Pacific Journal of Mathematics

ON THE SECOND K-GROUP OF A RATIONAL FUNCTION FIELD

KARIM JOHANNES BECHER AND MÉLANIE RACZEK

Volume 262 No. 1 March 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 262, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.262.1

ON THE SECOND K-GROUP OF A RATIONAL FUNCTION FIELD

KARIM JOHANNES BECHER AND MÉLANIE RACZEK

We give an optimal bound on the minimal length of a sum of symbols in the second Milnor K-group of a rational function field in terms of the degree of the ramification.

1. Introduction

Let E be an arbitrary field and F the function field of the projective line $\mathbb{P}^1_E$ over E. For $m \in \mathbb{N}$, there is a well-known exact sequence

$$0 \longrightarrow K_2^{(m)}E \longrightarrow K_2^{(m)}F \xrightarrow{\partial} \bigoplus_{x\in\mathbb{P}_E^{1(1)}} K_1^{(m)}E(x) \longrightarrow K_1^{(m)}E \longrightarrow 0, \tag{1.1}$$

due to Milnor and Tate; see [Milnor 1970, (2.3)]. Here, $K_1^{(m)}$ and $K_2^{(m)}$ are the functors that associate to a field its first and second K-groups modulo m, respectively, and $\mathbb{P}_E^{1(1)}$ is the set of closed points of $\mathbb{P}^1_E$. The map ∂ is called the *ramification map*. By [Gille and Szamuely 2006, (7.5.4)], for m prime to the characteristic of E, the sequence (1.1) translates into a sequence in Galois cohomology, and the proof of its exactness essentially goes back to [Faddeev 1951].

In this article we study how for a given element ρ in the image of ∂ one finds a good $\xi \in K_2^{(m)}F$ with $\partial(\xi) = \rho$. Our main result Theorem 3.10 states that there is such a ξ that is a sum of r symbols (canonical generators of $K_2^{(m)}F$) where r is bounded by half the degree of the support of ρ. This generalizes results from [Kunyavskiĭ et al. 2006; Rowen et al. 2005; Sivatski 2007], where the problem has been studied in terms of Brauer groups in the presence of a primitive m-th root of unity in E for $m > 0$. Developing further an idea in [Sivatski 2007, Proposition 2], we provide examples (Example 4.3) where the bound on r cannot be improved.

This work was done while Becher was a Fellow of the Zukunftskolleg and Raczek was a Postdoctoral Fellow of the Fonds de la Recherche Scientifique – FNRS. The project was further supported by the Deutsche Forschungsgemeinschaft (project "Quadratic Forms and Invariants", BE 2614/3).
MSC2010: 12Y05, 12E30, 12G05, 19D45.
Keywords: Milnor K-theory, field extension, valuation, ramification.

2. Milnor K-theory of a rational function field

We recall the basic terminology of K-theory for fields as introduced in [Milnor 1970], with slightly different notation. Let F be a field. For $m, n \in \mathbb{N}$, let $K_n^{(m)}F$ denote the abelian group generated by elements called *symbols*, which are of the form $\{a_1, \dots, a_n\}$ with $a_1, \dots, a_n \in F^\times$, subject to the defining relations that $\{\cdot, \dots, \cdot\} : (F^\times)^n \to K_n^{(m)}F$ is a multilinear map, that $\{a_1, \dots, a_n\} = 0$ whenever $a_i + a_{i+1} = 1$ in F for some $i < n$, and that $m \cdot \{a_1, \dots, a_n\} = 0$. For $a, b \in F^\times$ we have $\{ab\} = \{a\} + \{b\}$ in $K_1^{(m)}F$. The second relation above is void when $n = 1$, hence $K_1^{(m)}F$ is the same as $F^\times / F^{\times m}$, only with different notation for the elements and the group operation. As shown in [Milnor 1970, (1.1) and (1.3)], it follows from the defining relations that, for $a_1, \dots, a_n \in F^\times$, we have $\{a_{\sigma(1)}, \dots, a_{\sigma(n)}\} = \varepsilon\{a_1, \dots, a_n\}$ for any permutation σ of the numbers $1, \dots, n$ with signature $\varepsilon = \pm 1$, and furthermore $\{a_1, \dots, a_n\} = 0$ whenever $a_i + a_{i+1} = 0$ for some $i < n$.

