DROPLET CONDENSATION AND
ISOPERIMETRIC TOWERS

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We consider a variational problem in a planar convex domain, motivated by the statistical mechanics of crystal growth in a saturated solution. The minimizers are constructed explicitly and are completely characterized.

1. Introduction

In understanding the physical phenomenon of droplet condensation or crystal growth, the central issue is to explain how a particular macroscopic shape of the growing droplet or crystal is determined by microscopic interactions of its constituent particles.

According to Gibbs' formulation of statistical mechanics, the probability of a microscopic configuration σ is proportional to $\exp(-\beta H(\sigma))$, where $\beta > 0$ is the inverse temperature and $H(\,\cdot\,)$ is the Hamiltonian defining the energy of the system. Therefore the most probable configurations are the ones with minimal energy. In the "thermodynamical" limit of a large number of particles, this minimum becomes very sharp: the overall configuration of the system settles, up to minute fluctuations, to a well-defined deterministic structure.

It turns out that the microscopic laws of atomic interactions give rise to a certain macroscopic quantity, the surface tension, which determines the droplet shape via minimization of the surface energy. Phenomenology of surface tension was proposed by Gibbs in the late 1870s. In an important contribution, G. Wulff suggested in 1900 that for a growing crystal, its equilibrium shape is that of a ball in a metric generated by the surface tension (the *Wulff shape*).

It has been furthermore observed experimentally that flat facets of a growing crystal may carry macroscopic but monomolecular "islands", whose shape is also determined by the surface tension. A mathematical approach to explaining this phenomenon has been developed by S. Shlosman and collaborators in a series of works [Schonmann and Shlosman 1996a; Ioffe and Shlosman 2008; 2010], building upon his earlier work with R. L. Dobrushin and R. Kotecky [Dobrushin et al. 1992].

A typical setting in this approach is represented by the following discrete model of crystal growth, which is a variant of the Ising model: fix an open domain $\Omega \subset \mathbb{R}^2$ of unit area and consider the three-dimensional lattice obtained by intersecting the cylinder $\Omega \times [-1, 1] \subset \mathbb{R}^3$ with $(1/N)(\mathbb{Z}^3 + (0, 0, \frac{1}{2}))$, where N is a large integer parameter. At each node t of this lattice there is a variable σ_t (the *spin*) taking values $+1$ (interpreted as "t belongs to the free phase") and -1 (interpreted as "t belongs to the condensed phase"). The collection $\sigma = (\sigma_t)$ is called the microscopic configuration of the system.

Fix now the Ising Hamiltonian $H(\sigma) = -\sum_{s,t:|s-t|=1} \sigma_s \sigma_t$, which describes a "ferromagnetic" interaction between nearest neighbors (equal values have smaller energy than opposite ones), and consider the canonical probability distribution $p(\sigma) = \exp(-\beta H(\sigma))/Z$. Here the normalization coefficient $Z = \sum_{\sigma} \exp(-\beta H(\sigma))$ is defined by summation over all configurations that satisfy the so-called *Dobrushin boundary condition*: spins at outermost nodes (x, y, z) of the lattice have values $+1$ if $z > 0$ and -1 if $z < 0$.

It turns out that in the limit of large N the main contribution to probability comes from configurations where the lower and upper halves of the lattice are filled, respectively, with -1's and $+1$'s. In this equilibrium state, the numbers of $+1$'s and -1's are asymptotically equal, so that $S_N = \sum_t \sigma_t \sim 0$, and fluctuations of the flat surface dividing the two phases are logarithmic in N.

A more interesting situation occurs when, in addition to the Dobrushin boundary values, the system is conditioned to have macroscopically more -1's than $+1$'s:

$$S_N = \sum_t \sigma_t = -mN^2$$

with $m > 0$. In this case, depending on the value of m, the most probable state of the system may feature one or more monomolecular layers on top of the surface $z = 0$ in the box $\Omega \times [-1, 1]$. A detailed account of the observed equilibrium states as m changes can be found in [Ioffe and Shlosman 2010].

As proved in [Schonmann and Shlosman 1996a], the behavior of this model in the continuous limit $N \to \infty$ is closely related to the following variational problem: given an open set $\Omega \subset \mathbb{R}^n$ and a value $m \in [0, +\infty)$, find

$$(1\text{-}1) \quad \min\left\{ \int_{\Omega} \varphi^*(Du) : u \in BV(\mathbb{R}^n), u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, u(\cdot) \in \mathbb{N}, \int_{\Omega} u \, dx = m \right\},$$

where φ^* is some given general norm on $\mathbb{R}^n$. Of course, in the application to the Ising model we are discussing here one has $n = 2$, the two-dimensional case; however the case of generic dimension n of the ambient space $\mathbb{R}^n$ also makes sense from the mathematical point of view. The growth of a droplet and formation of new layers of the solid is described by the growth of profile u as m increases.

The norm $\varphi^*(\cdot)$ here is related to the surface tension as follows. The surface tension $\gamma^{3D}(\cdot)$ is a function defined over $\mathbb{S}^2$, the two-dimensional unit sphere in $\mathbb{R}^3$, and satisfying $\gamma^{3D}(\nu) \geq 0$ and $\gamma^{3D}(-\nu) = \gamma^{3D}(\nu)$ for all $\nu \in \mathbb{S}^2$. The surface energy of a closed surface $M^2 \subset \mathbb{R}^3$ is defined to be

$$H(M^2) = \int_{M^2} \gamma^{3D}(\nu_s)\, ds,$$

where ν_s is the unit normal to M^2 at $s \in M^2$. While γ^{3D} defines the $3D$ shape of a crystal growing in space, the shape of monolayers growing on facets is given by the restricted $2D$ surface tension defined for $n \in \mathbb{S}^1$ by

$$\gamma^{2D}(\nu) = \frac{\partial}{\partial \nu}\, \gamma^{3D}\Big|_{\nu_s=(0,0,1)},$$

where the derivatives are taken at the "north pole" $\nu_s = (0, 0, 1) \in \mathbb{S}^2$ along all tangents $\nu \in \mathbb{S}^1$ to $\mathbb{S}^2$ [Ioffe and Shlosman 2010]. The function γ^{2D} can then be extended to all of $\mathbb{R}^2$ by homogeneity of degree one, and $\varphi^*(\cdot)$ is defined as the convex hull of the thus defined $\gamma^{2D}(\cdot)$. However in the sequel φ^* will be fixed, without any assumptions of smoothness or strict convexity: indeed one of the examples in Section 5 corresponds to a crystalline norm.

It is easy to see that the functional minimized in (1-1) is the one-dimensional surface energy for the restricted surface tension. It turns out that minimization of this surface energy alone is sufficient to reconstruct most of the physics of monomolecular layer growth described in [Ioffe and Shlosman 2010]. In particular, if $\varphi^*(\cdot)$ is the Euclidean norm and Ω a unit square, then as m grows, the first four monomolecular layers start as Wulff circles and then develop into "Wulff plaquettes" while from the fifth layer on all new layers appear as Wulff plaquettes identical to underlying layers (Section 5).

In contrast, this simple variational model does not capture the thermodynamic fluctuations, which render Wulff circles below a certain size unstable and prevent their formation for small m. Neither does it capture the microscopic (that is, "finite-N") structure of the Wulff plaquettes, whose boundaries are in fact separated with gaps that vanish in the continuous limit. A first-principle approach that takes proper account of these phenomena is due to R. Dobrushin, S. Shlosman and their coauthors and is presented in [Dobrushin et al. 1992; Schonmann and Shlosman 1996a; 1996b; Ioffe and Shlosman 2008; 2010].

It is worth observing that a similar problem with the additional restriction that u be a characteristic function of some set (that is, that the droplet has exactly one layer) in the two-dimensional situation (that is when $n = 2$), the set Ω is convex, and the norm φ^* is Euclidean, has been studied in [Stredulinsky and Ziemer 1997], and for more general anisotropic norms (but for a somewhat different functional, namely, with penalization on the volume instead of the volume constraint), in [Novaga and Paolini 2005]. The latter problem will play an important role also in the present paper. Eventually, one has to mention that it is also very similar to the well-known Cheeger problem, the solutions of the latter being so-called Cheeger sets; see for instance [Buttazzo et al. 2007; Kawohl and Novaga 2008; Kawohl and Lachand-Robert 2006; Caselles et al. 2010].

