Pacific Journal of Mathematics

OPERATOR ALGEBRAS AND CONJUGACY PROBLEM FOR THE PSEUDO-ANOSOV AUTOMORPHISMS OF A SURFACE

IGOR NIKOLAEV

Volume 261 No. 2 February 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 261, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.261.445

OPERATOR ALGEBRAS AND CONJUGACY PROBLEM FOR THE PSEUDO-ANOSOV AUTOMORPHISMS OF A SURFACE

IGOR NIKOLAEV

In memory of W. P. Thurston

The conjugacy problem for the pseudo-Anosov automorphisms of a compact surface is studied. To each pseudo-Anosov automorphism ϕ, we assign an AF C^*-algebra $\mathbb{A}_\phi$ (an operator algebra). It is proved that the assignment is functorial, i.e., every ϕ', conjugate to ϕ, maps to an AF C^*-algebra $\mathbb{A}_{\phi'}$, which is stably isomorphic to $\mathbb{A}_\phi$. The new invariants of the conjugacy of the pseudo-Anosov automorphisms are obtained from the known invariants of the stable isomorphisms of the AF C^*-algebras. Namely, the main invariant is a triple $(\Lambda, [I], K)$, where Λ is an order in the ring of integers in a real algebraic number field K and $[I]$ an equivalence class of the ideals in Λ. The numerical invariants include the determinant Δ and the signature Σ, which we compute for the case of the Anosov automorphisms. A question concerning the p-adic invariants of the pseudo-Anosov automorphism is formulated.

Introduction

A. ***Conjugacy problem.*** Let $\mathrm{Mod}(X)$ be the mapping class group of a compact surface X, i.e., the group of orientation preserving automorphisms of X modulo the trivial ones. Recall that $\phi, \phi' \in \mathrm{Mod}(X)$ are conjugate automorphisms whenever $\phi' = h \circ \phi \circ h^{-1}$ for an $h \in \mathrm{Mod}(X)$. It is not hard to see that conjugation is an equivalence relation which splits the mapping class group into disjoint classes of conjugate automorphisms. The construction of invariants of the conjugacy

Partially supported by NSERC.
MSC2010: primary 46L85; secondary 57M27.
Keywords: mapping class group, AF C^*-algebras.

classes in $\mathrm{Mod}(X)$ is an important and difficult problem studied by Hemion [1979], Mosher [1986], and others. Any knowledge of such invariants leads to a topological classification of three-dimensional manifolds, which fiber over the circle with monodromy $\phi \in \mathrm{Mod}(X)$ [Thurston 1982].

B. ***Pseudo-Anosov automorphisms.*** It is known that any $\phi \in \mathrm{Mod}(X)$ is isotopic to an automorphism ϕ', such that either (i) ϕ' has a finite order, or (ii) ϕ' is a pseudo-Anosov (aperiodic) automorphism, or else (iii) ϕ' is reducible by a system of curves Γ surrounded by the small tubular neighborhoods $N(\Gamma)$, such that on $X \setminus N(\Gamma)$, ϕ' satisfies either (i) or (ii). Let ϕ be a representative of the equivalence class of a pseudo-Anosov automorphism. Then there exist a pair consisting of the stable $\mathcal{F}_s$ and unstable $\mathcal{F}_u$ mutually orthogonal measured foliations on the surface X, such that $\phi(\mathcal{F}_s) = (1/\lambda_\phi)\mathcal{F}_s$ and $\phi(\mathcal{F}_u) = \lambda_\phi \mathcal{F}_u$, where $\lambda_\phi > 1$ is called a dilatation of ϕ. The foliations $\mathcal{F}_s$, $\mathcal{F}_u$ are minimal, uniquely ergodic and describe the automorphism ϕ up to a power. In the sequel, we shall focus on the conjugacy problem for the pseudo-Anosov automorphisms of a surface X.

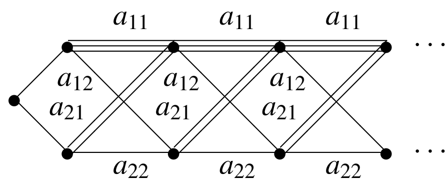
C. ***AF C*-algebras.*** A C^*-algebra is an algebra $\mathbb{A}$ over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $\|ab\| \le \|a\|\|b\|$ and $\|a^*a\| = \|a^2\|$ for all $a, b \in \mathbb{A}$. The C^*-algebras have been introduced by Murray and von Neumann as rings of bounded operators on a Hilbert space and are strongly connected with the geometry and topology of manifolds [Blackadar 1986, Section 24]. Any simple finite-dimensional C^*-algebra is isomorphic to the algebra $M_n(\mathbb{C})$ of the complex $n \times n$ matrices. A natural completion of the finite-dimensional semisimple C^*-algebras (as $n \to \infty$) is known as an *AF C^*-algebra* [Effros 1981]. An AF C^*-algebra is most conveniently given by an infinite graph, which records the inclusion of the finite-dimensional subalgebras into the AF C^*-algebra. The graph is called a *Bratteli diagram*. When the diagram is periodic, the AF C^*-algebra is *stationary*; this is an important special case. In addition to the usual isomorphism $\cong$, the C^*-algebras $\mathbb{A}$, $\mathbb{A}'$ are called *stably isomorphic* whenever $\mathbb{A} \otimes \mathcal{K} \cong \mathbb{A}' \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators.

D. ***Motivation.*** Let $\phi \in \mathrm{Mod}(X)$ be a pseudo-Anosov automorphism. The main idea of the present paper is to assign to ϕ an AF C^*-algebra, $\mathbb{A}_\phi$, so that for every $h \in \mathrm{Mod}(X)$ the following diagram commutes:

$$\begin{array}{ccc}
\phi & \xrightarrow{\text{conjugacy}} & \phi' = h \circ \phi \circ h^{-1} \\
\downarrow & & \downarrow \\
\mathbb{A}_\phi & \xrightarrow[\text{isomorphism}]{\text{stable}} & \mathbb{A}_{\phi'}
\end{array}$$

(In other words, if ϕ and ϕ' are conjugate pseudo-Anosov automorphisms, then the AF C^*-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are stably isomorphic.) For the sake of clarity, we shall consider an example illustrating the idea in the case $X = T^2$ (a torus).

E. ***Model example.*** Let $\phi \in \mathrm{Mod}(T^2)$ be the Anosov automorphism given by a nonnegative matrix $A_\phi \in \mathrm{SL}_2(\mathbb{Z})$. (The assumption is not restrictive; each A_ϕ with $\mathrm{Tr}(A_\phi) > 0$ is similar to a nonnegative matrix. The case $\mathrm{Tr}(A_\phi) < 0$ is treated likewise — by reduction to a nonpositive matrix; then the absolute value of all entries must be taken.) Consider a stationary AF C^*-algebra, $\mathbb{A}_\phi$, given by the following periodic Bratteli diagram:



$$A_\phi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Figure 1. The AF C^*-algebra $\mathbb{A}_\phi$.