With this notation, $K_n^{(0)}F$ is the full Milnor K-group K_nF introduced in [Milnor 1970], and $K_n^{(m)}F$ is its quotient modulo m for $m \geq 1$.

By a $\mathbb{Z}$-*valuation* we mean a valuation with value group $\mathbb{Z}$. Given a $\mathbb{Z}$-valuation v on F we denote by $\mathcal{O}_v$ its valuation ring and by κ_v its residue field. For $a \in \mathcal{O}_v$ let $\overline{a}$ denote the natural image of a in κ_v. By [ibid., (2.1)], for $n \geq 2$ and a $\mathbb{Z}$-valuation v on F, there is a unique homomorphism $\partial_v : K_n^{(m)}F \to K_{n-1}^{(m)}\kappa_v$ such that

$$\partial_v(\{f, g_2, \dots, g_n\}) = v(f) \cdot \{\overline{g}_2, \dots, \overline{g}_n\} \quad \text{for } f \in F^\times \text{ and } g_2, \dots, g_n \in \mathcal{O}_v^\times .$$

When $n = 2$, for $f, g \in F^\times$ we have $f^{-v(g)} g^{v(f)} \in \mathcal{O}_v^\times$ and

$$\partial_v(\{f, g\}) = \{(-1)^{v(f)v(g)} \overline{f^{-v(g)} g^{v(f)}}\} \quad \text{in } K_1^{(m)}\kappa_v .$$

We turn to the situation where F is the function field of $\mathbb{P}^1$ over E. By the choice of a generator, we identify F with the rational function field $E(t)$ in the variable t over E. Let $\mathcal{P}$ denote the set of monic irreducible polynomials in $E[t]$. Any $p \in \mathcal{P}$ determines a $\mathbb{Z}$-valuation v_p on $E(t)$ that is trivial on E and such that $v_p(p) = 1$. There is further a unique $\mathbb{Z}$-valuation v_∞ on $E(t)$ such that $v_\infty(f) = -\deg(f)$ for any $f \in E[t] \setminus \{0\}$. We set $\mathcal{P}' = \mathcal{P} \cup \{\infty\}$. For $p \in \mathcal{P}'$ we write ∂_p for ∂_{v_p} and we denote by E_p the residue field of v_p. Note that E_p is naturally isomorphic to $E[t]/(p)$ for $p \in \mathcal{P}$, and E_∞ is naturally isomorphic to E.

It follows from [ibid., Section 2] that the sequence

$$(2.1) \qquad 0 \longrightarrow K_n^{(m)}E \longrightarrow K_n^{(m)}E(t) \xrightarrow{\bigoplus \partial_p} \bigoplus_{p \in \mathcal{P}} K_{n-1}^{(m)}E_p \longrightarrow 0$$

is split exact. We are going to reformulate this fact for $n = 2$ and to relate the sequences (2.1) and (1.1). We set

$$\mathfrak{R}'_m(E) = \bigoplus_{p\in\mathcal{P}'} K_1^{(m)} E_p \,.$$

For $p \in \mathcal{P}'$, the norm map of the finite extension E_p/E yields a group homomorphism $K_1^{(m)}E_p \to K_1^{(m)}E$. Summation over these maps for all $p \in \mathcal{P}'$ yields a homomorphism $\mathrm{N} : \mathfrak{R}'_m(E) \to K_1^{(m)}E$. Let $\mathfrak{R}_m(E)$ denote the kernel of N. We set $\partial = \bigoplus_{p\in\mathcal{P}'} \partial_p$. By [Gille and Szamuely 2006, (7.2.4) and (7.2.5)] we obtain an exact sequence

$$(2.2) \qquad 0 \longrightarrow K_2^{(m)}E \longrightarrow K_2^{(m)}E(t) \xrightarrow{\partial} \mathfrak{R}'_m(E) \xrightarrow{\mathrm{N}} K_1^{(m)}E \longrightarrow 0\,.$$

In particular, $\mathfrak{R}_m(E)$ is equal to the image of $\partial : K_2^{(m)}E(t) \to \mathfrak{R}'_m(E)$.

