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CLASSIFYING ZEROS OF TWO-SIDED QUATERNIONIC POLYNOMIALS AND COMPUTING ZEROS OF TWO-SIDED POLYNOMIALS WITH COMPLEX COEFFICIENTS

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We improve the method of Janovská and Opfer for computing the zeros on the surface of a given sphere for a quaternionic two-sided polynomial. We classify the zeros of quaternionic two-sided polynomials into three types — isolated, spherical and circular — and characterize each type. We provide a method to find all quaternion zeros for two-sided polynomials with complex coefficients. We also establish standard formulae for roots of a quadratic two-sided polynomial with complex coefficients, which yields a simpler and more efficient algorithm to produce all zeros in the quadratic case.

1. Introduction

In this paper we will treat *two-sided* quaternionic polynomials, those of the form

$$p(x) := \sum_{j=0}^{n} a_j x^j b_j, \quad x, a_j, b_j \in \mathbb{H}, \quad a_n b_n \neq 0, \tag{1}$$

where $\mathbb{H}$ is the skew field of quaternions. These polynomials include also all *one-sided* polynomials, where all coefficients are located on the left side or the right side of the powers. For a long time, it has been known that one-sided quaternionic polynomials may have two classes of zeros: isolated zeros and spherical zeros (see for instance [Pogorui and Shapiro 2004; Topuridze 2003]), while a method to compute all zeros of such polynomials was developed in [Janovská and Opfer 2010b] and a more efficient means was found in [Feng and Zhao 2011].

A general quaternionic polynomial is a finite sum of terms of the form

$$t_j(x) := a_{0j} \cdot x \cdot a_{1j} \cdots a_{j-1,j} \cdot x \cdot a_{jj}, \quad x, a_{0j}, a_{1j}, \ldots, a_{jj} \in \mathbb{H}, \ j \geq 0. \tag{2}$$

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Such a term is called a *monomial of degree* j. The polynomial $p(z)$ in (1) is only a very special type of a general quaternionic polynomial. There are relatively few results on two-sided quaternionic polynomials; we list some that are relevant to our study.

In [De Leo et al. 2006], the authors gave an example of a two-sided polynomial and Opfer [2009] obtained that a general quaternionic polynomial of degree n has at least one zero provided the polynomial has only one monomial of degree n. More recently, for a quaternionic two-sided polynomial of type (1), Janovská and Opfer [2010a] showed that there may be five classes of zeros according to the five possible ranks of a certain real (4×4) matrix, and they provided a method to find the zeros in a given equivalence class.

This paper is organized as follows. In Section 2, by improving the method of [Janovská and Opfer 2010a], we classify the zeros of quaternionic two-sided polynomials into three types — isolated zeros, spherical zeros and circular zeros — and characterize each type of zero. In Section 3, we provide a method to compute all quaternion zeros of a two-sided polynomial with complex coefficients. In Section 4, for a quadratic two-sided polynomial with complex coefficients, we further establish the standard formulae for roots, so that a simpler and more efficient algorithm is given to produce all zeros for a quadratic two-sided polynomial with complex coefficients.

We will now give a short introduction to the quaternionic algebra. By $\mathbb{R}$, $\mathbb{C}$ we denote the fields of real and complex numbers, respectively, and by $\mathbb{N}$ the set of natural numbers. In the skew field $\mathbb{H}$ of quaternions, any element has the form

$$q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = (a_0 + a_1\boldsymbol{i}) + (a_2 + a_3\boldsymbol{i})\boldsymbol{j}, \tag{3}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ satisfy

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j};$$

the product is extended to $\mathbb{H}$ by $\mathbb{R}$-bilinearity. We call a_0 the *real part* of the quaternion q in (3), also written $\Re q$, while $q - \Re q = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is called the *imaginary part* and denoted by $\Im q$. The *modulus* $|q|$ of q is

$$|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

The *conjugate* of q, denoted by $\bar{q}$, is defined by $\bar{q} = a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} - a_3\boldsymbol{k}$.

Two quaternions q_1, q_2 are called *equivalent*, denoted by $q_1 \sim q_2$, if there is an $h \in \mathbb{H}\backslash\{0\}$ such that $q_1 = hq_2h^{-1}$. The set $[q] = \{hqh^{-1} : h \in \mathbb{H}\backslash\{0\}\}$ will be called the *equivalence class* of q or, for short, the *class* of q. Indeed "$\sim$" defines an equivalence relation on $\mathbb{H}$. So each quaternion is located in one and only one equivalence class. It is well known that

$$q_1 \sim q_2 \iff \Re q_1 = \Re q_2 \text{ and } |q_1| = |q_2|,$$

that is, $[q] = \{u \in \mathbb{H} : \Re u = \Re q,\ |u| = |q|\}$, which can be regarded as the surface of a ball in $\mathbb{R}^3 = \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$ if q is not real. It is easy to see that $[q] = \{q\}$ if q is real and $[q]$ contains infinitely many elements if q is not real. In the case that q is not real, the only two complex numbers contained in $[q]$ are ξ and $\bar{\xi}$, where $\xi = \Re q + \sqrt{|q|^2 - (\Re q)^2}\,\boldsymbol{i}$. Here we are calling a quaternion *complex* if it is of the form $a_0 + a_i\boldsymbol{i}$, with $a_0, a_i \in \mathbb{R}$.

There is a very useful tool to study the quaternion algebra, which is the so-called *derived matrix* (appeared in [Feng 2010])

The *derived mapping* $\sigma : \mathbb{H}^{n\times n} \to \mathbb{C}^{2n\times 2n}$ from the set of $n \times n$ quaternionic matrices into the set of $2n \times 2n$ complex matrices is defined by

$$A = A_1 + A_2\,\boldsymbol{j} \ \mapsto\ \sigma(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{pmatrix}, \tag{4}$$

where $A_1, A_2 \in \mathbb{C}^{n\times n}$. This mapping is injective, and obviously preserves addition and multiplication of matrices. We call $\sigma(A)$ the *derived matrix* of A (or, following [Zhang 1997], the *complex adjoint matrix* of A). We will be interested in the case where $n = 1$.

To conclude this introduction, we mention that it was Niven [1941; 1942] who made first steps in generalizing the fundamental theorem of algebra to the quaternionic situation. Since then many attempts have been made to compute roots of a quaternionic polynomial [Serôdio et al. 2001; Serôdio and Siu 2001; Pumplün and Walcher 2002; De Leo et al. 2006; Gentili and Stoppato 2008; Gentili and Struppa 2008; Gentili et al. 2008], most of which have focused on one-sided polynomials. In [Lam 2001, Section 16] and [Wang et al. 2009b] there are several general results on polynomials (of the one-sided type) over division rings. There are also a lot of recent studies on quaternionic matrices, for example [Farid et al. 2011; Wang et al. 2009a]. A large bibliography on quaternions can be found in [Gsponer and Hurni 2008].

2. Classifying zeros of two-sided quaternionic polynomials

To investigate the zeros of the polynomial in (1), we can assume (by left and right division, respectively) that $a_n = 1$ and $b_n = 1$, that is, the polynomial is monic:

$$p(x) := x^n + a_{n-1}x^{n-1}b_{n-1} + \cdots + a_1 x b_1 + a_0, \quad x, a_i, b_j \in \mathbb{H}. \tag{5}$$

Let ξ be a fixed complex number. In this section, we shall give a classification of the zeros of $p(x)$, which improves the results in [Janovská and Opfer 2010a].

Furthermore, we give a clear description of the structure of each class of zeros for a polynomial $p(x)$ whose coefficients are complex.