where a_{ij} indicate the multiplicity of the respective edges of the graph. We encourage the reader to verify that $F : \phi \mapsto \mathbb{A}_\phi$ is a well-defined function on the set of Anosov automorphisms given by the hyperbolic matrices with nonnegative entries. Let us show that if $\phi, \phi' \in \mathrm{Mod}(T^2)$ are conjugate Anosov automorphisms, then $\mathbb{A}_\phi$, $\mathbb{A}_{\phi'}$ are stably isomorphic AF C^*-algebras. Indeed, let $\phi' = h \circ \phi \circ h^{-1}$ for an $h \in \mathrm{Mod}(X)$. Then $A_{\phi'} = T A_\phi T^{-1}$ for a matrix $T \in \mathrm{SL}_2(\mathbb{Z})$. Note that

$$(A'_\phi)^n = (T A_\phi T^{-1})^n = T A_\phi^n T^{-1},$$

where $n \in \mathbb{N}$. We shall use the following criterion: the AF C^*-algebras $\mathbb{A}$, $\mathbb{A}'$ are stably isomorphic if and only if their Bratteli diagrams contain a common block of an arbitrary length (compare with [Effros 1981, Theorem 2.3]; recall that an order-isomorphism mentioned in the theorem is equivalent to the condition that the corresponding Bratteli diagrams have the same infinite tails — i.e., a common block of infinite length). Consider two sequences of matrices:

$$\underbrace{A_\phi A_\phi \cdots A_\phi}_{n},$$

which mimics the Bratteli diagram of $\mathbb{A}_\phi$, and

$$T \underbrace{A_\phi A_\phi \cdots A_\phi}_{n} T^{-1},$$

which mimics that of $\mathbb{A}_{\phi'}$. Letting $n \to \infty$, we conclude that $\mathbb{A}_\phi \otimes \mathcal{K} \cong \mathbb{A}_{\phi'} \otimes \mathcal{K}$.

F. ***Invariants of torus automorphisms obtained from the operator algebras.*** The conjugacy problem for the Anosov automorphisms can now be recast in terms of AF C^*-algebras: find invariants of stable isomorphism classes of the stationary AF C^*-algebras. One such invariant is due to Handelman [1981]. Consider an eigenvalue problem for the hyperbolic matrix $A_\phi \in SL_2(\mathbb{Z})$: $A_\phi v_A = \lambda_A v_A$, where $\lambda_A > 1$ is the Perron–Frobenius eigenvalue and $v_A = (v_A^{(1)}, v_A^{(2)})$ the corresponding eigenvector with the positive entries normalized so that $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. Denote by $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \mathbb{Z}v_A^{(2)}$ the $\mathbb{Z}$-module in the number field K. Recall that the coefficient ring, Λ, of module $\mathfrak{m}$ consists of the elements $\alpha \in K$ such that $\alpha\mathfrak{m} \subseteq \mathfrak{m}$. It is known that Λ is an order in K (i.e., a subring of K containing 1) and, with no restriction, one can assume that $\mathfrak{m} \subseteq \Lambda$. It follows from the definition that $\mathfrak{m}$ coincides with an ideal, I, whose equivalence class in Λ we shall denote by $[I]$. It has been proved by Handelman that the triple $(\Lambda, [I], K)$ is an arithmetic invariant of the stable isomorphism class of $\mathbb{A}_\phi$: the $\mathbb{A}_\phi$, $\mathbb{A}_{\phi'}$ are stably isomorphic AF C^*-algebras if and only if $\Lambda = \Lambda'$, $[I] = [I']$ and $K = K'$. It is interesting to compare the operator algebra invariants with the matrix invariants obtained in [Latimer and MacDuffee 1933] and [Wallace 1984].

G. ***AF C*-algebra*** $\mathbb{A}_\phi$ **(*pseudo-Anosov case*).** Denote by $\mathcal{F}_\phi$ the stable foliation of a pseudo-Anosov automorphism $\phi \in \mathrm{Mod}(X)$. For brevity, we assume that $\mathcal{F}_\phi$ is an oriented foliation given by the trajectories of a closed 1-form $\omega \in H^1(X; \mathbb{R})$. Let $v^{(i)} = \int_{\gamma_i} \omega$, where $\{\gamma_1, \dots, \gamma_n\}$ is a basis in the relative homology $H_1(X, \mathrm{Sing}\,\mathcal{F}_\phi; \mathbb{Z})$, such that $\theta = (\theta_1, \dots, \theta_{n-1})$ is a vector with positive coordinates $\theta_i = v^{(i+1)}/v^{(1)}$. (Note that the θ_i depend on a basis in the homology group, but a $\mathbb{Z}$-module generated by the θ_i does not — see Lemma 5.) Consider the (infinite) Jacobi–Perron continued fraction [Bernstein 1971] of θ:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$ is a vector of nonnegative integers, I the unit matrix and $\mathbb{I} = (0, \dots, 0, 1)^T$. By definition, $\mathbb{A}_\phi$ is an (isomorphism class of the) AF C^*-algebra given by the Bratteli diagram whose incidence matrices coincide with $B_k = \left(\begin{smallmatrix} 0 & 1 \\ I & b_k \end{smallmatrix}\right)$ for $k = 1, \dots, \infty$. Note that this yields the Bratteli diagram derived in the model example (the Anosov case).

H. ***Main results.*** For a matrix $A \in GL_n(\mathbb{Z})$ with positive entries, we denote by λ_A the Perron–Frobenius eigenvalue and let $(v_A^{(1)}, \dots, v_A^{(n)})$ denote the corresponding normalized eigenvector with $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. The coefficient (endomorphism) ring of the module $\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \cdots + \mathbb{Z}v_A^{(n)}$ will be denoted by Λ. The equivalence class of ideal I in Λ will be denoted $[I]$. Finally, we denote by $\Delta = \mathrm{Det}(a_{ij})$ and Σ

the determinant and signature of the symmetric bilinear form $q(x, y) = \sum_{i,j}^{n} a_{ij} x_i x_j$, where $a_{ij} = \mathrm{Tr}(v_A^{(i)} v_A^{(j)})$, with $\mathrm{Tr}(\,\cdot\,)$ the trace function. Our main results can be expressed as follows.

Theorem 1. *$\mathbb{A}_\phi$ is a stationary AF C^*-algebra.*

Let Φ be a category of all pseudo-Anosov (Anosov, respectively) automorphisms of a surface of the genus $g \geq 2$ ($g = 1$, respectively); the arrows (morphisms) are conjugations between the automorphisms. Likewise, let $\mathscr{A}$ be the category of all stationary AF C^*-algebras $\mathbb{A}_\phi$, where ϕ runs over the set Φ; the arrows of $\mathscr{A}$ are stable isomorphisms among the algebras $\mathbb{A}_\phi$.

Theorem 2. *Let $F : \Phi \to \mathscr{A}$ be a map given by the formula $\phi \mapsto \mathbb{A}_\phi$. Then*:

(i) *F is a functor*; *it maps conjugate pseudo-Anosov automorphisms to stably isomorphic AF C^*-algebras.*

(ii) $\mathrm{Ker}\, F = [\phi]$, *where $[\phi] = \{\phi' \in \Phi \mid (\phi')^m = \phi^n, m, n \in \mathbb{N}\}$ is the commensurability class of the pseudo-Anoov automorphism ϕ.*

Corollary 3. *The triple $(\Lambda, [I], K)$ and the integers Δ and Σ are invariants of the conjugacy classes of the pseudo-Anosov automorphisms.*

I. ***How can the invariants $(\Lambda, [I], K)$, Δ and Σ be calculated?*** There is no easy way; the problem is comparable to that of numerical invariants of the fundamental group of a knot. A step in this direction would be computation of the matrix A; the latter is similar to the matrix $\rho(\phi)$, where $\rho : \mathrm{Mod}(X) \to \mathrm{PIL}$ is a faithful representation of the mapping class group as a group of the piecewise-integral-linear transformations [Penner 1984, p. 45]. The entries of $\rho(\phi)$ are the linear combinations of the Dehn twists along the $(3g - 1)$ (Lickorish) curves on the surface X. Then one can effectively determine whether $\rho(\phi)$ and A are similar matrices (over $\mathbb{Z}$) by bringing the polynomial matrices $\rho(\phi) - xI$ and $A - xI$ to the Smith normal form; when the similarity is established, the numerical invariants Δ and Σ become the polynomials in the Dehn twists. A tabulation of the simplest elements of $\mathrm{Mod}(X)$ is possible in terms of Δ and Σ (see the Examples section, page 459); however, this task lies beyond the scope of present paper.