The choice of the generator of F over E fixes a bijection $\phi : \mathbb{P}_E^{1(1)} \to \mathcal{P}'$ and for any $x \in \mathbb{P}_E^{1(1)}$ a natural isomorphism between $E(x)$ and $E_{\phi(x)}$. This identifies $\bigoplus_{x\in\mathbb{P}_E^{1(1)}} K_1^{(m)}E(x)$ with $\mathfrak{R}'_m(E)$, and further the sequence (1.1) with (2.2). We will work with (2.2) in the sequel.

For $\rho = (\rho_p)_{p\in\mathcal{P}'} \in \mathfrak{R}'_m(E)$ we denote $\mathrm{Supp}(\rho) = \{p \in \mathcal{P}' \mid \rho_p \neq 0\}$ and $\deg(\rho) = \sum_{p\in\mathrm{Supp}(\rho)}[E_p : E]$, and call this the *support* and the *degree of* ρ. The degree of an element of $\mathfrak{R}'_m(E)$ is invariant under automorphisms of $E(t)/E$.

3. Bound for representation by symbols in terms of the degree

In this section we study the relation between the degree of $\rho \in \mathfrak{R}_m(E)$ and the properties of elements $\xi \in K_2^{(m)}E(t)$ with $\partial(\xi) = \rho$. In Theorem 3.10 we will show that there always exists such ξ that is a sum of r symbols where r is the integral part of $\deg(\rho)/2$. In particular, any ramification of degree at most three is realized by a symbol. This settles a question in [Kunyavskiĭ et al. 2006, (2.5)]. In some of the following statements, we consider elements of $\mathfrak{R}'_m(E)$, rather than only of $\mathfrak{R}_m(E)$.

Proposition 3.1. *If* $\rho \in \mathfrak{R}_m(E)$ *then* $\deg(\rho) \neq 1$.

Proof. Consider an element $\rho \in \mathfrak{R}'_m(E)$ with $\deg(\rho) = 1$. The support of ρ consists of one rational point $p \in \mathcal{P}'$. Hence $\mathrm{N}(\rho) = \rho_p \neq 0$ in $K_1^{(m)}E$, whereby $\rho \notin \mathfrak{R}_m(E)$. □

We say that $p \in \mathcal{P}'$ is *rational* if $[E_p : E] = 1$. We call a subset of $\mathcal{P}'$ *rational* if all its elements are rational. We give two examples showing how to realize a given ramification of small degree and with rational support by one symbol.

Examples 3.2. (1) Let $a, c \in E^\times$ and $c \notin E^{\times m}$. The symbol $\sigma = \{t - a, c\}$ in $K_2^{(m)}E(t)$ satisfies $\mathrm{Supp}(\sigma) = \{t - a, \infty\}$, $\partial_{t-a}(\sigma) = \{c\}$ and $\partial_\infty(\sigma) = \{c^{-1}\}$.

(2) For $a_1, a_2, c_1, c_2 \in E^\times$ with $a_1 \neq a_2$, we compute the ramification of the symbol

$$\sigma = \left\{ \frac{t - a_1}{c_2(a_2 - a_1)}, \frac{c_1(t - a_2)}{a_1 - a_2} \right\}$$

in $K_2^{(m)}E(t)$. It has $\mathrm{Supp}(\sigma) \subseteq \{t-a_1, t-a_2, \infty\}$, $\partial_{t-a_i}(\sigma) = \{c_i\}$ for $i = 1, 2$, and $\partial_\infty(\sigma) = \{(c_1c_2)^{-1}\}$.