Our aim in this paper is to study the variational problem (1-1) in the two-dimensional case (that is, when $n = 2$). This geometric optimization problem is considered without resort to the underlying lattice model or its continuous limit, allowing us to treat an arbitrary open domain Ω and an arbitrary norm φ^* that is not necessarily strictly convex. In this setting we completely characterize the minimizers and the possible levels of u when the domain Ω is convex. In particular it turns out that except some degenerate situation, which can however happen only when Ω is not strictly convex, the number of nonzero levels of u is at most two.

The basic tool we use is the auxiliary problem when u is a priori required to have a single nonzero level (that is, is requested to be a characteristic function); namely, we show that in the two-dimensional case ($n = 2$) when Ω is convex, the nonzero levels of solutions to the latter problem corresponding to different values of m as m grows can be arranged as a family of sets ordered by inclusion. Thus, solutions to problem (1-1) can be seen as "towers" with levels solving the auxiliary problem. The assumption of convexity of Ω is essential, as shown by a counterexample at the end of Section 4A. The main result of the paper is formulated as Theorem 4.10. We conclude with an explicit example of solutions to (1-1) for the case of a square $\Omega = [0, 1]^2$ with a strictly convex (Euclidean) norm and a crystalline norm.

This work was inspired by some seminar talks of Senya Shlosman. After it was completed, we learned that a full description of the solutions to the variational problem (1-1) when Ω is a square and φ^* is generated by a physical Hamiltonian (in particular, when it is the Euclidean norm) has been independently obtained by him and Ioffe by a rigorous continuous limit of a suitable lattice model (S. Shlosman, private communication, 2012). Their proof, together with an analysis of the microscopic structure of the solution and its behavior under thermal perturbations, has not yet been published.

2. Notation and preliminary results

For a set $E \subset \mathbb{R}^n$ we denote by $|E|$ its Lebesgue measure, by $\mathbf{1}_E$ its characteristic function, by $\bar{E}$ its closure, by ∂E its topological boundary, and by E^c its complement.

In the following φ will denote the given (not necessarily Euclidean) norm over $\mathbb{R}^n$. Given $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$\text{dist}_\varphi(x, E) := \inf_{y \in E} \varphi(x - y), \quad d_\varphi^E(x) := \text{dist}_\varphi(x, E) - \text{dist}_\varphi(x, E^c).$$

The value $d_\varphi^E(x)$ is the signed distance from x to ∂E and is positive outside E. Notice that at each point where d_φ^E is differentiable one has (see [Bellettini et al. 2001])

$$(2\text{-}1) \qquad \varphi^*(\nabla d_\varphi^E) = 1, \qquad \nu \cdot \nabla d_\varphi^E = 1 \quad \text{for all } \nu \in \partial \varphi^*(\nabla d_\varphi^E),$$

where φ^* denotes the dual norm of φ defined as

$$\varphi^*(\xi) := \max_{\eta: \varphi(\eta) \leq 1} \xi \cdot \eta$$

and $\partial \varphi^*$ denotes the subdifferential of φ^* in the sense of convex analysis. In particular

$$\nabla d_\varphi^E = \frac{\nu^E}{\varphi^*(\nu^E)}$$

where ν^E is the exterior Euclidean unit normal to ∂E.

We define the anisotropic perimeter of a set $E \subseteq \mathbb{R}^n$ as

$$(2\text{-}2) \quad P_\varphi(E) := \sup\left\{ \int_E \text{div } \eta \, dx : \eta \in C_0^1(\mathbb{R}^n), \varphi(\eta) \leq 1 \right\} = \int_{\partial^* E} \varphi^*(\nu^E) d\mathcal{H}^{n-1},$$

where $\partial^* E$ is the reduced boundary of E according to De Giorgi. We will usually identify a set E of finite perimeter with the set of its density points (that is, points of density 1).

Given an open set $\Omega \subset \mathbb{R}^n$ we define the BV-seminorm of $v \in BV(\Omega)$ as

$$\int_\Omega \varphi^*(Dv) := \sup\left\{ \int_\Omega v \, \text{div } \eta \, dx : \eta \in C_0^1(\mathbb{R}^n), \varphi(\eta) \leq 1 \right\},$$

where $C_0^1(\mathbb{R}^n)$ stands for the set of continuously differentiable functions with compact support is $\mathbb{R}^n$.

We let $W_\varphi := \{x \mid \varphi(x) < 1\}$, usually called the *Wulff shape*, be the unit ball of φ. Observe that $P_\varphi(W_\varphi) = n|W_\varphi|$.

In the sequel, given $x \in \mathbb{R}^n$ and $r > 0$, we set $W_r(x) := x + r W_\varphi$ (a *Wulff ball* of radius r with center x). In this notation the reference to a norm φ is not retained

for the sake of brevity, but always silently assumed. When φ is the Euclidean norm, we will use a more common notation $B_r(x)$ instead of $W_r(x)$ and P instead of P_φ.

Definition 2.1. Given an $r > 0$, we say that E satisfies the $r W_\varphi$-condition, if for every $x \in \partial E$ there exists an $y \in \mathbb{R}^n$ such that $W_r(y) \subset E$ and $x \in \partial W_r(y)$.

Observe that, if E is convex, then E^c satisfies the $r W_\varphi$-condition for all $r > 0$.

We conclude the section by recalling the following isoperimetric inequality [Taylor 1975].

Proposition 2.2. *For all $E \subset \mathbb{R}^n$ such that $|E| < +\infty$ there holds*

$$(2\text{-}3) \qquad P_\varphi(E) \geq \frac{|E|^{\frac{n-1}{n}}}{|W_\varphi|^{\frac{n-1}{n}}} P_\varphi(W_\varphi).$$

3. Existence of minimizers

Notice that, since the total variation is lower semicontinuous and the constraints are closed under weak BV convergence, by direct method one immediately gets existence of minimizers of (1-1).

Proposition 3.1. *For any $m \geq 0$ there exists a (possibly nonunique) minimizer of* (1-1).

For every $u \in L^1(\mathbb{R}^n)$ and $j \in \mathbb{N}$ we set

$$(3\text{-}1) \qquad E_j := \{u \geq j\}.$$

It is worth observing that whenever $u(\cdot)$ takes values in $\mathbb{N}$,

$$(3\text{-}2) \qquad u = \sum_{i=1}^\infty \mathbf{1}_{E_i}$$

and

$$(3\text{-}3) \qquad \int_{\mathbb{R}^n} \varphi^*(Du) = \sum_{i=1}^\infty P_\varphi(E_i).$$

Remark 3.2. If we let u_m be a minimizer of (1-1) for a given $m > 0$, then the normalized functions $v_m := u_m/m$ converge, as $m \to \infty$, up to a subsequence, to a minimizer of the problem

$$\min\left\{ \int_\Omega \varphi^*(Dv) : v \in BV(\mathbb{R}^n), \, v = 0 \text{ on } \Omega^c, \int_\Omega v \, dx = 1 \right\},$$

which is closely related to the *Cheeger problem* in Ω [Kawohl and Novaga 2008].

Proposition 3.3. *If u is a minimizer of* (1-1), *then $u \in L^\infty(\mathbb{R}^n)$.*

Proof. Assume by contradiction that $|E_j| > 0$ for all $j \in \mathbb{N}$. Notice that

$$\lim_{j \to \infty} |E_j| = 0$$

(since otherwise u would not be integrable). Given $x_0 \in \Omega$ we let

$$u_j := \min(u, j) + \mathbf{1}_{W_{R_j}(x_0)}$$

where the radius R_j is such that

$$\int_\Omega u_j = \int_\Omega u = m,$$

that is (keeping in mind (3-2)),

$$|W_\varphi| R_j^n = \sum_{i > j} |E_i|,$$

and choose $j \in \mathbb{N}$ big enough so that $W_{R_j}(x_0) \subset \Omega$.