From [Pogorui and Shapiro 2004], we know that all powers x^k, $k \in \mathbb{N}$, of a quaternion x have the form $x^k = \alpha_k x + \beta_k$, where α_k, β_k are real numbers. In order to determine the numbers α_k, β_k, Janovská and Opfer [2010b] gave two approaches. One is via the iteration

$$\begin{cases} \alpha_0 = 0, \quad \beta_0 = 1, \\ \alpha_{j+1} = 2\Re x\, \alpha_j + \beta_j, \\ \beta_{j+1} = -|x|^2 \alpha_j, \;\; j = 0, 1, \dots. \end{cases} \tag{6}$$

The other one relies on the formula

$$\begin{cases} \alpha_j = \Im(u_1^j)/\sqrt{|x|^2 - (\Re x)^2}, \\ \beta_0 = 1, \;\; \beta_{j+1} = -|x|^2 \alpha_j, \;\; j = 0, 1, \dots, \end{cases} \tag{7}$$

where u_1 is the complex solution of $u^2 - 2(\Re x)u + |x|^2 = 0$ with positive imaginary part. Formula (7) for α_j is of course easier to program than the iteration (6). However, since a power is involved, an economic use of (7) would also require an iteration.

For convenience of later use, we will first give a self-closed formula for α_k and β_k to improve the above formulas, that is, we give the following lemma, by which we can determine the real numbers α_k, β_k directly.

Lemma 2.1. *Suppose z is a quaternion, k is a natural number. Let*

$$\xi = \Re z + \sqrt{|z|^2 - (\Re z)^2}\, \boldsymbol{i}.$$

Then $z^k = \alpha_k z + \beta_k$, where

$$\alpha_k = \frac{\xi^k - \bar{\xi}^k}{\xi - \bar{\xi}} \in \mathbb{R}, \qquad \beta_k = |\xi|^2 \cdot \frac{\bar{\xi}^{k-1} - \xi^{k-1}}{\xi - \bar{\xi}} \in \mathbb{R}.$$

Remark 2.2. In this lemma, we set

$$\frac{\xi^k - \bar{\xi}^k}{\xi - \bar{\xi}} = 1 \quad \text{and} \quad \frac{\bar{\xi}^{k-1} - \xi^{k-1}}{\xi - \bar{\xi}} = 0$$

for $k = 1$, while

$$\frac{\xi^k - \bar{\xi}^k}{\xi - \bar{\xi}} = \xi^{k-1} + \xi^{k-2}\bar{\xi} + \cdots + \bar{\xi}^{k-1}, \quad \frac{\bar{\xi}^{k-1} - \xi^{k-1}}{\xi - \bar{\xi}} = -(\xi^{k-2} + \xi^{k-3}\bar{\xi} + \cdots + \bar{\xi}^{k-2})$$

for $k > 1$ if ξ is real. Actually ξ is a complex number contained in $[z]$.

Proof. Since $\Re z = \Re \xi$ and $|z| = |\xi|$, we see that $z \in [\xi]$. Let

$$g(t) = t^2 - (\xi + \bar{\xi})t + |\xi|^2.$$

Then $g(t)$ is a polynomial with real coefficients, that annihilates each element of $[\xi]$. Note that the polynomial t^k can be expressed as $t^k = h(t)g(t) + \alpha_k t + \beta_k$, where α_k and β_k are real constants, $h(t) \in \mathbb{R}[t]$. Consequently, we have

$$\begin{cases} \alpha_k \xi + \beta_k = \xi^k, \\ \alpha_k \overline{\xi} + \beta_k = \overline{\xi}^k. \end{cases} \tag{8}$$

If $\xi - \overline{\xi} = 0$, then ξ is a real number and $z = \xi$. A straightforward verification shows the statement of the lemma for this case. Now suppose $\xi - \overline{\xi} \neq 0$. By (8),

$$\alpha_k = \frac{\xi^k - \overline{\xi}^k}{\xi - \overline{\xi}}, \qquad \beta_k = \frac{\xi\overline{\xi}^k - \overline{\xi}\xi^k}{\xi - \overline{\xi}} = |\xi|^2 \cdot \frac{\overline{\xi}^{k-1} - \xi^{k-1}}{\xi - \overline{\xi}}.$$

Since $q^k = h(q)g(q) + \alpha_k q + \beta_k = \alpha_k q + \beta_k$ for all $q \in [\xi]$, the proof is complete. □

With Lemma 2.1 in hand, we now introduce the method to find all zeros in the sphere $[\xi]$ for $p(x)$ where ξ is a fixed complex (so $\Re\xi$ is fixed).

Now for the fixed complex number ξ, and for any $z \in [\xi]$, $p(z)$ can be represented by

$$\begin{aligned} p(z) &= (\alpha_n z + \beta_n) + a_{n-1}(\alpha_{n-1} z + \beta_{n-1}) b_{n-1} + \cdots + a_1(\alpha_1 z + \beta_1) b_1 + a_0 \\ &= (\alpha_n z + a_{n-1}\alpha_{n-1} z b_{n-1} + \cdots + a_1 \alpha_1 z b_1) + (\beta_n + \cdots + a_1 \beta_1 b_1 + a_0) \\ &= A(z) + B, \end{aligned}$$

where

$$\begin{aligned} A(z) &= \alpha_n z + \alpha_{n-1} a_{n-1} z b_{n-1} + \cdots + \alpha_1 a_1 z b_1, \\ B &= \beta_n + \beta_{n-1} a_{n-1} b_{n-1} + \cdots + a_1 \beta_1 b_1 + a_0 \in \mathbb{H}. \end{aligned}$$

It is clear that the coefficients α_j, β_j $(j = 1, \ldots, n)$ are given in Lemma 2.1. So, solving the equation $p(z) = 0$ in $[\xi]$ is equivalent to finding the solutions in the sphere surface $[\xi]$ of the following equation:

$$A(z) = -B. \tag{9}$$

Let $z = \Re\xi + x_1 \boldsymbol{i} + x_2 \boldsymbol{j} + x_3 \boldsymbol{k}$, $x_1, x_2, x_3 \in \mathbb{R}$. Regard z as the vector

$$\begin{pmatrix} \Re\xi \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and regard the surface of the sphere $[\xi]$ as

$$\Sigma = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1^2 + x_2^2 + x_3^2 = |\xi|^2 - (\Re\xi)^2,\ x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Unfolding the left side of (9) leads to the following linear system consisting of four equations in three variables,

$$M\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}e_0\\e_1\\e_2\\e_3\end{pmatrix},\tag{10}$$

where M is a known real 4×3 matrix, e_0, e_1, e_2, and e_3 are known real numbers. Suppose S is the solution set of the linear system (10). Then the set of zeros of $p(x)$ contained in $[\xi]$ is

$$\left\{\begin{pmatrix}\Re\xi\\s\end{pmatrix}:s\in S\cap\Sigma\right\}.$$

If (10) has no solution, that is, $S=\varnothing$, then $[\xi]$ contains no zero of $p(x)$.

When (10) has a solution, its solution set can be represented by $\mathcal{N}+X_0$, where $\mathcal{N}$ is the solution space of the system of homogeneous linear equations $MX=0$, while X_0 is a particular solution of (10). Now we analyze the set $\mathcal{N}+X_0$ as follows.

If $\dim\mathcal{N}=0$, then (10) has only one solution X_0, so $[\xi]$ contains at most one zero of $p(x)$.

If $\dim\mathcal{N}=1$, then S becomes a straight line in the three-dimensional $\{x_1,x_2,x_3\}$-space. So $[\xi]$ contains no zero of $p(x)$ when S is separated from the sphere $[\xi]$, $[\xi]$ contains only one zero when S is tangent to the sphere $[\xi]$, and $[\xi]$ contains two zeros if the straight line S pierces the sphere $[\xi]$.