J. ***Structure of the paper.*** Proofs of the main results can be found in Section 3. Sections 1 and 2 consist of lemmas used to prove the main results. Section 4 includes some examples, open problems and conjectures. Since the paper does not include a formal section on the preliminaries, we encourage the reader to consult [Blackadar 1986; Effros 1981; Krieger 1980] (operator algebras and dynamics), [Hubbard and Masur 1979; Thurston 1988] (measured foliations) and [Bernstein 1971; Perron 1907] (Jacobi–Perron continued fractions).

1. The jacobian of a measured foliation

Let $\mathcal{F}$ be a measured foliation on a compact surface X [Thurston 1988]. For the sake of brevity, we shall always assume that $\mathcal{F}$ is an oriented foliation, i.e., given by the trajectories of a closed 1-form ω on X. (The assumption is not a restriction; by [Hubbard and Masur 1979], every measured foliation is oriented on a double cover $\widetilde{X}$ of X ramified at the singular points of the half-integer index of the nonoriented foliation.) Let $\{\gamma_1, \dots, \gamma_n\}$ be a basis in the relative homology group $H_1(X, \operatorname{Sing}\mathcal{F}; \mathbb{Z})$, where $\operatorname{Sing}\mathcal{F}$ is the set of singular points of the foliation $\mathcal{F}$. It is well known that $n = 2g + m - 1$, where g is the genus of X and $m = |\operatorname{Sing}(\mathcal{F})|$. The periods of ω in this basis will be written

$$\lambda_i = \int_{\gamma_i} \omega.$$

The real numbers λ_i are coordinates of $\mathcal{F}$ in the space of all measured foliations on X (with the fixed set of singular points) [Douady and Hubbard 1975].

Definition 4. By the jacobian $\operatorname{Jac}(\mathcal{F})$ of the measured foliation $\mathcal{F}$, we understand the $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ regarded as a subset of the real line $\mathbb{R}$.

The importance of the jacobian stems from the observation that although the periods, λ_i, depend on the choice of a basis in $H_1(X, \operatorname{Sing}\mathcal{F}; \mathbb{Z})$, the jacobian does not. Moreover, up to a scalar multiple, the jacobian is an invariant of the equivalence class of the foliation $\mathcal{F}$. We formalize these observations in the following two results.

Lemma 5 (invariance of the jacobian). *The $\mathbb{Z}$-module $\mathfrak{m}$ is independent of the choice of a basis in $H_1(X, \operatorname{Sing}\mathcal{F}; \mathbb{Z})$ and depends solely on the foliation $\mathcal{F}$.*

Proof. Indeed, let $A = (a_{ij}) \in GL_n(\mathbb{Z})$ and let

$$\gamma_i' = \sum_{j=1}^n a_{ij}\gamma_j$$

be a new basis in $H_1(X, \operatorname{Sing}\mathcal{F}; \mathbb{Z})$. Then using the integration rules,

$$\lambda_i' = \int_{\gamma_i'} \omega = \int_{\sum_{j=1}^n a_{ij}\gamma_j} \omega = \sum_{j=1}^n \int_{\gamma_j} \omega = \sum_{j=1}^n a_{ij}\lambda_j.$$

To prove that $\mathfrak{m} = \mathfrak{m}'$, consider the following equations:

$$\mathfrak{m}' = \sum_{i=1}^n \mathbb{Z}\lambda_i' = \sum_{i=1}^n \mathbb{Z} \sum_{j=1}^n a_{ij}\lambda_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\mathbb{Z} \right) \lambda_j \subseteq \mathfrak{m}.$$

Let $A^{-1} = (b_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ be an inverse to the matrix A. Then $\lambda_i = \sum_{j=1}^{n} b_{ij}\lambda'_j$ and

$$\mathfrak{m} = \sum_{i=1}^{n} \mathbb{Z}\lambda_i = \sum_{i=1}^{n} \mathbb{Z} \sum_{j=1}^{n} b_{ij}\lambda'_j = \sum_{j=1}^{n}\Bigl(\sum_{i=1}^{n} b_{ij}\mathbb{Z}\Bigr)\lambda'_j \subseteq \mathfrak{m}'.$$

Since both $\mathfrak{m}' \subseteq \mathfrak{m}$ and $\mathfrak{m} \subseteq \mathfrak{m}'$, we conclude that $\mathfrak{m}' = \mathfrak{m}$. Lemma 5 follows. □

Now recall that two measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent* if there exists an automorphism $h \in \mathrm{Mod}(X)$ that sends the leaves of the foliation $\mathcal{F}$ to the leaves of the foliation $\mathcal{F}'$. This equivalence deals with topological foliations, i.e., projective classes of measured foliations; see [Thurston 1988] for an explanation.

Lemma 6 (projective invariance). *Let $\mathcal{F}$, $\mathcal{F}'$ be the equivalent measured foliations on a surface X. Then*

$$\mathrm{Jac}(\mathcal{F}') = \mu\,\mathrm{Jac}(\mathcal{F}),$$

where $\mu > 0$ is a real number.

Proof. Let $h : X \to X$ be an automorphism of the surface X. Denote by h_* its action on $H_1(X, \mathrm{Sing}(\mathcal{F}); \mathbb{Z})$ and by h^* on $H^1(X; \mathbb{R})$ connected by the formula

$$\int_{h_*(\gamma)} \omega = \int_{\gamma} h^*(\omega), \quad \text{for all } \gamma \in H_1(X, \mathrm{Sing}(\mathcal{F}); \mathbb{Z}) \text{ and } \omega \in H^1(X; \mathbb{R}).$$

Let $\omega, \omega' \in H^1(X; \mathbb{R})$ be the closed 1-forms whose trajectories define the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Since $\mathcal{F}$, $\mathcal{F}'$ are equivalent measured foliations,

$$\omega' = \mu h^*(\omega)$$

for a $\mu > 0$.

Let $\mathrm{Jac}(\mathcal{F}) = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ and $\mathrm{Jac}(\mathcal{F}') = \mathbb{Z}\lambda'_1 + \cdots + \mathbb{Z}\lambda'_n$. Then

$$\lambda'_i = \int_{\gamma_i} \omega' = \mu \int_{\gamma_i} h^*(\omega) = \mu \int_{h_*(\gamma_i)} \omega, \quad 1 \le i \le n.$$

By Lemma 5, we have

$$\mathrm{Jac}(\mathcal{F}) = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega = \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega.$$

Therefore

$$\mathrm{Jac}(\mathcal{F}') = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega' = \mu \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega = \mu\,\mathrm{Jac}(\mathcal{F}).$$

Lemma 6 follows. □

2. Equivalent foliations are stably isomorphic

Let $\mathscr{F}$ be a measured foliation on the surface X. We introduce an AF C^*-algebra, $\mathbb{A}_{\mathscr{F}}$, corresponding to the foliation $\mathscr{F}$ as explained in Section G of the Introduction (for the foliation $\mathscr{F}_\phi$). The goal of this section is to prove the commutativity of the following diagram:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\text{equivalence}"] \arrow[d] & \mathscr{F}' \arrow[d] \\
\mathbb{A}_{\mathscr{F}} \arrow[r, "\text{stable}", "\text{isomorphism}"'] & \mathbb{A}_{\mathscr{F}'}
\end{tikzcd}$$

We start with a simple property of Jacobi–Perron fractions [Bernstein 1971].