Letting

$$f(t) := n|W_\varphi|^{\frac{1}{n}} t^{\frac{n-1}{n}} \quad \text{so that} \quad P_\varphi(W_{R_j}(x_0)) = f(|W_\varphi| R_j^n),$$

we have

$$\int_\Omega \varphi^*(Du_j) \leq \int_\Omega \varphi^*(D\min(u, j)) + P_\varphi(W_{R_j}(x_0))$$

$$= \int_\Omega \varphi^*(D\min(u, j)) + f(|W_\varphi| R_j^n)$$

$$\leq \int_\Omega \varphi^*(D\min(u, j)) + \sum_{i > j} f(|E_i|) \quad \text{by the concavity of } f$$

$$\leq \int_\Omega \varphi^*(D\min(u, j)) + \sum_{i > j} P_\varphi(E_i) \quad \text{by (2-3)}$$

$$= \int_\Omega \varphi^*(Du) \quad \text{by (3-3)},$$

the second inequality being strict unless $|E_i| = |E_k|$ for all $i > j$, $k > j$, thus leading to a contradiction. $\qquad\square$

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be star-shaped. Then the problem (1-1) is equivalent to the following relaxed problem:*

$$(3\text{-}4) \quad \min\left\{ \int_\Omega \varphi^*(Du) : u \in BV(\mathbb{R}^n), u = 0 \text{ on } \Omega^c, u(\cdot) \in \mathbb{N}, \int_\Omega u \, dx \geq m \right\}.$$

Namely, the minimum values and the minimizers are the same for both problems.

Proof. It is enough to show that any minimizer u of (3-4) satisfies

$$(3\text{-}5) \qquad\qquad \int_\Omega u \, dx = m.$$

To this aim let Ω be star-shaped with respect to x_0 and assume by contradiction that (3-5) is violated. Let $u_\lambda(x) := u(x_0 + \lambda(x - x_0))$ for any $\lambda > 0$, so that $u_\lambda \in BV(\mathbb{R}^n)$, $u_\lambda(\cdot) \in \mathbb{N}$, while, by star-shapedness of Ω, one has $u_\lambda = 0$ outside of Ω for every $\lambda \geq 1$. Then there exists a $\lambda > 1$ such that (3-5) holds with u replaced by u_λ. However,

$$\int_\Omega \varphi^*(Du_\lambda) = \lambda^{1-n} \int_{x_0 + \lambda(\Omega - x_0)} \varphi^*(Du) = \lambda^{1-n} \int_\Omega \varphi^*(Du) < \int_\Omega \varphi^*(Du)$$

(the second equality is due to the fact that $\Omega \subset x_0 + \lambda(\Omega - x_0)$ for $\lambda > 1$, while $u = 0$ outside of Ω), contradicting the minimality of u. $\qquad\square$

4. The convex two-dimensional case

In this section we shall assume that $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a convex open set.

Given $E \subset \mathbb{R}^2$ and an $r > 0$ we define the set $E^r \subset E$ by the formula

$$(4\text{-}1) \qquad\qquad E^r := \begin{cases} \bigcup\{W_r(x): \ W_r(x) \subset E\} & \text{if } r > 0, \\ E & \text{if } r = 0. \end{cases}$$

Notice that, if E is a convex set, then E^r is convex and satisfies the rW_φ-condition. The set E^r is called the *Wulff plaquette* of radius r relative to E.

The following assertion holds:

Lemma 4.1. *Let $E \subset \mathbb{R}^2$ be a convex open set satisfying the rW_φ-condition for some $r > 0$. Then $E = E^r$.*

Proof. One has $E^r \subset E$. On the other hand, $\partial E \subset \partial E^r$ because E satisfies the rW_φ-condition. Minding that E, and hence E^r, is convex, we get $E = E^r$. $\qquad\square$

The convexity of set E is essential in Lemma 4.1. In fact, if A, B and C are the vertices of an equilateral triangle $\triangle ABC$ with side length 1, then letting

$$E := B_{1/2}(A) \cup B_{1/2}(B) \cup B_{1/2}(C) \cup \triangle ABC$$

we have that E satisfies the $\frac{1}{2}W_\varphi$-condition with respect to the Euclidean norm, but

$$E^{1/2} = B_{1/2}(A) \cup B_{1/2}(B) \cup B_{1/2}(C) \neq E.$$

4A. *Isoperimetric sets.* We consider the constrained isoperimetric problem

$$(4\text{-}2) \qquad \min\{P_\varphi(E) : E \subset \Omega, |E| = m \in [0, |\Omega|]\},$$

which corresponds to the problem (1-1) under the additional constraint that u is a characteristic function. Clearly, the minimizers of this problem exist and the assertion of Proposition 3.4 remains valid for this problem.

Let $R_\Omega > 0$ be the maximal radius R such that $W_R(x) \subseteq \Omega$ for some $x \in \Omega$, and let $r_\Omega \in [0, R_\Omega]$ be the maximal radius r such that Ω satisfies the $r W_\varphi$-condition (we set for convenience $r_\Omega = 0$ if Ω does not satisfy any $r W_\varphi$-condition). Observe that in the Euclidean case one has

$$r_\Omega = \frac{1}{\|\kappa\|_{L^\infty(\partial\Omega)}}$$

where κ stands for the curvature of $\partial\Omega$.

Lemma 4.2. *Let $m \in (0, |\Omega|)$, and let E be a minimizer of* (4-2). *Then E is convex and there exists an $r > 0$ (depending on m) such that E satisfies the $r W_\varphi$-condition and each connected component of $\partial E \cap \Omega$ is contained in $\partial W_r(x)$, for some Wulff ball $W_r(x) \subset \Omega$ (with x depending on the connected component of $\partial E \cap \Omega$).*

Remark 4.3. Recall that here and in the sequel when speaking of the properties of a set E of finite perimeter we actually refer to the respective properties of the set of its density points. In particular, a minimizer E of (4-2) is not necessarily convex, but the set of its density points is (and hence, in particular, the closure $\bar{E}$ is convex).

Proof. STEP 1. We first show the convexity of E. As in [Ambrosio et al. 2001, Theorem 2] we can uniquely decompose E as a union of (measure theoretic) connected components $\{E_i\}_{i \in I}$, where I is finite or countable, such that

$$|E| = \sum_{i \in I} |E_i| \quad \text{and} \quad P_\varphi(E) = \sum_{i \in I} P_\varphi(E_i).$$

As in [Ambrosio et al. 2002, Proposition 6.12], one shows by the isoperimetric inequality and the minimality of E that the number of connected components is finite and the boundary of each connected component E_i is parametrized by a finite number of pairwise disjoint Jordan curves. In particular, the boundaries of two different connected components do not intersect. Further, using Lemma 6.9 from [Ambrosio et al. 2002], one has that the perimeter $P_\varphi(E_i)$ of a measure theoretic connected component E_i that has its boundary parametrized by Jordan curves $\{\theta_i^j\}_{j=1}^{N_i}$ (all parametrized, say, over [0, 1]) is given by

$$P_\varphi(E_i) = \sum_{j=1}^{N_i} \int_0^1 \psi(\dot{\theta}_i^j(t)) \, dt,$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is some convex and 1-homogeneous function (in fact, $\psi := \varphi^* \circ R$, R being the clockwise rotation of $\mathbb{R}^2$ by $\pi/2$; see Corollary 6.10 from [Ambrosio et al. 2002]). Hence, using Jensen inequality one shows that the convex envelope of E_i has lower (anisotropic) perimeter than E_i itself, and minding that is also has greater volume (as well as the fact that the assertion of Proposition 3.4 is valid for the problem (4-2)), one has that each E_i is convex.

Finally, if E is not connected, recalling that Ω is convex we can translate a connected component inside Ω in such a way that its boundary touches the boundary of another connected component (this changes neither the perimeter nor the volume), and taking the convex envelope of the resulting set we obtain again a set with greater volume and strictly lower anisotropic perimeter, hence a contradiction which shows that E is convex.

STEP 2. Reasoning as in [Novaga and Paolini 2005, Theorem 4.5], where the related problem

$$\min\{P_\varphi(E) - \lambda|E| : E \subset \Omega, \lambda \geq 0\}$$

was considered instead of (4-2), one gets that each connected component of $\partial E \cap \Omega$ is contained in $\partial W_r(x)$, for some $x \in \mathbb{R}^2$ and $r > 0$.

Moreover, as in [Ambrosio et al. 2002, Theorem 6.19] one can show the existence of a (possibly nonunique) Lipschitz continuous vector field $n : \partial E \to \mathbb{R}^2$ such that $n(x) \in \partial\varphi^*(\nu(x))$ for $\mathcal{H}^1$-a.e. $x \in \partial E$. In particular $\mathrm{div}_\tau n \in L^\infty(\partial E)$, where $\mathrm{div}_\tau n := \partial_\tau(n \cdot \tau)$ denotes the tangential divergence of n and corresponds to the anisotropic curvature of ∂E; see [Taylor 1975; Bellettini et al. 2001]. (Here and below τ and ν denote respectively the Euclidean unit tangent and exterior normal vectors to ∂E.)