If $\dim\mathcal{N}=2$, S is a plane in the three-dimensional $\{x_1,x_2,x_3\}$-space, there are three possible position relationships between the plane and the sphere: separated, tangent and intersected. Then $[\xi]$ contains no zero of $p(x)$ for the separated situation, contains only one zeros for the tangent situation. With respect to the intersected situation, the intersection of the plane and the sphere is a circular curve, so the zeros of $p(x)$ contained in $[\xi]$ form a circle in the three-dimensional $\{x_1,x_2,x_3\}$-space.

Finally, if $\dim\mathcal{N}=3$, then $S=\mathbb{R}^3$, and each point in $[\xi]$ is a zero of $p(x)$.

To sum up the above arguments, we have obtained:

Theorem 2.3. *Let $p(x)$ be as in* (5), *and let ξ be a complex number. If $Z_{[\xi]}(p)$ is the set of zeros of $p(x)$ contained in $[\xi]$ and $|Z_{[\xi]}(p)|$ is its cardinality, we have the following possibilities*:

- $|Z_{[\xi]}(p)|\le 2$.
- $Z_{[\xi]}(p)$ *is a circle on the surface of the sphere* $[\xi]$.
- $Z_{[\xi]}(p)=[\xi]$.

Definition 2.4. Let $p(x)$ be as in (5), and let z_0 be a zero of $p(x)$. If z_0 is not real and $Z_{[z_0]}(p)=[z_0]$, we say that z_0 *generates a spherical zero*, or simply that it is a spherical zero. If z_0 is real or $|Z_{[z_0]}(p)|\le 2$, it is called an isolated zero. If z_0 is not real and has the property that $Z_{[z_0]}(p)$ is a circle on the sphere $[z_0]$, we say that z_0 *generates a circular zero*, or is a circular zero.

Thus Theorem 2.3 classifies the zeros of quaternionic two-sided polynomials into three types: isolated zeros, spherical zeros and circular zeros.

Now we apply Theorem 2.3 to two-sided polynomials with complex coefficients.

Theorem 2.5. *Let* $p(x) := x^n + a_{n-1}x^{n-1}b_{n-1} + \cdots + a_1xb_1 + a_0$, *where all the* a_i, b_i $(i = 0, 1, \ldots, n-1)$ *are complex numbers. Let* ξ *be a complex number with* $Z_{[\xi]}(p) \neq \varnothing$. *We have the following possibilities*:

- $Z_{[\xi]}(p) \subseteq \{\xi, \bar{\xi}\}$.
- $Z_{[\xi]}(p) = \{z_1 + z_2\boldsymbol{j} : z_1 \in \mathbb{C}$ *fixed*, $z_2 \in \mathbb{C},\ |z_2|^2 = |\xi|^2 - |z_1|^2 > 0\}$.
- $Z_{[\xi]}(p) = [\xi]$.

Proof. If all a_i, b_i are complex, in (9) we set

$$\begin{aligned} &p_n = \alpha_n, \quad p_{n-1} = \alpha_{n-1}a_{n-1}, \quad \ldots, \quad p_1 = \alpha_1 a_1, \\ &q_n = 1, \quad q_{n-1} = b_{n-1}, \quad \ldots, \quad q_1 = b_1, \quad q_0 = -B. \end{aligned}$$

Then all p_i, q_i are known complex numbers. Writing the point z in $[\xi]$ as

$$z = z_1 + z_2\boldsymbol{j}, \quad z_1, z_2 \in \mathbb{C},$$

and using the derived mapping, we can write (9) as

$$\begin{pmatrix} p_n & \\ & \bar{p}_n \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} q_n & \\ & \bar{q}_n \end{pmatrix} + \cdots + \begin{pmatrix} p_1 & \\ & \bar{p}_1 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} q_1 & \\ & \bar{q}_1 \end{pmatrix} = \begin{pmatrix} q_0 & \\ & \bar{q}_0 \end{pmatrix},$$

which is equivalent to

$$\text{(11)} \qquad \Big(\sum_{i=1}^n p_i q_i\Big) z_1 = q_0, \quad \Big(\sum_{i=1}^n p_i \bar{q}_i\Big) z_2 = 0.$$

Since $Z_{[\xi]}(p) \neq \varnothing$, (11) is consistent.

If $\sum_{i=1}^n p_i\bar{q}_i = 0$ and $\sum_{i=1}^n p_i q_i \neq 0$, then

$$z_1 = \frac{q_0}{\sum_{i=1}^n p_i q_i}, \quad \text{and} \quad |z_2|^2 = |\xi|^2 - \left|\frac{q_0}{\sum_{i=1}^n p_i q_i}\right|^2.$$

When

$$|\xi|^2 - \left|\frac{q_0}{\sum_{i=1}^n p_i q_i}\right|^2 > 0,$$

the zeros of $p(x)$ contained in $[\xi]$ are circular zeros, and

$$Z_{[\xi]}(p) = \left\{\frac{q_0}{\sum_{i=1}^n p_i q_i} + z_2\boldsymbol{j} : z_2 \in \mathbb{C}, |z_2|^2 = |\xi|^2 - \left|\frac{q_0}{\sum_{i=1}^n p_i q_i}\right|^2\right\}.$$

Otherwise,

$$Z_{[\xi]}(p)=\left\{\frac{q_0}{\sum_{i=1}^{n} p_i q_i}\right\}\subseteq\{\xi,\bar{\xi}\}.$$

If $\sum_{i=1}^{n} p_i\bar{q}_i=0$ and $\sum_{i=1}^{n} p_i q_i=0$, then $q_0=0$ and each point in $[\xi]$ is a zero of $p(x)$. So it is a spherical zero, that is, $Z_{[\xi]}(p)=[\xi]$.

Finally, if $\sum_{i=1}^{n} p_i\bar{q}_i\neq 0$, then $z_2=0$ and $z=z_1+z_2\boldsymbol{j}=z_1$, which has at most two values in $[\xi]$: ξ and $\bar{\xi}$. □

Janovská and Opfer [2010a] classified the zeros of quaternionic two-sided polynomials $p(x)$ into five classes according to the five possible ranks of a real (4×4) matrix obtained from the coefficients of $p(x)$. In their notation, type 0 and type 1 solutions are isolated solutions, a type 2 solution can be an isolated solution or a circular solution, a type 3 solution is a circular solution or a spherical solution, while a type 4 solution is a spherical solution.

We can understand Theorem 2.3 from the view of point of geometry as follows. The set of isolated zeros in $[\xi]$ is of dimension 0, the set of circular zeros in $[\xi]$ is of dimension 1 because they form a circular line, and the set of spherical zeros in $[\xi]$ is of dimension 2 because these zeros form a surface of a ball.

Remark 2.6. (a) Since one-sided polynomials, as in (5), belong to the class we are considering, isolated zeros and spherical zeros in fact occur (actually these two types are the only solutions; see [Feng and Zhao 2011; Janovská and Opfer 2010b]). From the study of the quadratic case in Section 4 of this paper, we shall see that the polynomial $p(x)=x^2+\boldsymbol{i}x\boldsymbol{i}+2$ has circular zeros and two conjugate isolated zeros.

(b) From Theorem 2.5 we see that, for a two-sided polynomial $p(x)$ with complex coefficients, an isolated zero (if exists) of $p(x)$ should be a complex number, and the equivalence class $[z]$ for an arbitrary circular zero z (if it exists) should contain no complex roots of $p(x)$. These facts will be used in the sequel.

