# Pacific Journal of Mathematics


## *G*-BUNDLES OVER ELLIPTIC CURVES FOR NON-SIMPLY LACED LIE GROUPS AND CONFIGURATIONS OF LINES IN RATIONAL SURFACES


MANG XU AND JIAJIN ZHANG



Volume 261 No. 2 February 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 261, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.261.497

# $G$-BUNDLES OVER ELLIPTIC CURVES FOR NON-SIMPLY LACED LIE GROUPS AND CONFIGURATIONS OF LINES IN RATIONAL SURFACES

MANG XU AND JIAJIN ZHANG

**We study the relation between the moduli space of flat $G$-bundles over a fixed elliptic curve $\Sigma$ and the moduli space of rational surfaces with $G$-configurations containing $\Sigma$ as a fixed anticanonical curve, where $G$ is a non-simply laced, compact, simple and simply connected Lie group. Our method is to reduce $G$ to a simply laced maximal subgroup $G'$.**

## 1. Introduction

This paper is a continuation of our earlier study, briefly recapitulated below, on the identification between the moduli space of flat $G$-bundles over a fixed elliptic curve $\Sigma$ and the moduli space of rational surfaces with $G$-configurations containing $\Sigma$ as an anticanonical curve. For the case of $G = E_n$, the rational surfaces are exactly del Pezzo surfaces, and the identification was predicted by a duality argument in physics and proved in [Looijenga 1976; Donagi 1997; 1998; Friedman et al. 1997]. The essential reason for this identification in this case is the existence of an $E_n$-structure on del Pezzo surfaces [Demazure et al. 1980; Manin 1974], which turns out to be related to Gosset polytopes [Lee 2010; 2012].

This structure on rational surfaces was extended to the cases $A_n$ and $D_n$ in [Leung 2000]. Starting from Leung's result, we obtained in [Leung and Zhang 2009a] an analogous identification for all simply laced Lie groups $G$. In [Leung et al. 2012; Leung and Zhang 2009b], we extended this identification further to the non-simply laced cases and the affine Kac–Moody $\widetilde{E}_n$ case. The method in that last paper consists in reducing non-simply laced cases to simply laced cases, by considering a non-simply laced Lie group $G$ as the fixed subgroup of a bigger simply laced group $G'$, under the action of the outer automorphism group of $G'$.

In this paper, we consider another reduction. From Lie theory (see [Bourbaki 2005], for example), a non-simply laced Lie group $G$ is uniquely determined by

Xu is partially supported by National Natural Science Foundation of China (grant 11126264). Zhang is partially supported by Doctoral Fund of Ministry of Education of China (grant 20110181120091).
*MSC2010:* primary 14J26; secondary 17B10, 14H60.
*Keywords:* $G$-bundle, rational surface, elliptic curve.

a simply laced maximal subgroup $G'$ determined by the long roots of $G$. Hence it is natural to apply our earlier results for the simply laced cases in [Leung and Zhang 2009a] to the current situation. In this way, we establish the identification between flat $G$-bundles over a fixed elliptic curve $\Sigma$ and rational surfaces with $\Sigma$ as an anticanonical curve for non-simply laced Lie groups $G$ ($G \neq F_4$), by considering the maximal simply laced subgroup $G'$ determined by the long roots of $G$. Unfortunately, this method is not very effective for the case $G = F_4$. In the following, we assume that $G \neq F_4$. Similar to the simply laced cases, we define *$G$-surfaces* and *rational surfaces with $G$-configurations* (see Definitions 5, 12, and 16). Let $\mathrm{Out}(G')$ be the finite group defined in Proposition 2. Our result is this:

**Theorem 1** (Propositions 10, 14 and 19). *Let $\Sigma$ be an elliptic curve with identity element $0 \in \Sigma$, and let $G$ be any simple, compact, simply connected Lie group of $B_n$, $C_n$ or $G_2$ type. Denote by $\mathcal{S}_\Sigma^G$ the moduli space of the pairs $(S, \Sigma)$, where $S$ is a $G$-surface such that $\Sigma \in |-K_S|$. Denote by $\mathcal{M}_\Sigma^G$ the moduli space of flat $G$-bundles over $\Sigma$.*

(i) *$\mathcal{S}_\Sigma^G$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.*

(ii) *This embedding can extend to an isomorphism from $\overline{\mathcal{S}_\Sigma^G}$ onto $\mathcal{M}_\Sigma^G$ by including all rational surfaces with $G$-configurations, and this gives us a natural and explicit compactification $\overline{\mathcal{S}_\Sigma^G}$ of $\mathcal{S}_\Sigma^G$.*

This study is motivated by a certain duality in physics. When $G = E_n$ is considered as a simple subgroup of $E_8 \times E_8$, these $G$-bundles are related to the duality between $F$-theory and string theory. Among other things, this duality predicts the identification between the moduli of flat $E_n$-bundles over a fixed elliptic curve $\Sigma$ and the moduli of del Pezzo surfaces with the fixed anticanonical curve $\Sigma$. For more details, one can see [Donagi 1997; 1998; Friedman et al. 1997]. Our result can be considered as a test of this duality for other Lie groups.

As an application, this identification provides us with an intuitive explanation for $\mathcal{M}_\Sigma^G$. We also provide an interesting geometric realization of root system theory, and we can see very clearly how the Weyl group acts on the moduli space of (marked) flat $G$-bundles over $\Sigma$.

**Notation.** Let $G$ be a compact, simple and simply connected Lie group. We preserve the notation of in [Leung and Zhang 2009a], which is as follows.

| | | | |
|---|---|---|---|
| $r(G)$ | the rank of $G$ | $\Lambda(G)$ | the root lattice |
| $R(G)$ | the root system | $\Lambda_c(G)$ | the coroot lattice |
| $R_c(G)$ | the coroot system | $\Lambda_w(G)$ | the weight lattice |
| $W(G)$ | the Weyl group | $\mathrm{ad}(G)$ | the adjoint group of $G$ (= $G/C(G)$) |
| $T(G)$ | a maximal torus | $\Delta(G)$ | the set of simple roots of $G$ |
| $C(G)$ | the center of $G$ | $\mathrm{Out}(G)$ | the outer automorphism group of $G$ |

Recall that the outer automorphism group of $G$ is defined as the quotient of the automorphism group of $G$ by its inner automorphism group. As is well-known, it is isomorphic to the diagram automorphism group of the Dynkin diagram of $G$.

