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GLOBAL WELL-POSEDNESS FOR THE 3D ROTATING NAVIER–STOKES EQUATIONS WITH HIGHLY OSCILLATING INITIAL DATA

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We prove the global well-posedness for the 3D rotating Navier–Stokes equations in the critical functional framework. This result allows us to construct global solutions for a class of highly oscillating initial data.

1. Introduction

In this paper, we study the 3D rotating Navier–Stokes equations

$$\begin{cases} u_t - \nu\Delta u + \Omega e_3 \times u + u\cdot\nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1-1}$$

where ν denotes the viscosity coefficient of the fluid, Ω the speed of rotation, e_3 the unit vector in the x_3 direction and $\Omega e_3 \times u$ the Coriolis force. We refer to [Chemin et al. 2006; Majda 2003; Pedlosky 1987] for its background in geophysical fluid dynamics. If the Coriolis force is neglected, the equations (1-1) become the classical 3D incompressible Navier–Stokes equations

$$\begin{cases} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1-2}$$

The global existence of a weak solution of (1-1) can be proved by the classical compactness method, since we still have the energy estimate

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\,ds \le \|u_0\|_{L^2}^2.$$

As in 3D Navier–Stokes equations, the uniqueness and regularity of weak solutions are also open problems. Recently, Giga et al. [2006; 2007; 2008] studied the local

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existence of a mild solution for a class of nondecaying initial data which includes a class of almost periodic functions, as well as global existence for small data. On the other hand, when the speed Ω of rotation is fast enough, the global existence of smooth solution was proved in [Babin et al. 1997; 1999; Chemin et al. 2000; 2006].

For the 3D Navier–Stokes equations, Fujita and Kato [1964; Kato 1984] proved the local well-posedness for large initial data and the global well-posedness for small initial data in the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}$ and the Lebesgue space L^3, respectively. These spaces are all the critical ones, which are relevant to the scaling of the Navier–Stokes equations: if (u, p) solves (1-2), then

$$\big(u_\lambda(t,x), p_\lambda(t,x)\big) := \big(\lambda u(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x)\big) \tag{1-3}$$

is also a solution of (1-2). The so-called *critical space* is the one such that the associated norm is invariant under the scaling of (1-3). Recently, Cannone [1997] (see also [Cannone 1995; 2004; Cannone et al. 1994]) generalized it to Besov spaces with negative index of regularity. More precisely, he showed that if the initial data satisfies

$$\|u_0\|_{\dot{B}^{-1+\frac{3}{p}}_{p,\infty}} \le c, \quad p > 3$$

for some small constant c, then the Navier–Stokes equations (1-2) are globally well-posed. Let us emphasize that this result allows us to construct global solutions for highly oscillating initial data which may have a large norm in $\dot{H}^{\frac{1}{2}}$ or L^3. A typical example is

$$u_0(x) = \sin\frac{x_3}{\varepsilon}\big(-\partial_2\phi(x), \partial_1\phi(x), 0\big)$$

where $\phi \in \mathcal{S}(\mathbb{R}^3)$ and $\varepsilon > 0$ is small enough. We refer to [Chemin and Gallagher 2006; Chemin and Zhang 2007; Chen et al. 2010a] for some relevant results. A natural question is then to prove a theorem of this type for the rotating Navier–Stokes equations.

We know that Kato's method heavily relies on the uniform boundedness of the Stokes semigroup in L^p and global $L^p - L^q$ estimates, but the Stokes–Coriolis semigroup is not uniformly bounded in L^p for $p \neq 2$; see Theorems 5 and 6 in [Dragičević et al. 2006]. Standard techniques allow us to prove these estimates only locally for the Stokes–Coriolis semigroup, hence one can obtain the local existence of mild solution in L^3 by Kato's method. Whether one can extend this solution to a global one for small data in L^3 is a very interesting problem.

Very recently, based on the global $L^p - L^q$ estimates with $q \le 2 \le p$ and $L^q - H^{\frac{1}{2}}$ estimates with $q > 3$ for the Stokes–Coriolis semigroup, Hieber and Shibata [2010] proved the following global result for small data in $H^{\frac{1}{2}}$.

Theorem 1.1. *Let $q > 3$. Then there exists $c > 0$ independent of Ω such that for any $u_0 \in H_\sigma^{\frac{1}{2}}$ with $\|u_0\|_{H^{\frac{1}{2}}} \le c$, the equations (1-1) admit a unique mild solution $u \in C\big([0,\infty), H_\sigma^{\frac{1}{2}}\big)$ satisfying*

$$
\begin{aligned}
&u \in C((0,\infty), L^q) \quad \text{and} \quad \lim_{t\to 0+} \sup_{0<s<t} s^{\frac{1}{2}-\frac{3}{2q}} \|u(s,\cdot)\|_{L^q} = 0,\\
&\nabla u \in C((0,\infty), L^2) \quad \text{and} \quad \lim_{t\to 0+} \sup_{0<s<t} s^{\frac{1}{4}} \|\nabla u(s,\cdot)\|_{L^2} = 0.
\end{aligned}
\tag{1-4}
$$

Here $H_\sigma^{\frac{1}{2}}$ denotes the closure of the set $\{u \in C_c^\infty(\mathbb{R}^3)^3, \operatorname{div} u = 0\}$ in the norm of $\|\cdot\|_{H^{\frac{1}{2}}}$.

The goal of this paper is to prove the global existence of a solution of (1-1) for a class of highly oscillating initial velocity. Thus we need to solve the system (1-1) for the initial data in a critical functional framework whose regularity index is negative, for example, $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ for $p > 3$. However, Cannone's proof [1997] doesn't work for our case, since it also relies on the global $L^p - L^q$ estimates for the Stokes semigroup. Indeed, for the Stokes–Coriolis semigroup $\mathcal{G}(t)$, one has

$$
\|\mathcal{G}(t)u_0\|_{L^p} \le C_{p,\Omega} t^2 \|u_0\|_{L^p}, \quad \text{if } p \ne 2;
$$

see Proposition 2.2 in [Hieber and Shibata 2010]. Then we can infer from the definition of the Besov space that

$$
\|\mathcal{G}(t)u_0\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}} \le C t^2 \|u_0\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}}.
$$

This means that even if the initial data u_0 is small in $\dot{B}_{p,q}^{-1+\frac{3}{p}}$, the linear part of the solution, $\|\mathcal{G}(t)u_0\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}}$, may become large after some time $t_0 > 0$.

Fortunately, we have the following important observation: if u is an element of L^p with $\operatorname{supp} \hat{u} \in \{\xi : |\xi| \gtrsim \lambda\}$, then

$$
\|\mathcal{G}(t)u\|_{L^p} \le C_{p,\Omega} e^{-t\lambda^2} \|u\|_{L^p}
$$

for any $p \in [1,\infty]$ and $t \in [0,\infty]$, while for any $u \in L^2$,

$$
\|\mathcal{G}(t)u\|_{L^2} \le \|u\|_{L^2}.
$$

This motivates us to introduce the hybrid-Besov spaces $\dot{\mathcal{B}}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}$ (see Definition 2.2). Roughly speaking, if $u \in \dot{\mathcal{B}}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}$, the low frequency part of u belongs to $\dot{H}^{\frac{1}{2}}$ and the high frequency part belongs to $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$. So, $\dot{\mathcal{B}}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}$ is still a critical space. A remarkable property of $\dot{\mathcal{B}}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}$ is that if $p > 3$, then

$$
\|u_0(x)\|_{\dot{\mathcal{B}}_{2,p}^{\frac{1}{2},\frac{3}{p}-1}} \le C\varepsilon^{1-\frac{3}{p}},
$$

for $u_0(x) = \sin(x_1/\varepsilon)\phi(x)$, with $\phi(x) \in \mathcal{S}(\mathbb{R}^3)$; see Proposition 2.4. That is, the highly oscillating function is still small in the norm of $\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$.