3. Finding all zeros of quaternionic two-sided polynomials with complex coefficients

Consider a quaternionic two-sided polynomial with complex coefficients:

$$p(x):=x^n+a_{n-1}x^{n-1}b_{n-1}+\cdots+a_1xb_1+a_0,\quad x\in\mathbb{H},\ a_i,b_j\in\mathbb{C}. \tag{12}$$

We will find a method to compute all the zeros of $p(x)$. We introduce the notation

$$\begin{aligned}\widetilde{p}(x)&:=x^n+a_{n-1}b_{n-1}x^{n-1}+\cdots+a_1b_1x+a_0,\\ \bar{\widetilde{p}}(x)&:=x^n+\bar{a}_{n-1}\bar{b}_{n-1}x^{n-1}+\cdots+\bar{a}_1\bar{b}_1x+\bar{a}_0,\\ \overleftarrow{p}(x)&:=x^n+a_{n-1}\bar{b}_{n-1}x^{n-1}+\cdots+a_1\bar{b}_1x+a_0.\end{aligned}$$

Theorem 3.1 (characterization of spherical zeros). *Let ξ be a complex number. Then each point of $[\xi]$ is a zero of $p(x)$ if and only if*

$$\widetilde{p}(\xi) = \widetilde{p}(\bar{\xi}) = 0 \quad \textit{and} \quad \overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi}).$$

Proof. For any $z \in [\xi]$, we can write z as $z = q\xi q^{-1}$ for some $q = z_1 + z_2\boldsymbol{j}$ with $z_1, z_2 \in \mathbb{C}$, and $|z_1|^2 + |z_2|^2 = 1$. Then $p(z) = 0$ is equivalent to

$$q\xi^n q^{-1} + a_{n-1}q\xi^{n-1}q^{-1}b_{n-1} + \cdots + a_1 q\xi q^{-1}b_1 + a_0 = 0.$$

By the derived mapping, we get

$$\begin{aligned}&\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} \xi^n & \\ & \bar{\xi}^n \end{pmatrix}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix} \\ &\quad + \begin{pmatrix} a_{n-1} & \\ & \bar{a}_{n-1} \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} \xi^{n-1} & \\ & \bar{\xi}^{n-1} \end{pmatrix}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix}\begin{pmatrix} b_{n-1} & \\ & \bar{b}_{n-1} \end{pmatrix} + \cdots \\ &\qquad + \begin{pmatrix} a_1 & \\ & \bar{a}_1 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} \xi & \\ & \bar{\xi} \end{pmatrix}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix}\begin{pmatrix} b_1 & \\ & \bar{b}_1 \end{pmatrix} + \begin{pmatrix} a_0 & \\ & \bar{a}_0 \end{pmatrix} = 0,\end{aligned}$$

which is equivalent to the system of the following two equations:

$$|z_1|^2(\widetilde{p}(\xi) - a_0) + |z_2|^2(\widetilde{p}(\bar{\xi}) - a_0) + a_0 = 0, \tag{13}$$

$$z_1 z_2(\overleftarrow{p}(\bar{\xi}) - \overleftarrow{p}(\xi)) = 0. \tag{14}$$

The above argument will be also used in later proofs.

$\Rightarrow$) Suppose each point of $[\xi]$ is a zero of $p(x)$. Then (13) and (14) hold for any $z_1, z_2 \in \mathbb{C}$ with $|z_1|^2 + |z_2^2| = 1$. Note that (13) can also be written as

$$|z_1|^2(\widetilde{p}(\xi) - \widetilde{p}(\bar{\xi})) + \widetilde{p}(\bar{\xi}) = 0. \tag{15}$$

The equalities (15) and (14) hold for arbitrary complex z_1, z_2 with $|z_1|^2 + |z_2|^2 = 1$, yielding that $\widetilde{p}(\xi) - \widetilde{p}(\bar{\xi}) = 0$, $\widetilde{p}(\bar{\xi}) = 0$, and $\overleftarrow{p}(\bar{\xi}) - \overleftarrow{p}(\xi) = 0$. The rest follow easily for this direction.

$\Leftarrow$) Obvious. □

Theorem 3.2 (characterization of isolated zeros). *Let T be the set of nonreal, isolated zeros of $p(x)$. Then*

$$T = \{\xi \in \mathbb{C} : \widetilde{p}(\xi) = 0,\ \overleftarrow{p}(\xi) \neq \overleftarrow{p}(\bar{\xi})\} \cup \{\xi \in \mathbb{C} : \widetilde{p}(\xi) = 0,\ \widetilde{p}(\bar{\xi}) \neq 0\};$$

the set of all isolated zeros of $p(x)$ is $\{$the real roots of $\widetilde{p}(x)\} \cup T$.

Proof. By Remark 2.6, we see that the set of isolated zeros of $p(x)$ is contained in $\{\xi \in \mathbb{C} : \widetilde{p}(\xi) = 0\}$. Let ξ be a nonreal complex root of $\widetilde{p}$. If $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$ and

$\widetilde{p}(\bar{\xi}) = 0$, then ξ becomes a spherical zero from Theorem 3.1. Hence,

$$T = \{\xi \in \mathbb{C} : \widetilde{p}(\xi) = 0,\ \overleftarrow{p}(\xi) \neq \overleftarrow{p}(\bar{\xi})\} \cup \{\xi \in \mathbb{C} : \widetilde{p}(\xi) = 0,\ \widetilde{p}(\bar{\xi}) \neq 0\},$$

and the set of all isolated zeros of $p(x)$ is {the real roots of $\widetilde{p}$} $\cup\, T$. □

Theorem 3.3 (characterization of circular zeros). *Let ξ be a given complex number. Then $[\xi]$ contains a circular zero of $p(x)$ if and only if*

$$\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0 \quad \text{and} \quad \overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi}). \tag{16}$$

Moreover, if $[\xi]$ contains a circular zero of $p(x)$, then

$$(\Im\xi)^2\left(1 - \frac{\widetilde{p}(\bar{\xi}) + \widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi}) - \widetilde{p}(\xi)}\right)^2 > 0$$

and the set of circular zeros in $[\xi]$ is

$$\left\{\Re\xi + \frac{\widetilde{p}(\bar{\xi}) + \widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi}) - \widetilde{p}(\xi)}\Im\xi\,\boldsymbol{i} + z_2\boldsymbol{j} : z_2 \in \mathbb{C},\ |z_2|^2 = (\Im\xi)^2\left(1 - \frac{\widetilde{p}(\bar{\xi}) + \widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi}) - \widetilde{p}(\xi)}\right)^2\right\}.$$

Proof. $\Rightarrow$) Suppose $[\xi]$ contains a circular zero of $p(x)$. By Theorem 2.5, the circular zeros in $[\xi]$ contain no complex zeros of p. Using the first part of the proof in Theorem 3.1, we know that there exist $z_1, z_2 \in \mathbb{C}$ with $z_1z_2 \neq 0$ and $|z_1|^2 + |z_2|^2 = 1$, such that $z = (z_1 + z_2\boldsymbol{j})\xi(z_1 + z_2\boldsymbol{j})^{-1}$ is a zero of $p(x)$. So (13) and (14) hold for this z, which yield $|z_1|^2\widetilde{p}(\xi) + |z_2|^2\widetilde{p}(\bar{\xi}) = 0$ and $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$. From $|z_1|^2\widetilde{p}(\xi) + |z_2|^2\widetilde{p}(\bar{\xi}) = 0$, we have

$$|z_1|^2\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) = -|z_2|^2\widetilde{p}(\bar{\xi})\overline{\widetilde{p}}(\xi) = -|z_2|^2|\overline{\widetilde{p}}(\xi)|^2 < 0,$$

that is $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0$, as desired.