A ramification of degree two can, under some extra conditions, be realized by a symbol one of whose entries is a constant:

Proposition 3.3. *Let $\rho \in \mathfrak{R}_m(E)$ be such that* $\deg(\rho) = 2$. *If* $\mathrm{Supp}(\rho)$ *is rational or* $\mathrm{char}(E) \neq m = 2$, *there exist $e \in E^\times$ and $f \in E(t)^\times$ such that $\rho = \partial(\{e, f\})$.*

Proof. Suppose first that the support of ρ is rational. We choose $a, e \in E^\times$ such that $t - a \in \mathrm{Supp}(\rho)$ and $\rho_{t-a} = \{e\}$ in $K_1^{(m)}E$. Then $\mathrm{Supp}(\rho) = \{t-a, p\}$ where $p \in \mathcal{P}'$ is rational. As $N(\rho) = 0$ we obtain that $\rho_p = \{e^{-1}\}$ in $K_1^{(m)}E_p$. If $p = \infty$, we set $f = 1/(t-a)$. Otherwise $p = t - b$ for some for $b \in E$, and we set $f = (t-b)/(t-a)$. In either case we obtain $\rho = \partial(\{e, f\})$.

It remains to consider the case where $\mathrm{char}(E) \neq m = 2$ and $\mathrm{Supp}(\rho) = \{p\}$ for a quadratic polynomial $p \in \mathcal{P}$. Then E_p/E is a separable quadratic extension. Let $x \in E_p^\times$ be such that $\rho_p = \{x\}$. As $\mathrm{Supp}(\rho) = \{p\}$ and $\mathrm{N}(\rho) = 0$, we obtain that the norm of x with respect to the extension E_p/E lies in $E^{\times 2}$, and therefore $xE_p^{\times 2} = eE_p^{\times 2}$ for some $e \in E^\times$; see [Lam 2005, Chapter VII, (3.9)]. Hence, $\rho_p = \{x\} = \{e\}$ in $K_1^{(2)}E_p$, and we obtain $\rho = \partial(\{e, p\})$. □

In Proposition 3.3 the rationality of the support when $m \neq 2$ is not a superfluous condition; the following example was pointed out to us by J.-P. Tignol.

Example 3.4. Let k be a field. We consider the rational function field in two variables u and v over k. Let τ denote the k-automorphism of $k(u, v)$ satisfying $\tau(u) = v$ and $\tau(v) = u$. Then τ^2 is the identity map on $k(u, v)$, and $E = \{x \in k(u, v) \mid \tau(x) = x\}$ is a subfield of $k(u, v)$ such that $[k(u, v) : E] = 2$. Consider the element $y = v/u \in k(u, v)$. Since $y \notin E$, the quadratic polynomial

$$p = (t-y)(t-\tau(y)) = t^2 - \frac{u^2+v^2}{uv}t + 1$$

is irreducible over E.

Let m be an odd positive integer. We consider the symbol $\sigma = \{p, t\}$ in $K_2^{(m)}E(t)$. Note that the support of $\partial(\sigma)$ is contained in $\{p\}$ and $\partial_p(\sigma) = \{\bar{t}\}$. Moreover, mapping t to y induces an E-isomorphism $E_p \to k(u, v)$. Since y is not an m-th power in $k(u, v)$, it follows that $\partial_p(\sigma) \neq 0$. Hence, $\mathrm{Supp}(\partial(\sigma)) = \{p\}$ and $\deg(\partial(\sigma)) = 2$.

We claim that $\partial_p(\sigma) \neq \partial_p(\{e, f\})$ for any $e \in E^\times$ and $f \in E(t)^\times$. Suppose on the contrary that there exist $e \in E^\times$ and $f \in E(t)^\times$ such that $\partial_p(\sigma) = \partial_p(\{e, f\})$. Then we obtain that $e^{v_p(f)}y$ is an m-th power in $k(u, v)$, and taking norms with respect to the extension $k(u, v)/E$ yields that $e^{2v_p(f)} \in E^{\times m}$. Since m is odd, it follows that $e^{v_p(f)} \in E^{\times m}$, and thus $\partial_p(\{e, f\}) = 0$, a contradiction.

The remainder of this section builds up to our main result, Theorem 3.10.

Lemma 3.5. *Let* $\rho \in \mathfrak{R}'_m(E)$ *with* $\deg(\rho) \geq 2$. *There exists a symbol* σ *in* $K_2^{(m)}E(t)$ *such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$ *and where this inequality is strict if* $\deg(\rho) \geq 3$ *and* $\rho_\infty \neq 0$. *More precisely, one may choose* $\sigma = \{fh, g\}$ *where* f *is the product of the polynomials in* $\mathrm{Supp}(\rho)$ *and where* $g, h \in E[t] \setminus \{0\}$ *are such that* $\deg(g) < \deg(f)$ *and, either* $\deg(h) < \deg(g)$, *or* $gh \in E^\times$.