Lemma 7 (modules and continued fractions). *Let* $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ *and* $\mathfrak{m}' = \mathbb{Z}\lambda_1' + \cdots + \mathbb{Z}\lambda_n'$ *be two* $\mathbb{Z}$*-modules, such that* $\mathfrak{m}' = \mu\mathfrak{m}$ *for a* $\mu > 0$*. Then the Jacobi–Perron continued fractions of the vectors* λ *and* λ' *coincide except, possibly, at a finite number of terms.*

Proof. Let $\mathfrak{m} = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_n$ and $\mathfrak{m}' = \mathbb{Z}\lambda_1' + \cdots + \mathbb{Z}\lambda_n'$. Since $\mathfrak{m}' = \mu\mathfrak{m}$, where μ is a positive real, one gets the following identity of the $\mathbb{Z}$-modules:

$$\mathbb{Z}\lambda_1' + \cdots + \mathbb{Z}\lambda_n' = \mathbb{Z}(\mu\lambda_1) + \cdots + \mathbb{Z}(\mu\lambda_n).$$

One can always assume that λ_i and λ_i' are positive reals. For obvious reasons, there exists a basis $\{\lambda_1'', \ldots, \lambda_n''\}$ of the module $\mathfrak{m}'$, such that

$$\begin{cases} \lambda'' = A(\mu\lambda), \\ \lambda'' = A'\lambda', \end{cases}$$

where $A, A' \in GL_n^+(\mathbb{Z})$ are the matrices whose entries are nonnegative integers. In view of Proposition 3 of [Bauer 1996], we have

$$A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} 0 & 1 \\ I & b_1' \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_l' \end{pmatrix},$$

where b_i, b_i' are nonnegative integer vectors. Since the (Jacobi–Perron) continued fraction for the vectors λ and $\mu\lambda$ coincide for any $\mu > 0$ [Bernstein 1971], we conclude that

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \square \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ \theta' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ I & b_1' \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_l' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \square \end{pmatrix},$$

where

$$\begin{pmatrix} 1 \\ \theta'' \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & a_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}.$$

In other words, the continued fractions of the vectors λ and λ' coincide except at a finite number of terms. $\square$

Lemma 8 (main lemma). *Let $\mathscr{F}$ and $\mathscr{F}'$ be equivalent measured foliations on a surface X. Then the AF C^*-algebras $\mathbb{A}_{\mathscr{F}}$ and $\mathbb{A}_{\mathscr{F}'}$ are stably isomorphic.*

Proof. Notice that Lemma 6 implies that equivalent measured foliations $\mathscr{F}$, $\mathscr{F}'$ have proportional jacobians, i.e., $\mathfrak{m}' = \mu\mathfrak{m}$ for a $\mu > 0$. On the other hand, by Lemma 7 the continued fraction expansion of the basis vectors of the proportional jacobians must coincide, except a finite number of terms. Thus, the AF C^*-algebras $\mathbb{A}_{\mathscr{F}}$ and $\mathbb{A}_{\mathscr{F}'}$ are given by the Bratteli diagrams, which are identical, except a finite part of the diagram. It is well known [Effros 1981, Theorem 2.3] that AF C^*-algebras that have such a property are stably isomorphic. $\square$

3. Proofs

Proof of Theorem 1. Let $\phi \in \mathrm{Mod}(X)$ be a pseudo-Anosov automorphism of the surface X. Denote by $\mathscr{F}_\phi$ the invariant foliation of ϕ. By definition of such a foliation, $\phi(\mathscr{F}_\phi) = \lambda_\phi \mathscr{F}_\phi$, where $\lambda_\phi > 1$ is the dilatation of ϕ.

Consider the jacobian $\mathrm{Jac}(\mathscr{F}_\phi) = \mathfrak{m}_\phi$ of $\mathscr{F}_\phi$. Since $\mathscr{F}_\phi$ is an invariant foliation of the pseudo-Anosov automorphism ϕ, one gets the following equality of the $\mathbb{Z}$-modules:

$$\mathfrak{m}_\phi = \lambda_\phi \mathfrak{m}_\phi, \quad \lambda_\phi \neq \pm 1. \tag{1}$$

Let $\{v^{(1)}, \dots, v^{(n)}\}$ be a basis in module $\mathfrak{m}_\phi$, such that $v^{(i)} > 0$. In view of (1), one obtains the following system of linear equations:

$$\begin{cases} \lambda_\phi v^{(1)} = a_{11} v^{(1)} + a_{12} v^{(2)} + \cdots + a_{1n} v^{(n)}, \\ \lambda_\phi v^{(2)} = a_{21} v^{(1)} + a_{22} v^{(2)} + \cdots + a_{2n} v^{(n)}, \\ \qquad \vdots \\ \lambda_\phi v^{(n)} = a_{n1} v^{(1)} + a_{n2} v^{(2)} + \cdots + a_{nn} v^{(n)}, \end{cases} \tag{2}$$

where $a_{ij} \in \mathbb{Z}$. The matrix $A = (a_{ij})$ is invertible. Indeed, since the foliation $\mathscr{F}_\phi$ is minimal, the real numbers $v^{(1)}, \dots, v^{(n)}$ are linearly independent over $\mathbb{Q}$. So are the numbers $\lambda_\phi v^{(1)}, \dots, \lambda_\phi v^{(n)}$, which therefore can be taken for a basis of the module $\mathfrak{m}_\phi$. Thus, there exists an integer matrix $B = (b_{ij})$, such that $v^{(j)} = \sum_{i,j} w^{(i)}$, where $w^{(i)} = \lambda_\phi v^{(i)}$. Clearly, B is an inverse to matrix A. Therefore, $A \in \mathrm{GL}_n(\mathbb{Z})$.

Moreover, without loss of generality one can assume that $a_{ij} \geq 0$. Indeed, if this is not yet the case, consider the conjugacy class $[A]$ of the matrix A. Since

$v^{(i)} > 0$, there exists a matrix $A^+ \in [A]$ whose entries are nonnegative integers. One has to replace basis $v = (v^{(1)}, \dots, v^{(n)})$ in the module $\mathfrak{m}_\phi$ by a basis Tv, where $A^+ = TAT^{-1}$. It will be further assumed that $A = A^+$.