Definition 1.2. Let $1 \le p \le \infty$, we denote by E_p the space of functions such that

$$E_p = \{u : \operatorname{div} u = 0, \|u\|_{E_p} < +\infty\},$$

where

$$\|u\|_{E_p} := \|u\|_{\widetilde{L}^\infty\left(\mathbb{R}^+;\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}\right)} + \|u\|_{\widetilde{L}^1\left(\mathbb{R}^+;\dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}\right)}.$$

Definition 1.3. We denote by $C_*\big([0,\infty);\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}\big)$ the set of functions u such that u is continuous from $(0,\infty)$ to $\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$, but weakly continuous at $t = 0$; i.e.,

$$\lim_{t\to 0+} \sup_{0<s<t} \big\langle u(s,\cdot),\, g(\cdot)\big\rangle = 0 \quad \text{for all } g \in \mathcal{S} \text{ with } \|g\|_{\dot{\mathcal{B}}^{-\frac{1}{2},1-\frac{3}{p}}_{2,p}} \le 1.$$

Our main results are stated as follows.

Theorem 1.4. *Let $p \in [2,4]$. There exists a positive constant c independent of Ω such that if $\|u_0\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \le c$, then there exists a unique solution $u \in E_p$ of (1-1) such that*

$$u \in C_*\big([0,\infty);\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}\big).$$

Remark 1.5. Due to the inclusion map

$$H^{\frac{1}{2}} \subseteq \dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p} \quad \text{for } p \ge 2,$$

Theorem 1.4 is an improvement on Theorem 1.1. The importance of this is that it allows us to construct global solutions of (1-1) for a class of highly oscillating initial velocity u_0, for example,

$$u_0(x) = \sin\Big(\frac{x_3}{\varepsilon}\Big)(-\partial_2\phi(x), \partial_1\phi(x), 0) \tag{1-5}$$

where $\phi \in \mathcal{S}(\mathbb{R}^3)$ and $\varepsilon > 0$ is small enough. This type of data is large in the Sobolev norm; however, it is small in the norms of Besov spaces with negative regularity index.

Remark 1.6. As shown in Section 4.2 of [Cannone 2004], for the classical Navier–Stokes equations (1-2), there exists the following "highly oscillating" initial data: $u_0(x) \in \mathcal{S}'(\mathbb{R}^3)$ is such that $\hat{u}_0(\xi) = 0$ if $|\xi| \le 1/\varepsilon$. Then

$$\|u_0\|_{\dot{H}^{1/2}} \le \varepsilon^{1/2}\|u_0\|_{\dot{H}^1}. \tag{1-6}$$

We point out that examples like (1-5) are not included in such initial data. In fact, if $\operatorname{supp}\hat{\phi}(\xi) \subset \{|\xi| \le 1/2\varepsilon\}$, then the above estimate is satisfied, while if $\hat{\phi}(\xi)$ has no support, it is not sure that (1-6) holds, which implies the norm of $\|u_0\|_{\dot{H}^{1/2}}$ may not be small enough.

Remark 1.7. The inhomogeneous part of the solution has more regularity:

$$u-\mathcal{G}(t)u_0 \in C\big(\mathbb{R}^+;\dot{B}^{\frac{1}{2}}_{2,\infty}\big),$$

which can be proved by following the proof of Proposition 4.1.

If u_0 lies in $\dot{H}^{\frac{1}{2}}$, we can obtain the following global well-posedness result.

Theorem 1.8. *Let $p \in [2,4]$. There exists a positive constant c independent of Ω such that, if u_0 belongs to $\dot{H}^{\frac{1}{2}}$ with $\|u_0\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \le c$, then there exists a unique global solution of (1-1) in $C(\mathbb{R}^+,\dot{H}^{\frac{1}{2}})$.*

Remark 1.9. Since we only impose the smallness condition of the initial data in the norm of $\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}$, this allows us to obtain the global well-posedness of (1-1) for a class of highly oscillating initial velocity u_0. Moreover, the uniqueness holds in the class $C(\mathbb{R}^+,\dot{H}^{\frac{1}{2}})$; i.e., it is unconditional.

The structure of this paper is as follows. In Section 2, we recall some basic facts about Littlewood–Paley theory and the functional spaces. In Section 3, we recall some results concerning the Stokes–Coriolis semigroup's regularizing effect. Section 4 is devoted to the important bilinear estimates. In Section 5, we prove Theorem 1.4 and Theorem 1.8.

2. Littlewood–Paley theory and the function spaces

First of all, we introduce the Littlewood–Paley decomposition. Choose two radial functions $\varphi,\chi\in\mathcal{S}(\mathbb{R}^3)$ supported in $\mathcal{C}=\{\xi\in\mathbb{R}^3,\ \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$, $\mathcal{B}=\{\xi\in\mathbb{R}^3,\ |\xi|\le\frac{4}{3}\}$, respectively, such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1 \quad \text{for all } \xi\neq 0.$$

For $f\in\mathcal{S}'(\mathbb{R}^3)$, the frequency localization operators Δ_j and $S_j\,(j\in\mathbb{Z})$ are defined by

$$\Delta_j f=\varphi(2^{-j}D)f,\quad S_j f=\chi(2^{-j}D)f,\quad D=\frac{\nabla_x}{i}.$$

Moreover, we have

$$S_j f=\sum_{k=-\infty}^{j-1}\Delta_k f \quad \text{in } \mathcal{Z}'(\mathbb{R}^3).$$

Here we denote by $\mathcal{Z}'(\mathbb{R}^3)$ the dual space of

$$\mathcal{Z}(\mathbb{R}^3)=\{f\in\mathcal{S}(\mathbb{R}^3): D^\alpha\hat{f}(0)=0 \text{ for all multiindices } \alpha\in(\mathbb{N}\cup 0)^3\}.$$

With our choice of φ, it is easy to verify that

(2-1) $\Delta_j \Delta_k f = 0 \quad \text{if } |j-k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geq 5.$

In the sequel, we will constantly use Bony's decomposition [1981]:

$$fg = T_f g + T_g f + R(f,g), \tag{2-2}$$

with

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g, \quad R(f,g) = \sum_{j\in\mathbb{Z}} \Delta_j f \widetilde{\Delta}_j g, \quad \widetilde{\Delta}_j g = \sum_{|j'-j|\leq 1} \Delta_{j'} g.$$

Definition 2.1 (homogeneous Besov space). Let $s \in \mathbb{R}$, $1 \leq p, q \leq +\infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} := \{ f \in \mathscr{Z}'(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < +\infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} := \big\| 2^{ks} \|\Delta_k f\|_{L^p} \big\|_{l^q}.$$

If $p = q = 2$, $\dot{B}^s_{2,2}$ is equivalent to the homogeneous Sobolev space $\dot{H}^s$.

