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TOPOLOGICAL PRESSURES FOR ϵ-STABLE AND STABLE SETS

XIANFENG MA AND ERCAI CHEN

In this paper, topological pressures of the preimages of ϵ-stable sets and certain closed subsets of stable sets in positive entropy systems are investigated. It is shown that the topological pressure of any topological system can be calculated in terms of the topological pressure of the preimages of ϵ-stable sets. For the constructed closed subset (W. Huang, Commun. Math. Phys. 279, 535–557 (2008)) of the stable set or the unstable set of any point in a measure-theoretic "rather big" set of a topological system with positive entropy, especially for the weakly mixing subset contained in the closure of the stable and unstable sets, it is proved that topological pressures of these subsets can be no less than the measure-theoretic pressure.

1. Introduction

Let (X, T) be a *topological dynamical system* (TDS) in the sense that X is a compact metric space with a compatible metric d and $T : X \to X$ is a homeomorphism. A TDS is said to be *noninvertible* if the map is surjective and continuous but not one-to-one. For $x \in X$ and $\epsilon > 0$, the ϵ-*stable* set of x under T is the set of points whose forward orbit ϵ-shadows that of x:

$$W^s_\epsilon(x, T) = \{y \in X : d(T^n x, T^n y) \le \epsilon \text{ for all } n \ge 0\}.$$

The preimages of these sets can be nontrivial and hence disperse at a nonzero exponent rate. the dispersal rate function $h_s(T, x, \epsilon)$ was introduced in [Fiebig et al. 2003]. The relationship between $h_s(T, x, \epsilon)$ and the topological entropy $h_{\text{top}}(T)$ was also investigated. It was proved that when X has finite covering dimension, for all $\epsilon > 0$,

$$\sup_{x\in X} h_s(T, x, \epsilon) = h_{\text{top}}(T).$$

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In [Huang 2008], the finite-dimensionality hypothesis turns out to be redundant. This equality is proved to be always true for any noninvertible TDS.

It is known that certain results concerning topological entropy can be generalized to topological pressure. For any $f \in C(X, \mathbb{R})$, consider the *topological pressure of the preimages of the ϵ-stable set of x*:

$$P(T, f, x, \epsilon) = \lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(T, f, \delta, T^{-n} W^s_\epsilon(x, T)),$$

where

$$P_n(T, f, \delta, T^{-n} W^s_\epsilon(x, T)) \\ = \sup \left\{ \sum_{x \in E} \exp f_n(x) : E \text{ is an } (n, \delta)\text{-separated subset of } T^{-n} W^s_\epsilon(x, T) \right\},$$

and $f_n(x) = \sum_{i=0}^{n-1} f \circ T^i(x)$. We show that the topological pressure of any noninvertible TDS with positive metric entropy can be calculated in terms of the topological pressure of the preimages of ϵ-stable sets. That is, for all $\epsilon > 0$,

$$\sup_{x \in X} P(T, f, x, \epsilon) = P(T, f),$$

where $P(T, f)$ is the standard notion of the topological pressure. For the null function f, this equality is the above one for the topological entropy.

For $x \in X$, the *stable set* $W^s(x, T)$ and the *unstable set* $W^u(x, T)$ of x are defined as

$$W^s(x, T) = \{y \in X : \lim_{n \to +\infty} d(T^n x, T^n y) = 0\},$$
$$W^u(x, T) = \{y \in X : \lim_{n \to +\infty} d(T^{-n} x, T^{-n} y) = 0\}.$$

For Anosov diffeomorphisms on a compact manifold, pairs belonging to the stable set are asymptotic under T and tend to diverge under T^{-1}. However, Blanchard et al. [2002] showed that in most case, this phenomenon does not happen in a TDS with positive metric entropy. N. Sumi [2003] investigated the stable and unstable sets of C^2 diffeomorphisms of C^∞ manifolds with positive metric entropy. He showed that the closure of the stable set $W^s(x, T)$ of “many points” is a perfect $*$-chaotic set and the closure of the unstable set $W^u(x, T)$ contains a perfect $*$-chaotic set. W. Huang [2008] got further information in the general noninvertible TDS with positive metric entropy. He proved that there exists a measure-theoretically “rather big” set such that the closure of the stable or unstable sets of points in the set contains a weakly mixing set. The Bowen entropies of these sets were also estimated there. It was proved that the lower bound is the usual metric entropy $h_\mu(T)$ for the ergodic invariant measure μ.