Lemma 9. *The vector* $(v^{(1)}, \dots, v^{(n)})$ *is the limit of a periodic Jacobi–Perron continued fraction.*

Proof. It follows from the discussion above that there exists a nonnegative integer matrix A, such that $Av = \lambda_\phi v$. In view of [Bauer 1996, Proposition 3], matrix A admits a unique factorization:

$$A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}, \tag{3}$$

where $b_i = (b_1^{(i)}, \dots, b_n^{(i)})^T$ are vectors of nonnegative integers. Let us consider the periodic Jacobi–Perron continued fraction:

$$\mathrm{Per}\,\overline{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}. \tag{4}$$

According to [Perron 1907, Satz XII], the above fraction converges to a vector

$$w = (w^{(1)}, \dots, w^{(n)})$$

satisfying the equation $(B_1 B_2 \cdots B_k) w = Aw = \lambda_\phi w$. In view of the equation $Av = \lambda_\phi v$, we conclude that vectors v and w are collinear. Therefore, the Jacobi–Perron continued fractions of v and w must coincide. □

It is now straightforward to prove that the AF C^*-algebra attached to foliation $\mathcal{F}_\phi$ is stationary. Indeed, by Lemma 9, the vector of periods $v^{(i)} = \int_{\gamma_i} \omega$ unfolds into a periodic Jacobi–Perron continued fraction. By definition, the Bratteli diagram of the AF C^*-algebra $\mathbb{A}_\phi$ is periodic as well. In other words, the AF C^*-algebra $\mathbb{A}_\phi$ is stationary. □

Proof of Theorem 2. (i) For completeness, we give a proof of the following well-known lemma.

Lemma 10. *If* ϕ *and* ϕ' *are conjugate pseudo-Anosov automorphisms of a surface* X, *their invariant foliations* $\mathcal{F}_\phi$ *and* $\mathcal{F}_{\phi'}$ *are equivalent as measured foliations.*

Proof. Let $\phi, \phi' \in \mathrm{Mod}(X)$ be conjugate, i.e., $\phi' = h \circ \phi \circ h^{-1}$ for an automorphism $h \in \mathrm{Mod}(X)$. Since ϕ is the pseudo-Anosov automorphism, there exists a measured foliation $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$. Let us evaluate the automorphism ϕ' on the foliation $h(\mathcal{F}_\phi)$:

$$\phi'(h(\mathcal{F}_\phi)) = h\phi h^{-1}(h(\mathcal{F}_\phi)) = h\phi(\mathcal{F}_\phi) = h\lambda_\phi \mathcal{F}_\phi = \lambda_\phi(h(\mathcal{F}_\phi)). \tag{5}$$

Thus, $\mathscr{F}_{\phi'} = h(\mathscr{F}_\phi)$ is the invariant foliation for the pseudo-Anosov automorphism ϕ' and $\mathscr{F}_\phi$, $\mathscr{F}_{\phi'}$ are equivalent foliations. Note also that the pseudo-Anosov automorphism ϕ' has the same dilatation as the automorphism ϕ. $\square$

Suppose that ϕ and ϕ' are conjugate pseudo-Anosov automorphisms. The functor F acts by the formulas $\phi \mapsto \mathbb{A}_\phi$ and $\phi' \mapsto \mathbb{A}_{\phi'}$, where $\mathbb{A}_\phi$, $\mathbb{A}_{\phi'}$ are the AF C^*-algebras corresponding to the invariant foliations $\mathscr{F}_\phi$, $\mathscr{F}_{\phi'}$. In view of Lemma 10, $\mathscr{F}_\phi$ and $\mathscr{F}_{\phi'}$ are equivalent measured foliations. Then, by Lemma 8, the AF C^*-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are stably isomorphic AF C^*-algebras. Item (i) follows.

(ii) We start with an elementary observation. Let $\phi \in \mathrm{Mod}(X)$ be a pseudo-Anosov automorphism. Then there exists a unique measured foliation, $\mathscr{F}_\phi$, such that $\phi(\mathscr{F}_\phi) = \lambda_\phi \mathscr{F}_\phi$, where $\lambda_\phi > 1$ is an algebraic integer. Let us evaluate automorphism $\phi^2 \in \mathrm{Mod}(X)$ on the foliation $\mathscr{F}_\phi$:

$$\phi^2(\mathscr{F}_\phi) = \phi(\phi(\mathscr{F}_\phi)) = \phi(\lambda_\phi \mathscr{F}_\phi) = \lambda_\phi \phi(\mathscr{F}_\phi) = \lambda_\phi^2 \mathscr{F}_\phi = \lambda_{\phi^2} \mathscr{F}_\phi, \tag{6}$$

where $\lambda_{\phi^2} := \lambda_\phi^2$. Thus, foliation $\mathscr{F}_\phi$ is an invariant foliation for the automorphism ϕ^2 as well. By induction, one concludes that $\mathscr{F}_\phi$ is an invariant foliation of the automorphism ϕ^n for any $n \geq 1$.

Even more is true. Suppose that $\psi \in \mathrm{Mod}(X)$ is a pseudo-Anosov automorphism, such that $\psi^m = \phi^n$ for some $m \geq 1$ and $\psi \neq \phi$. Then $\mathscr{F}_\phi$ is an invariant foliation for the automorphism ψ. Indeed, $\mathscr{F}_\phi$ is invariant foliation of the automorphism ψ^m. If there exists $\mathscr{F}' \neq \mathscr{F}_\phi$ such that the foliation $\mathscr{F}'$ is an invariant foliation of ψ, then the foliation $\mathscr{F}'$ is also an invariant foliation of the pseudo-Anosov automorphism ψ^m. Thus, by uniqueness, $\mathscr{F}' = \mathscr{F}_\phi$. We have just proved the following lemma.

Lemma 11. *Let ϕ be the pseudo-Anosov automorphism of a surface X. Denote by $[\phi]$ a set of the pseudo-Anosov automorphisms ψ of X, such that $\psi^m = \phi^n$ for some positive integers m and n. Then the pseudo-Anosov foliation $\mathscr{F}_\phi$ is an invariant foliation for every pseudo-Anosov automorphism $\psi \in [\phi]$.*

In view of Lemma 11, one arrives at the following identities among the AF C^*-algebras:

$$\mathbb{A}_\phi = \mathbb{A}_{\phi^2} = \cdots = \mathbb{A}_{\phi^n} = \mathbb{A}_{\psi^m} = \cdots = \mathbb{A}_{\psi^2} = \mathbb{A}_\psi. \tag{7}$$

Thus, functor F is not an injective functor: the preimage, $\mathrm{Ker}\, F$, of algbera $\mathbb{A}_\phi$ consists of a countable set of the pseudo-Anosov automorphisms $\psi \in [\phi]$, commensurable with the automorphism ϕ. This proves Theorem 2(ii). $\square$

Proof of Corollary 3.

Proof that $(\Lambda, [I], K)$ *is an invariant.* (i) It follows from Theorem 1 that $\mathbb{A}_\phi$ is a stationary AF C^*-algebra. An arithmetic invariant of the stable isomorphism classes of the stationary AF C^*-algebras has been found by D. Handelman [1981]. Summing up his results, the invariant is as follows.

Let $A \in \mathrm{GL}_n(\mathbb{Z})$ be a matrix with strictly positive entries, such that A is equal to the minimal period of the Bratteli diagram of the stationary AF C^*-algebra. (In case the matrix A has zero entries, it is necessary to take a proper minimal power of the matrix A.) By the Perron–Frobenius theory, matrix A has a real eigenvalue $\lambda_A > 1$, which exceeds the absolute values of other roots of the characteristic polynomial of A. Note that λ_A is an invertible algebraic integer (the unit). Consider the real algebraic number field $K = \mathbb{Q}(\lambda_A)$ obtained as an extension of the field of the rational numbers by the algebraic number λ_A. Let $(v_A^{(1)}, \dots, v_A^{(n)})$ be the eigenvector corresponding to the eigenvalue λ_A. One can normalize the eigenvector so that $v_A^{(i)} \in K$.