When there is no danger of confusion, we can omit the letter $G$.

## 2. Reductions to the simply laced cases

Let $G$ be a simple, compact and simply connected Lie group. Then $G$ is classified into the following 7 types according to its Lie algebra.

(1) $A_n$-type, $G = \mathrm{SU}(n+1)$;

(2) $B_n$-type, $G = \mathrm{Spin}(2n+1)$;

(3) $C_n$-type, $G = \mathrm{Sp}(n)$;

(4) $D_n$-type, $G = \mathrm{Spin}(2n)$;

(5) $E_n$-type, $n = 6, 7, 8$;

(6) $F_4$-type;

(7) $G_2$-type.

Among these, $A_n$, $D_n$ and $E_n$ are called of simply laced type, while $B_n$, $C_n$, $F_4$ and $G_2$ are called of non-simply laced type. $A_n$, $B_n$, $C_n$, $D_n$ are called classical Lie groups, while $E_n$, $F_4$ and $G_2$ are called exceptional Lie groups.

From now on, we always assume that $G$ is a compact, simple, simply connected Lie group of non-simply laced type, that is, of type $B_n$, $C_n$, $F_4$, $G_2$. There are two natural approaches to reduce these situations to the simply laced cases. One is embedding $G$ into a simply laced Lie group $G''$ such that $G$ is the subgroup fixed by the outer automorphism group of $G''$. Another is taking the simply laced subgroup $G'$ of maximal rank.

In [Leung and Zhang 2009b] we explained the first reduction. In this paper we concentrate on the second.

**Proposition 2** [Bourbaki 2005]. *There exists canonically a simply laced Lie subgroup $G'$ of maximal rank of $G$ determined by the long roots of $G$, such that $G'$ and $G$ share a common maximal torus. There is a short exact sequence*

$$1 \to W(G') \to W(G) \to \mathrm{Out}(G') \to 1.$$

*This exact sequence is split, that is,*

$$W(G) \cong W(G') \ltimes \mathrm{Out}(G').$$

We write the moduli space of flat $G$-bundles on $\Sigma$ as $\mathcal{M}_\Sigma^G$.

**Corollary 3.**
$$\mathcal{M}_\Sigma^G \cong \mathcal{M}_\Sigma^{G'} / \mathrm{Out}(G').$$

*Proof.* Let $T$ be the common maximal torus of $G$ and $G'$. Then

$$\mathcal{M}_\Sigma^G \cong \mathrm{Hom}(\pi_1(\Sigma), G)/\mathrm{ad}(G) \cong \mathrm{Hom}(\pi_1(\Sigma), T)/W(G) \cong T \times T/W(G).$$

Similarly, $\mathcal{M}_\Sigma^{G'} \cong T \times T/W(G')$. Therefore

$$\mathcal{M}_\Sigma^G \cong T \times T/W(G) \cong (T \times T/W(G'))/(W(G)/W(G')) \cong \mathcal{M}_\Sigma^{G'}/\mathrm{Out}(G'). \quad \square$$

We defined in [Leung and Zhang 2009a] (rational) $G'$-surfaces and rational surfaces with $G'$-configurations. Let $\mathcal{S}_\Sigma^{G'}$ be the moduli space of $G'$-surfaces containing $\Sigma$ as a fixed anticanonical curve. As shows in the same paper, we have the following identification of moduli spaces

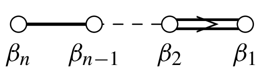
$$\mathcal{S}_\Sigma^{G'} \cong \mathcal{M}_\Sigma^{G'}.$$

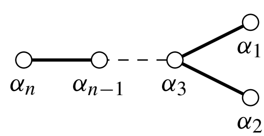
Let $\mathrm{Out}(G')$ act on $\mathcal{S}_\Sigma^{G'}$ via the above isomorphism. In the next section, we shall see explicitly how $\mathrm{Out}(G')$ acts on $\mathcal{S}_\Sigma^{G'}$.

Thus we have a natural question: How can we define $G$-configurations on rational surfaces when $G$ is non-simply laced, in such a way that $\mathcal{S}_\Sigma^G \cong \mathcal{S}_\Sigma^{G'}/\mathrm{Out}(G')$? We answer this question in the next section.

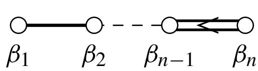
**Remark 4** [Bourbaki 2005; Humphreys 1978]. We give the construction, the root system, and the finite group $\mathrm{Out}(G')$ of $G'$ for non-simply laced Lie group $G$ in each case. We also give the Dynkin diagrams of $G$ and $G'$.

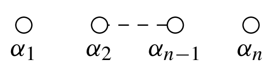
(1) For $G = \mathrm{Spin}(2n+1)$, we take $G' = \mathrm{Spin}(2n)$.
$\Delta(G') = \{\alpha_i, i = 1, \dots, n\}$.
$\Delta(G) = \{\beta_i, i = 1, \dots, n\}$, where $\beta_1 = \frac{1}{2}(\alpha_2 - \alpha_1)$, $\beta_2 = \alpha_1$, $\beta_i = \alpha_i$, $i = 3, \dots, n$.
$\mathrm{Out}(G')$ is the group $\mathbb{Z}_2$ that exchanges the two spin representations of $\mathrm{Spin}(2n)$. In fact, $\mathrm{Out}(G') = \{1, \rho\}$, where $\rho(\alpha_i) = \alpha_i$, $i = 3, \dots, n$, $\rho(\alpha_1) = \alpha_2$, and $\rho(\alpha_2) = \alpha_1$.

$B_n$ 


$D_n$ 


(2) For $G = \mathrm{Sp}(n)$, we take $G' = \mathrm{SU}(2)^n$.
$\Delta(G') = \{\alpha_i, i = 1, \dots, n\}$.
$\Delta(G) = \{\beta_i, i = 1, \dots, n\}$, where $\beta_i = \frac{1}{2}(\alpha_i - \alpha_{i+1})$, $i = 1, \dots, n-1$, $\beta_n = \alpha_n$.
$\mathrm{Out}(G')$ is the symmetry group $S_n$ of the $n$ copies of $\mathrm{SU}(2)$ in $G'$.