By introducing the topological pressure for the closed subset and using the excellent partition formed in Lemma 4 of [Blanchard et al. 2002], we show that, for the constructed closed subsets of stable and unstable sets in [Huang 2008], the topological pressure of these sets can also be estimated. More precisely, we prove that if μ is an ergodic invariant measure of a TDS (X, T) with $h_\mu(T) > 0$, then, for μ-a.e. $x \in X$, the closed subsets

$$A(x) \subseteq W^s(x, T), \quad B(x) \subseteq W^u(x, T)$$

and the weakly mixing subset

$$E(x) \subseteq \overline{W^s(x, T)} \cap \overline{W^u(x, T)}$$

constructed in [Huang 2008] have the following properties:

(a) $\lim_{n\to+\infty} \operatorname{diam} T^n A(x) = 0$ and $P(T^{-1}, f, A(x)) \geq P_\mu(T, f)$,

(b) $\lim_{n\to+\infty} \operatorname{diam} T^{-n} B(x) = 0$ and $P(T, f, B(x)) \geq P_\mu(T, f)$,

(c) $P(T, f, E(x)) \geq P_\mu(T, f)$ and $P(T^{-1}, f, E(x)) \geq P_\mu(T, f)$,

where $P_\mu(T, f)$ is the measure-theoretic pressure.

The paper is organized as follows. In Section 2, the topological pressure for the closed subset of a TDS is introduced. Some related notions and results about entropy are also listed. In Section 3, the topological pressure of the preimages of an ϵ-stable set is introduced. Using the tool formed in [Blanchard et al. 2002], we show that the topological pressure of any TDS can be calculated in terms of the topological pressure of the preimages of an ϵ-stable set. As a generalization of the entropy point, the notion of the pressure point is also introduced. In Section 4, results (a)–(c) above are proved. In Section 5, the results in sections 3 and 4 are stated and proved for the noninvertible TDS.

2. Preliminaries

Let (X, T) be a TDS and $\mathcal{B}_X$ be the σ-algebra of all Borel subsets of X. Recall that a *cover* of X is a finite family of Borel subsets of X whose union is X, and a *partition* of X is a cover of X whose elements are pairwise disjoint. We denote the set of covers, partitions, and open covers, of X by $\mathcal{C}_X$, $\mathcal{P}_X$, and $\mathcal{C}_X^o$. Given a partition α of X and $x \in X$, denote by $\alpha(x)$ the atom of α containing x. For two given covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denoted by $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Given integers M, N with $0 \leq M \leq N$ and $\mathcal{U} \in \mathcal{C}_X$, we set

$$\mathcal{U}_M^N = \bigvee_{n=M}^{N} T^{-n}\mathcal{U}.$$

Given $\mathcal{U} \in \mathcal{C}_X$ and $K \subset X$, put

$$N(\mathcal{U}, K) = \min\Big\{\text{the cardinality of } \mathcal{F} : \mathcal{F} \subset \mathcal{U}, \bigcup_{F \in \mathcal{F}} F \supset K\Big\}$$

and $H(\mathcal{U}, K) = \log N(\mathcal{U}, K)$. Then the topological entropy of $\mathcal{U}$ with respect to T for the compact subset K is

$$h_{\text{top}}(T, \mathcal{U}, K) = \lim_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}, K) = \inf_{n \geq 1} \frac{1}{n} H(\mathcal{U}_0^{n-1}, K).$$

The topological entropy of T for the compact subset K is defined by $h_{\text{top}}(T, K) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} h_{\text{top}}(T, \mathcal{U}, K)$; and the topological entropy of T is defined by $h_{\text{top}}(T) = \sup_K h_{\text{top}}(T, K)$.

Let (X, T) be a TDS, K a closed subset of X, $\mathcal{U} \in \mathcal{C}_X^o$, and $f \in C(X, \mathbb{R})$, where $C(X, \mathbb{R})$ is the Banach space of all continuous, real-valued functions on X endowed with the supremum norm. We set

$$P_n(T, f, \mathcal{U}, K) = \inf\Big\{\sum_{V \in \mathcal{V}} \sup_{x \in V \cap K} \exp f_n(x) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1}\Big\}, \tag{1}$$

where $f_n(x) = \sum_{j=0}^{n-1} f(T^j x)$. When $V \cap K = \varnothing$, we let $\sup_{x \in V \cap K} \exp f_n(x) = 0$. Then the above definition is well defined. It is clear that if f is the null function, $P_n(T, 0, \mathcal{U}, K) = N(\mathcal{U}_0^{n-1}, K)$.