$\Leftarrow$) Note that $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0$ implies $\Im\xi \neq 0$ and $|\overline{\widetilde{p}}(\xi)|^2 - \widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) > 0$. Let $z_1, z_2 \in \mathbb{C}$ be given by

$$|z_1|^2 = \frac{|\overline{\widetilde{p}}(\xi)|^2}{|\overline{\widetilde{p}}(\xi)|^2 - \widetilde{p}(\xi)\overline{\widetilde{p}}(\xi)}, \quad |z_2|^2 = 1 - |z_1|^2. \tag{17}$$

Then $z_1z_2 \neq 0$, $|z_1|^2 + |z_2|^2 = 1$ and it is easy to verify that (13) and (14) hold simultaneously. So $(z_1 + z_2\boldsymbol{j})\xi(z_1 + z_2\boldsymbol{j})^{-1}$ is a zero of $p(x)$. From

$$\begin{aligned}(z_1 + z_2\boldsymbol{j})\xi(z_1 + z_2\boldsymbol{j})^{-1} &= |z_1|^2\xi + |z_2|^2\bar{\xi} - 2z_1z_2\Im\xi\,\boldsymbol{j}\\ &= \Re\xi + (2|z_1|^2 - 1)(\Im\xi)\boldsymbol{i} - 2z_1z_2\Im\xi\,\boldsymbol{j},\end{aligned} \tag{18}$$

we see that $[\xi]$ contains a noncomplex zero of $p(x)$. The inequality $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0$ also implies $\widetilde{p}(\xi) \neq 0$, so from Theorem 3.1 we know $[\xi]$ contains no spherical zeros of $p(x)$. Now combining with Theorem 2.5 we see $[\xi]$ contains a circular zero of $p(x)$. The proof of this direction is completed.

Finally, if $[\xi]$ contains a circular zero, let z_1 and z_2 be defined by (17). Then

$$(2|z_1|^2-1)\Im\xi = \frac{\widetilde{p}(\bar{\xi})+\widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi)}\Im\xi,$$

$$\frac{\widetilde{p}(\bar{\xi})+\widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi)}-1 = \frac{2\widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi)} = \frac{2\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi)}{(\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi))\overline{\widetilde{p}}(\xi)} < 0.$$

Therefore,

$$r := (\Im\xi)^2\left(1-\frac{\widetilde{p}(\bar{\xi})+\widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi)}\right)^2 > 0$$

and by Theorem 2.5 and (18), the set of circular zeros in $[\xi]$ is

$$\left\{\Re\xi+\frac{\widetilde{p}(\bar{\xi})+\widetilde{p}(\xi)}{\widetilde{p}(\bar{\xi})-\widetilde{p}(\xi)}\Im\xi\boldsymbol{i}+z\boldsymbol{j} : z\in\mathbb{C},\ |z|^2=r\right\}. \qquad \square$$

From Theorems 3.1 and 3.2 we have actually given a method to find all isolated zeros and spherical zeros:

Let the complex solution set of $\widetilde{p}(x)=0$ be

$$\{\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_k,\zeta_1,\bar{\zeta}_1,\dots,\zeta_l,\bar{\zeta}_l,\zeta_{l+1},\bar{\zeta}_{l+1},\dots,\zeta_t,\bar{\zeta}_t\},$$

where $\xi_1,\dots,\xi_s$ are distinct real numbers, $\eta_1,\dots,\eta_k,\zeta_1,\dots,\zeta_t$ are distinct nonreal complex numbers (each $\bar{\eta}_i$ is no longer a root of $\widetilde{p}(x)$), $\overleftarrow{p}(\zeta_i)\neq\overleftarrow{p}(\bar{\zeta}_i)$ for $i=1,\dots,l$ and $\overleftarrow{p}(\zeta_i)=\overleftarrow{p}(\bar{\zeta}_i)$ for $i=l+1,\dots,t$. Then the set of all spherical zeros of $p(x)$ is

$$[\zeta_{l+1}]\cup\cdots\cup[\zeta_t],$$

and the set of all isolated zeros of $p(x)$ is

$$\{\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_k,\zeta_1,\bar{\zeta}_1,\dots,\zeta_l,\bar{\zeta}_l\}.$$

Next we consider how to find all circular zeros of $p(x)$. From Theorem 3.3 we need only to find all complex numbers ξ with $[\xi]$ containing a circular zero of $p(x)$. First we give a necessary condition for $p(x)$ to have a circular zero.

Proposition 3.4. *Let $p(x)$ be a two-sided polynomial of the form of* (12). *If $p(x)$ has a circular zero, then*

$$\begin{pmatrix}1\\ a_{n-1}b_{n-1}\\ \vdots\\ a_1b_1\end{pmatrix}\neq\begin{pmatrix}1\\ a_{n-1}\bar{b}_{n-1}\\ \vdots\\ a_1\bar{b}_1\end{pmatrix}\neq\begin{pmatrix}1\\ \bar{a}_{n-1}\bar{b}_{n-1}\\ \vdots\\ \bar{a}_1\bar{b}_1\end{pmatrix};$$

that is, $p(x)$ cannot be essentially written as a one-sided polynomial.

Proof. Suppose contrarily

$$\begin{pmatrix} 1 \\ a_{n-1}b_{n-1} \\ \vdots \\ a_1 b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix}.$$

Then $\widetilde{p}(x) = \overleftarrow{p}(x)$. Let ξ be a complex number and $[\xi]$ contain a circular zero of $p(x)$. Then by Theorem 3.3 we have

$$\widetilde{p}(\xi) = \overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi}) = \widetilde{p}(\bar{\xi}) \quad \text{and} \quad \widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) = \widetilde{p}(\xi)\overline{\widetilde{p}(\bar{\xi})} = |\widetilde{p}(\xi)|^2 \geq 0,$$

a contradiction to $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0$. The proof for the other inequality is similar. □

Lemma 3.5 [Zhang 1998, Theorem 3.16]. *Let* $f = a_n(x)y^n + \cdots + a_1(x)y + a_0(x)$ *and* $g = b_m(x)y^m + \cdots + b_1(x)y + b_0(x) \in \mathbb{C}[x, y]$, *where* $a_i(x), b_j(x) \in \mathbb{C}[x]$ $(i = 0, \ldots, n,\ j = 0, 1, \ldots, m)$ *with* $a_n(x)b_m(x) \neq 0$. *Let* $R(f, g; x)$ *be the resultant of* f *and* g *of order* $m + n$, *given by*

$$\begin{vmatrix}
a_n(x) & a_{n-1}(x) & \ldots & \ldots & \ldots & a_0(x) & & & \\
 & a_n(x) & a_{n-1}(x) & \ldots & \ldots & \ldots & a_0(x) & & \\
 & & \ddots & & & & & \ddots & \\
 & & & a_n(x) & a_{n-1}(x) & \ldots & \ldots & \ldots & a_0(x) \\
b_m(x) & b_{m-1}(x) & \ldots & \ldots & b_0(x) & & & & \\
 & b_m(x) & b_{m-1}(x) & \ldots & \ldots & b_0(x) & & & \\
 & & \ddots & & & & \ddots & & \\
 & & & \ddots & & & & \ddots & \\
 & & & & b_m(x) & b_{m-1}(x) & \ldots & \ldots & b_0(x)
\end{vmatrix}$$

where $a_n(x), \ldots, a_1(x), a_0(x)$ *are located in the first* m *rows, and the coefficients* $b_m(x), \ldots, b_1(x), b_0(x)$ *are located in the lower* n *rows. Then a complex* x_0 *is a zero of* $R(f, g; x)$ *if and only if the system*

$$\begin{cases} f(x_0, y) = 0 \\ g(x_0, y) = 0 \end{cases}$$

has a solution $y_0 \in \mathbb{C}$ *or the system*

$$\begin{cases} a_n(x_0) = 0 \\ b_m(x_0) = 0 \end{cases}$$

holds.