The departure point of Handelman's invariant is the $\mathbb{Z}$-module

$$\mathfrak{m} = \mathbb{Z}v_A^{(1)} + \cdots + \mathbb{Z}v_A^{(n)}.$$

The module $\mathfrak{m}$ brings in two new arithmetic objects: (i) the ring Λ of the endomorphisms of $\mathfrak{m}$ and (ii) an ideal I in the ring Λ, such that $I = \mathfrak{m}$ after a scaling [Borevich and Shafarevich 1966, Lemma 1, p. 88]. The ring Λ is an order in the algebraic number field K and therefore one can talk about the ideal classes in Λ. The ideal class of I is denoted by $[I]$. Omitting the embedding question for the field K, the triple $(\Lambda, [I], K)$ is an invariant of the stable isomorphism class of the stationary AF C^*-algebra $\mathbb{A}_\phi$ [Handelman 1981, Section 5]. □

Proof that Δ and Σ ae invariants. Numerical invariants of the stable isomorphism classes of the stationary AF C^*-algebras can be derived from the triple $(\Lambda, [I], K)$. These invariants are rational integers — called the determinant and signature — and can be obtained as follows.

Let $\mathfrak{m}$, $\mathfrak{m}'$ be the full $\mathbb{Z}$-modules in an algebraic number field K. It follows from (i) that if $\mathfrak{m} \neq \mathfrak{m}'$ are distinct as the $\mathbb{Z}$-modules, then the corresponding AF C^*-algebras cannot be stably isomorphic. We wish to find the numerical invariants, which discern the case $\mathfrak{m} \neq \mathfrak{m}'$. It is assumed that a $\mathbb{Z}$-module is given by the set of generators $\{\lambda_1, \dots, \lambda_n\}$. Therefore, the problem can be formulated as follows: find a number attached to the set of generators $\{\lambda_1, \dots, \lambda_n\}$, which does not change on the set of generators $\{\lambda_1', \dots, \lambda_n'\}$ of the same $\mathbb{Z}$-module.

One such invariant is associated with the trace function on the algebraic number field K. Recall that $\mathrm{Tr} : K \to \mathbb{Q}$ is a linear function on K, that is, $\mathrm{Tr}(\alpha + \beta) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\beta)$ and $\mathrm{Tr}(a\alpha) = a\,\mathrm{Tr}(\alpha)$ for all $\alpha, \beta \in K$ and all $a \in \mathbb{Q}$.

Let $\mathfrak{m}$ be a full $\mathbb{Z}$-module in the field K. The trace function defines a symmetric bilinear form $q(x, y) : \mathfrak{m} \times \mathfrak{m} \to \mathbb{Q}$ by the formula

$$(x, y) \longmapsto \mathrm{Tr}(xy) \quad \text{for all } x, y \in \mathfrak{m}. \tag{8}$$

The form $q(x, y)$ depends on the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$:

$$q(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i y_j, \quad \text{where } a_{ij} = \mathrm{Tr}(\lambda_i \lambda_j). \tag{9}$$

However, the general theory of bilinear forms (over the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or the ring of rational integers $\mathbb{Z}$) tells us that certain numerical quantities will not depend on the choice of such a basis.

Namely, one such invariant is as follows. Consider a symmetric matrix A corresponding to the bilinear form $q(x, y)$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}. \tag{10}$$

It is known that the matrix A, written in a new basis, will take the form $A' = U^T A U$, where $U \in \mathrm{GL}_n(\mathbb{Z})$. Then $\mathrm{Det}(A') = \mathrm{Det}(U^T A U) = \mathrm{Det}(U^T)\,\mathrm{Det}(A)\,\mathrm{Det}(U) = \mathrm{Det}(A)$. Therefore, the rational integer number

$$\Delta = \mathrm{Det}(\mathrm{Tr}(\lambda_i \lambda_j)), \tag{11}$$

called a *determinant* of the bilinear form $q(x, y)$, does not depend on the choice of the basis $\{\lambda_1, \dots, \lambda_n\}$ in the module $\mathfrak{m}$. We conclude that the determinant Δ discerns[1] the modules $\mathfrak{m} \neq \mathfrak{m}'$.

Finally, recall that the form $q(x, y)$ can be brought by an integer linear transformation to the diagonal form:

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2, \tag{12}$$

where $a_i \in \mathbb{Z} \setminus \{0\}$. We let a_i^+ be the positive and a_i^- the negative entries in the diagonal form. In view of the law of inertia for bilinear forms, the integer number $\Sigma = (\#a_i^+) - (\#a_i^-)$, called a *signature*, does not depend on a particular choice of the basis in the module $\mathfrak{m}$. Thus, Σ discerns the modules $\mathfrak{m} \neq \mathfrak{m}'$. Corollary 3 follows. □

[1]Note that if $\Delta = \Delta'$ for the modules $\mathfrak{m}$, $\mathfrak{m}'$, one cannot conclude that $\mathfrak{m} = \mathfrak{m}'$. The problem of equivalence of symmetric bilinear forms over $\mathbb{Q}$ (i.e., the existence of a linear substitution over $\mathbb{Q}$ that transforms one form to the other), is a fundamental question of number theory. The Minkowski–Hasse theorem says that two such forms are equivalent if and only if they are equivalent over the field $\mathbb{Q}_p$ for every prime number p and over the field $\mathbb{R}$. Clearly, the resulting p-adic quantities will give new invariants of the stable isomorphism classes of the AF C^*-algebras. The question is similar to the Minkowski units attached to knots; see, e.g., [Reidemeister 1932]. We will not pursue this topic here and refer the reader to the section on open problems, on page 460.

4. Examples, open problems and conjectures

In the present section we shall calculate invariants Δ and Σ for the Anosov automorphisms of the two-dimensional torus. Examples of two nonconjugate Anosov automorphisms with the same Alexander polynomial, but different determinants Δ are constructed. Recall that isotopy classes of the orientation-preserving diffeomorphisms of the torus T^2 are bijective with the 2×2 matrices with integer entries and determinant $+1$, i.e., $\mathrm{Mod}(T^2)\cong SL(2,\mathbb{Z})$. Under the identification, the nonperiodic automorphisms correspond to the matrices $A\in SL(2,\mathbb{Z})$ with $|\mathrm{Tr}\,A|>2$.

Full modules and orders in the quadratic field. Let $K=\mathbb{Q}(\sqrt{d})$ be a quadratic extension of the field of rational numbers $\mathbb{Q}$. Further we suppose that d is a positive square free integer. Let

$$\omega=\begin{cases}\dfrac{1+\sqrt{d}}{2} & \text{if } d\equiv 1 \mod 4,\\ \sqrt{d} & \text{if } d\equiv 2,3 \mod 4.\end{cases}\tag{13}$$

Proposition 12. *Let f be a positive integer. Every order in K has form $\Lambda_f=\mathbb{Z}+(f\omega)\mathbb{Z}$, where f is the conductor of Λ_f.*

Proof. See [Borevich and Shafarevich 1966, pp. 130–132]. □

Proposition 12 allows to classify the similarity classes of the full modules in the field K. Indeed, there exists a finite number of $\mathfrak{m}_f^{(1)},\dots,\mathfrak{m}_f^{(s)}$ of the nonsimilar full modules in the field K, whose coefficient ring is the order Λ_f; cf. [Borevich and Shafarevich 1966, Theorem 3, Chapter 2.7]. Thus, Proposition 12 gives a finite-to-one classification of the similarity classes of full modules in the field K.