For $\mathcal{V} \in \mathcal{C}_X$, we let α be the Borel partition generated by $\mathcal{V}$ and define

$$\mathcal{P}^*(\mathcal{V}) = \{\beta \in \mathcal{P}_X : \beta \succeq \mathcal{V} \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha\}.$$

Lemma 2.1 [Ma et al. 2010, Lemma 2.1]. *Let M be a compact subset of X and let $f \in C(X, \mathbb{R})$, $\mathcal{V} \in \mathcal{C}_X$. Then*

$$\inf_{\substack{\beta \in \mathcal{C}_X \\ \beta \succeq \mathcal{V}}} \sum_{B \in \beta} \sup_{x \in B \cap M} f(x) = \min\Big\{\sum_{B \in \beta} \sup_{x \in B \cap M} f(x) : \beta \in \mathcal{P}^*(\mathcal{V})\Big\}.$$

Let $\mathcal{K}(X)$ be the collection of all nonempty closed subsets of X. For any nonempty subset A of X and $\epsilon > 0$, let $N(A, \epsilon) = \{x \in X : \text{dist}(x, A) < \epsilon\}$, where $\text{dist}(x, A) = \inf\{d(x, y) : y \in A\}$. The *Hausdorff metric* H_d on the space $\mathcal{K}(X)$ induced by the metric d is defined as

$$H_d(A, B) = \inf\{\epsilon : A \subset N(B, \epsilon) \text{ and } B \subset N(A, \epsilon)\} \quad \text{for any } A, B \subset X.$$

Then $(\mathcal{K}(X), H_d)$ constitutes a compact metric space.

Lemma 2.2. *Let (X, T) be a TDS, $\mathcal{U} \in \mathcal{C}_X^o$, and $f \in C(X, \mathbb{R}^+)$. Then the function*

$$F : K \to \inf\Big\{\sum_{V \in \mathcal{V}} \sup_{x \in V \cap K} f(x) : \mathcal{V} \in \mathcal{C}_X \textit{ and } \mathcal{V} \succeq \mathcal{U}\Big\}$$

is measurable from $\mathcal{K}(X)$ to $\mathbb{R}^+$, where $\sup_{x \in V \cap K} f(x) = 0$ for $V \cap K = \varnothing$.

Proof. By Lemma 2.1 it suffices to prove that for each $B \in \beta$, where $\beta \in \mathcal{P}^*(\mathcal{U})$, the function $F_B : K \to \sup_{x \in B \cap K} f(x)$ is measurable.

For each $r \in \mathbb{R}$, let $\mathcal{E}_r = \{K : \sup_{x \in B \cap K} f(x) > r\}$. Let $U = f^{-1}(r, +\infty)$. Then U is an open subset of X. For $r \geq 0$, if $B \cap U = \varnothing$, $\mathcal{E}_r = \varnothing$. If $B \cap U \neq \varnothing$, $\mathcal{E}_r = \{K : K \cap (B \cap U) \neq \varnothing\}$. Let α be the Borel partition generated by the open cover $\mathcal{U} = \{U_i\}_{i=1}^{s}$. Then each $A \in \alpha$ has the form $\big(\bigcap_{i \in L} U_i\big) \cap \big(\bigcap_{j \in M} U_j^c\big)$, where $L, M \subset \{1, \ldots, s\}$ and $L \cap M = \varnothing$. Note that, for each open subset W of X, the sets $\{K : K \cap (W \cap U) \neq \varnothing\}$ and $\{K : K \cap (W^c \cap U) \neq \varnothing\}$ — which equals $\{K : K \cap U \neq \varnothing\} \cap \big(\mathcal{K}(X) \backslash \{K : K \subset W\}\big)$ — are both measurable subsets of $\mathcal{K}(X)$. Then the set $\{K : K \cap (A \cap U) \neq \varnothing\}$ is measurable for each $A \in \alpha$. Since each atom B of β is the finite union of elements of α, it follows that $\mathcal{E}_r$ is a measurable subset of $\mathcal{K}(X)$. For $r < 0$, $\mathcal{E}_r = \mathcal{E}_0 \cup \{K : \sup_{x \in B \cap K} f(x) = 0\} = \mathcal{E}_0 \cup \{K : B \cap K = \varnothing\}$. Since $\{K : B \cap K = \varnothing\} = \mathcal{K}(X) \backslash \{K : B \cap K \neq \varnothing\}$ and $\{K : B \cap K \neq \varnothing\}$ is measurable, $\mathcal{E}_r$ is also measurable. Thus F_B is a measurable function. □

Let $K \in \mathcal{K}(X)$, $\mathcal{U} \in \mathcal{C}_X^o$, and $f \in C(X, \mathbb{R})$. We define $P(T, f, \mathcal{U}, K) = \limsup_{n \to \infty} (1/n) \log P_n(T, f, \mathcal{U}, K)$.