Now suppose $[\xi]$ contains a circular zero of $p(x)$, where ξ is a given complex number. Then by Theorem 3.3 we have $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$ and $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi) < 0$. The later inequality implies the imaginary part of $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi)$ is 0.

Set $a_0 = c_0 + d_0\boldsymbol{i}$, where c_0, d_0 are the real part and imaginary part of a_0, respectively. Also set $(\xi^n, \xi^{n-1}, \cdots, \xi^2, \xi) = \alpha + \beta\boldsymbol{i}$ and

$$\begin{pmatrix} 1 \\ a_{n-1}b_{n-1} \\ \vdots \\ a_1 b_1 \end{pmatrix} = U + V\boldsymbol{i},$$

where α, β are both real row vectors (real $1 \times n$ matrices), U and V are both real column vectors (real $n \times 1$ matrices). It is easy to see the first component of U is 1 while the first component of V is 0. From $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$ we get

$$\beta \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix} = 0. \tag{19}$$

And note that the imaginary part of $\widetilde{p}(\xi)\overline{\widetilde{p}}(\xi)$ is 0, so we get

$$\beta U \alpha U + \beta V \alpha V + c_0 \beta U + d_0 \beta V = 0. \tag{20}$$

Let $\xi = u + y\boldsymbol{i}$ with $u, y \in \mathbb{R}$. Remark 2.6(b) ensures that $y \neq 0$. Then

$$\xi^n = u^n + C_n^1 u^{n-1}(y\boldsymbol{i}) + \cdots + C_n^{n-1} u (y\boldsymbol{i})^{n-1} + (y\boldsymbol{i})^n.$$

When n is even, we have

$$\begin{cases} \Re\xi^n = u^n - C_n^2 u^{n-2} y^2 + \cdots + y^n (-1)^{\frac{n}{2}}, \\ \Im\xi^n = C_n^1 u^{n-1} y + \cdots + C_n^{n-1} u y^{n-1} (-1)^{\frac{n-2}{2}}, \end{cases}$$

and when n is odd we have

$$\begin{cases} \Re\xi^n = u^n - C_n^2 u^{n-2} y^2 + \cdots + C_n^{n-1} u y^{n-1} (-1)^{\frac{n-1}{2}}, \\ \Im\xi^n = C_n^1 u^{n-1} y + \cdots + C_n^{n-2} u^2 y^{n-2} (-1)^{\frac{n-3}{2}} + y^n (-1)^{\frac{n-1}{2}}. \end{cases}$$

For convenience we take n to be an odd number, $n = 2k+1$ since it is similar for the case that n is even. In this case (19) becomes

$$\left(C_n^1 u^{n-1} y + \cdots + C_n^{n-2} u^2 y^{n-2} (-1)^{\frac{n-3}{2}} + y^n (-1)^{\frac{n-1}{2}}, \ldots, y\right) \cdot \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix} = 0. \tag{21}$$

Since $y \neq 0$, we can write this as

$$(22)\quad \left(C_n^1 u^{n-1} + \cdots + C_n^{n-2} u^2 y^{n-3} (-1)^{\frac{n-3}{2}} + y^{n-1} (-1)^{\frac{n-1}{2}}, \ldots, 1\right) \cdot \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix} = 0.$$

It is easy to see that (22) can be rewritten as

$$(23)\qquad z^k + d_1(u) z^{k-1} + \cdots + d_{k-1}(u) z + d_k(u) = 0,$$

where $z := y^2$, $k = \frac{n-1}{2}$, $d_1(u), \ldots, d_k(u) \in \mathbb{C}[u]$, $\deg d_k(u) = 2k$ (implying that $d_k(u) \neq 0$).

We treat (20) in a similar manner. Note that $y \neq 0$, the first component of U is 1 and the first component of V is 0, then we obtain from (20) the following equation:

$$(24)\qquad h_1(u) z^{2k} + h_2(u) z^{2k-1} + \cdots + h_n(u) = 0,$$

where $z := y^2$, $k = \frac{n-1}{2}$, $h_1(u), \ldots, h_n(u) \in \mathbb{C}[u]$, $\deg h_n(u) = 2n - 1$.

Up to now, we have shown that, if $[\xi]$ contains a circular zero of $p(x)$, then the real part and imaginary part of ξ must satisfy (23) and (24). Let

$$f := z^k + d_1(u) z^{k-1} + \cdots + d_{k-1}(u) z + d_k(u),$$
$$g := h_1(u) z^{2k} + h_2(u) z^{2k-1} + \cdots + h_n(u).$$

We denote by R_p the resultant of f and g. Then

$$R_p = \begin{vmatrix} 1 & d_1(u) & \ldots & \ldots & \ldots & d_k(u) & & & \\ & 1 & d_1(u) & \ldots & \ldots & \ldots & d_k(u) & & \\ & & \ddots & & & & & \ddots & \\ & & & 1 & d_1(u) & \ldots & \ldots & \ldots & d_k(u) \\ h_1(u) & h_2(u) & \ldots & \ldots & h_n(u) & & & & \\ & h_1(u) & h_2(u) & \ldots & \ldots & h_n(u) & & & \\ & & \ddots & & & & \ddots & & \\ & & & \ddots & & & & \ddots & \\ & & & & h_1(u) & h_2(u) & \ldots & \ldots & h_n(u) \end{vmatrix},$$

which is a polynomial in the variable u with complex coefficients. Let $x_1, \ldots, x_s$ be the real roots of R_p (if R_p has no real root, then $p(x)$ has no circular zero, by Lemma 3.5). Then substitute x_l for u in (23) to get corresponding nonzero solutions for y. In this way we get at most finitely many complex numbers $x_l + y_{lj}\boldsymbol{i}$, where y_{lj} is the real solution of

$$(y^2)^k + d_1(x_l)(y^2)^{k-1} + \cdots + d_{k-1}(x_l)y^2 + d_k(x_l) = 0,$$

$l = 1, \ldots, s$, $j = 1, \ldots, n_l$. (If such a y_{lj} does not exist, this also shows $p(x)$ has no circular zeros.) Now if $[\xi]$ contains a circular zero, then from Lemma 3.5 we know ξ must be equal to some $x_l + y_{lj}\boldsymbol{i}$. Therefore, for the finitely many complex numbers $x_l + y_{lj}\boldsymbol{i}$ ($l = 1, \ldots, s$, $j = 1, \ldots, n_l$), using Theorem 3.3 we can find all circular zeros of $p(x)$.

This method for finding circular zeros will be valid so long as the resultant R_p is not the zero polynomial. Since we have excluded the cases

$$\begin{pmatrix} 1 \\ a_{n-1}b_{n-1} \\ \vdots \\ a_1 b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ a_{n-1}\bar{b}_{n-1} \\ \vdots \\ a_1 \bar{b}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \bar{a}_{n-1}\bar{b}_{n-1} \\ \vdots \\ \bar{a}_1 \bar{b}_1 \end{pmatrix}$$

(Proposition 3.4 ensures that $p(x)$ has no circular zeros under such circumstances), generally speaking the resultant R_p obtained at the moment cannot vanish. We have done a lot of tests, and have never discovered a two-sided polynomial $p(x)$ of form (12) with the conditions in Proposition 3.4 such that $R_p = 0$.

Example 3.6. Find all zeros of $p(z) = z^3 - \boldsymbol{i}z^2\boldsymbol{i} - \boldsymbol{i}z\boldsymbol{i} + 1$ in $\mathbb{H}$.