Definition 2.2 (hybrid-Besov space). Let $s, \sigma \in \mathbb{R}$, $1 \leq p \leq +\infty$. The hybrid-Besov space $\dot{\mathscr{B}}^{s,\sigma}_{2,p}$ is defined by

$$\dot{\mathscr{B}}^{s,\sigma}_{2,p} := \{ f \in \mathscr{Z}'(\mathbb{R}^3) : \|f\|_{\dot{\mathscr{B}}^{s,\sigma}_{2,p}} < +\infty \},$$

where

$$\|f\|_{\dot{\mathscr{B}}^{s,\sigma}_{2,p}} := \sup_{2^k \leq \Omega} 2^{ks} \|\Delta_k f\|_{L^2} + \sup_{2^k > \Omega} 2^{k\sigma} \|\Delta_k f\|_{L^p}.$$

The norm of the space $\widetilde{L}^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})$ is defined by

$$\|f\|_{\widetilde{L}^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})} := \sup_{2^k \leq \Omega} 2^{ks} \|\Delta_k f\|_{L^r_T L^2} + \sup_{2^k > \Omega} 2^{k\sigma} \|\Delta_k f\|_{L^r_T L^p}.$$

It is easy to check that $L^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p}) \subseteq \widetilde{L}^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})$, where the norm of $L^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})$ is defined by

$$\|f\|_{L^r_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})} := \big\| \|f(t)\|_{\dot{\mathscr{B}}^{s,\sigma}_{2,p}} \big\|_{L^r_T}.$$

Bernstein's lemma will be repeatedly used throughout this paper:

Lemma 2.3 [Chemin 1995]. *Let $1 \leq p \leq q \leq +\infty$. Then for any $\beta, \gamma \in (\mathbb{N} \cup \{0\})^3$, there exists a constant C independent of f, j such that, for any $f \in L^p$,*

$$\operatorname{supp} \hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Rightarrow \|\partial^\gamma f\|_{L^q} \leq C 2^{j|\gamma| + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},$$

$$\operatorname{supp} \hat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \Rightarrow \|f\|_{L^p} \leq C 2^{-j|\gamma|} \sup_{|\beta|=|\gamma|} \|\partial^\beta f\|_{L^p}.$$

Proposition 2.4. *Let $\phi \in \mathcal{S}(\mathbb{R}^3)$ and $p > 3$. If $\phi_\varepsilon(x) := e^{i\frac{x_1}{\varepsilon}}\phi(x)$, then, for any $0 < \varepsilon \le \Omega^{-1}$,*

$$\|\phi_\varepsilon\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \le C\varepsilon^{1-\frac{3}{p}},$$

where C is a constant independent of ε.

Proof. Let $j_0 \in \mathbb{N}$ be such that $\Omega \le 2^{j_0} \sim \varepsilon^{-1}$. By Lemma 2.3, we have

$$\sup_{j \ge j_0} 2^{(\frac{3}{p}-1)j}\|\Delta_j\phi_\varepsilon\|_{L^p} \le C2^{(\frac{3}{p}-1)j_0} \le C\varepsilon^{1-\frac{3}{p}}.$$

Noting that $e^{i\frac{x_1}{\varepsilon}} = (-i\varepsilon\partial_1)^N e^{i\frac{x_1}{\varepsilon}}$ for any $N \in \mathbb{N}$, we get, by integration by parts,

$$\Delta_j\phi_\varepsilon(x) = (i\varepsilon)^N 2^{3j}\int_{\mathbb{R}^3} e^{i\frac{y_1}{\varepsilon}}\partial_{y_1}^N\big(h(2^j(x-y))\phi(y)\big)\,dy, \quad h(x) := (\mathcal{F}^{-1}\varphi)(x).$$

By the Leibnitz formula, we have

$$|\Delta_j\phi_\varepsilon(x)| \le C\varepsilon^N 2^{3j}\sum_{k=0}^N 2^{kj}\int_{\mathbb{R}^3}\big|(\partial_{y_1}^k h)(2^j(x-y))\big|\big|\partial_{y_1}^{N-k}\phi(y)\big|\,dy,$$

from which, along with Young's inequality, we infer that, for $j \ge 0$,

$$\|\Delta_j\phi_\varepsilon\|_{L^q} \le C\varepsilon^N\sum_{k=0}^N 2^{kj}2^{3j}\big\|(\partial_{y_1}^k h)(2^j y)\big\|_{L^1}\big\|\partial_{y_1}^{N-k}\phi(y)\big\|_{L^q} \le C\varepsilon^N 2^{jN},$$

and for $j \le 0$,

$$\|\Delta_j\phi_\varepsilon\|_{L^q} \le C\varepsilon^N\sum_{k=0}^N 2^{kj}2^{3j}\big\|(\partial_{y_1}^k h)(2^j y)\big\|_{L^q}\big\|\partial_{y_1}^{N-k}\phi(y)\big\|_{L^1} \le C\varepsilon^N 2^{(1-\frac{1}{q})3j}.$$

Thus we have

$$\sup_{\Omega < 2^j < 2^{j_0}} 2^{(\frac{3}{p}-1)j}\|\Delta_j\phi_\varepsilon\|_{L^p} \le C\varepsilon^N 2^{(N-1+\frac{3}{p})j_0} \le C\varepsilon^{1-\frac{3}{p}},$$

$$\sup_{2^j \le \Omega} 2^{\frac{j}{2}}\|\Delta_j\phi_\varepsilon\|_{L^2} \le C\Omega^{\frac{1}{2}}\varepsilon^N \le C\varepsilon^{N-\frac{1}{2}}.$$

Summing up the above estimates yields that

$$\|\phi_\varepsilon\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \le C\varepsilon^{1-\frac{3}{p}}.$$

The proof of Proposition 2.4 is completed. ∎

3. Regularizing effect of the Stokes–Coriolis semigroup

We consider the linear system

$$\begin{cases} u_t - \nu\Delta u + \Omega e_3 \times u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{3-1}$$

From [Giga et al. 2005; Hieber and Shibata 2010, Proposition 2.1], we know that

$$\hat{u}(t,\xi) = \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} I \widehat{u_0}(\xi) + \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} R(\xi) \widehat{u_0}(\xi), \tag{3-2}$$

for $t \geq 0$ and $\xi \in \mathbb{R}^3$, where I is the identity matrix and

$$R(\xi) = \begin{pmatrix} 0 & \xi_3/|\xi| & -\xi_2/|\xi| \\ -\xi_3/|\xi| & 0 & \xi_1/|\xi| \\ \xi_2/|\xi| & -\xi_1/|\xi| & 0 \end{pmatrix}.$$

The Stokes–Coriolis semigroup is explicitly represented by

$$\mathcal{G}(t) f = \big[\cos(\Omega R_3 t) I + \sin(\Omega R_3 t) R\big] e^{\nu t \Delta} f, \quad \text{for } t \geq 0,\ f \in L^p_\sigma, \tag{3-3}$$

where $\widehat{R_3 f}(\xi) := (\xi_3/|\xi|)\hat{f}(\xi)$ for $\xi \neq 0$.