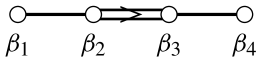
$C_n$ 


$A_1^n$ 

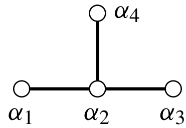

(3) For $G = F_4$, we take $G' = \mathrm{Spin}(8)$.
$\Delta(G') = \{\alpha_i, i = 1, \dots, 4\}$.
$\Delta(G) = \{\beta_i, i = 1, \dots, 4\}$, where $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3$, $\beta_3 = \frac{1}{2}(\alpha_4 - \alpha_3)$, $\beta_4 = \frac{1}{2}(\alpha_1 - \alpha_4)$.

Out($G'$) is the triality group $S_3$ that permutes the three 8-dimensional representations of Spin(8).

$F_4$

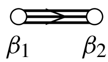



$D_4$

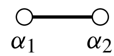



(4) For $G = G_2$, we take $G' = \mathrm{SU}(3)$.
$\Delta(G') = \{\alpha_i, i = 1, 2\}$.
$\Delta(G) = \{\beta_i, i = 1, 2\}$, where $\beta_1 = \alpha_1$, $\beta_2 = -1/3(\alpha_1 + \alpha_2)$.
Out($G'$) is the group $\mathbb{Z}_2$ that exchanges the 3-dimensional representation of SU(3) with its dual. In fact, Out($G'$) is generated by $-1 \in \mathrm{Aut}(\Lambda(G'))$.

$G_2$




$A_2$




In the following we let $\Sigma$ be a fixed elliptic curve with the identity element 0, and we fix a primitive $d$-th root of $\mathrm{Jac}(\Sigma) \cong \Sigma$ (equivalently, a level $d$ structure on $\Sigma$), where $d = 2$ for $G = D_n, B_n$, $d = 9 - n$ for $G = E_n$, and $d = n + 1$ for $G = A_n, C_n, G_2$, respectively; see [Leung and Zhang 2009a] for the $ADE$ cases. Recall from the same reference (for instance) that for any compact, simple and simply connected Lie group $H$, we have

$$\mathcal{M}_\Sigma^H \cong (\Lambda_c(H) \otimes \Sigma)/W(H),$$

where $\mathcal{M}_\Sigma^H$ is the moduli space of flat $H$-bundles over $\Sigma$.

## 3. Flat $G$-bundles over elliptic curves and rational surfaces: the non-simply laced cases

In this section, we study case by case the $G$-bundles over elliptic curves and corresponding rational surfaces for a non-simply laced Lie group $G$ ($G \neq F_4$).

**3.1.** ***$B_n$-bundles* ($n \geq 2$).** According to the last section, for $G = \mathrm{Spin}(2n+1)$ we take $G' = \mathrm{Spin}(2n) \subseteq G$.

Let $S$ be a $D_n$ surface containing $\Sigma$ as a smooth anticanonical curve. Recall from [Leung and Zhang 2009a] that $S$ is a blow-up of $\mathbb{F}_1$ at $n$ points $x_1, \dots, x_n$ on $\Sigma$ that are in general position,[1] with corresponding exceptional classes $l_1, \dots, l_n$. Let $f$ and $s$ be the classes of fibers and the section in $\mathbb{F}_1$. The Picard group of $S$ is isomorphic to $H^2(S, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \dots, l_n$. The canonical class is $K = -\left(2s + 3f - \sum_{i=1}^n l_i\right)$.

We know from [ibid.] that the set

$$\{x \in H^2(S, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

[1] This means that the $x_i$ are all distinct and that $x_i + x_j \neq 0$ for all $i, j$.

is a root lattice of $D_n$ type. We take a simple root system of $G' = D_n$ as

$$\Delta(D_n) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \dots, \alpha_n = l_{n-1} - l_n\}.$$

Let $\rho$ be the generator of $\mathrm{Out}(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_1) = \alpha_2$, $\rho(\alpha_2) = \alpha_1$ and $\rho(\alpha_i) = \alpha_i$ for $i = 3, \dots, n$.

Recall that a $D_n$-configuration on $S$ is an $n$-tuple $\zeta = (e_1, \dots, e_n)$ where $e_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ such that $\sum e_i \cdot s \equiv 0 \pmod 2$. Equivalently, a $D_n$-configuration on $S$ is an $n$-tuple $\zeta = (e_1, \dots, e_n)$ such that after blowing down $e_n, \dots, e_1$ successively, we obtain $\mathbb{F}^1$ with a fibration $\mathbb{F}^1 \to \mathbb{P}^1$ defined by the fiber $f$.

On the other hand, the exceptional system $\zeta' = (e'_1, \dots, e'_n)$ where $e'_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ such that $\sum e'_i \cdot s \equiv 1 \pmod 2$ also determines $\Lambda(D_n)$. The condition $\sum e'_i \cdot s \equiv 1 \pmod 2$ is equivalent to the fact that after blowing down $e'_n, \dots, e'_1$ successively, we obtain $\mathbb{P}^1 \times \mathbb{P}^1$ with a fibration $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by $f$. It is easy to see that the map which interchanges $l_1$ and $f - l_1$, and preserves all other $l_i$ and $f - l_i$, plays the role of the generator of $\mathrm{Out}(D_n) \cong \mathbb{Z}_2$. Therefore we have the following natural definition of $B_n$-configurations.

Let $S$ be a rational surface with a ruling $f : S \to \mathbb{P}^1$ [ibid.], and $\Sigma \in |-K_S|$, such that $f|_\Sigma : \Sigma \to \mathbb{P}^1$ is a double covering with $0 \in \Sigma$ as a ramification point. Recall that an *exceptional system of length n* on $S$ is an $n$-tuple $\zeta = (e_1, e_2, \dots, e_n)$ where the $e_i$'s are exceptional divisors such that $e_i \cdot e_j = -\delta_{ij}$, $e_i \cdot K_S = -1$, $1 \le i, j \le n$. A divisor defining the ruling $f : S \to \mathbb{P}^1$ is still denoted by $f$, which is effective of arithmetic genus 0.

**Definition 5.** A *$B_n$-configuration* on $S$ is an exceptional system of length $n$ (if exists) $\zeta = (e_1, e_2, \dots, e_n)$ with $e_i \cdot f = 0$ for all $i$, such that we can consider $S$ as a blow-up of $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ at $n$ points $x_1, x_2, \dots, x_n$ on $\Sigma$, with corresponding exceptional divisors $e_1, e_2, \dots, e_n$. When such a $\zeta$ exists, we call $S$ a (*rational*) *surface with a $B_n$-configuration*. Let $\rho \in \mathrm{Out}(D_n)$ be the diagram automorphism. Define $\rho(\zeta) := (f - e_1, e_2, \dots, e_n)$.