Solution. $\widetilde{p}(z) = z^3 + z^2 + z + 1$, $\overleftarrow{p}(z) = z^3 - z^2 - z + 1$. The complex roots of $\widetilde{p}(z)$ are $-1, \boldsymbol{i}, -\boldsymbol{i}$. $\overleftarrow{p}(\boldsymbol{i}) = 2 - 2\boldsymbol{i}$, $\overleftarrow{p}(\bar{\boldsymbol{i}}) = 2 + 2\boldsymbol{i}$. Thus, p has no spherical zero, and the set of isolated zeros is $\{-1, \boldsymbol{i}, -\boldsymbol{i}\}$.

Now we seek the circular zeros. We have

$$U = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad V = 0, \quad c_0 = 1, \quad d_0 = 0,$$

$$\alpha = (x^3 - 3xy^2, x^2 - y^2, x), \quad \beta = (3x^2y - y^3, 2xy, y),$$

In this case, (24) and (23) become

$$(3x+1)t^2 - (10x^3 + 10x^2 + 6x + 2)t + (3x^5 + 5x^4 + 6x^3 + 6x^2 + 3x + 1) = 0, \tag{25}$$

$$t + (-3x^2 + 2x + 1) = 0, \tag{26}$$

where $t := y^2$. The resultant R_p is

$$R_p = \begin{vmatrix} 1 & -3x^2 + 2x + 1 & 0 \\ 0 & 1 & -3x^2 + 2x + 1 \\ 3x + 1 & -(10x^3 + 10x^2 + 6x + 2) & 3x^5 + 5x^4 + 6x^3 + 6x^2 + 3x + 1 \end{vmatrix}$$
$$= -32x^4 + 32x^2 + 20x + 4.$$

The real roots of R_p are (from MATLAB)

$$x_1 = -0.500000000000000, \quad x_2 = 1.255773570847266,$$

and from (26) we get the positive roots

$$y_{x_1} = 0.866025403784440, \quad y_{x_2} = 1.104243923243840$$

and their opposites We investigate the complex numbers

$$\xi_1 = x_1 + y_{x_1}\boldsymbol{i}, \quad \xi_2 = \bar{\xi}_1, \quad \xi_3 = x_2 + y_{x_2}\boldsymbol{i}, \quad \xi_4 = \bar{\xi}_3.$$

For ξ_1, we have $\widetilde{p}(\xi_1) = 1$, $\widetilde{p}(\xi_1)\overline{\widetilde{p}}(\xi_1) = 1 > 0$. So, $[\xi_1]$ $(= [\xi_2])$ contains no circular zeros of p.

Since $\widetilde{p}(\xi_3) = 7.7552\boldsymbol{i}$, $\widetilde{p}(\xi_3)\overline{\widetilde{p}}(\xi_3) < 0$, and $\overleftarrow{p}(\xi_3) = \overleftarrow{p}(\bar{\xi}_3)$, then $[\xi_3]$ $(= [\xi_4])$ contains a circular zero of p, and the set of circular zeros of p is

$$\Upsilon = \{1.255773570847266 + z\boldsymbol{j} : z \in \mathbb{C}, |z|^2 = (1.104243923243840)^2\}.$$

Hence the zero set of p is $\{-1, \boldsymbol{i}, -\boldsymbol{i}\} \cup \Upsilon$.

4. Formulae of zeros for quadratic two-sided polynomials with complex coefficients

In this section we concentrate on the case where p is quadratic with complex coefficients. We establish formulae for finding its spherical, circular and isolated zeros, and spell out a simple and efficient algorithm to find all zeros. So let

$$p(x) := x^2 + (a + b\boldsymbol{i})x(c + d\boldsymbol{i}) + (e + f\boldsymbol{i}), \tag{27}$$

where a, b, c, d, e, f are real numbers. In the notation introduced at the beginning of Section 3, we then have

$$\widetilde{p}(x) := x^2 + (a + b\boldsymbol{i})(c + d\boldsymbol{i})x + (e + f\boldsymbol{i}),$$
$$\overleftarrow{p}(x) := x^2 + (a + b\boldsymbol{i})(c - d\boldsymbol{i})x + (e + f\boldsymbol{i}),$$

Recall that a complex ξ is said to be a spherical zero of $p(x)$ if ξ is nonreal and each point of $[\xi]$ is a zero of $p(x)$.

Theorem 4.1 (existence of spherical zeros). *The polynomial $p(x)$ in* (27) *has a spherical zero if and only if one of the following conditions is met*:

- $b = d = f = 0$ *and* $(ac)^2 < 4e$.
- $a = b = f = 0$ *and* $e > 0$.
- $c = d = f = 0$ *and* $e > 0$.

Furthermore, *in this case the set of all zeros of $p(x)$ is*

$$\left[\frac{-ac + \sqrt{4e - (ac)^2}\,\boldsymbol{i}}{2}\right]. \tag{28}$$

Proof. $\Leftarrow$) When one of the conditions is met, $p(x)$ becomes $x^2 + axc + e$, which is a real polynomial. So each nonreal zero of $p(x)$ is a spherical zero. In this case the complex roots of $\widetilde{p}$ are

$$\frac{-ac + \sqrt{4e - (ac)^2}\,\boldsymbol{i}}{2}, \qquad \frac{-ac - \sqrt{4e - (ac)^2}\,\boldsymbol{i}}{2}.$$

By the method provided in Section 3 to find all spherical zeros and isolated zeros, we conclude that the set of all zeros of $p(x)$ is given by (28).

$\Rightarrow$) Suppose the complex ξ is a spherical zero of $p(x)$. Then by Theorem 3.1 we have $\widetilde{p}(\xi) = \widetilde{p}(\bar{\xi}) = 0$ and $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$. Consequently, $\widetilde{p}$ should be a polynomial with real coefficients, from which we get

$$f = 0, \quad ad = -bc, \tag{29}$$

and $\widetilde{p}(x) = x^2 + (ac - bd)x + e$. This forces ξ to equal one of the two conjugate numbers

$$\frac{(bd - ac) \pm \sqrt{4e - (ac - bd)^2}\,\boldsymbol{i}}{2},$$

where $(ac - bd)^2 < 4e$. We may assume that

$$\xi = \frac{(bd - ac) + \sqrt{4e - (ac - bd)^2}\,\boldsymbol{i}}{2}. \tag{30}$$

Since $\overleftarrow{p}(\xi) = \overleftarrow{p}(\bar{\xi})$, we have

$$\xi^2 + \big((ac + bd) + (bc - ad)\boldsymbol{i}\big)\xi = \bar{\xi}^2 + \big((ac + bd) + (bc - ad)\boldsymbol{i}\big)\bar{\xi}.$$

Substituting (30), simplifying and comparing real and imaginary parts, we obtain

$$((bd - ac) + (ac + bd))\sqrt{4e - (ac - bd)^2} = 0, \qquad (bc - ad)\sqrt{4e - (ac - bd)^2} = 0,$$

which yields

$$bd = 0, \quad bc = ad. \tag{31}$$

From (29) and (31) it is easy to see $a = b = f = 0$, or $c = d = f = 0$, or $b = d = f = 0$. If $a = b = f = 0$ or $c = d = f = 0$, then from $(ac - bd)^2 < 4e$ we find $e > 0$. If $b = d = f = 0$, then by $(ac - bd)^2 < 4e$ we get $(ac)^2 < 4e$. ☐