Proposition 3.1 (smoothing effect of the Stokes–Coriolis semigroup). *Let $\mathscr{C}$ be a ring centered at 0 in $\mathbb{R}^3$. Then there exist positive constants c and C depending only on ν such that if* $\operatorname{supp} \hat{u} \subset \lambda\mathscr{C}$, *then we have*:

(i) *for any* $\lambda > 0$,

$$\|\mathcal{G}(t)u\|_{L^2} \leq C e^{-c\lambda^2 t} \|u\|_{L^2}; \tag{3-4}$$

(ii) *if* $\lambda \gtrsim \Omega$, *then, for any* $1 \leq p \leq \infty$,

$$\|\mathcal{G}(t)u\|_{L^p} \leq C e^{-c\lambda^2 t} \|u\|_{L^p}. \tag{3-5}$$

Proof. (i) Thanks to (3-2) and the Plancherel theorem, we get

$$\|\mathcal{G}(t)u\|_{L^2} = \big\|\widehat{\mathcal{G}}(t,\xi)\hat{u}(\xi)\big\|_{L^2} \leq C \big\|e^{-\nu|\xi|^2 t}\hat{u}(\xi)\big\|_2 \leq C e^{-\nu\lambda^2 t}\|u\|_2,$$

where we have used the support property of $\hat{u}(\xi)$.

(ii) Let $\phi \in \mathscr{D}(\mathbb{R}^3 \setminus \{0\})$, which equals 1 near the ring $\mathscr{C}$. Set

$$g(t,x) := (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \phi(\lambda^{-1}\xi) \widehat{\mathcal{G}}(t,\xi)\, d\xi.$$

To prove (3-5), it suffices to show

$$\|g(x,t)\|_{L^1} \leq C e^{-c\lambda^2 t}. \tag{3-6}$$

Thanks to (3-3), we infer that

$$\int_{|x|\le\lambda^{-1}}|g(x,t)|dx\le C\int_{|x|\le\lambda^{-1}}\int_{\mathbb{R}^3}\big|\phi(\lambda^{-1}\xi)\big|\big|\widehat{\mathcal{G}}(t,\xi)\big|\,d\xi\,dx\le Ce^{-c\lambda^2 t}. \tag{3-7}$$

Set $L:=x\cdot\nabla_\xi/(i|x|^2)$. Noting that $L(e^{ix\cdot\xi})=e^{ix\cdot\xi}$, we get, using integration by parts,

$$g(x,t)=\int_{\mathbb{R}^3}L^N(e^{ix\cdot\xi})\phi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi)\,d\xi=\int_{\mathbb{R}^3}e^{ix\cdot\xi}(L^*)^N\big(\phi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi)\big)\,d\xi,$$

where $N\in\mathbb{N}$ is chosen later. Using the Leibnitz formula, it is easy to verify that

$$\big|\partial^\gamma(e^{\pm i\Omega\frac{\xi_3}{|\xi|}t})\big|\le C|\xi|^{-|\gamma|}(1+\Omega t)^{|\gamma|},\quad \big|\partial^\gamma(e^{-\nu|\xi|^2t})\big|\le C|\xi|^{-|\gamma|}e^{-\frac{\nu}{2}|\xi|^2t}.$$

Thus we obtain

$$\begin{aligned}&\big|(L^*)^N\big(\phi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi)\big)\big|\\ &\le C|x|^{-N}\sum_{\substack{|\alpha_1|+|\alpha_2|\\+|\alpha_3|=|\alpha|\\|\alpha|\le N}}\lambda^{-N+\alpha}\big|(\nabla^{N-\alpha}\phi)(\lambda^{-1}\xi)\partial^{\alpha_1}\big(e^{\pm i\Omega\frac{\xi_3}{|\xi|}t}\big)\partial^{\alpha_2}\big(e^{-\nu|\xi|^2t}\big)\partial^{\alpha_3}(I+R(\xi))\big|\\ &\le C|\lambda x|^{-N}\sum_{\substack{|\alpha_1|+|\alpha_2|\\+|\alpha_3|=|\alpha|\\|\alpha|\le N}}\lambda^{\alpha}\big|\big(\nabla^{N-\alpha}\phi\big)(\lambda^{-1}\xi)\big||\xi|^{-|\alpha_1|-|\alpha_2|-|\alpha_3|}e^{-\frac{\nu}{2}|\xi|^2t}(1+\Omega t)^{|\alpha_1|}.\end{aligned}$$

Taking $N=4$, for any $\xi\in\{\xi:A^{-1}\lambda\le|\xi|\le A\lambda\}$ and for some constant A depending on the ring $\mathscr{C}$ and $\lambda\gtrsim\Omega$,

$$\big|(L^*)^4\big(\phi(\lambda^{-1}\xi)\widehat{\mathcal{G}}(t,\xi)\big)\big|\le C|\lambda x|^{-4}e^{-\frac{\nu}{4}|\xi|^2t},$$

which implies that

$$\int_{|x|\ge\frac{1}{\lambda}}|g(x,t)|\,dx\le Ce^{-c\lambda^2t}\lambda^3\int_{|x|\ge\frac{1}{\lambda}}|\lambda x|^{-4}\,dx\le Ce^{-c\lambda^2t},$$

which, together with (3-7), gives (3-6). Then the inequality (3-5) is proved. □

The following proposition is a direct consequence of Proposition 3.1.

Proposition 3.2. *Let $s,\sigma\in\mathbb{R}$, and $(p,q)\in[1,\infty]$. Then, for any $u\in\dot{\mathscr{B}}^{s-\frac{2}{q},\sigma-\frac{2}{q}}_{2,p}$, we have*

$$\|\mathcal{G}(t)u\|_{\widetilde{L}^q_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})}\le C\|u\|_{\dot{\mathscr{B}}^{s-\frac{2}{q},\sigma-\frac{2}{q}}_{2,p}}, \tag{3-8}$$

and for any $f\in\widetilde{L}^1_T\mathscr{B}^{s,\sigma}_{2,p}$, we have

$$\Big\|\int_0^t\mathcal{G}(t-\tau)f(\tau)\,d\tau\Big\|_{\widetilde{L}^q_T\big(\dot{\mathscr{B}}^{s+\frac{2}{q},\sigma+\frac{2}{q}}_{2,p}\big)}\le C\|f(t)\|_{\widetilde{L}^1_T(\dot{\mathscr{B}}^{s,\sigma}_{2,p})}. \tag{3-9}$$

Proof. Here we only prove (3-9). For any $2^j \geq \Omega$, we get by Proposition 3.1 that

$$\Big\| \Delta_j \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \Big\|_{L^p} \leq C \int_0^t e^{-c(t-\tau)2^{2j}} \|\Delta_j f(\tau)\|_{L^p}\, d\tau,$$

from which, along with Young's inequality, it follows that

$$\Big\| \Delta_j \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \Big\|_{L^q_T L^p} \leq C \big\| e^{-ct2^{2j}} \big\|_{L^q_T} \|\Delta_j f(\tau)\|_{L^1_T L^p} \leq C 2^{-\frac{2}{q} j} \|\Delta_j f(\tau)\|_{L^1_T L^p}. \tag{3-10}$$

Similarly, we also have

$$\Big\| \Delta_j \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \Big\|_{L^q_T L^2} \leq C \big\| e^{-ct2^{2j}} \big\|_{L^q_T} \|\Delta_j f(\tau)\|_{L^1_T L^2} \leq C 2^{-\frac{2}{q} j} \|\Delta_j f(\tau)\|_{L^1_T L^2}.$$

Then the inequality (3-9) follows from (3-10) and (3-11). □

4. Bilinear estimates

We study the continuity of the inhomogeneous term in the space $E_{p,T}$ whose norm is defined by

$$\|u\|_{E_{p,T}} := \|u\|_{\widetilde{L}^\infty\left(0,T;\dot{\mathcal{B}}^{\frac12,\frac3p-1}_{2,p}\right)} + \|u\|_{\widetilde{L}^1\left(0,T;\dot{\mathcal{B}}^{\frac52,\frac3p+1}_{2,p}\right)}.$$

We define

$$B(u,v) := \int_0^t \mathcal{G}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes v)\, d\tau,$$

where $\mathbb{P}$ denotes the Helmholtz projection which is bounded in the L^p space for $1 < p < \infty$.