Proof. Let f be the product of the polynomials in $\mathrm{Supp}(\rho)$. By the Chinese Remainder Theorem, we may choose $g \in E[t]$ prime to f with $\deg(g) < \deg(f)$ such that $\partial_p(\{f, g\}) = \rho_p$ for all monic irreducible polynomials $p \in \mathrm{Supp}(\rho)$. If g is constant, let $h = 1$. If g is not square-free, let h be the product of the different monic irreducible factors of g. If g is square-free and not constant, then using the Chinese remainder theorem we choose $h \in E[t]$ prime to g with $\deg(h) < \deg(g)$ such that

$$\partial_p(\{f, g\}) - \rho_p = \{\bar{h}\}$$

in $K_1^{(m)}E_p$ for every monic irreducible factor p of g. For $\sigma = \{fh, g\}$ we obtain that $\mathrm{Supp}(\rho - \partial(\sigma)) \setminus \{\infty\}$ is contained in the set of monic irreducible factors of h, whereby g, h, and σ have the desired properties. □

Lemma 3.6. *Let* $d \in \mathbb{N} \setminus \{0\}$ *and* $f \in E[t]$ *nonconstant and square-free such that* $\deg(p) \geq d$ *for every irreducible factor* p *of* f. *Let* $F = E[t]/(f)$ *and let* ϑ *denote the class of* t *in* F. *For any* $a \in F^\times$ *there exist nonzero polynomials* $g, h \in E[t]$ *with* $\deg(h) \leq d - 1$ *and* $\deg(g) \leq \deg(f) - d$ *such that* $a = g(\vartheta)/h(\vartheta)$.

Proof. Let

$$V = \bigoplus_{i=0}^{d-1} E\vartheta^i \quad \text{and} \quad W = \bigoplus_{i=0}^{e-d} E\vartheta^i,$$

where $e = \deg(f)$. By the choice of d and the Chinese Remainder Theorem, we have $V \setminus \{0\} \subseteq F^\times$, where $F^\times$ denotes the group of invertible elements of F. As $a \in F^\times$ we have $\dim_E(Va) = \dim_E(V) = d$ and $\dim_E(Va) + \dim_E(W) = e + 1 > e = [F : E]$, so $Va \cap W \neq 0$. Therefore $h(\vartheta)a = g(\vartheta)$ for certain $h, g \in E[t] \setminus \{0\}$ with $\deg(h) \leq d - 1$ and $\deg(g) \leq e - d$. Thus $h(\vartheta) \in V \setminus \{0\} \subseteq F^\times$ and $a = g(\vartheta)/h(\vartheta)$. □

Lemma 3.7. *Let* $\rho \in \mathfrak{R}'_m(E)$ *and* $q \in \mathrm{Supp}(\rho)$ *such that* $\deg(q) = 2n + 1$ *with* $n \geq 1$. *There exists a symbol* σ *in* $K_2^{(m)}E(t)$ *such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$. *More precisely, one may choose* $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$ *with* $f, g, h \in E[t] \setminus \{0\}$ *such that* $\deg(f), \deg(g) \leq n$ *and* $\deg(h) \leq 2n - 1$.

Proof. Applying Lemma 3.6 for $d = n + 1$ we find $f, g \in E[t] \setminus \{0\}$ with $\deg(f), \deg(g) \leq n$ such that $\partial_q(\{q, g^{-1}f\}) = \rho_q$. Then q is prime to fg. If fg is constant, let $h = 1$. If fg is not square-free, let h be the product of the different monic irreducible factors of fg. If fg is square-free and not constant, we choose $h \in E[t]$ prime to fg and with $\deg(h) < \deg(fg)$ such that

$\partial_p(\{h, g^{-1}f\}) = \partial_p(\{q^{-1}f^2g^2, g^{-1}f\})$ for every monic irreducible factor p of fg. In any case $\deg(h) \le 2n-1 = \deg(q)-2$.