Without loss of generality we may assume that $\mathrm{div}_\tau n$ is constant along every maximal segment contained in ∂E (if not, we can substitute n over the segment by a convex combination of its values on the endpoints of the segment; one would then still have $n \in \partial\varphi^*(\nu)$ along the segment because ν is constant there and $\partial\varphi^*(\cdot)$ is convex). In particular, if a connected component Σ of $\partial E \cap \Omega$ is contained in $\partial W_r(x)$, then $n(y) = (y - x)/(r\varphi(y - x))$ for $\mathcal{H}^1$-a.e. $y \in \Sigma$.

STEP 3. We now prove that E satisfies the rW_φ-condition for some $r > 0$. Since E is convex, it is enough to show that

(4-3) $$\mathrm{div}_\tau n \leq \frac{1}{r} \quad \mathcal{H}^1\text{-a.e. on } \partial E.$$

This follows by a local variation argument as in the proof of Lemma 4.9 below. Let us fix $x_1 \in \Sigma$, where Σ is a connected component of $\partial E \cap \Omega$, and $x_2 \in \partial E \setminus \bar{\Sigma}$. We know from the previous step that Σ is contained in $\partial W_r(x)$ for some $x \in \mathbb{R}^2$ and $r > 0$. We distinguish four cases.

<u>Case 1.</u> There are two disjoint open sets U_i, $i = 1, 2$, such that $x_i \in U_i$ and $U_i \cap \partial E$ do not contain segments. Let ψ_1, ψ_2 be two nonnegative smooth functions, with support on U_1, U_2 respectively, such that

$$(4\text{-}4) \qquad \int_{U_1 \cap \partial E} \psi_1(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z) = \int_{U_2 \cap \partial E} \psi_2(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z).$$

We consider a family of diffeomorphisms such that

$$\Psi(\varepsilon, x) := x + \varepsilon \psi_1(x) n(x) - \varepsilon \psi_2(x) n(x) + o(\varepsilon)$$

for $\varepsilon > 0$ small enough. By (4-4), the term $o(\varepsilon)$ can be chosen in such a way that

$$(4\text{-}5) \qquad\qquad |E^\varepsilon| = |E| \quad \text{for all } \varepsilon > 0 \text{ small enough,}$$

with $E^\varepsilon := \Psi(\varepsilon, E) \subset \Omega$. We then have

$$P_\varphi(E^\varepsilon) = P_\varphi(E) + \frac{\varepsilon}{r} \int_{U_1 \cap \partial E} \psi_1(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z)$$

$$- \varepsilon \int_{U_2 \cap \partial E} \psi_2(z) \mathrm{div}_\tau n(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z) + o(\varepsilon),$$

where ν stands for the exterior Euclidean unit normal to ∂E. As $\varepsilon \to 0^+$, by minimality of E, we get

$$\frac{1}{r} \int_{U_1 \cap \partial E} \psi_1(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z) \geq \int_{U_2 \cap \partial E} \psi_2(z) \mathrm{div}_\tau n(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z),$$

which in view of (4-4) gives (4-3).

<u>Case 2.</u> We can find two maximal segments ℓ_1, $\ell_2 \subset \partial E$ such that $x_i \in \ell_i$, and we define E^ε by shifting ℓ_1 by $c_1 \varepsilon$ parallel to itself outside E, and by shifting ℓ_2 by $c_2 \varepsilon$ inside of E, with c_1, c_2 so that (4-5) holds, that is

$$(4\text{-}6) \qquad\qquad c_1 |\ell_1| = c_2 |\ell_2|.$$

By [Novaga and Paolini 2005, Lemma 4.4] we have

$$P_\varphi(E^\varepsilon) = P_\varphi(E) + c_1 \alpha_1 \varepsilon - c_2 \alpha_2 \varepsilon + o(\varepsilon),$$

where α_1, α_2 are respectively the (Euclidean) length of the face of W_φ parallel to ℓ_1, ℓ_2. By minimality of E, letting $\varepsilon \to 0^+$ we obtain $c_1 \alpha_1 \geq c_2 \alpha_2$. Recalling (4-6), we finally get

$$\frac{1}{r} = \frac{\alpha_1}{|\ell_1|} \geq \frac{\alpha_2}{|\ell_2|} = \mathrm{div}_\tau n(z) \quad \text{for } z \in \ell_2.$$

Case 3. There is a maximal segment $\ell_1 \subset \partial E$ and an open set U_2 such that $x_1 \in \ell_1$, $x_2 \in U_2$ and $U_2 \cap \partial E$ does not contain segments. We proceed by combining the previous strategies and we define the set E^ε by shifting ℓ_1 by ε parallel to itself outside E, and then taking the image of the resulting set through the diffeomorphism

$$\Psi(\varepsilon, x) := x - \varepsilon \psi_2(x) n(x) + o(\varepsilon),$$

where ψ_2 is a nonnegative smooth function supported on U_2 satisfying

$$(4\text{-}7) \qquad \int_{U_2 \cap \partial E} \psi_2(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z) = |\ell_1|.$$

This condition guarantees that the volume change after these two operations is of order $o(\varepsilon)$, so that the extra term $o(\varepsilon)$ in the definition of Ψ is chosen in such a way that (4-5) holds. Reasoning as above, we get

$$P_\varphi(E^\varepsilon) = P_\varphi(E) + \alpha_1 \varepsilon - \varepsilon \int_{U_2 \cap \partial E} \psi_2(z) \mathrm{div}_\tau n(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z) + o(\varepsilon),$$

which gives, by minimality of E,

$$\alpha_1 = \frac{|\ell_1|}{r} \geq \int_{U_2 \cap \partial E} \psi_2(z) \mathrm{div}_\tau n(z) \varphi^*(\nu(z)) \, d\mathcal{H}^1(z),$$

which gives (4-3), recalling (4-7).

Case 4. There is a maximal segment $\ell_2 \subset \partial E$ and an open set U_1 such that $x_1 \in U_1$, $x_2 \in \ell_2$ and $U_1 \cap \partial E$ does not contain segments. This case can be dealt with reasoning as in the previous case, by shifting ℓ_2 by ε inside E and defining

$$\Psi(\varepsilon, x) := x + \varepsilon \psi_1(x) n_2(x) + o(\varepsilon).$$

STEP 4. From (4-3) it follows that the radius r in Step 3 does not depend on the connected component Σ. In particular, every connected component of $\partial E \cap \Omega$ is contained in $\partial W_r(x)$, for a fixed $r > 0$ (while x depends in general on the connected component). $\qquad \square$

Consider now the function $v(r) := |\Omega^r|$. It is clearly constantly equal to $|\Omega|$ for $r \leq r_\Omega$ and to zero for $r > R_\Omega$, while over $[r_\Omega, R_\Omega]$ it is continuous and monotone decreasing. In particular, for all $m \in [|\Omega^{R_\Omega}|, |\Omega|]$ there exists a unique value $r_m \in [r_\Omega, R_\Omega]$ such that $v(r_m) = m$.

From the isoperimetric inequality (2-3) and Lemma 4.2, we get the following statement.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^2$ be convex, and let E be a minimizer of* (4-2) *with $m \in [0, |\Omega|]$. Then either*

(a) $\bar{E} = \bar{\Omega}^{r_m}$, *if $m > |\Omega^{R_\Omega}|$, or*

(b) $\bar{E}$ is the closure of some convex union of Wulff balls of radius R_Ω, if $m \in [R_\Omega^2 |W_\varphi|, |\Omega^{R_\Omega}|]$, or

(c) $\bar{E} = \bar{W}_{\sqrt{m/|W_\varphi|}}(x)$ for some $x \in \Omega$, if $m \leq R_\Omega^2 |W_\varphi|$.

Proof. We can assume $m \in (0, |\Omega|)$. By Lemma 4.2, there exists an $r > 0$ (depending on m) such that $\bar{E}$ is the closure of a union of Wulff balls of radius r, hence $\bar{E} \subset \bar{\Omega}^r$ and $r \leq R_\Omega$.

If $m > |\Omega^{R_\Omega}|$, then necessarily $r < R_\Omega$ and $\bar{E} = \bar{\Omega}^r$, since otherwise we could find a connected component of $\partial E \cap \Omega$ that is not contained in the boundary of a Wulff ball, contradicting Lemma 4.2. In particular, we have $r = r_m$.