Proposition 4.1. *Let $p \in [2,4]$. Assume that $u, v \in E_{p,T}$. There exists a constant C independent of Ω, u, v such that, for any $T > 0$,*

$$\|B(u,v)\|_{E_{p,T}} \leq C \|u\|_{E_{p,T}} \|v\|_{E_{p,T}}. \tag{4-1}$$

Proof. Thanks to Proposition 3.2, it suffices to show that

$$\|uv\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac32,\frac3p}_{2,p}} \leq C \|u\|_{E_{p,T}} \|v\|_{E_{p,T}}. \tag{4-2}$$

From Bony's decomposition (2-2) and (2-1), we have

$$\Delta_j(uv) = \sum_{|k-j|\leq 4} \Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\leq 4} \Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\geq j-2} \Delta_j(\Delta_k u \widetilde{\Delta}_k v) =: I_j + II_j + III_j.$$

Set $J_j := \{(k', k) : |k - j| \le 4, k' \le k - 2\}$. Then for $2^j > \Omega$,

$$\|I_j\|_{L^1_T L^p} \le \sum_{J_j} \|\Delta_j(\Delta_{k'} u \Delta_k v)\|_{L^1_T L^p}$$
$$\le \Big(\sum_{J_{j,ll}} + \sum_{J_{,lh}} + \sum_{J_{j,hh}}\Big) \|\Delta_j(\Delta_{k'} u \Delta_k v)\|_{L^1_T L^p} := I_{j,1} + I_{j,2} + I_{j,3},$$

where

$$J_{j,ll} = \{(k', k) \in J_j : 2^{k'} \le \Omega, 2^k \le \Omega\},$$
$$J_{j,lh} = \{(k', k) \in J_j : 2^{k'} \le \Omega, 2^k > \Omega\},$$
$$J_{j,hh} = \{(k', k) \in J_j : 2^{k'} > \Omega, 2^k > \Omega\}.$$

We get by using Lemma 2.3 that

$$I_{j,1} \le C \sum_{(k',k)\in J_{j,ll}} \|\Delta_{k'} u\|_{L^\infty_T L^\infty} 2^{k(\frac{3}{2}-\frac{3}{p})} \|\Delta_k v\|_{L^1_T L^2}$$
$$\le C \sum_{(k',k)\in J_{j,ll}} 2^{\frac{k'}{2}} \|\Delta_{k'} u\|_{L^\infty_T L^2} 2^{k'} \|\Delta_k v\|_{L^1_T L^2} 2^{k(\frac{3}{2}-\frac{3}{p})}$$
$$\le C \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2}, \frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}} \sum_{(k',k)\in J_{j,ll}} 2^{(k'-k)} 2^{-\frac{3}{p}k}$$
$$\le C 2^{-\frac{3}{p}j} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2}, \frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}},$$

where we used in the last inequality the fact that

$$\sum_{(k',k)\in J_{j,ll}} 2^{(k'-k)} 2^{-\frac{3}{p}k} \le \sum_{k'\le k-2} 2^{(k'-k)} \sum_{|k-j|\le 4} 2^{-\frac{3}{p}k} \le C 2^{-\frac{3}{p}j},$$

with C independent of j. Similarly, we have

$$I_{j,2} \le \sum_{(k',k)\in J_{j,lh}} \|\Delta_{k'} u\|_{L^\infty_T L^\infty} \|\Delta_k v\|_{L^1_T L^p}$$
$$\le C \sum_{(k',k)\in J_{j,lh}} 2^{\frac{k'}{2}} \|\Delta_{k'} u\|_{L^\infty_T L^2} 2^{k'} \|\Delta_k v\|_{L^1_T L^p}$$
$$\le C 2^{-\frac{3}{p}j} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2}, \frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2}, \frac{3}{p}+1}_{2,p}}$$

and

$$\begin{aligned}
I_{j,3} &\le \sum_{(k',k)\in J_{j,hh}} \|\Delta_{k'}u\|_{L^\infty_T L^\infty} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C \sum_{(k',k)\in J_{j,hh}} 2^{k'(\frac{3}{p}-1)} \|\Delta_{k'}u\|_{L^\infty_T L^p} 2^{k'} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C 2^{-\frac{3}{p}j} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}}.
\end{aligned}$$

On the other hand, for $2^j \le \Omega$, we have

$$\begin{aligned}
\|I_j\|_{L^1_T L^2} &\le \sum_{J_j} \|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_T L^2} \\
&\le \Big(\sum_{J_{j,ll}} + \sum_{J_{j,lh}} + \sum_{J_{j,hh}}\Big) \|\Delta_j(\Delta_{k'}u\Delta_k v)\|_{L^1_T L^2} := I_{j,4} + I_{j,5} + I_{j,6}.
\end{aligned}$$

We get by using Lemma 2.3 that

$$\begin{aligned}
I_{j,4} &\le C \sum_{(k,k')\in J_{j,ll}} 2^{\frac{k'}{2}} \|\Delta_{k'}u\|_{L^\infty_T L^2} 2^{k'} \|\Delta_k v\|_{L^1_T L^2} \\
&\le C 2^{-\frac{3j}{2}} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}},
\end{aligned}$$

and, noting that $p \le 4$,

$$\begin{aligned}
I_{j,5} &\le C \sum_{(k,k')\in J_{j,lh}} \|\Delta_{k'}u\|_{L^\infty_T L^{\frac{2p}{p-2}}} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C \sum_{(k,k')\in J_{j,lh}} 2^{\frac{k'}{2}} \|\Delta_{k'}u\|_{L^\infty_T L^2} 2^{k'(\frac{3}{p}-\frac{1}{2})} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C 2^{-\frac{3j}{2}} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}},
\end{aligned}$$

and

$$\begin{aligned}
I_{j,6} &\le C \sum_{(k,k')\in J_{j,hh}} \|\Delta_{k'}u\|_{L^\infty_T L^{\frac{2p}{p-2}}} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C \sum_{(k,k')\in J_{j,hh}} 2^{k'(\frac{3}{p}-1)} \|\Delta_{k'}u\|_{L^\infty_T L^p} 2^{k'(\frac{3}{p}-\frac{1}{2})} \|\Delta_k v\|_{L^1_T L^p} \\
&\le C 2^{-\frac{3j}{2}} \|u\|_{\widetilde{L}^\infty_T \dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \|v\|_{\widetilde{L}^1_T \dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}}.
\end{aligned}$$

Summing up the estimates for $I_{j,1}$ through $I_{j,6}$ yields that

$$\sup_{2^j>1} 2^{j\frac{3}{p}} \|I_j\|_{L^1_T L^p} + \sup_{2^j\le 1} 2^{\frac{3j}{2}} \|I_j\|_{L^1_T L^2} \le C\|u\|_{E_{p,T}} \|v\|_{E_{p,T}}. \tag{4-3}$$