Numerical invariants of Anosov automorphisms. Let Λ_f be an order in K with the conductor f. Under the addition operation, the order Λ_f is a full module, which we denote by $\mathfrak{m}_f$. Let us evaluate the invariants $q(x,y)$, Δ and Σ on the module $\mathfrak{m}_f$. To calculate $(a_{ij})=\mathrm{Tr}(\lambda_i\lambda_j)$, we let $\lambda_1=1$, $\lambda_2=f\omega$. Then

$$\begin{aligned}&a_{11}=2,\quad a_{12}=a_{21}=f,\quad a_{22}=\tfrac{1}{2}f^2(d+1) && \text{if } d\equiv 1 \mod 4,\\ &a_{11}=2,\quad a_{12}=a_{21}=0,\quad a_{22}=2f^2d && \text{if } d\equiv 2,3 \mod 4,\end{aligned}\tag{14}$$

and

$$\begin{aligned}&q(x,y)=2x^2+2fxy+\tfrac{1}{2}f^2(d+1)y^2 && \text{if } d\equiv 1 \mod 4,\\ &q(x,y)=2x^2+2f^2dy^2 && \text{if } d\equiv 2,3 \mod 4.\end{aligned}\tag{15}$$

Therefore

$$\Delta=\begin{cases}f^2d & \text{if } d\equiv 1 \mod 4,\\ 4f^2d & \text{if } d\equiv 2,3 \mod 4,\end{cases}\tag{16}$$

and $\Sigma = +2$ in both cases, where $\Sigma = \#(\text{positive}) - \#(\text{negative})$ entries in the diagonal normal form of $q(x, y)$.

Examples. Let us consider some numerical examples, which illustrate advantages of our invariants in comparison to the classical Alexander polynomials.

Example 13. Denote by M_A and M_B the hyperbolic 3-dimensional manifolds obtained as a torus bundle over the circle with the monodromies

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \tag{17}$$

respectively. The Alexander polynomials of M_A and M_B are identical: $\Delta_A(t) = \Delta_B(t) = t^2 - 6t + 1$. However, the manifolds M_A and M_B are *not* homotopy equivalent. Indeed, the Perron–Frobenius eigenvector of matrix A is $v_A = (1, \sqrt{2} - 1)$ while of the matrix B is $v_B = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $\mathfrak{m}_A = \mathbb{Z} + (\sqrt{2} - 1)\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (2\sqrt{2} - 2)\mathbb{Z}$ can be written as

$$q_A(x, y) = 2x^2 - 4xy + 6y^2, \quad q_B(x, y) = 2x^2 - 8xy + 24y^2, \tag{18}$$

respectively. The modules $\mathfrak{m}_A, \mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(\mathfrak{m}_A) = 8$ and $\Delta(\mathfrak{m}_B) = 32$ are not equal. Therefore, matrices A and B are not conjugate[2] in the group $\mathrm{SL}(2, \mathbb{Z})$. Note that the class number $h_K = 1$ for the field K.

Example 14 [Handelman 2009, p. 12]. Let M_A and M_B be 3-dimensional manifolds corresponding to matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix}, \tag{19}$$

respectively. The Alexander polynomials of M_A and M_B are identical: $\Delta_A(t) = \Delta_B(t) = t^2 - 8t + 1$. Yet the manifolds M_A and M_B are not homotopy equivalent. Indeed, the Perron–Frobenius eigenvector of matrix A is $v_A = (1, \frac{1}{3}\sqrt{15})$ while of the matrix B is $v_B = (1, \frac{1}{15}\sqrt{15})$. The corresponding modules are $\mathfrak{m}_A = \mathbb{Z} + (\frac{1}{3}\sqrt{15})\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (\frac{1}{15}\sqrt{15})\mathbb{Z}$; note that $d = 15 \equiv 3 \mod 4$ in both cases, but the corresponding conductors are $f_A = 3$ and $f_B = 15$. Using formulas (15) one finds

$$q_A(x, y) = 2x^2 + 18y^2, \quad q_B(x, y) = 2x^2 + 450y^2, \tag{20}$$

[2] The reader may verify this fact using the method of periods, which dates back to Gauss. First we have to find the fixed points $Ax = x$ and $Bx = x$, which gives us $x_A = 1 + \sqrt{2}$ and $x_B = (1 + \sqrt{2})/2$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_A = [2, 2, 2, \dots]$ and $x_B = [1, 4, 1, 4, \dots]$. Since the period (2) of x_A differs from the period (1, 4) of B, the matrices A and B belong to different conjugacy classes in $\mathrm{SL}(2, \mathbb{Z})$.

respectively. The modules $\mathfrak{m}_A$, $\mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{15})$, since formulas (16) imply that their determinants $\Delta(\mathfrak{m}_A) = 36$ and $\Delta(\mathfrak{m}_B) = 900$ are not equal. Therefore, matrices A and B are not conjugate in the group $\mathrm{SL}(2, \mathbb{Z})$.

Example 15 [Handelman 2009, p. 12]. Let a, b be positive integers satisfying the Pell equation $a^2 - 8b^2 = 1$; the latter has infinitely many solutions, e.g., $a = 3$, $b = 1$, etc. Denote by M_A and M_B the 3-dimensional manifolds corresponding to matrices

$$A = \begin{pmatrix} a & 4b \\ 2b & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 8b \\ b & a \end{pmatrix}. \tag{21}$$

M_A and M_B have the same Alexander polynomial, $\Delta_A(t) = \Delta_B(t) = t^2 - 2at + 1$, yet they are not homotopy equivalent. Indeed, the Perron–Frobenius eigenvector of matrix A is $v_A = (1, \frac{1}{4b}\sqrt{a^2-1})$ while of the matrix B is $v_B = (1, \frac{1}{8b}\sqrt{a^2-1})$. The corresponding modules are $\mathfrak{m}_A = \mathbb{Z} + (\frac{1}{4b}\sqrt{a^2-1})\mathbb{Z}$ and $\mathfrak{m}_B = \mathbb{Z} + (\frac{1}{8b}\sqrt{a^2-1})\mathbb{Z}$. It is easy to see that the discriminant $d = a^2 - 1 \equiv 3 \mod 4$ for all $a \geq 2$. Indeed, $d = (a-1)(a+1)$, so the integer a satisfies $a \not\equiv 1; 3 \mod 4$; hence $a \equiv 2 \mod 4$, so that $a - 1 \equiv 1 \mod 4$ and $a + 1 \equiv 3 \mod 4$ and, thus, $d = a^2 - 1 \equiv 3 \mod 4$. Therefore the corresponding conductors are $f_A = 4b$ and $f_B = 8b$, and

$$q_A(x, y) = 2x^2 + 32b^2(a^2-1)y^2, \quad q_B(x, y) = 2x^2 + 128b^2(a^2-1)y^2. \tag{22}$$

The modules $\mathfrak{m}_A$, $\mathfrak{m}_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{a^2-1})$, because their determinants $\Delta(\mathfrak{m}_A) = 64b^2(a^2-1)$ and $\Delta(\mathfrak{m}_B) = 256b^2(a^2-1)$ are not equal. Therefore, the matrices A and B are not conjugate in $\mathrm{SL}(2, \mathbb{Z})$.

Open problems and conjectures. This section is devoted to some questions and conjectures in connection with the invariants $(\Lambda, [I], K)$, $q(x, y)$, Δ and Σ.