If $m \in [R_\Omega^2 |W_\varphi|, |\Omega^{R_\Omega}|]$ then $r = R_\Omega$, since otherwise $\bar{E}$ would coincide with the set $\bar{\Omega}^r$ (with $r < R_\Omega$), which has volume strictly greater than $|\Omega^{R_\Omega}|$.

If $m \leq R_\Omega^2 |W_\varphi|$ the result follows by the isoperimetric inequality (2-3). □

Remark 4.5. It is worth noticing that, if Ω is strictly convex, then there exists a *unique* ball $W_{R_\Omega}(x) \subset \Omega$, and thus $\Omega^{R_\Omega} = W_{R_\Omega}(x)$. In other words, the case (b) of the above Proposition 4.4 reduces to case (c). Therefore, either $\bar{E} = \bar{\Omega}^{r_m}$, if $m \geq |\Omega^{R_\Omega}|$, or $\bar{E} = \bar{W}_{\sqrt{m/|W_\varphi|}}(x)$ for some $x \in \Omega$, if $m \leq |\Omega^{R_\Omega}|$.

We now state an easy consequence of Proposition 4.4 showing that solutions to the problem (4-2) with decreasing volumes may be arranged as a decreasing sequence of sets.

Corollary 4.6. *Let Ω be convex and let m_j be a decreasing sequence such that $m_j \in (0, |\Omega|)$, for all j. There exists a sequence of sets E_j such that $E_{j+1} \subset E_j \subset \Omega$, $|E_j| = m_j$ and each E_j is a minimizer of (4-2) with $m := m_j$.*

Note that the convexity assumption on the set Ω is essential in the above result. In fact, reasoning as in [Kawohl and Lachand-Robert 2006, Section 6] with the example of Ω a couple of circles connected by a thin tube (like a barbell considered in [Kawohl and Lachand-Robert 2006, Section 6]), one provides a family of minimizers of (4-2) with decreasing volumes which cannot be arranged as a decreasing sequence of sets (see Figure 1).

4B. *Isoperimetric towers.* We return now to the original problem (1-1). Here and below we let $u \in L^1(\mathbb{R}^2)$ be an arbitrary minimizer of this problem and E_j be its level set corresponding to a $j \in \mathbb{N}$, as defined by (3-1). The following result follows directly from Corollary 4.6.

Proposition 4.7. *If Ω is convex, then for all $j \in \mathbb{N}$ the set E_j is a minimizer of the problem (4-2) with $m := |E_j|$ (in particular E_j is convex).*

Proof. If the assertion is not true, then considering a sequence of sets E'_j of minimizers of (4-2) (with $m := |E_j|$) such that $E'_{j+1} \subset E'_j \subset \Omega$, $|E'_j| := |E_j|$ (the
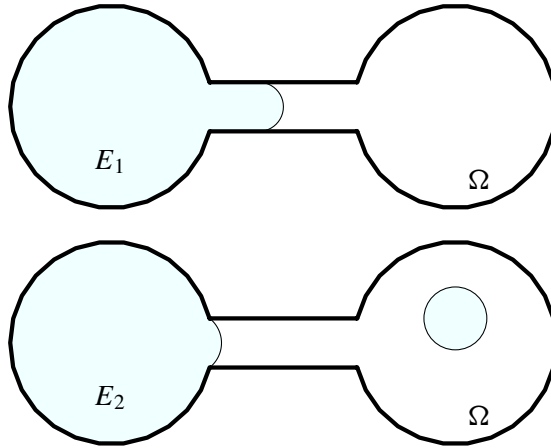
Figure 1. Example of $\Omega \subset \mathbb{R}^2$ nonconvex: two circles connected with a thin tube. E_1 and E_2 (which has two connected components) are two minimizers of (4-2) that are not included into one another.

existence of such a sequence is guaranteed by Corollary 4.6), and setting

$$u' := \sum_j \mathbf{1}_{E'_j},$$

we get

$$\int_{\mathbb{R}^2} \varphi^*(Du') = \sum_j P_\varphi(E'_j) < \sum_j P_\varphi(E_j) = \int_{\mathbb{R}^2} \varphi^*(Du),$$

the strict inequality being due to the fact that one of E_j is not a minimizer of (4-2) (with $m := |E_j|$) by assumption. On the other hand,

$$\int_\Omega u' \, dx = \int_\Omega u \, dx = m,$$

since the level sets of u' and u have the same volume by construction. This would mean that u is not a solution to the problem (1-1). $\qquad\square$

Remark 4.8. By Proposition 4.7 and Lemma 4.2, each set E_i is convex and each connected component of $\partial E_i \cap \Omega$ is contained in $\partial W_{r_i}(x_i)$ for some Wulff ball $W_{r_i}(x_i) \subset \Omega$.

Lemma 4.9. *Let S_i, S_j be connected components of $\partial E_i \cap \Omega$ and $\partial E_j \cap \Omega$, respectively, with $j > i$, such that*

$$(4\text{-}8) \quad S_i \subset \partial W_{r_i}(x_i) \subset \bar{\Omega}, \quad S_j \subset \partial W_{r_j}(x_j) \subset \bar{\Omega}, \quad \frac{1}{r_i}(S_i - x_i) \subset \frac{1}{r_j}(S_j - x_j),$$

for some $x_i, x_j \in \mathbb{R}^2$, $r_i, r_j > 0$. Then $r_i \geq r_j$.

Proof. It is enough to consider the case $j = i + 1$. We can also assume $S_i \neq S_{i+1}$, otherwise there is nothing to prove. As in Figure 2, there are two cases to consider.

<u>Case 1.</u> There are two points $y_i \in S_i$, $y_{i+1} \in S_{i+1}$ and two disjoint open sets $U_i \subset \Omega$ and $U_{i+1} \subset \Omega$ such that $y_i \in U_i$, $y_{i+1} \in U_{i+1}$, and that $U_i \cap S_i$ and $U_{i+1} \cap S_{i+1}$ do not contain segments. Consider a smooth function ψ_i with support on U_i. It generates a one-parameter family of diffeomorphisms of E_i defined by

$$\Psi_i(\varepsilon, x) := x - \varepsilon \psi_i(x) n_i(x)$$

for all sufficiently small $\varepsilon > 0$, where

$$n_i(x) := \frac{x - x_i}{r_i \varphi(x - x_i)} .$$

Consider now a one-parameter family $\{\Psi_{i+1}(\varepsilon, \cdot)\}$ of diffeomorphisms of E_{i+1} such that $\Psi_{i+1}(0, x) = x$ for all $x \in E_{i+1}$, $\Psi_{i+1}(\varepsilon, \cdot) - \mathrm{Id}$ is supported in U_{i+1} for all $\varepsilon > 0$, while

$$\Psi_{i+1}(\varepsilon, x) := x + \varepsilon \psi_{i+1}(x) n_{i+1}(x) + o(\varepsilon)$$

as $\varepsilon \to 0^+$, where ψ_{i+1} is some smooth function (with support in U_{i+1}), and

$$n_{i+1}(x) := \frac{x - x_{i+1}}{r_{i+1} \varphi(x - x_{i+1})} .$$

We choose Ψ_{i+1} so that the sets $E_i^\varepsilon := \Psi_i(\varepsilon, E_i)$ and $E_{i+1}^\varepsilon := \Psi_{i+1}(\varepsilon, E_{i+1})$ satisfy

$$|E_i^\varepsilon| + |E_{i+1}^\varepsilon| = |E_i| + |E_{i+1}|$$

for all sufficiently small $\varepsilon > 0$. Denote by ν_j the exterior Euclidean unit normal to ∂E_j. Since

$$|E_i^\varepsilon| = |E_i| - \varepsilon \int_{\partial E_i \cap U_i} \psi_i(z) \varphi^*(\nu_i(z)) \, d\mathcal{H}^1(z) + o(\varepsilon),$$

$$|E_{i+1}^\varepsilon| = |E_{i+1}| + \varepsilon \int_{\partial E_{i+1} \cap U_{i+1}} \psi_{i+1}(z) \varphi^*(\nu_{i+1}(z)) \, d\mathcal{H}^1(z) + o(\varepsilon),$$

as $\varepsilon \to 0^+$, we have

$$(4\text{-}9) \quad \int_{\partial E_i \cap U_i} \psi_i(z) \varphi^*(\nu_i(z)) \, d\mathcal{H}^1(z) = \int_{\partial E_{i+1} \cap U_{i+1}} \psi_{i+1}(z) \varphi^*(\nu_{i+1}(z)) \, d\mathcal{H}^1(z).$$