By the same procedure as the one used to derive (4-3), we have

$$\sup_{2^j>1} 2^{j\frac{3}{p}}\|II_j\|_{L^1_TL^p} + \sup_{2^j\le 1} 2^{\frac{3j}{2}}\|II_j\|_{L^1_TL^2} \le C\|u\|_{E_{p,T}}\|v\|_{E_{p,T}}. \tag{4-4}$$

Set $K_j := \{(k,k') : k \ge j-3, |k'-k| \le 1\}$. Then we have

$$III_j = \Big(\sum_{K_{j,ll}} + \sum_{K_{j,lh}} + \sum_{K_{j,hl}} + \sum_{K_{j,hh}}\Big)\Delta_j(\Delta_k u\Delta_{k'}v) := III_{j,1}+III_{j,2}+III_{j,3}+III_{j,4},$$

where

$$K_{j,ll} = \{(k,k') \in K_j : 2^k \le \Omega, 2^{k'} \le \Omega\},$$
$$K_{j,lm} = \{(k,k') \in K_j : 2^k \le \Omega, 2^{k'} > \Omega\},$$
$$K_{j,hm} = \{(k,k') \in K_j : 2^k > \Omega, 2^{k'} \le \Omega\},$$
$$K_{j,hh} = \{(k,k') \in K_j : 2^k > \Omega, 2^{k'} > \Omega\}.$$

We get by Lemma 2.3 that

$$\begin{aligned}
\|III_{j,1}\|_{L^1_TL^p} &\le C2^{3j(1-\frac{1}{p})} \sum_{(k,k')\in K_{j,ll}} \|\Delta_k u\Delta_{k'}v\|_{L^1_TL^1}\\
&\le C2^{3j(1-\frac{1}{p})} \sum_{(k,k')\in K_{j,ll}} 2^{\frac{k}{2}}\|\Delta_k u\|_{L^\infty_TL^2}2^{-\frac{k}{2}}2^{k'\frac{5}{2}}\|\Delta_{k'}v\|_{L^1_TL^2}2^{-k'\frac{5}{2}}\\
&\le C2^{3j(1-\frac{1}{p})}\|u\|_{\widetilde{L}^\infty_T\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}}\|v\|_{\widetilde{L}^1_T\dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} \sum_{(k,k')\in K_{j,ll}} 2^{-\frac{k}{2}-\frac{5}{2}k'}\\
&\le C2^{-\frac{3}{p}j}\|u\|_{\widetilde{L}^\infty_T\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}}\|v\|_{\widetilde{L}^1_T\dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}} \sum_{k\ge j-3} 2^{-3(k-j)}\\
&\le C2^{-\frac{3}{p}j}\|u\|_{E_{p,T}}\|v\|_{E_{p,T}},
\end{aligned}$$

and

$$\|III_{j,1}\|_{L^1_TL^2} \le C2^{\frac{3j}{2}} \sum_{(k,k')\in K_{j,ll}} \|\Delta_k u\Delta_{k'}v\|_{L^1_TL^1} \le C2^{-\frac{3j}{2}}\|u\|_{E_{p,T}}\|v\|_{E_{p,T}}.$$

Similarly, we obtain

$$\begin{aligned}
&\|III_{j,2}+III_{j,3}\|_{L^1_TL^p}\\
&\quad\le C2^{\frac{3j}{2}} \sum_{(k,k')\in K_{j,lh}\cup K_{j,hl}} \|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{2p}{2+p}}}\\
&\quad\le C2^{\frac{3j}{2}}\Big(\sum_{K_{j,lh}} \|\Delta_k u\|_{L^\infty_TL^2}\|\Delta_{k'}v\|_{L^1_TL^p} + \sum_{K_{j,hl}} \|\Delta_k u\|_{L^1_TL^p}\|\Delta_{k'}v\|_{L^\infty_TL^2}\Big)\\
&\quad\le C2^{-\frac{3}{p}j}\|u\|_{E_{,T}}\|v\|_{E_{p,T}}
\end{aligned}$$

and

$$\|III_{j,2}+III_{j,3}\|_{L^1_TL^2}\le C2^{\frac{3}{p}j}\sum_{(k,k')\in K_{j,lh}\cup K_{j,hl}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{2p}{2+p}}}$$
$$\le C2^{-\frac{3j}{2}}\|u\|_{E_{p,T}}\|v\|_{E_{p,T}}.$$

Finally, due to $2\le p\le 4$, we have

$$\|III_{j,4}\|_{L^1_TL^p}\le C2^{\frac{3}{p}j}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{p}{2}}}$$
$$\le C2^{\frac{3}{p}j}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\|_{L^\infty_TL^p}\|\Delta_{k'}v\|_{L^1_TL^p}$$
$$\le C2^{-\frac{3}{p}j}\|u\|_{E_{p,T}}\|v\|_{E_{p,T}},$$

and

$$\|III_{j,4}\|_{L^1_TL^2}\le C2^{3j(\frac{2}{p}-\frac{1}{2})}\sum_{(k,k')\in K_{j,hh}}\|\Delta_k u\Delta_{k'}v\|_{L^1_TL^{\frac{p}{2}}}\le C2^{-\frac{3j}{2}}\|u\|_{E_{p,T}}\|v\|_{E_{p,T}}.$$

Summing up the estimates of $III_{j,1}$–$III_{j,4}$, we obtain

$$\sup_{2^j>1}2^{\frac{3}{p}j}\|III_j\|_{L^1_TL^p}+\sup_{2^j\le 1}2^{\frac{3j}{2}}\|III_j\|_{L^1_TL^2}\le C\|u\|_{E_{p,T}}\|v\|_{E_{p,T}}. \tag{4-5}$$

Then the inequality (4-2) can be deduced from (4-3)–(4-5). □

In order to prove the uniqueness of the solution in $C(\mathbb{R}^+;\dot H^{\frac{1}{2}})$, we establish the following new bilinear estimate in the weighted time-space Besov space introduced in [Chen et al. 2008; 2010b].

Proposition 4.2. *Assume that $u,v\in L^\infty_T(\dot B^{\frac{1}{2}}_{2,\infty})$. Then, for any $T>0$, we have*

$$\|B(u,v)\|_{L^\infty_T\dot B^{\frac{1}{2}}_{2,\infty}}\le C\|u\|_{L^\infty_T\dot B^{\frac{1}{2}}_{2,\infty}}\big\|\omega_{j,T}2^{\frac{j}{2}}\|\Delta_j v\|_{L^\infty_TL^2}\big\|_{l^\infty},$$

where

$$\omega_{j,T}:=\sup_{k\ge j}e_{k,T}2^{\frac{1}{2}(j-k)},\quad e_{j,T}:=1-e^{-c2^{2j}T}.$$

Remark 4.3. The inequality $e_{j,T}\le\omega_{j,T}$ (top of page 277) is important to the following estimates. On the other hand, due to the fact $\lim_{T\to 0}\omega_{j,T}=0$, it can be proved that if $u\in C([0,T];\dot H^{\frac{1}{2}})$, then, for any $\varepsilon>0$, one has

$$\big\|\omega_{j,T}2^{\frac{j}{2}}\|\Delta_j v\|_{L^\infty_TL^2}\big\|_{l^\infty}<\varepsilon\quad\text{if } T \text{ is small enough.}$$

This point is important in the proof of uniqueness.