**Lemma 6.** *Let $\zeta = (e_1, e_2, \dots, e_n)$ be a $B_n$-configuration. Then*

$$\rho(\zeta) = (f - e_1, e_2, \dots, e_n)$$

*is also a $B_n$-configuration.*

*Proof.* By [Leung and Zhang 2009a], if after blowing down $e_n, \dots, e_1$ successively we obtain $\mathbb{F}_1$, then after blowing down $e_n, \dots, e_2, f - e_1$ we shall obtain $\mathbb{P}^1 \times \mathbb{P}^1$. Conversely, if after blowing down $e_n, \dots, e_1$ successively we obtain $\mathbb{P}^1 \times \mathbb{P}^1$, then after blowing down $e_n, \dots, e_2, f - e_1$ we shall obtain $\mathbb{F}_1$. The result follows. □

When $x_1, \dots, x_n \in \Sigma$ are in general position (footnote 1), the surface $S$ in Definition 5 is called a *$B_n$-surface*.

**Lemma 7.** *Let $S$ be a $B_n$-surface.*

(i) *Any $B_n$-configuration on $S$ consists of exceptional curves.*

(ii) *The Weyl group $W(D_n)$ acts on all $B_n$-configurations with two orbits and acts on each orbit simply transitively.*

(iii) *These two orbits are exchanged by* $\mathrm{Out}(D_n)$.

(iv) *The group $W(D_n) \ltimes \mathrm{Out}(D_n)$ acts on all $B_n$-exceptional systems simply transitively*

*Proof.* Let $S$ be a $B_n$-surface with a ruling $f : S \to \mathbb{P}^1$. Then by definition, $S$ is a blow-up of $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ at $n$ points $x_1, x_2, \ldots, x_n \in \Sigma$. Let $l_1, \ldots, l_n$ be the corresponding exceptional divisors. Then we have

$$\begin{aligned} &\{\, x \in \mathrm{Pic}(S) \mid x^2 = xK = -1, xf = 0 \,\} \\ &= \{\, l_1, \ldots, l_n, f - l_1, \ldots, f - l_n \,\}. \end{aligned}$$

Thus a $B_n$-configuration must be of the form: $\zeta = (e_1, \ldots, e_n)$ where $e_i = l_{\sigma(i)}$ or $e_i = f - l_{\sigma(i)}$ and $\sigma$ is a permutation of $1, \ldots, n$. Obviously, $x_1, \ldots, x_n$ are in general position if and only if all the $l_i$ and $f - l_i$ are exceptional curves. Therefore, (i) is true.

(iii) This follows from Definition 5.

(iv) This is a consequence of (ii) and (iii).

(ii) Let $(e_1, e_2, \ldots, e_n)$ be a $B_n$-configuration on $S$. Then $e_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ for $1 \le i \le n$, where $\sigma$ is a permutation of $\{1, \ldots, n\}$. The Weyl group $W(D_n)$ acts as the group generated by permutations of the $n$ pairs $\{(l_i, f - l_i) \mid i = 1, \ldots, n\}$ and interchanges of $l_i$ and $f - l_i$ simultaneously in two pairs in $\{(l_i, f - l_i) \mid 1 \le i \le n\}$. Therefore $W(D_n)$ acts on the set $\big\{(e_1, \ldots, e_n) \mid \sum e_i \cdot s \equiv 0 \pmod 2\big\}$ simply transitively. Similarly the condition $\sum e_i \cdot s \equiv 1 \pmod 2$ determines another orbit on which $W(D_n)$ acts simply transitively. □

**Remark 8.** Although we know the $B_n$-configurations on $S$, unfortunately, we can not single out the $B_n$-root system within the Picard lattice $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$. However, according to Section 2, we have a root system of $B_n$ type consisting of $\mathbb{Q}$-divisors on $S$:

$$R(B_n) \triangleq \Big\{\pm\big(\tfrac{1}{2} f - l_i\big), \pm(l_i - l_j), \pm(f - l_i - l_j) \,\Big|\, i \ne j, 1 \le i, j \le n\Big\}.$$

It is easy to see that the corresponding root lattice is

$$\Lambda(B_n) \triangleq \Big\{x \in \mathbb{Z}\big(\tfrac{1}{2} f\big) \oplus \bigoplus_{i=1}^{n} \mathbb{Z}(l_i) \,\Big|\, xf = xK = 0\Big\}$$

and

$$R(B_n) = \{x \in \Lambda(B_n) \mid x^2 = -2 \text{ or } x^2 = -1\}.$$

The set of simple roots of $B_n$ is

$$\Delta(B_n) = \left\{\beta_1 = \tfrac{1}{2} f - l_1, \beta_i = l_{i-1} - l_i, i = 2, \ldots, n\right\}.$$

Recall that the Weyl group $W(B_n)$ is the subgroup of $\mathrm{Aut}(\Lambda(B_n))$ generated by the reflections $\sigma_\alpha$ with $\alpha \in R(B_n)$.

**Corollary 9.** *Let $R(B_n)$ be defined as above. Then $W(B_n)$ acts on the set of all $B_n$-configurations simply transitively.*

Let $\mathcal{S}_\Sigma^{B_n}$ be the moduli space of pairs $(S, \Sigma)$ where $S$ is a $B_n$-surface (so the blown-up points $x_1, x_2, \ldots, x_n$ are in general position), and $\Sigma \in |-K_S|$, where two pairs $(S, \Sigma)$ and $(S', \Sigma)$ are said to be isomorphic to each other if there is an isomorphism $f : S \xrightarrow{\sim} S'$ such that $f|_\Sigma = \mathrm{id}_\Sigma$. Denote $\mathcal{M}_\Sigma^{B_n}$ the moduli space of flat $B_n$-bundles over $\Sigma$. Let $\underline{\mathcal{S}}_\Sigma^{B_n}$ be the (marked) moduli space of the triples $(S, \Sigma, \zeta = (l_1, \ldots, l_n))$. By Lemma 7, we have

$$\mathcal{S}_\Sigma^{B_n} \cong \underline{\mathcal{S}}_\Sigma^{B_n} / W(B_n) \cong \underline{\mathcal{S}}_\Sigma^{B_n} / (W(D_n) \ltimes \mathrm{Out}(D_n)).$$

