stability means that $m_c(N) < 0$ holds automatically. These strengthen the Theorem 5.4 in [Ross and Thomas 2006], which is only concerned with when N is smooth and $0 < c < e$ is rational.

In the remainder of this section, we want to see that the strict slope stability is the same as the existence for type I manifolds.

To make things simpler, let us take care of the $F(\mathrm{OP}_n)$ fiber case first. In our setting, we only need to deal with the case in which N is the closed orbit. In this case, by [Guan 2011b, Section 3], we have $\dim N = n - 1$. Let us calculate the number e for our case. By [ibid., Section 3] we see that the curvature of the exceptional divisor has eigenvalues $D(\mathbb{C}P^n) = 2$ times the coefficient of u. Therefore, $\omega - aE$ has the first zero eigenvalues when $a = (D(F))^{-1}(-l)$. That is, $e = -2^{-1}l$.

$$\omega^n m_c(N) - m(M)m_c = \int_0^{-l} Q\, du\Big[\int_0^c (-K_M)((\omega - xE)^{n-1} - \omega^{n-1}) - E(\omega - xE)^{n-1} - R_0((\omega - xE)^n - \omega^n)\, dx\Big].$$

This is proportional to

$$\int_0^c \Big[\int_0^x \big[(n-1)K_M E(\omega - uE)^{n-2} + nR_0 E(\omega - uE)^{n-1}\big]\, du - E(\omega - xE)^{n-1}\Big]\, dx.$$

Letting $y = -l - 2x$ and $v = -l - 2u$, $d = -l - 2c$, we obtain that the integral is proportional to

$$\int_d^{-l}\Big[\int_y^{-l}\big[(n-1)K_M E(\omega + 2^{-1}(v+l)E)^{n-2} + nR_0 E(\omega + 2^{-1}(v+l)E)^{n-1}\big]\, dv - 2E(\omega + 2^{-1}(y+l)E)^{n-1}\Big]\, dy = \int_d^{-l} h_l\, dy.$$

By taking the derivative twice we have

$$h_l' = -(n-1)K(E)E(\omega + 2^{-1}(y+l)E)^{n-2} - nR_0 E(\omega + 2^{-1}(y+l)E)^{n-1}.$$

By the argument in [Guan 2011a] after (14) and in the proof of Lemma 6, we see that h_l' is proportional to g_l' there. Therefore, we only need to check for a point 0, the function h_l is right. To prove our conclusion, we only need to check that

$$h_l(0) = \int_0^{-l} \big[(n-1)K_M E(\omega+2^{-1}(v+l)E)^{n-2} + nR_0 E(\omega+2^{-1}(v+l)E)^{n-1}\big]\,dv = 0,$$

since $g_l(0)=0$. Notice that $nE(\omega+2^{-1}(v+l)E)^{n-1}$ is related to ω^n there.

The exact same argument works for the case in which the fiber $F = \mathrm{Gr}(2k, 2)$.

For the case in which the fiber $F = Q^n$, we have $D(F) = 1$. Therefore, we could let $y = -l - x$, $v = -l - u$, $d = -l - c$ instead and we notice

$$-K(E) = -K_M - (n - \dim N)E.$$

The same proof goes through.

Theorem 13. *On a type I compact almost homogeneous manifold of cohomogeneity one there is a Kähler metric of constant scalar curvature in a given Kähler class if and only if the Kähler class is strictly slope stable with respect to the closed orbit.*

This is also true for general compact Kähler almost homogeneous manifolds of cohomogeneity one. But it will take some time for us to publish the detailed results and proofs.

6.3. In the case of Fano manifolds, our discussion in Section 6.1 shows:

Theorem 14. *Let M be any Fano manifold. If a smooth submanifold N destabilizes the Ricci class, then N, the blowing-up manifold $M(N)$ of M along N and the exceptional divisor E are all Fano manifolds.*

One could also consider the case where N is a union of smooth submanifolds. We expect that each of them should be Fano also. Similarly, it should be easy to obtain some results similar to those of Nadel [1990] and to check out the unstable Fano threefolds.

For the compact Kähler manifolds with a zero or negative first Chern class we showed at the beginning of Section 6.2 that:

Theorem 15. *Let M be any compact Kähler manifold with a negative first Chern class. Then the negative Ricci class is strictly slope stable.*

Theorem 16. *Let M be any compact Kähler manifold with a zero first Chern class. Then any Kähler class is strictly slope stable.*

Theorems 14, 15, 16 give a good reason why the Calabi conjecture is true for the negative and zero case but not true in general for the positive case.

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Received November 4, 2011. Revised September 6, 2012.

DANIEL GUAN
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF CALIFORNIA AT RIVERSIDE
RIVERSIDE CA 92521
UNITED STATES

zguan@math.ucr.edu

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 2 February 2013