Proof. First we note that $e_{j,T} \le \omega_{j,T}$ for any $j \in \mathbb{Z}$ and that

$$\omega_{j,T} \le 2^{\frac{1}{2}(j-j')}\omega_{j',T} \quad \text{if } j' \le j, \qquad \omega_{j,T} \le 2\omega_{j',T} \quad \text{if } j \le j'. \tag{4-6}$$

We get by Proposition 3.1 that

$$\begin{aligned} \|B(u,v)\|_{\dot B^{\frac{1}{2}}_{2,\infty}} &\le \sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \int_0^t \big\|\mathcal{G}(t-\tau)\Delta_j \mathbb{P}\nabla\cdot(u\otimes v)\big\|_{L^2}\,d\tau \\ &\le \sup_{j\in\mathbb{Z}} 2^{\frac{3j}{2}} \big\|e^{-c2^{2j}t}\big\|_{L^1_T}\|\Delta_j(u\otimes v)\|_{L^\infty_T L^2} \\ &\le C\sup_{j\in\mathbb{Z}} 2^{-\frac{j}{2}} e_{j,T}\|\Delta_j(uv)\|_{L^\infty_T L^2}. \end{aligned} \tag{4-7}$$

We use Bony's decomposition to estimate $\|\Delta_j(uv)\|_{L^\infty_T L^2}$. Since $e_{j,T} \le \omega_{j,T}$ and thanks to (4-6), we have

$$\begin{aligned} &\sum_{|k-j|\le 4} \|\Delta_j(S_{k-1}u\Delta_k v)\|_{L^\infty_T L^2} \\ &\qquad \le C\|u\|_{L^\infty_T \dot B^{\frac{1}{2}}_{2,\infty}} \sum_{|k-j|\le 4} 2^k \|\Delta_k v\|_{L^\infty_T L^2} \\ &\qquad \le C\omega_{j,T}^{-1} 2^{\frac{j}{2}} \|u\|_{L^\infty_T \dot B^{\frac{1}{2}}_{2,\infty}} \big\|\omega_{k,T} 2^{\frac{k}{2}} \|\Delta_k v\|_{L^\infty_T L^2}\big\|_{l^\infty}, \end{aligned} \tag{4-8}$$

and, again by the same properties of $\omega_{j,T}$,

$$\begin{aligned} \|S_{k-1}v\|_{L^\infty} &\le \sum_{k'\le k-2} \|\Delta_{k'}v\|_{L^2} 2^{\frac{3}{2}k'} \le \big\|\omega_{k',T} 2^{\frac{k'}{2}} \|\Delta_{k'}v\|_{L^\infty_T L^2}\big\|_{l^\infty} \sum_{k'\le k-2} 2^{k'}\omega_{k',T}^{-1} \\ &\le 2^k \omega_{k,T}^{-1} \big\|\omega_{k',T} 2^{\frac{k'}{2}} \|\Delta_{k'}v\|_{L^\infty_T L^2}\big\|_{l^\infty}, \end{aligned}$$

which implies that

$$\begin{aligned} &\sum_{|k-j|\le 4} \|\Delta_j(S_{k-1}v\Delta_k u)\|_{L^\infty_T L^2} \\ &\qquad \le 2^{\frac{k}{2}} \omega_{k,T}^{-1} \|u\|_{L^\infty_T \dot B^{\frac{1}{2}}_{2,\infty}} \big\|\omega_{k',T} 2^{\frac{k'}{2}} \|\Delta_{k'}v\|_{L^\infty_T L^2}\big\|_{l^\infty}, \end{aligned} \tag{4-9}$$

and for the remainder term,

$$\begin{aligned} &\sum_{k\ge j-2} \big\|\Delta_j(\Delta_k u\widetilde{\Delta}_k v)\big\|_{L^\infty_T L^2} \\ &\qquad \le \sum_{k\ge j-2} 2^{\frac{3}{2}j} \big\|\Delta_j(\Delta_k u\widetilde{\Delta}_k v)\big\|_{L^\infty_T L^1} \\ &\qquad \le C\sum_{k\ge j-2} 2^{\frac{3}{2}j} \|\Delta_k u\|_{L^\infty_T L^2} \big\|\widetilde{\Delta}_k v\big\|_{L^\infty_T L^2} \\ &\qquad \le C\omega_{j,T}^{-1} 2^{\frac{j}{2}} \|u\|_{L^\infty_T \dot B^{\frac{1}{2}}_{2,\infty}} \big\|\omega_{k,T} 2^{\frac{k}{2}} \|\Delta_k v\|_{L^\infty_T L^2}\big\|_{l^\infty}. \end{aligned} \tag{4-10}$$

Substituting (4-8)–(4-10) into (4-7) concludes the proof. □

5. Proofs of Theorem 1.4 and Theorem 1.8

The proof of Theorem 1.4 is based on the following classical lemma.

Lemma 5.1 [Cannone 1995]. *Let X be an abstract Banach space and $B : X\times X \to X$ a bilinear operator, $\|\cdot\|$ being the X-norm, such that for any $x_1 \in X$ and $x_2 \in X$, we have*

$$\|B(x_1, x_2)\| \le \eta\|x_1\|\|x_2\|.$$

Then for any $y \in X$ such that

$$4\eta\|y\| < 1,$$

the equation

$$x = y + B(x, x)$$

has a solution x in X. Moreover, this solution x is the only one such that

$$\|x\| \le \frac{1-\sqrt{1-4\eta\|y\|}}{2\eta}.$$

Proof of Theorem 1.4. Using the Stokes–Coriolis semigroup, we rewrite the system (1-1) as the integral form

$$u(x,t) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)\,d\tau := \mathcal{G}(t)u_0 + B(u,u). \tag{5-1}$$

Thanks to Proposition 3.2, we have

$$\|\mathcal{G}(t)u_0\|_{E_p} \le C\|u_0\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} \le Cc.$$

Obviously, $B(u, v)$ is bilinear, and we get by Proposition 4.1 that

$$\|B(u,v)\|_{E_p} \le C\|u\|_{E_p}\|v\|_{E_p}.$$

Taking c such that $4C^2c < \frac{3}{4}$, Lemma 5.1 ensures that the equation

$$u = \mathcal{G}(t)u_0 + B(u,u)$$

has a unique solution in the ball $\{u \in E_p : \|u\|_{E_p} \le \frac{1}{4C}\}$. □

Now we prove Theorem 1.8.