Let $(S, \Sigma, \zeta = (l_1, \ldots, l_n)) \in \underline{\mathcal{S}}_\Sigma^{B_n}$ be as above. For all $\alpha = \frac{a_0}{2} f + \sum a_i l_i \in \Lambda(B_n) \subseteq \mathrm{Pic}(S)_\mathbb{Q} = \mathrm{Pic}(S) \otimes \mathbb{Q}$ with $a_i \in \mathbb{Z}, i = 0, \ldots, n$, the invertible sheaf induced by restriction to $\Sigma$

$$\mathcal{O}_\Sigma(\alpha) := \mathcal{O}_\Sigma(a_0(0)) \otimes \mathcal{O}\left(\sum a_i l_i\right)|_\Sigma$$

is well-defined. Moreover, $\deg(\mathcal{O}_\Sigma(\alpha)) = \alpha \cdot (-K_S) = 0$. Then

$$\mathcal{O}_\Sigma(\alpha) \in \mathrm{Jac}(\Sigma) \cong \Sigma.$$

Thus there is a morphism

$$\underline{\phi} : \underline{\mathcal{S}}_\Sigma^{B_n} \to \mathrm{Hom}(\Lambda(B_n), \Sigma),$$

which is induced by the restriction: for all $\alpha \in \Lambda(B_n) \subseteq \mathrm{Pic}(S)_\mathbb{Q}$,

$$\underline{\phi}((S, \Sigma, \zeta))(\alpha) := \mathcal{O}_\Sigma(\alpha) \in \mathrm{Jac}(\Sigma) \cong \Sigma.$$

**Proposition 10.** (i) *$\mathcal{S}_\Sigma^{B_n}$ is embedded into $\mathcal{M}_\Sigma^{B_n}$ as an open dense subset.*

(ii) *This embedding can be extended naturally to an isomorphism*

$$\overline{\mathcal{S}_\Sigma^{B_n}} \cong \mathcal{M}_\Sigma^{B_n},$$

*by including all rational surfaces with $B_n$-configurations.*

*Proof.* Similarly as in [Leung and Zhang 2009a], we have

$$\mathcal{M}_\Sigma^{B_n} \cong \mathrm{Hom}(\Lambda(B_n), \Sigma) / W(B_n).$$

Then by Lemma 7 or Corollary 9, since two different sets of simple roots differ by a $W(B_n)$-action, we just need to show that the map

$$\underline{\phi}: \underline{\mathcal{S}}_{\Sigma}^{B_n} \hookrightarrow \operatorname{Hom}(\Lambda(B_n), \Sigma)$$

is an open dense embedding and that $\underline{\phi}$ can be extended to an isomorphism $\overline{\underline{\phi}}$ from the natural compactification $\overline{\underline{\mathcal{S}}_{\Sigma}^{B_n}}$ of $\underline{\mathcal{S}}_{\Sigma}^{B_n}$ to $\operatorname{Hom}(\Lambda(B_n), \Sigma)$:

$$\overline{\underline{\phi}}: \overline{\underline{\mathcal{S}}_{\Sigma}^{B_n}} \xrightarrow{\sim} \operatorname{Hom}(\Lambda(B_n), \Sigma).$$

The map $\underline{\phi}$ is injective. For this, we take a simple root system of $D_n$ as

$$\beta_1 = \tfrac{1}{2} f - l_1, \qquad \beta_i = l_{i-1} - l_i \quad \text{for } 2 \le i \le n.$$

Then the restriction induces an element $u \in \operatorname{Hom}(\Lambda(B_n), \Sigma)$. For

$$\beta = a_0\left(\tfrac{1}{2} f\right) + \sum a_i l_i \in \Lambda(B_n),$$

let $x_i = l_i \cap \Sigma$ and $p = u(\beta) \in \Sigma$. Then we have an equation

$$\sum a_i x_i = p,$$

where $+$ is the addition on the elliptic curve $\Sigma$. Taking $\beta = \beta_i, i = 1, \ldots, n$ respectively, and setting $p_i = u(\beta_i)$ accordingly, we obtain the following system of linear equations

$$\begin{cases} -x_1 = p_1, \\ x_{i-1} - x_i = p_i, \ \ i = 2, \ldots, n. \end{cases}$$

Obviously, the solution of this system of linear equations exists uniquely for given $p_i$ with $1 \le i \le n$.

The open dense property of the image of the embedding $\underline{\phi}$ is obvious.

Finally, the statement (ii) comes from the existence of the solutions to the above system of linear equations. □

**3.2.** ***$C_n$-bundles.*** We take $G' = A_1^n \subseteq G = C_n$, where $C_n = \operatorname{Sp}(n)$ and $A_1 = \operatorname{SU}(2)$. Note that $\operatorname{Out}(A_1^n) \cong S_n$.

Let $S$ be a rational surface with an $A_1^n$-configuration that contains $\Sigma$ as a smooth anticanonical curve. Recall from [Leung and Zhang 2009a] that $S$ is a (successive) blow-up of $\mathbb{P}^2$ at $2n$ points $x_1, y_1, \ldots, x_n, y_n$ on $\Sigma$, with corresponding exceptional classes $l_1, l_1', \ldots, l_n, l_n'$, where $x_i + y_i = 0 \in \Sigma$. The Picard group of $S$ is $H^2(S, \mathbb{Z})$, which is a lattice with basis $h, l_1, l_1', \ldots, l_n, l_n'$. The canonical divisor is $K = -\left(3h - \sum_{i=1}^n (l_i + l_i')\right)$.