Let $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$. Then we have $\partial_q(\sigma) = \rho_q$ and $\partial_p(\sigma) = 0$ for every monic irreducible polynomial $p \in E[t]$ prime to h and not contained in $\mathrm{Supp}(\rho)$. It follows that $q \in \mathrm{Supp}(\rho) \setminus \mathrm{Supp}(\rho - \partial(\sigma))$ and that every polynomial in $\mathrm{Supp}(\rho - \partial(\sigma)) \setminus \mathrm{Supp}(\rho)$ divides h. Furthermore, if $\deg(h) = 2n-1$, then $\deg(f) = \deg(g) = n$, so that $\deg(qh) = 4n = 2\deg(fg)$ and thus $\partial_\infty(\sigma) = 0$. We conclude that $\deg(\rho - \partial(\sigma)) \le \deg(\rho) - 2$ in any case. □

Proposition 3.8. *Let $\rho \in \mathfrak{R}'_m(E)$ with $\deg(\rho) \ge 2$. There exists a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \le \deg(\rho) - 1$. Moreover, if $\deg(\rho) \ge 3$ and $\mathrm{Supp}(\rho)$ contains an element of odd degree, then there exists a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \le \deg(\rho) - 2$.*

Proof. In view of Lemma 3.5 only the second part of the statement remains to be proven. If $\mathrm{Supp}(\rho)$ contains a nonrational point of odd degree, the statement follows from Lemma 3.7. Suppose now that $\mathrm{Supp}(\rho)$ contains a rational point. Note that the statement is invariant under E-automorphisms of $E(t)$. Hence, we may assume that $\infty \in \mathrm{Supp}(\rho)$, in which case the statement follows from Lemma 3.5. □

Question 3.9. Given $\rho \in \mathfrak{R}_m(E)$ with $\deg(\rho) \ge 3$, does there always exist a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \le \deg(\rho) - 2$?

For $x \in \mathbb{R}$, the unique $z \in \mathbb{Z}$ such that $z \le x < z+1$ is denoted by $\lfloor x \rfloor$.

Theorem 3.10. *For $\rho \in \mathfrak{R}_m(E)$ and $n = \lfloor \deg(\rho)/2 \rfloor$, there exist symbols $\sigma_1, \dots, \sigma_n$ in $K_2^{(m)}E(t)$ such that $\rho = \partial(\sigma_1 + \cdots + \sigma_n)$.*

Proof. We proceed by induction on n. If $n = 0$ then $\rho = 0$ by Proposition 3.1 and the statement is trivial. Assume that $n > 0$. We have either $\deg(\rho) = 2n+1$, in which case ρ contains a point of odd degree, or $\deg(\rho) = 2n$. Hence, by Proposition 3.8 there exists a symbol σ in $K_2^{(m)}E(t)$ with $\deg(\rho - \partial(\sigma)) \le 2n-1$. By the induction hypothesis there exist symbols $\sigma_1, \dots, \sigma_{n-1}$ in $K_2^{(m)}E(t)$ with $\rho - \partial(\sigma) = \partial(\sigma_1 + \cdots + \sigma_{n-1})$. Then $\rho = \partial(\sigma_1 + \cdots + \sigma_{n-1} + \sigma)$. □

If we knew that for $m \ge 1$ every element of $\mathfrak{R}_m(E)$ had a lift to $\mathfrak{R}_0(E)$ of the same degree, it would be sufficient to formulate and prove Theorem 3.10 for $m = 0$.

4. Example showing that the bound is sharp

In this section we show that the bound in Theorem 3.10 is sharp for all m and in all degrees. In order to obtain an example in Example 4.3 where the bound of Theorem 3.10 is an equality, we adapt Sivatski's argument [2007, Proposition 2].

For any $a \in E$, there is a unique homomorphism $s_a : K_n^{(m)}E(t) \to K_n^{(m)}E$ such that $s_a(\{f_1, \dots, f_n\}) = \{f_1(a), \dots, f_n(a)\}$ for any $f_1, \dots, f_n \in E[t]$ prime to $t-a$ and such that $s_a(\{t-a, \cdot, \dots, \cdot\}) = 0$; see [Gille and Szamuely 2006, (7.1.4)].