Proof of Theorem 1.8. We introduce a Banach space F_p whose norm is defined by

$$\|u\|_{F_p} := \|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})} + \|u\|_{E_p}.$$

Step 1: existence in F_p. We define the map

$$\mathcal{T}u := \mathcal{G}(t)u_0 + B(u,u).$$

Next we prove that, if c is small enough, the map $\mathcal{T}$ has a unique fixed point in the ball

$$B_A := \{u \in F_p : \|u\|_{E_p} \le Ac, \|u\|_{F_p} \le A\|u_0\|_{\dot{H}^{\frac{1}{2}}}\},$$

for some $A > 0$ to be determined later. From Proposition 3.2 and Proposition 4.1, we infer that

$$\|\mathcal{T}u\|_{E_p} \le C\|u_0\|_{\mathcal{B}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}} + C\|u\|^2_{E_p}. \tag{5-2}$$

On the other hand, we get by Proposition 3.1 that

$$\begin{aligned} \|B(u,u)\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})} &\le \left\| \int_0^t \mathcal{G}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\,d\tau \right\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})} \\ &\le C\left(\sum_{j\in\mathbb{Z}} 2^j \left(\sup_{t\in\mathbb{R}^+}\int_0^t \|\mathcal{G}(t-\tau)\Delta_j\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^2}\,d\tau\right)^2\right)^{\frac{1}{2}} \\ &\le C\left\| 2^{\frac{3}{2}j} \sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j}t}\|\Delta_j(u\otimes u)\|_{L^2}\,d\tau \right\|_{l^2}. \end{aligned} \tag{5-3}$$

In the following, we denote by $\{c_j\}_{j\in\mathbb{Z}}$ a sequence in l^2 with norm $\|\{c_j\}\|_{l^2(\mathbb{Z})} \le 1$. We get by Lemma 2.3 that

$$\begin{aligned} \sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j}t}\|\Delta_j(T_u u)\|_{L^2}\,d\tau &\le \|e^{-\tilde{c}2^{2j}t}\|_{L^1(\mathbb{R}^+)} \sum_{|k-j|\le 4} \|\Delta_j(S_{k-1}u\Delta_k u)\|_{L^\infty(\mathbb{R}^+;L^2)} \\ &\le C2^{-2j}\|S_{k-1}u\|_{L^\infty(\mathbb{R}^+;L^\infty)} \sum_{|k-j|\le 4} \|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)} \\ &\le C\|u\|_{\widetilde{L}^\infty\left(\mathbb{R}^+;\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}\right)} 2^k 2^{-2j} \sum_{|k-j|\le 4} \|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)} \\ &\le C2^{-\frac{3}{2}j}\|u\|_{E_p} \sum_{|k-j|\le 4} 2^{\frac{(k-j)}{2}} 2^{\frac{k}{2}} \|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)} \\ &\le C2^{-\frac{3}{2}j} c_j \|u\|_{E_p}\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})}. \end{aligned} \tag{5-4}$$

The remainder term of uv is estimated by

$$
\begin{aligned}
(5\text{-}5)\quad \sup_{t\in\mathbb{R}^+}\int_0^t e^{-\tilde{c}2^{2j}t}\|\Delta_j R(u,u)\|_{L^2}\,d\tau
&\le \|e^{-\tilde{c}2^{2j}t}\|_{L^\infty(\mathbb{R}^+)}\sum_{k\ge j-2}\|\Delta_j(\Delta_k u\tilde{\Delta}_k u)\|_{L^1(\mathbb{R}^+;L^2)}\\
&\le C\sum_{k\ge j-2}\|\tilde{\Delta}_k u\|_{L^1(\mathbb{R}^+;L^\infty)}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\\
&\le C\|u\|_{\tilde{L}^1\dot{\mathcal{B}}^{\frac{5}{2},\frac{3}{p}+1}_{2,p}}\sum_{k\ge j-2}2^{-k}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\\
&\le C\|u\|_{E_p}\sum_{k\ge j-2}2^{-\frac{3}{2}k}2^{\frac{1}{2}k}\|\Delta_k u\|_{L^\infty(\mathbb{R}^+;L^2)}\\
&\le C2^{-\frac{3}{2}j}c_j\|u\|_{E_p}\|u\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})}.
\end{aligned}
$$

Combining (5-4)–(5-5) with (5-3) yields that

$$\|B(u,u)\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})}\le C\|u\|_{E_p}\|u\|_{\tilde{L}^\infty(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})}.$$

It is easy to verify that

$$\|\mathcal{G}(t)u_0\|_{\tilde{L}^\infty_T\dot{H}^{\frac{1}{2}}}\le C\|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Consequently by (5-2) and the estimate

$$\|u_0\|_{\dot{\mathcal{B}}^{\frac{1}{2},\frac{3}{p}-1}_{2,p}}\le C\|u_0\|_{\dot{H}^{\frac{1}{2}}}$$

(which follows from Lemma 2.3 and the definition of the Besov space), we obtain

$$(5\text{-}6)\qquad \|\mathcal{T}u\|_{F_p}\le C\|u_0\|_{\dot{H}^{\frac{1}{2}}}+C\|u\|_{E_p}\|u\|_{F_p}.$$

Taking $A=2C$ and $c>0$ such that $2C^2c\le\frac{1}{2}$, it follows from (5-2) and (5-6) that the map $\mathcal{T}$ is a map from B_A to B_A. Similarly, it can be proved that $\mathcal{T}$ is also a contraction in B_A. Thus, the Banach fixed point theorem ensures that the map $\mathcal{T}$ has a unique fixed point in B_A.

Step 2: uniqueness in $C(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})$. Let u_1 and u_2 be two solutions of (1-1) in $C(\mathbb{R}^+;\dot{H}^{\frac{1}{2}})$ with the same initial data u_0. We consider

$$
\begin{aligned}
u_1-u_2=B\big(u_1-\mathcal{G}(t)u_0,u_1-u_2\big)+B\big(\mathcal{G}(t)u_0,u_1-u_2\big)&\\
+B\big(u_1-u_2,u_2-\mathcal{G}(t)u_0\big)+B\big(u_1-u_2,\mathcal{G}(t)u_0\big)&.
\end{aligned}
$$

Then we get by Proposition 4.2 that

$$\begin{aligned}(5\text{-}7)\quad \sup_{t\in[0,T]} \|(u_1-u_2)(t)\|_{\dot B^{\frac12}_{2,\infty}} &\le C \sup_{t\in[0,T]} \|(u_1-u_2)(t)\|_{\dot B^{\frac12}_{2,\infty}} \Big(\big\|\omega_{j,T} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \big\|_{l^\infty} \\ &\quad + \sup_{t\in[0,T]} \|u_1(t)-\mathcal{G}(t)u_0\|_{\dot H^{\frac12}} + \sup_{t\in[0,T]} \|u_2(t)-\mathcal{G}(t)u_0\|_{\dot H^{\frac12}} \Big),\end{aligned}$$

where we used the fact $\omega_{j,T} \le 1$ so that

$$\big\|\omega_{j,T} 2^{\frac{j}{2}} \|\Delta_j u\|_{L^\infty_T L^2} \big\|_{l^\infty} \le \sup_{t\in[0,T]} \|u(t)\|_{\dot H^{\frac12}}.$$

Noticing that $\omega_{j,0}=0$ and $u_0 \in \dot H^{\frac12}$, we have

$$\big\|\omega_{j,T} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \big\|_{l^\infty} \le \frac{1}{3C},$$

for T small enough. On the other hand, since $u_1, u_2 \in C(\mathbb{R}^+; \dot H^{\frac12})$, we also have

$$\sup_{t\in[0,T]} \|u_1-\mathcal{G}(t)u_0\|_{\dot H^{\frac12}} + \sup_{t\in[0,T]} \|u_2-\mathcal{G}(t)u_0\|_{\dot H^{\frac12}} \le \frac{1}{3C},$$

for T small enough. Then (5-7) ensures that $u_1(t)=u_2(t)$ for T small enough. Then, by a standard continuity argument, we conclude that $u_1=u_2$ on $[0,\infty)$. □

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