A simple root system of $A_1^n$ can be taken as

$$\Delta(A_n^1) = \{\alpha_i = l_i - l_i' \mid 1 \le i \le n\}.$$

When the above simple root system is chosen, the pair $(S, \Sigma)$ determines a homomorphism $u \in \mathrm{Hom}(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) := \mathbb{O}(\alpha)|_\Sigma.$$

**Lemma 11.** *Let $u \in \mathrm{Hom}(\Lambda(G'), \Sigma)$ be an element corresponding to a triple $(S, \Sigma, \zeta)$, where $S$ is a surface with an $A_1^n$-configuration $\zeta = (l_1, l_1', \dots, l_n, l_n')$. Let $\rho \in \mathrm{Out}(G') \cong S_n$. Then $\rho \cdot u$ corresponds to the triple $(S, \Sigma, \rho(\zeta))$, where $\rho(\zeta) = (l_{\rho(1)}, l'_{\rho(1)}, \dots, l_{\rho(n)}, l'_{\rho(n)})$.*

*Proof.* Since $u$ is the restriction map: $\alpha_i \mapsto \mathbb{O}(\alpha_i)|_\Sigma$, $u(\alpha_i) = \mathbb{O}(l_i - l_i')|_\Sigma = x_i - y_i$ for $i = 1, \dots, n$. Hence $\rho \cdot u(\alpha_i) = u(\alpha_{\rho(i)}) = x_{\rho(i)} - y_{\rho(i)}$. Therefore we have the result, since $x_{\rho(i)} + y_{\rho(i)} = 0$. □

Thus, it is natural to define a $C_n$-configuration on $S$ to be the form

$$\zeta = ((l_1, l_1'), \dots, (l_n, l_n')).$$

More precisely, denote $S$ the blow-up of $\mathbb{P}_2$ at $n$ pairs of points $(x_1, -x_1), \dots, (x_n, -x_n)$ on $\Sigma$, with $n$ pairs of corresponding exceptional divisors $(l_1, l_1'), \dots, (l_n, l_n')$, where $l_i$ and $l_i'$ are the exceptional divisors corresponding to the blowing up at $x_i$ and $-x_i$, respectively.

**Definition 12.** A *$C_n$-exceptional system* on $S$ is an $n$-tuple of pairs

$$((e_1, e_1'), \dots, (e_n, e_n'))$$

where $(e_i, e_i') = (l_{\sigma(i)}, l'_{\sigma(i)})$ or $(l'_{\sigma(i)}, l_{\sigma(i)})$, $i = 1, \dots, n$, and $\sigma$ is a permutation of $1, \dots, n$. A *$C_n$-configuration* on $S$ is a $C_n$-exceptional system $\zeta_{C_n} = ((e_1, e_1'), \dots, (e_n, e_n'))$ such that after blowing down successively $e_n', e_n, \dots, e_1', e_1$, we obtain the surface $\mathbb{P}^2$.

It can be shown that $x_1, x_2, \dots, x_n \in \Sigma \subseteq \mathbb{P}^2$ are in general position (in the sense of footnote 1) if and only if any $C_n$-exceptional system on $S$ consists of smooth exceptional curves. Such a surface is called a *$C_n$-surface*.

**Lemma 13.** (i) *Let $S$ be a surface with a $C_n$-configuration. The group $W(A_1^n) \ltimes S_n$ acts on all $C_n$-exceptional systems on $S$ simply transitively.*

(ii) *Let $S$ be a $C_n$-surface. The group $W(A_1^n) \ltimes S_n$ acts on all $C_n$-configurations on $S$ simply transitively.*

*Proof.* It suffices to prove (i). The Weyl group $W(A_1^n) \ltimes S_n$ acts as the group generated by permutations of the $n$ pairs $\{(l_i, l_i') \mid i = 1, \dots, n\}$ and interchanging of $l_i$ and $l_i'$ for each $i$. From this, we see that $W(A_1^n) \ltimes S_n$ acts on all $C_n$-configurations simply transitively. □

Denote by $\mathcal{S}_{\Sigma}^{G'}$ the moduli space of $G' = A_1^n$-surfaces with a fixed anticanonical curve $\Sigma$, and by $\overline{\mathcal{S}_{\Sigma}^{G'}}$ the natural compactification by including all rational surfaces with $A_1^n$-configurations. From [Leung and Zhang 2009a] we know that there is an isomorphism $\phi : \overline{\mathcal{S}_{\Sigma}^{G'}} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{G'}$.

Denote by $\mathcal{S}_{\Sigma}^{C_n}$ the moduli space of pairs $(S, \Sigma)$, where $S$ is a $C_n$-surface, that is, $S$ is the blow-up of $\mathbb{P}^2$ at $2n$ points $\pm x_1, \dots, \pm x_n \in \Sigma$ such that $x_1, \dots, x_n$ are in general position, and two pairs $(S, \Sigma)$ and $(S', \Sigma)$ are said to be isomorphic to each other if there is an isomorphism $f : S \xrightarrow{\sim} S'$ such that $f|_{\Sigma} = \mathrm{id}_{\Sigma}$. Denote by $\mathcal{M}_{\Sigma}^{C_n}$ the moduli space of flat $C_n$-bundles over $\Sigma$.

**Proposition 14.** (i) *$\mathcal{S}_{\Sigma}^{C_n}$ is embedded into $\mathcal{M}_{\Sigma}^{C_n}$ as an open dense subset.*

(ii) *This embedding can be extended naturally to an isomorphism*

$$\overline{\mathcal{S}_{\Sigma}^{C_n}} \cong \mathcal{M}_{\Sigma}^{C_n},$$

*by including all rational surfaces with $C_n$-configurations.*

*Proof.* By Corollary 3, $\mathcal{M}_{\Sigma}^{C_n} \cong \mathcal{M}_{\Sigma}^{A_1^n}/S_n \cong \overline{\mathcal{S}_{\Sigma}^{A_1^n}}/S_n$. Therefore it is sufficient to show that $\mathcal{S}_{\Sigma}^{C_n} \cong \mathcal{S}_{\Sigma}^{A_1^n}/S_n$. This follows from Lemma 13. □

**Remark 15.** Obviously, this description in Proposition 14 coincides with the well-known description of flat $C_n$-bundles over elliptic curves [Friedman et al. 1997]. A flat $C_n = \mathrm{Sp}(n)$-bundle over $\Sigma$ corresponds to $n$ pairs (unordered) of points $(x_i, -x_i)$, $i = 1, \dots, n$ on $\Sigma$, uniquely up to isomorphism. One pair $(x_i, -x_i)$ will determine exactly one point on $\mathbb{CP}^1$, since the rational map determined by the linear system $|2(0)|$ induces a double covering from $\Sigma$ onto $\mathbb{CP}^1$. The moduli space of flat $\mathrm{SU}(2)$-bundles over $\Sigma$ is isomorphic to $\mathbb{P}^1$. So the moduli space of flat $C_n$-bundles over $\Sigma$ is precisely isomorphic to $S^n(\mathbb{CP}^1) = \mathbb{CP}^n$, the ordinary projective $n$ space.