Lemma 4.1. *The homomorphism* $s = s_0 - s_1 : K_n^{(m)}E(t) \to K_n^{(m)}E$ *has the following properties*:

(a) $s(K_n^{(m)}E) = 0$.

(b) $s(\{(1-a)t + a, b_2, \dots, b_n\}) = \{a, b_2, \dots, b_n\}$ *for any* $a, b_2, \dots, b_n \in E^\times$.

(c) *Any symbol in* $K_n^{(m)}E(t)$ *is mapped under* s *to a sum of two symbols in* $K_n^{(m)}E$.

Proof. Since s_0 and s_1 both restrict to the identity on $K_n^{(m)}E$, part (a) is clear. For $a, b_2, \dots, b_n \in E^\times$ and $\sigma = \{(1-a)t + a, b_2, \dots, b_n\}$, we have $s_1(\sigma) = 0$ and thus $s(\sigma) = s_0(\sigma) = \{a, b_2, \dots, b_n\}$. This shows (b). Part (c) follows from the observation that both s_0 and s_1 map symbols to symbols. □

Proposition 4.2. *Let* $d \in \mathbb{N}$, $a_1, \dots, a_d \in E^\times$, *and* $\sigma_1, \dots, \sigma_d$ *symbols in* $K_{n-1}^{(m)}E$. *Assume that* $\sum_{i=1}^d \{a_i\} \cdot \sigma_i \in K_n^{(m)}E$ *is not equal to a sum of less than* d *symbols and let*

$$\xi = \sum_{i=1}^{d} \{(1-a_i)t + a_i\} \cdot \sigma_i \in K_n^{(m)}E(t)\,.$$

Then $\deg(\partial(\xi)) = d+1$, *and if* $r \in \mathbb{N}$ *is such that* $\partial(\xi) = \partial(\tau_1 + \cdots + \tau_r)$ *for symbols* $\tau_1, \dots, \tau_r$ *in* $K_n^{(m)}E(t)$, *then* $r \geq \lfloor (d+1)/2 \rfloor$.

Proof. The hypothesis that $\sum_{i=1}^d \{a_i\} \cdot \sigma_i \in K_n^{(m)}E$ cannot be written as a sum of less than d symbols has a few consequences. For $i = 1, \dots, d$, it follows that $\{a_i\} \cdot \sigma_i \neq 0$, so in particular $a_i \neq 1$, and with $p = t + a_i/(1-a_i)$ we get that $\partial_p(\xi) = \sigma_i \neq 0$ in $K_{n-1}^{(m)}E$. Furthermore, since

$$\sum_{i=1}^{d} \{a_i\} \cdot \sigma_i \neq \sum_{i=1}^{d-1} \{a_i a_d^{-1}\} \cdot \sigma_i\,,$$

we have $\partial_\infty(\xi) = -\sum_{i=1}^d \sigma_i \neq 0$ in $K_{n-1}^{(m)}E$. Therefore we obtain

$$\mathrm{Supp}(\partial(\xi)) = \left\{ t + \frac{a_i}{1-a_i} \;\middle|\; 1 \leq i \leq d \right\} \cup \{\infty\}$$

and thus $\deg(\partial(\xi)) = d+1$.

Assume now that $r \in \mathbb{N}$ and $\partial(\xi) = \partial(\tau_1 + \cdots + \tau_r)$ for symbols $\tau_1, \dots, \tau_r$ in $K_n^{(m)}E(t)$. Then $\tau_1 + \cdots + \tau_r - \xi$ is defined over E. Let s be the map from Lemma 4.1. By Lemma 4.1 we obtain that $s(\tau_1 + \cdots + \tau_r - \xi) = 0$ and thus

$$\sum_{i=1}^{d} \{a_i\} \cdot \sigma_i = s(\xi) = s(\tau_1) + \cdots + s(\tau_r) \in K_n^{(m)}E\,,$$