Letting now

$$u_\varepsilon := u - \mathbf{1}_{E_i} - \mathbf{1}_{E_{i+1}} + \mathbf{1}_{E_i^\varepsilon} + \mathbf{1}_{E_{i+1}^\varepsilon} = \sum_{\substack{k \neq i \\ k \neq i+1}} \mathbf{1}_{E_k} + \mathbf{1}_{E_i^\varepsilon} + \mathbf{1}_{E_{i+1}^\varepsilon},$$

we have $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$ for all sufficiently small $\varepsilon > 0$. Recall that

$$\int_\Omega \varphi^*(Du_\varepsilon) = \int_\Omega \varphi^*(Du) - \varepsilon \int_{\partial E_i \cap U_i} \frac{1}{r_i} \psi_i(z) \varphi^*(\nu_i(z)) \, d\mathcal{H}^1(z)$$

$$+ \varepsilon \int_{\partial E_{i+1} \cap U_{i+1}} \frac{1}{r_{i+1}} \psi_{i+1}(z) \varphi^*(\nu_{i+1}(z)) \, d\mathcal{H}^1(z) + o(\varepsilon).$$

As $\varepsilon \to 0^+$, by minimality of u, we get

$$-\frac{1}{r_i} \int_{\partial E_i \cap U_i} \psi_i(z) \varphi^*(\nu_i(z)) \, d\mathcal{H}^1(z) + \frac{1}{r_{i+1}} \int_{\partial E_{i+1} \cap U_{i+1}} \psi_{i+1}(z) \varphi^*(\nu_{i+1}(z)) \, d\mathcal{H}^1(z)$$
$$\geq 0,$$

which together with (4-9) implies the thesis.

<u>Case 2</u>. We can find two maximal line segments $\ell_i \subset S_i$ and $\ell_{i+1} \subset S_{i+1}$. We define then E_i^ε by shifting the segment ℓ_i by $c_i \varepsilon$ parallel to itself inside E_i and E_{i+1}^ε by shifting the segment ℓ_{i+1} parallel to itself outside of E_{i+1} by $c_{i+1}\varepsilon$ with c_i and c_{i+1} so as to satisfy

$$|E_i^\varepsilon| + |E_{i+1}^\varepsilon| = |E_i| + |E_{i+1}|$$

for all $\varepsilon > 0$ sufficiently small. Since

$$|E_i^\varepsilon| = |E_i| - c_i|\ell_i|\varepsilon + o(\varepsilon),$$
$$|E_{i+1}^\varepsilon| = |E_{i+1}| + c_{i+1}|\ell_{i+1}|\varepsilon + o(\varepsilon),$$

as $\varepsilon \to 0^+$, we have

(4-10) $$c_i|\ell_i| = c_{i+1}|\ell_{i+1}|.$$

Letting again, as in Case 1,

$$u_\varepsilon := u - \mathbf{1}_{E_i} - \mathbf{1}_{E_{i+1}} + \mathbf{1}_{E_i^\varepsilon} + \mathbf{1}_{E_{i+1}^\varepsilon} = \sum_{\substack{k \neq i \\ k \neq i+1}} \mathbf{1}_{E_k} + \mathbf{1}_{E_i^\varepsilon} + \mathbf{1}_{E_{i+1}^\varepsilon},$$

we have $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$ for all sufficiently small $\varepsilon > 0$. On the other hand, by [Novaga and Paolini 2005, Lemma 4.4],

$$\int_\Omega \varphi^*(Du_\varepsilon) = \int_\Omega \varphi^*(Du) - c_i \alpha_i \varepsilon + c_{i+1} \alpha_{i+1} \varepsilon + o(\varepsilon),$$

where α_i, α_{i+1} are the (Euclidean) lengths of the face of W_φ parallel to ℓ_i, ℓ_{i+1}, respectively. By minimality of u, letting $\varepsilon \to 0^+$ we obtain $c_i \alpha_i \leq c_{i+1} \alpha_{i+1}$. Recalling (4-10), we get $r_i = |\ell_i|/\alpha_i \geq |\ell_{i+1}|/\alpha_{i+1} = r_{i+1}$.

Notice that in this proof we do not have to deal with the situation depicted in Cases 3 and 4 of the proof of Lemma 4.2 due to condition (4-8). In fact, the latter implies that if S_i contains a line segment ℓ_i, then the line segment $\ell_j :=$

$x_j + (l_i - x_i)r_j/r_i$ is contained in S_j. Otherwise, if there is a neighborhood U_i of a point of S_i such that $S_i \cap U_i$ does not contain any line segment, then $U_j := x_j + (U_i - x_i)r_j/r_i$ is a neighborhood of a point in S_j such that $S_j \cap U_j$ does not contain any line segment. □
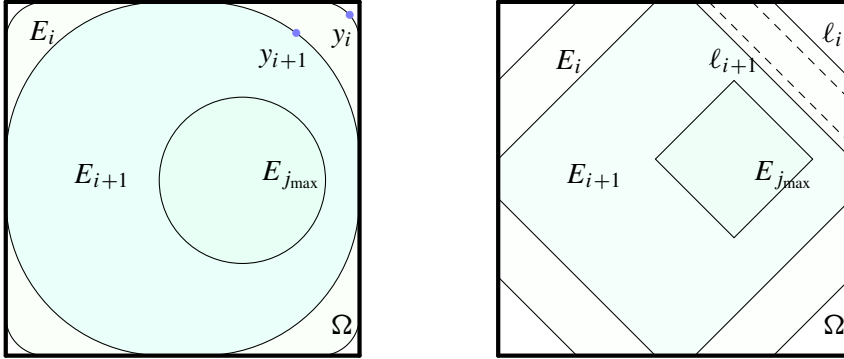
Figure 2. The two possible cases in the proof of Lemma 4.9.

We are now able to prove the following result giving the complete characterization of solutions to the problem (1-1).

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^2$ be convex and set $j_{\max} := \|u\|_\infty$. Then one of the following cases holds.*

(a) *There exists an $\bar{r} \in [r_\Omega, R_\Omega)$ such that $\bar{E}_j = \bar{\Omega}^{\bar{r}}$ for all $j \le j_{\max}$. In this case*

$$u = j_{\max} \mathbf{1}_{\Omega^{\bar{r}}}$$

   (in particular, if $\bar{r} = r_\Omega$, then $u = j_{\max} \mathbf{1}_\Omega$).

(b) *There exists an $\bar{r} \in (r_\Omega, R_\Omega)$ such that $\bar{E}_{j_{\max}} = \bar{W}_{\bar{r}}(x)$ for some $x \in \Omega$ such that $W_{\bar{r}}(x) \subset \Omega^{\bar{r}}$, and $\bar{E}_j = \bar{\Omega}^{\bar{r}}$ for all $j < j_{\max}$. In this case*

$$u = \mathbf{1}_{W_{\bar{r}}(x)} + (j_{\max} - 1)\mathbf{1}_{\Omega^{\bar{r}}}.$$

(c) *There exists an $\bar{r} \in (0, r_\Omega]$ such that $\bar{E}_{j_{\max}} = \bar{W}_{\bar{r}}(x)$ for some $x \in \Omega$ such that $W_{\bar{r}}(x) \subset \Omega$, and $\bar{E}_j = \bar{\Omega}$ for all $j < j_{\max}$. In this case*

$$u = \mathbf{1}_{W_{\bar{r}}(x)} + (j_{\max} - 1)\mathbf{1}_\Omega$$

   (note that this condition may hold only when $r_\Omega > 0$).

(d) *Every $\bar{E}_j$ is the closure of a convex union of Wulff balls of radius R_Ω for all $j \le j_{\max}$. In this case*

$$u = \sum_{j=1}^{j_{\max}} \mathbf{1}_{E_j}.$$

Remark 4.11. Observe that Case (d) of Theorem 4.10 is the only case where the number of nonzero level sets of the minimizer may be bigger than two.

Proof. We may assume $j_{\max} > 1$, since otherwise the result follows directly from Proposition 4.4.