**3.3. *$G_2$-bundles.*** For $G = G_2$, we take $G' = A_2 = \mathrm{SU}(3)$.

Let $S$ be a rational surface with an $A_2$-configuration (see [Leung and Zhang 2009a]) containing $\Sigma$ as a smooth anticanonical curve. Recall [ibid.] that $S$ is a (successive) blow-up of $\mathbb{P}^2$ at 3 points $x_1, x_2, x_3$ on $\Sigma$, with corresponding exceptional classes $l_1, l_2, l_3$, where $x_1 + x_2 + x_3 = 0 \in \Sigma$. Let $h$ be the class of lines in $\mathbb{P}^2$. The Picard group of $S$ is $Pic(S) \cong H^2(S, \mathbb{Z})$, which is a lattice with basis $h, l_1, l_2, l_3$. The canonical line bundle $K = -\big(3h - \sum_{i=1}^{3} l_i\big)$.

Recall that

$$\{x \in H^2(S, \mathbb{Z}) \mid x \cdot K = x \cdot h = 0\}$$

is a root lattice of $A_2$ type. We can take a simple root system of $A_2$ as

$$\Delta(A_2) = \{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3\}.$$

Let $\rho \in \mathrm{Out}(A_2) \cong \mathbb{Z}_2$ be the generator of order 2 (we can take $\rho = -1$, that is, $\rho(\alpha_i) = -\alpha_i$).

Denote by $\mathcal{S}_\Sigma^{A_2}$ the moduli space of $A_2$-surfaces with a fixed anticanonical curve $\Sigma$, and $\overline{\mathcal{S}_\Sigma^{A_2}}$ the natural compactification by including all rational surfaces with $A_2$-configurations. From [Leung and Zhang 2009a] we know that $\overline{\mathcal{S}_\Sigma^{A_2}} \xrightarrow{\sim} \mathcal{M}_\Sigma^{A_2}$. Let $\phi$ be the isomorphism.

**Definition 16.** Let $S$ be as immediately above. A *$G_2$-exceptional system* on $S$ is an ordered triple $(e_1, e_2, e_3)$ of exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot h$, $i \neq j$ and $y_1 + y_2 + y_3 = 0$ where $y_i = e_i \cap \Sigma$. A *$G_2$-configuration* on $S$ is a $G_2$-exceptional system $\zeta_{G_2} = (e_1, e_2, e_3)$ such that we can consider $S$ as a blow-up of $\mathbb{P}^2$ at these 3 points $y_1, y_2, y_3$ on $\Sigma$, with corresponding exceptional divisors $e_1, e_2, e_3$. When $S$ has a $G_2$-configuration (of course $\Sigma \in |-K_S|$), we call $S$ a *(rational) surface with a $G_2$-configuration*.

When $x_1, x_2, x_3$ are nonzero distinct points on $\Sigma$, any $G_2$-exceptional system on $S$ consists of exceptional curves. Such a surface is called a *$G_2$-surface*. These 3 points $x_1, x_2, x_3 \in \Sigma$ are said to be *in general position*.

Let $S, S'$ be two surfaces with $G_2$-configurations $\zeta, \zeta'$ respectively. We say that $(S, \Sigma, \zeta) \cong (S', \Sigma, \zeta')$ if there exists an isomorphism $f : S \xrightarrow{\sim} S'$ such that $f|_\Sigma : \Sigma \to \Sigma$ is the identity or the involution of $\Sigma$.

A triple $(S, \Sigma, \zeta)$ determines an element $u$ of $\mathrm{Hom}(\Lambda(A_2), \Sigma)$ by the restriction

$$u(\alpha) := \mathcal{O}(\alpha)|_\Sigma.$$

**Lemma 17.** *Let $u \in \mathrm{Hom}(\Lambda(A_2), \Sigma)$ correspond to the triple $(S, \Sigma, \zeta)$, where $S$ is a surface with a $G_2$-configuration $\zeta = \{l_1, l_2, l_3\}$. Then $\rho \cdot u$ corresponds to $(S', \Sigma, \zeta')$, where $S'$ is another surface with a $G_2$-configuration $\zeta' = (l_1', l_2', l_3')$ with $l_i' \cap \Sigma = -x_i$. Moreover, we have $(S, \Sigma, \zeta) \cong (S', \Sigma, \zeta')$.*

*Proof.* Since $u$ is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_\Sigma$, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = x_2 - x_3$. Hence $\rho \cdot u = v \Leftrightarrow v(\alpha_i) = -u(\alpha_i) \Leftrightarrow x_1 - x_2 = y_2 - y_1$, $x_2 - x_3 = y_3 - y_2 \Leftrightarrow y_i = -x_i$.

Next we prove the second assertion. We first fix an embedding $\iota : \Sigma \hookrightarrow \mathbb{P}^2$ such that (the image of) $\Sigma$ is defined by the equation $zy^2 = 4x^3 + axz^2 + bz^3$ and $0 = [0, 1, 0] \in \Sigma$, where $[x, y, z]$ is the coordinate system of $\mathbb{P}^2$. Then the automorphism of $\mathbb{P}^2$ defined by $[x, y, z] \mapsto [x, -y, z]$ induces an isomorphism $f$ of the triple $(0, \Sigma, \mathbb{P}^2)$, which is the involution on $\Sigma$ that maps $x \in \Sigma$ to $-x$. On the other hand, for $x_1, x_2, x_3 \in \Sigma$, we have obviously $(-x_1) + (-x_2) + (-x_3) = 0$. Thus we have the isomorphism $\phi$ defined by $f$. □

**Lemma 18.** (i) *Let $S$ be a surface with a $G_2$-configuration. The Weyl group $W(A_2)$ acts on all $G_2$-exceptional systems on $S$ simply transitively.*

(ii) *Let $S$ be a $G_2$-surface. The Weyl group $W(A_2)$ acts on all $G_2$-configurations on $S$ simply transitively.*

(iii) *Let $[(S, \Sigma, \zeta)]$ be the isomorphism class of $(S, \Sigma, \zeta)$. Then $W(A_2) \ltimes \mathbb{Z}_2$ acts on the set $[(S, \Sigma, \zeta)]$ simply transitively.*