which is a sum of $2r$ symbols. Hence $2r \geq d$, by the hypothesis on d. □

Example 4.3. Let p be a prime dividing m. Let k be a field containing a primitive p-th root of unity ω and $a_1, \dots, a_d \in k^\times$ such that the Kummer extension $k(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_d})$ of k has degree p^d. Let $b_1, \dots, b_d$ be indeterminates over k and set $E = k(b_1, \dots, b_d)$. Using [Tignol 1987, (2.10)] and [Becher and Hoffmann 2004, (2.1)], it follows that $\sum_{i=1}^d \{a_i, b_i\}$ is not equal to a sum of less than d symbols in $K_2^{(p)}E$. Since p divides m, it follows immediately that $\sum_{i=1}^d \{a_i, b_i\} \in K_2^{(m)}E$ is not a sum of less than d symbols in $K_2^{(m)}E$. Consider

$$\xi = \sum_{i=1}^{d} \{(1-a_i)t + a_i, b_i\}$$

in $K_2^{(m)}E(t)$. By Proposition 4.2, for $\rho = \partial(\xi)$ we have that $\deg(\rho) = d+1$ and $\rho \neq \partial(\xi')$ for any $\xi' \in K_2^{(m)}E(t)$ that is a sum of less than $r = \lfloor \deg(\rho)/2 \rfloor$ symbols.

Acknowledgements

We wish to express our gratitude to Jean-Pierre Tignol for his interest in our work and all his support in its course. We further would like to thank the referee for several very valuable remarks.

References

[Becher and Hoffmann 2004] K. J. Becher and D. W. Hoffmann, "Symbol lengths in Milnor K-theory", *Homology Homotopy Appl.* **6**:1 (2004), 17–31. MR 2005b:19001 Zbl 1069.19004

[Faddeev 1951] D. K. Faddeev, "Simple algebras over a field of algebraic functions of one variable", *Trudy Mat. Inst. Steklov.* **38** (1951), 321–344. In Russian; translated in *AMS Transl. Ser. 2* **3** (1956), 15–38. MR 13,905c Zbl 0053.35602

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, Cambridge, 2006. MR 2007k:16033 Zbl 1137.12001

[Kunyavskiĭ et al. 2006] B. È. Kunyavskiĭ, L. H. Rowen, S. V. Tikhonov, and V. I. Yanchevskiĭ, "Bicyclic algebras of prime exponent over function fields", *Trans. Amer. Math. Soc.* **358**:6 (2006), 2579–2610. MR 2007d:16034 Zbl 1101.16013

[Lam 2005] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics **67**, American Mathematical Society, Providence, RI, 2005. MR 2005h:11075 Zbl 1068.11023

[Milnor 1970] J. Milnor, "Algebraic K-theory and quadratic forms", *Invent. Math.* **9** (1970), 318–344. MR 41 #5465 Zbl 0199.55501

[Rowen et al. 2005] L. H. Rowen, A. S. Sivatski, and J.-P. Tignol, "Division algebras over rational function fields in one variable", pp. 158–180 in *Algebra and number theory*, edited by R. Tandon, Hindustan Book Agency, Delhi, 2005. MR 2006i:16029 Zbl 1089.16015

[Sivatski 2007] A. S. Sivatski, "On the Faddeev index of an algebra over the function field of a curve", preprint 255, Universität Bielefeld, 2007, http://www.math.uni-bielefeld.de/lag/man/255.

[Tignol 1987] J.-P. Tignol, "Algèbres indécomposables d'exposant premier", *Adv. Math.* **65**:3 (1987), 205–228. MR 88h:16028 Zbl 0642.16015

Received February 8, 2012. Revised June 1, 2012.

KARIM JOHANNES BECHER
UNIVERSITEIT ANTWERPEN
DEPARTMENT MATHEMATICS AND COMPUTER SCIENCE
MIDDELHEIMLAAN 1
B-2020 ANTWERPEN
BELGIUM

and

UNIVERSITÄT KONSTANZ
ZUKUNFTSKOLLEG / FB MATHEMATIK UND STATISTIK
D-78457 KONSTANZ
GERMANY

becher@maths.ucd.ie

MÉLANIE RACZEK
UNIVERSITÉ CATHOLIQUE DE LOUVAIN
ICTEAM
CHEMIN DU CYCLOTRON 2
1348 LOUVAIN-LA-NEUVE
BELGIUM

melanie.raczek@uclouvain.be

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 262 No. 1 March 2013