By Remark 4.8, for all $i \le j_{\max}$ the set E_i is convex and each connected component of $\partial E_i \cap \Omega$ is contained, up to a translation, in $\partial W_{r_i}(x_i)$ for some $r_i > 0$, $x_i \in \mathbb{R}^2$. Moreover, if $\partial E_i \cap \Omega$ and $\partial E_{i+1} \cap \Omega$ are nonempty, from the inclusion $E_{i+1} \subset E_i$ it follows that we can always find two connected components $S_i \subset \partial E_i \cap \Omega$ and $S_{i+1} \subset \partial E_{i+1} \cap \Omega$ satisfying the assumptions of Lemma 4.9. By Lemma 4.9 we then get $r_i \ge r_{i+1}$ for all $i < j_{\max}$.

Recalling Propositions 4.7 and 4.4, this leaves only the following possibilities:

(i) $\bar{E}_i = \bar{\Omega}^{r_i}$, $\bar{E}_{i+1} = \bar{\Omega}^{r_{i+1}}$ with $r_i \ge r_{i+1}$. If $r_i > r_\Omega$ (hence $\Omega^{r_i} \ne \Omega$), then minding $\bar{E}_{i+1} \subset \bar{E}_i$ we have in this case $r_i = r_{i+1}$ hence $\bar{E}_i = \bar{E}_{i+1} = \bar{\Omega}^{r_i}$, while if $r_{i+1} \le r_i \le r_\Omega$ we have $\bar{E}_i = \bar{E}_{i+1} = \bar{\Omega}$, and we may just set $r_i = r_{i+1} := r_\Omega$ so that still $\bar{E}_i = \bar{E}_{i+1} = \bar{\Omega}^{r_i}$ (because $\Omega^{r_\Omega} = \Omega$).

(ii) $\bar{E}_i = \bar{\Omega}^{r_i}$, $\bar{E}_{i+1} = \bar{W}_{r_{i+1}}(x_{i+1})$ with $r_i \ge r_{i+1}$.

(iii) $\bar{E}_i = \bar{W}_{r_i}(x_i)$, $\bar{E}_{i+1} = \bar{W}_{r_{i+1}}(x_{i+1})$ with $r_i \ge r_{i+1}$.

(iv) $\bar{E}_i$ is a closure of some convex union of Wulff balls of radius R_Ω and $E_{i+1} = \bar{W}_{r_{i+1}}(x_{i+1})$ with $R_\Omega > r_{i+1}$.

(v) Both $\bar{E}_i$ and $\bar{E}_{i+1}$ are closures of some convex unions of Wulff balls of radius R_Ω.

(Note that the case when $\bar{E}_{i+1}$ is a closure of a convex union of Wulff balls of radius R_Ω and $\bar{E}_i = \bar{W}_{r_i}(x_i)$ with $R_\Omega < r_i$ is impossible.) Thus there is a $\bar{j} \in \{0, \dots, j_{\max}\}$ and an $r_1 \in [r_\Omega, R_\Omega)$ such that either

(A) for every $i \le \bar{j}$ one has $\bar{E}_i = \bar{\Omega}^{r_1}$ or

(B) for every $i \le \bar{j}$ each $\bar{E}_i$ is a closure of a convex union of Wulff balls of radius R_Ω (in particular, just a single closed Wulff ball),

while $\bar{E}_i = \bar{W}_{r_i}(x_i)$, $r_i < R_\Omega$ for all $i > \bar{j}$, with $\{r_i\}$ decreasing.

Consider now an arbitrary $i > \bar{j}$ such that $E_i \ne \emptyset$. Note that either $r_i \le r_1 < R_\Omega$ (Case A) or $r_i < R_\Omega$ (Case B).

It remains to show that $E_j = \emptyset$ for all $j > i$. Suppose the contrary, namely, that $E_{i+1} \ne \emptyset$. We may assume without loss of generality all level sets are open convex

by Proposition 4.7, and, further, $\bar{E}_{l+1} \subset E_l$ for all $l \in \{\bar{j}, \dots, j_{\max}\}$) (if not, from what has been already proven it follows that we may just shift appropriately all the respective level sets, which would maintain both $\int_\Omega \varphi^*(Du)$ and $\int_\Omega u$). Choose now $\varepsilon > 0$ and $\varepsilon' > 0$ sufficiently small so that for $r'_{i+1} := r_{i+1} - \varepsilon'$ and $r_i := r_i + \varepsilon$ one would have

$$(4\text{-}11) \qquad r_i'^2 + r_{i+1}'^2 = r_i^2 + r_{i+1}^2$$

and $\bar{W}_{r'_i}(x_i) \subset E_{i-1}$, $\bar{E}_{i+2} \subset W_{r'_{i+1}}(x_{i+1})$. From (4-11) one gets $\varepsilon' = (r_i/r_{i+1})\varepsilon + o(\varepsilon)$, and hence

$$\frac{P_\varphi(W_{r'_{i+1}}(x_{i+1})) + P_\varphi(W_{r'_i}(x_i))}{P_\varphi(W_{r_{i+1}}(x_{i+1})) + P_\varphi(W_{r_i}(x_i))} = \frac{r'_{i+1} + r'_i}{r_{i+1} + r_i} = 1 - \varepsilon \frac{r_{i+1} - r_i}{r_{i+1}(r_{i+1} + r_i)} + o(\varepsilon),$$

where the error term $o(\varepsilon)$ is negative when $r_i = r_{i+1}$. Therefore, representing u as $u = \tilde{u} + \mathbf{1}_{E_{i+1}} + \mathbf{1}_{E_i}$, and letting

$$u'_\varepsilon := \tilde{u} + \mathbf{1}_{W_{r'_{i+1}}(x_{i+1})} + \mathbf{1}_{W_{r'_i}(x_i)},$$

we get $\int_\Omega \varphi^*(Du'_\varepsilon) < \int_\Omega \varphi^*(Du)$ for sufficiently small $\varepsilon > 0$; but $\int_{\mathbb{R}^2} u \, dx = \int_{\mathbb{R}^2} u'_\varepsilon \, dx$, contrary to the optimality of u, which proves the claim.

One has therefore either $\bar{j} = j_{\max} - 1$ or $\bar{j} = j_{\max}$, which concludes the proof. $\square$

5. An explicit example

5A. *A square with Euclidean norm.* Let now $\Omega := [0, 1]^2$ and let φ be the Euclidean norm on $\mathbb{R}^2$. From Theorem 4.10 we obtain the following characterization for the minimizers of (1-1).

Proposition 5.1. *Let* $\Omega = [0, 1]^2$.

(1) *If* $m \in (n - 1, n\pi/4)$, *with* $1 \le n \le 4$, *we have* $j_{\max} = n$, $\bar{E}_{j_{\max}} = \bar{B}_r(x_0) \subset \bar{\Omega}$ *and* $\bar{E}_j = \bar{\Omega}^r$ *for* $j < j_{\max}$, *with*

$$r = \sqrt{\frac{n - m - 1}{4(n - 1) - n\pi}}\,.$$

(2) *If* $m \in [n\pi/4, n]$, *with* $1 \le n \le 4$, *we have* $j_{\max} = n$ *and* $\bar{E}_j = \bar{\Omega}^r$ *for* $j \le j_{\max}$, *with* $r = \sqrt{(1 - m/n)/(4 - \pi)}$.

(3) *If* $m > 4$ *we have*

$$(5\text{-}1) \qquad j_{\max} \in \left\{ \left\lfloor \frac{2 + \sqrt{\pi}}{2\sqrt{\pi}} m \right\rfloor, \left\lfloor \frac{2 + \sqrt{\pi}}{2\sqrt{\pi}} m \right\rfloor + 1 \right\}$$

and $\bar{E}_j = \bar{\Omega}^r$ *for* $j \le j_{\max}$, *with* $r = \sqrt{(1 - m/j_{\max})/(4 - \pi)}$.

Proof. By Theorem 4.10 for all $m > 0$ we have one of the following two possibilities.