*Proof.* Let $f : (S', \Sigma, \zeta') \xrightarrow{\sim} (S, \Sigma, \zeta)$. If $f|_\Sigma = \mathrm{id}_\Sigma$, then $S = S'$ and $f = \mathrm{id}_S$. In this case, $W(A_2)$ acts on the $G_2$-configurations on $S$ simply transitively. On the other hand, by Lemma 17, the involution on $\Sigma$ can be extended to an isomorphism from $S'$ onto $S$. In this case the involution $-\mathrm{id}_\Sigma$ acts on the set $[(S, \Sigma, \zeta)]$. Thus the result follows. □

**Proposition 19.** *Let $\mathcal{S}_\Sigma^{G_2}$ be the moduli space of pairs $(S, \Sigma)$ where $S$ is a $G_2$-surface, and $\mathcal{M}_\Sigma^{G_2}$ be the moduli space of flat $G_2$-bundles over $\Sigma$. Then we have*

(i) *$\mathcal{S}_\Sigma^{G_2}$ is embedded into $\mathcal{M}_\Sigma^{G_2}$ as an open dense subset.*

(ii) *This embedding can be extended naturally to an isomorphism*

$$\overline{\mathcal{S}_\Sigma^{G_2}} \cong \mathcal{M}_\Sigma^{G_2},$$

*by including all rational surfaces with $G_2$-configurations.*

*Proof.* By Corollary 3 we have $\mathcal{M}_\Sigma^{G_2} \cong \mathcal{M}_\Sigma^{A_2} / \operatorname{Out}(A_2) \cong \overline{\mathcal{S}_\Sigma^{A_2}} / \mathbb{Z}_2$. Thus it suffices to show that $\mathcal{S}_\Sigma^{G_2} \cong \mathcal{S}_\Sigma^{A_2} / \mathbb{Z}_2$. This follows from Lemma 18. □

**Remark 20** [Friedman et al. 1997]. A SU(3)-bundles over $\Sigma$ is determined by a section of $H^0(\mathbb{O}_\Sigma(3(0)))$, which is a meromorphic function with the only pole 0 of order at most 3. Let $x$, $y$ be the local coordinates of $\Sigma$ around 0, then this function is $a_0 + a_1 x + a_2 y$ up to nonzero constant. Thus the moduli space $\mathcal{M}_\Sigma^{A_2}$ is isomorphic to $\mathbb{P}^2$. By the proof of Lemma 17, the function $a_0 + a_1 x + (-a_2) y$ defines the same $G_2$-bundle over $\Sigma$. Thus we have $\mathcal{M}_\Sigma^{G_2} \cong \mathbb{WP}^2_{1,1,2}$.

**Remark 21.** For the $F_4$ case, unfortunately, the method used in this paper is not very effective. We can not find a suitable definition for $F_4$-configurations. Thus in this case, the method used in [Leung and Zhang 2009b] is the better one.

## Acknowledgments

We would like to express our heartfelt gratitude to the referee for a very careful reading of our paper and valuable suggestions that improved the paper greatly.

## References

[Bourbaki 2005] N. Bourbaki, *Lie groups and Lie algebras*, Chapter 7–9, Springer, Berlin, 2005. MR 2005h:17001 Zbl 1139.17002

[Demazure et al. 1980] M. Demazure, H. C. Pinkham, and B. Teissier (editors), *Séminaire sur les Singularités des Surfaces* (Palaiseau, 1976–1977), Lecture Notes in Mathematics **777**, Springer, Berlin, 1980. MR 82d:14021 Zbl 0415.00010

[Donagi 1997] R. Y. Donagi, "Principal bundles on elliptic fibrations", *Asian J. Math.* **1**:2 (1997), 214–223. MR 99d:14010 Zbl 0927.14006 arXiv alg-geom/9702002

[Donagi 1998] R. Y. Donagi, "Taniguchi lectures on principal bundles on elliptic fibrations", pp. 33–46 in *Integrable systems and algebraic geometry* (Kobe/Kyoto, 1997), edited by M.-H. Saito et al., World Scientific, River Edge, NJ, 1998. MR 2000a:14015 Zbl 0963.14004

[Friedman et al. 1997] R. Friedman, J. W. Morgan, and E. Witten, "Vector bundles and F theory", *Comm. Math. Phys.* **187**:3 (1997), 679–743. MR 99g:14052 Zbl 0919.14010

[Humphreys 1978] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics **9**, Springer, New York, 1978. MR 81b:17007 Zbl 0447.17001

[Lee 2010] J.-H. Lee, "Configuration of lines in del Pezzo surfaces with Gosset polytopes", preprint, 2010. arXiv 1001.4174

[Lee 2012] J.-H. Lee, "Gosset polytopes in Picard groups of del Pezzo surfaces", *Canad. J. Math.* **64**:1 (2012), 123–150. MR 2932172 Zbl 06023029

[Leung 2000] N. C. Leung, "ADE-bundle over rational surfaces, configuration of lines and rulings", preprint, 2000. arXiv math/0009192

[Leung and Zhang 2009a] N. C. Leung and J. Zhang, "Moduli of bundles over rational surfaces and elliptic curves, I: Simply laced cases", *J. Lond. Math. Soc.* (2) **80**:3 (2009), 750–770. MR 2011e:14018 Zbl 1188.14025

[Leung and Zhang 2009b] N. C. Leung and J. Zhang, "Moduli of bundles over rational surfaces and elliptic curves, II: Nonsimply laced cases", *Int. Math. Res. Not.* **2009**:24 (2009), 4597–4625. MR 2011e:14019 Zbl 1222.14023

[Leung et al. 2012] N. C. Leung, M. Xu, and J. Zhang, "Kac–Moody $\widetilde{E}_k$-bundles over elliptic curves and del Pezzo surfaces with singularities of type $A$", *Math. Ann.* **352**:4 (2012), 805–828. MR 2892453 Zbl 1242.14036

[Looijenga 1976] E. Looijenga, "Root systems and elliptic curves", *Invent. Math.* **38**:1 (1976), 17–32. MR 57 #6015 Zbl 0358.17016

[Manin 1974] Y. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland Mathematical Library **4**, North-Holland, Amsterdam, 1974. MR 57 #343 Zbl 0277.14014

Received September 25, 2011.

MANG XU
DEPARTMENT OF MATHEMATICS
SOUTHWEST JIAOTONG UNIVERSITY
CHENGDU, 610031
CHINA
xumang@home.swjtu.edu.cn

JIAJIN ZHANG
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, 610065
CHINA
jjzhang@scu.edu.cn

# PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)





See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

# PACIFIC JOURNAL OF MATHEMATICS

Volume 261 No. 2 February 2013