1. *P-adic invariants of pseudo-Anosov automorphisms*

A. Let $\phi \in \mathrm{Mod}(X)$ be a pseudo-Anosov automorphism of a surface X. If λ_ϕ is the dilatation of ϕ, then one can consider a $\mathbb{Z}$-module $\mathfrak{m} = \mathbb{Z}v^{(1)} + \cdots + \mathbb{Z}v^{(n)}$ in the number field $K = \mathbb{Q}(\lambda_\phi)$ generated by the normalized eigenvector $(v^{(1)}, \ldots, v^{(n)})$ corresponding to the eigenvalue λ_ϕ. The trace function on the number field K gives rise to a symmetric bilinear form $q(x, y)$ on the module $\mathfrak{m}$. The form is defined over the field $\mathbb{Q}$. It has been shown that a pseudo-Anosov automorphism ϕ', conjugate to ϕ, yields a form $q'(x, y)$, equivalent to $q(x, y)$, i.e., $q(x, y)$ can be transformed to $q'(x, y)$ by an invertible linear substitution with the coefficients in $\mathbb{Z}$.

B. Recall that two rational bilinear forms $q(x, y)$ and $q'(x, y)$ are equivalent whenever the following conditions are met:

(i) $\Delta = \Delta'$, where Δ is the determinant of the form.

(ii) For each prime number p (including $p = \infty$), certain p-adic equations between the coefficients of forms q, q' must be satisfied; see, e.g., [Borevich and Shafarevich 1966, Chapter 1, Section 7.5]. (In fact, only a *finite* number of such equations have to be verified.)

Condition (i) has already been used to discern between the conjugacy classes of the pseudo-Anosov automorphisms. One can use condition (ii) to discern between the pseudo-Anosov automorphisms with $\Delta = \Delta'$. The following question can be posed: *find the p-adic invariants of the pseudo-Anosov automorphisms.*

2. *Signature of pseudo-Anosov automorphism*

The signature is an important and well-known invariant connected to the chirality and knotting number of knots and links [Reidemeister 1932]. It will be interesting to find a geometric interpretation of the signature Σ for the pseudo-Anosov automorphisms. One can ask the following question: *find a geometric meaning of the invariant* Σ.

3. *Number of conjugacy classes of pseudo-Anosov automorphisms with the same dilatation*

The dilatation λ_ϕ is an invariant of the conjugacy class of the pseudo-Anosov automorphism $\phi \in \mathrm{Mod}(X)$. On the other hand, it is known that there exist nonconjugate pseudo-Anosov's with the same dilatation and the number of such classes is finite [Thurston 1988]. It is natural to expect that the invariants of operator algebras can be used to evaluate the number. We conclude with the following conjecture.

Conjecture 16. Let $(\Lambda, [I], K)$ be the triple corresponding to a pseudo-Anosov automorphism $\phi \in \mathrm{Mod}(X)$. Then the number of the conjugacy classes of the pseudo-Anosov automorphisms with the dilatation λ_ϕ is equal to the class number $h_\Lambda = |\Lambda/[I]|$ of the integral order Λ.

Acknowledgments

I thank the referee for helpful comments and Daniel Silver and Susan Williams for their interest and hospitality.

References

[Bauer 1996] M. Bauer, "A characterization of uniquely ergodic interval exchange maps in terms of the Jacobi-pPerron algorithm", *Bol. Soc. Brasil. Mat.* (*N.S.*) **27**:2 (1996), 109–128. MR 98a:58100 Zbl 0877.11044

[Bernstein 1971] L. Bernstein, *The Jacobi–Perron algorithm: Its theory and application*, Lecture Notes in Mathematics **207**, Springer, Berlin, 1971. MR 44 #2696 Zbl 0213.05201

[Blackadar 1986] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications **5**, Springer, New York, 1986. MR 88g:46082 Zbl 0597.46072

[Borevich and Shafarevich 1966] A. I. Borevich and I. R. Shafarevich, *Number theory*, Pure and Applied Mathematics **20**, Academic, New York, 1966. MR 33 #4001 Zbl 0145.04902

[Douady and Hubbard 1975] A. Douady and J. Hubbard, "On the density of Strebel differentials", *Invent. Math.* **30**:2 (1975), 175–179. MR 53 #796 Zbl 0371.30017

[Effros 1981] E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conference Series in Mathematics **46**, Conference Board of the Mathematical Sciences, Washington, DC, 1981. MR 84k:46042 Zbl 0475.46050

[Handelman 1981] D. Handelman, "Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains", *J. Operator Theory* **6**:1 (1981), 55–74. MR 84i:46058 Zbl 0495.06011

[Handelman 2009] D. Handelman, "Matrices of positive polynomials", *Electron. J. Linear Algebra* **19** (2009), 2–89. MR 2011g:46105 Zbl 1195.37010

[Hemion 1979] G. Hemion, "On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds", *Acta Math.* **142**:1-2 (1979), 123–155. MR 80f:57003 Zbl 0402.57027

[Hubbard and Masur 1979] J. Hubbard and H. Masur, "Quadratic differentials and foliations", *Acta Math.* **142**:3-4 (1979), 221–274. MR 80h:30047 Zbl 0415.30038

[Krieger 1980] W. Krieger, "On dimension functions and topological Markov chains", *Invent. Math.* **56**:3 (1980), 239–250. MR 81m:28018 Zbl 0431.54024

[Latimer and MacDuffee 1933] C. G. Latimer and C. C. MacDuffee, "A correspondence between classes of ideals and classes of matrices", *Ann. of Math.* (2) **34**:2 (1933), 313–316. MR 1503108 Zbl 0006.29002

[Mosher 1986] L. Mosher, "The classification of pseudo–Anosovs", pp. 13–75 in *Low-dimensional topology and Kleinian groups* (Coventry/Durham, 1984), edited by D. B. A. Epstein, London Math. Soc. Lecture Note Ser. **112**, Cambridge Univ. Press, 1986. MR 89f:57016 Zbl 0623.57004

[Penner 1984] R. C. Penner, "The action of the mapping class group on curves in surfaces", *Enseign. Math.* (2) **30**:1-2 (1984), 39–55. MR 85j:57012 Zbl 0554.57007

[Perron 1907] O. Perron, "Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus", *Math. Ann.* **64**:1 (1907), 1–76. MR 1511422 JFM 38.0262.01

[Reidemeister 1932] K. Reidemeister, *Knotentheorie*, Ergebnisse der Mathematik (1) **1**, Springer, Berlin, 1932. In German. Zbl 0005.12001

[Thurston 1982] W. P. Thurston, "Three-dimensional manifolds, Kleinian groups and hyperbolic geometry", *Bull. Amer. Math. Soc.* (*N.S.*) **6**:3 (1982), 357–381. MR 83h:57019 Zbl 0496.57005

[Thurston 1988] W. P. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. Amer. Math. Soc.* (*N.S.*) **19**:2 (1988), 417–431. MR 89k:57023 Zbl 0674.57008

[Wallace 1984] D. I. Wallace, "Conjugacy classes of hyperbolic matrices in $\mathrm{Sl}(n, \mathbf{Z})$ and ideal classes in an order", *Trans. Amer. Math. Soc.* **283**:1 (1984), 177–184. MR 85h:11024 Zbl 0562.12010

Received February 22, 2010. Revised September 11, 2012.

IGOR NIKOLAEV
THE FIELDS INSTITUTE
TORONTO, ON M5T 3J1
CANADA
Current address:
616-315 HOLMWOOD AVENUE
OTTAWA, ON K1S 2R2
CANADA
igor.v.nikolaev@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 2 February 2013

0030-8730(201302)261:2;1-B