Case A. $\bar{E}_j = \bar{\Omega}^r$ for all $j \leq j_{\max}$ with

$$m = j_{\max}|\Omega^r| = j_{\max}\left(1 - (4-\pi)r^2\right) \quad r \in \left[0, \tfrac{1}{2}\right].$$

It then follows that

$$r = r_A(j_{\max}) := \sqrt{\frac{j_{\max} - m}{(4-\pi)j_{\max}}}$$

and $\sum_{j=1}^{j_{\max}} P(E_j) = F_A(j_{\max})$, where

$$F_A(x) := x\,P(\Omega^{r_A(x)}) = 4x - 2\sqrt{4-\pi}\,\sqrt{x(x-m)}\,.$$

Notice that

$$F_A'(x) = 4 - \sqrt{4-\pi}\,\frac{2x-m}{\sqrt{x(x-m)}}\,,$$

which implies that $F_A(x)$ is increasing for $x > ((2+\sqrt{\pi})/2\sqrt{\pi})m$, while it is decreasing for $m \leq x < ((2+\sqrt{\pi})/2\sqrt{\pi})m$. As a consequence we have

$$(5\text{-}2) \qquad j_{\max} \in \{j^A, j^A+1\}, \quad \text{where } j^A := \left\lfloor \frac{2+\sqrt{\pi}}{2\sqrt{\pi}}m \right\rfloor.$$

Case B. $\bar{E}_{j_{\max}} = \bar{B}_r(x_0) \subset \bar{\Omega}$ and $\bar{E}_j = \bar{\Omega}^r$ for all $j < j_{\max}$ with

$$m = \pi r^2 + (j_{\max} - 1)|\Omega^r|$$

$$= (j_{\max} - 1)\left(1 - \left(4 - \frac{j_{\max}}{j_{\max} - 1}\pi\right)r^2\right) \quad r \in \left(0, \tfrac{1}{2}\right).$$

It follows

$$r_B(j_{\max}) := \sqrt{\frac{j_{\max} - 1 - m}{(4-\pi)(j_{\max} - 1) - \pi}} > r_A(j_{\max} - 1)$$

and $\sum_{j=1}^{j_{\max}} P(E_j) = F_B(j_{\max})$, where

$$F_B(x) := (x-1)P(\Omega^{r_B(x)}) + 2\pi r_B(x)$$

$$= 4(x-1) - 2\sqrt{4-\pi}\,\sqrt{\left(x - 1 - \frac{\pi}{4-\pi}\right)(x-1-m)}\,.$$

Notice that the derivative

$$F_B'(x) = 4 - \sqrt{4-\pi}\,\frac{2(x-1) - m - \dfrac{\pi}{4-\pi}}{\sqrt{\left(x - 1 - \dfrac{\pi}{4-\pi}\right)(x-1-m)}}\,.$$

Assuming $x > 4/(4 - \pi)$, we then have that F_B is increasing for

$$x - 1 > \frac{2 + \sqrt{\pi}}{2\sqrt{\pi}} m - \frac{\sqrt{\pi}}{2(2 + \sqrt{\pi})}$$

and decreasing otherwise, so that

(5-3) $\qquad j_{\max} \in \{j^B, j^B + 1\}, \quad$ where $\; j^B := \left\lfloor \dfrac{2 + \sqrt{\pi}}{2\sqrt{\pi}} m + \dfrac{4 + \sqrt{\pi}}{2(2 + \sqrt{\pi})} \right\rfloor$

as soon as $j_{\max} \geq 5$.

Observe that, if $m < 5\pi/4$, we have $j_{\max} \leq 4$ and there is only one choice for the minimizers E_j. In particular, we are in Case A or Case B depending on the value of m. On the other hand, when $m > 5\pi/4$, we have to determine which one between Cases A and B is energetically more convenient. However, since $\min\{F_B(j^B), F_B(j^B + 1)\} > \min\{F_A(j^A), F_A(j^A + 1)\}$ for all $m > 5\pi/4$, it follows that Case B can never occur as a minimizer, thus implying the thesis. $\qquad \square$

It is worth remarking that $2\pi/(2 + \pi)$ is the volume of the (unique) Cheeger set C_Ω of Ω, so that Proposition 5.1 implies that the functions $u_m/j_{\max}$ converge to the characteristic function of C_Ω, according to the Remark 3.2.

5B. *A square with a crystalline norm.* Now we set $\Omega = [0, 1]^2$ as above and $\varphi(v) = \max\{|v_1|, |v_2|\}$. Notice that φ is a crystalline norm with Wulff shape

$$W_\varphi = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}.$$

As before, we are able to characterize completely the minimizers of (1-1).

Proposition 5.2. *Let Ω and φ be as above.*

(i) *If $m \in \left(0, \frac{1}{2}\right]$, we have $j_{\max} = 1$ and $\bar{E}_1 = \bar{W}_r(x_0) \subset \bar{\Omega}$, with $r = \sqrt{m/2}$.*

(ii) *If $m \in \left[\frac{1}{2}, 1\right)$, we have $j_{\max} = 1$ and $\bar{E}_1 = \bar{\Omega}^r$, with $r = \sqrt{(1 - m)/2}$.*

(iii) *If $m = 1$, then either $j_{\max} = 1$ and $\bar{E}_1 = \Omega$, or $j_{\max} = 2$, $\bar{E}_1 = \bar{\Omega}^r$ and $\bar{E}_2 = \bar{W}_r(x_0) \subset \Omega$, with $r \in \left(0, \frac{1}{2}\right]$.*

(iv) *If $m > 1$, we have*

(5-4) $\qquad j_{\max} \in \left\{ \left\lfloor \dfrac{1 + \sqrt{2}}{2} m \right\rfloor, \left\lfloor \dfrac{1 + \sqrt{2}}{2} m \right\rfloor + 1 \right\}$

and $\bar{E}_j = \bar{\Omega}^r$ for $j \leq j_{\max}$, with $r = \sqrt{(1 - m/j_{\max})/2}$.

Proof. The proof is similar to that of Proposition 5.1.

If $m \leq \frac{1}{2}$, then $j_{\max} = 1$ and $\bar{E}_1 = \bar{W}_r(x_0) \subset \Omega$, since the (rescaled) Wulff shape solves the isoperimetric problem. By Theorem 4.10, for all $m \geq \frac{1}{2}$ we have one of the following two possibilities.

<u>Case A.</u> $\bar{E}_j = \bar{\Omega}^r$ for all $j \le j_{\max}$ with

$$m = j_{\max}|\Omega^r| = j_{\max}(1 - 2r^2), \quad r \in \left[0, \tfrac{1}{2}\right],$$

which gives

$$r = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{m}{j_{\max}}},$$

and $\sum_{j=1}^{j_{\max}} P_\varphi(E_j) = F_A(j_{\max})$, where

$$F_A(x) = x(4 - 4r) = 4x - 4\sqrt{\frac{x^2 - mx}{2}}.$$

Since the function F_A is increasing for $x > (1 + \sqrt{2})m/2$ and decreasing for $m \le x < (1 + \sqrt{2})m/2$, we have

$$j_{\max} \in \{j^A, j^A + 1\}, \quad \text{where } j^A := \left\lfloor \frac{1 + \sqrt{2}}{2}m \right\rfloor.$$

<u>Case B.</u> $\bar{E}_{j_{\max}} = \bar{W}_r(x_0) \subset \bar{\Omega}$ and $\bar{E}_j = \bar{\Omega}^r$ for all $j < j_{\max}$, with $r \in \left(0, \tfrac{1}{2}\right]$ and

$$m = 2r^2 + (j_{\max} - 1)\left(1 - 2r^2\right),$$

and hence $m \ge 1$ because $j_{\max} \ge 2$ and $r \le \tfrac{1}{2}$.

If $m = 1$ then $j_{\max} = 2$ and we can take any $r \in \left(0, \tfrac{1}{2}\right]$.

If $m > 1$ then $j_{\max} \ge m + 1$ and we get

$$r = \sqrt{\frac{j_{\max} - m - 1}{2(j_{\max} - 2)}}$$

and $\sum_{j=1}^{j_{\max}} P_\varphi(E_j) = F_B(j_{\max})$, where

$$F_B(x) = 4(x - 1) - 4(x - 2)\sqrt{\frac{x - m - 1}{2(x - 2)}}.$$

Since the function F_B is increasing for $x > (1+\sqrt{2})m/2 + (3-\sqrt{2})/2$ and decreasing otherwise, we have

$$j_{\max} \in \{j^B, j^B + 1\}, \quad \text{where } j^B := \left\lfloor \frac{1 + \sqrt{2}}{2}m + \frac{3 - \sqrt{2}}{2} \right\rfloor.$$

As in the proof of Proposition 5.1, when $m > 1$ we have to determine which of Cases A and B is energetically more convenient. Since $\min\{F_B(j^B), F_B(j^B+1)\} > \min\{F_A(j^A), F_A(j^A + 1)\}$ (by a calculation as in the example with the Euclidean norm), it follows that Case B can never occur. $\qquad \square$

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