Corollary 4.2. *Let $p(x)$ be a polynomial of the form in* (27). *Then $p(x)$ has a spherical zero if and only if $p(x)$ can be written as $p(x) = x^2 + rx + s$, where r, s are real numbers with $r^2 - 4s < 0$. Moreover, in this case, the set of zeros of $p(x)$ is*

$$\left[\frac{-r + \sqrt{4s - r^2}\boldsymbol{i}}{2}\right].$$

Theorem 4.3 (existence of circular zeros). *The polynomial* $p(x)$ *in* (27) *has a circular zero if and only if* $bd \neq 0$, $ad = bc$, *and*

$$\tfrac{3}{4}\left((ac)^2+2(bc)^2+(bd)^2\right)+e-\frac{a}{b}f > \left(\frac{f-(ac+bd)bc}{2bd}\right)^2. \tag{32}$$

Moreover, in this case the set of all circular zeros of $p(x)$ *is*

$$\left\{-\frac{ac+bd}{2}+\frac{f-(ac+bd)bc}{2bd}\,\boldsymbol{i}+z\,\boldsymbol{j} : z\in\mathbb{C},\ |z|^2=\Delta-\left(\frac{f-(ac+bd)bc}{2bd}\right)^2\right\}, \tag{33}$$

where

$$\Delta := \tfrac{3}{4}((ac)^2+2(bc)^2+(bd)^2)+e-\tfrac{a}{b}f.$$

Proof. $\Rightarrow$) Let $[\xi]$ contain a circular zero of $p(x)$, where ξ is a complex number. Then $[\xi]$ contains no complex zeros of $p(x)$ (see Remark 2.6), and ξ satisfies (17). From Proposition 3.4 we see that $bd \neq 0$.

Let $\xi = u + y\boldsymbol{i}$ where $u, y \in \mathbb{R}$ with $y \neq 0$. From the second equation in (17) we deduce that $u = -(ac+bd)/2$ and $ad = bc$.

Now from Theorem 3.3 we may assume $p(x)$ has a solution $x = u + w\boldsymbol{i} + v\boldsymbol{j}$ with $u, w, v \in \mathbb{R}$ and $v \neq 0$. Substitute x in $p(x)$ with $u + w\boldsymbol{i} + v\boldsymbol{j}$. Then we get

$$u^2-w^2-v^2+acu-bcw-bdu-adw+e=0, \tag{34}$$

$$2uw+bcu+acw+adu-bdw+f=0. \tag{35}$$

From (34) it follows that $u^2+(ac-bd)u-w^2-2bcw+e=v^2>0$. So,

$$u^2+(ac-bd)u-2bcw+e>w^2. \tag{36}$$

From (35) we have $w = (f-(ac+bd)bc)/2bd$. Substituting this value in (36) yields (32). And in this case it's easy to see by Theorem 2.5 that the set of circular zeros in $[\xi]$ is as given in (33), since

$$u=-\frac{ac+bd}{2}, \quad w=\frac{f-(ac+bd)bc}{2bd},$$

$$v^2=u^2+(ac-bd)u-w^2-2bcw+e=\Delta-\left(\frac{f-(ac+bd)bc}{2bd}\right)^2,$$

and $x = u + w\boldsymbol{i} + v\boldsymbol{j}$ is a circular zero of p.

$\Leftarrow$) When the conditions $bd \neq 0$, $ad = bc$, and (32) are satisfied, we can verify directly that each element of the set in (33) is a zero of $p(x)$. Note that (33) has infinitely many elements, and Theorem 4.1 implies that $p(x)$ has no spherical zeros, since $bd \neq 0$. Again by Theorem 2.5 we know that $p(x)$ has a circular zero. $\square$

Next we give a consequence of Theorems 4.1 and 4.3.

Corollary 4.4. (1) *The polynomial* $x^2 + r(t + \boldsymbol{i})x(t + \boldsymbol{i}) + e$, *where* $r, t, e \in \mathbb{R}$, *has a circular zero if and only if* $r \neq 0$ *and* $4e/r^2 + t^4 + 5t^2 + 3 - t^6 > 0$.

(2) *No quadratic polynomial with two-sided complex coefficients can have a spherical zero and a circular zero simultaneously.*

From Theorem 2.5 we know that the set of isolated zeros of $p(x)$ is contained in the nonempty set $\{z : z \in \mathbb{C}, \widetilde{p}(z) = 0\}$ in this case. Using Theorem 4.1 and Theorem 4.3 we have:

Theorem 4.5. *The polynomial* $p(x)$ *in* (27) *has an isolated zero if and only if it either has a circular zero, or has no circular zero or spherical zero. In either case, the set of isolated zeros of* $p(x)$ *is* $\{z : z \in \mathbb{C}, \widetilde{p}(z) = 0\}$, *where* $\widetilde{p}$ *is regarded as a complex polynomial* (*so the classical formula can be used*).

Corollary 4.6. *The zeros of* $p(x)$ *are distributed in at most* 3 *equivalence classes, and* $p(x)$ *has finitely many zeros if and only if* $p(x)$ *has neither circular zeros nor spherical zeros.*

Summary of the algorithm to find all zeros of a quadratic two-sided quaternionic polynomial with complex coefficients. Given a polynomial $a_2x^2b_2 + a_1xb_1 + a_0$, with $x \in \mathbb{H}$, $a_i, b_i \in \mathbb{C}$, $a_2b_2 \neq 0$, first divide it by a_2 and b_2, so as to reduce it to the form

$$p(x) := x^2 + (a + b\boldsymbol{i})x(c + d\boldsymbol{i}) + e + f\boldsymbol{i}.$$

Step 1. Test the three conditions of Theorem 4.1. If any of them is met, the set of zeros of p is

$$\left[\frac{-ac + \sqrt{4e - (ac)^2}\,\boldsymbol{i}}{2}\right].$$

Otherwise, go to the next step.

Step 2. Compute the (real and complex) zeros of the polynomial

$$\widetilde{p}(x) := x^2 + (a + b\boldsymbol{i})(c + d\boldsymbol{i})x + e + f\boldsymbol{i}.$$

Denote them by z_1 and z_2. Test the three conditions of Theorem 4.3. If they are all met, the set of zeros of q is the union of $\{z_1, z_2\}$ with the set (33) of the same theorem. Otherwise, the set of zeros of $q(x)$ is $\{z_1, z_2\}$.

Example 4.7. For the polynomial $p(x) := x^2 + \boldsymbol{i}x\boldsymbol{i} + 2$, none of the conditions in Theorem 4.1 is met, so there are no spherical zeros. In Step 2 we get two (conjugate) isolated zeros and a circular zero. The complete set of zeros is

$$\left\{\frac{1+\sqrt{7}\boldsymbol{i}}{2}, \frac{1-\sqrt{7}\boldsymbol{i}}{2}\right\} \cup \left\{-\tfrac{1}{2} + z\boldsymbol{j} : z \in \mathbb{C}, |z|^2 = \tfrac{11}{4}\right\}.$$

The zeros fall into two equivalence classes.

Example 4.8. For $p(x) := x^2 + (1+\boldsymbol{i})x(1+\boldsymbol{i}) + 1$, again there are no spherical zeros. The algorithm (or Corollary 4.4) gives a circular zero, and two (nonconjugate) isolated zeros, so the set of zeros is

$$\big\{(\sqrt{2}-1)\boldsymbol{i},\, -(\sqrt{2}+1)\boldsymbol{i}\big\} \cup \big\{-1-\boldsymbol{i}+z\boldsymbol{j} : z \in \mathbb{C},\, |z|^2 = 3\big\}.$$

The zeros fall into three equivalence classes.

Example 4.9. The polynomial x^2+1 has a spherical zero; hence (by Step 1 or Corollary 4.2) its set of zeros is $[\boldsymbol{i}] = \{a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} : a_1^2 + a_2^2 + a_3^2 = 1\}$, forming a single equivalence class.

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