Let (X, T) be a TDS. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X, by $\mathcal{M}(X, T)$ the set of T-invariant measures, and by $\mathcal{M}^e(X, T)$ the set of ergodic measures. Then $\mathcal{M}^e(X, T) \subset \mathcal{M}(X, T) \subset \mathcal{M}(X)$, and $\mathcal{M}(X)$, $\mathcal{M}(X, T)$ are convex, compact metric spaces endowed with the weak*-topology.

Since the map f is a homeomorphism, it induces in a natural way a homeomorphism $\widehat{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ by $\widehat{T}(A) = T(A)$ for each $A \in \mathcal{K}(X)$. Then $(\mathcal{K}(X), \widehat{T})$ constitutes a TDS induced by (X, T).

For each $\hat{\mu} \in \mathcal{M}(\mathcal{K}(X), \widehat{T})$, the following lemma shows that the limit superior in the above definition can be obtained by the limit for $\hat{\mu}$-a.e. $K \in \mathcal{K}(X)$.

Lemma 2.3. *Let (X, T) be a TDS, $\mathcal{U} \in \mathcal{C}_X^o$, $f \in C(X, \mathbb{R})$, and $\hat{\mu} \in \mathcal{M}(\mathcal{K}(X), \widehat{T})$. Then, for $\hat{\mu}$-a.e. $K \in \mathcal{K}(X)$, $P(T, f, \mathcal{U}, K) = \lim_{n \to +\infty} (1/n) \log P_n(T, f, \mathcal{U}, K)$ exists.*

Proof. For any $n, m \in \mathbb{N}$, $\mathcal{V}_1 \succeq \mathcal{U}_0^{n-1}$, $\mathcal{V}_2 \succeq \mathcal{U}_0^{m-1}$, we have $\mathcal{V}_1 \vee T^{-n}\mathcal{V}_2 \succeq \mathcal{U}_0^{n+m-1}$. It follows that

$$\begin{aligned}
P_{n+m}(T, f, \mathcal{U}, K) &\leq \sum_{V_1 \in \mathcal{V}_1} \sum_{V_1 \in \mathcal{V}_2} \sup_{x \in V_1 \cap T^{-n} V_2 \cap K} \exp f_{n+m}(x) \\
&= \sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \in \mathcal{V}_2} \sup_{x \in V_1 \cap T^{-n} V_2 \cap K} \exp(f_n(x) + f_m(T^n x)) \\
&\leq \sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \in \mathcal{V}_2} \Big(\sup_{x \in V_1 K} \exp f_n(x) \cdot \sup_{z \in V_2 \cap T^n K} \exp f_m(z)\Big) \\
&= \Big(\sum_{V_1 \in \mathcal{V}_1} \sup_{x \in V_1 \cap K} \exp f_n(x)\Big)\Big(\sum_{V_2 \in \mathcal{V}_2} \sup_{z \in V_2 \cap T^n K} \exp f_m(z)\Big).
\end{aligned}$$

Since $\mathcal{V}_i$, $i = 1, 2$ is arbitrary,

$$P_{n+m}(T, f, \mathcal{U}, K) \le P_n(T, f, \mathcal{U}, K) \cdot P_m(T, f, \mathcal{U}, T^n K).$$

By the definition of $\widehat{T}$ and Lemma 2.2, we have that

$$\log P_n(T, f, \mathcal{U}, K) : \mathcal{K}(X) \to \mathbb{R} \cup \{-\infty\}$$

is a subadditive sequence of measurable functions. Then, by Kingman's subadditive ergodic theorem (see [Walters 1982]), we complete the proof. □

When $K = X$, $P(T, f, \mathcal{U}, X) = P(T, f, \mathcal{U})$, which is the local topological pressure defined by Huang and Yi [2007], clearly, $P(T, 0, \mathcal{U}, K) = h_{\mathrm{top}}(T, \mathcal{U}, K)$.

Given a partition $\alpha \in \mathcal{P}(X)$, $\mu \in \mathcal{M}(X)$ and a sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}_\mu$, let

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A) \log \mu(A),$$

$$H_\mu(\alpha \mid \mathcal{C}) = \sum_{A\in\alpha} \int_X -\mathbb{E}(1_A \mid \mathcal{C}) \log \mathbb{E}(1_A \mid \mathcal{C})\, d\mu,$$

where $\mathbb{E}(1_A \mid \mathcal{C})$ is the expectation of 1_A with respect to $\mathcal{C}$. One standard fact states that $H_\mu(\alpha \mid \mathcal{C})$ increases with respect to α and decreases with respect to $\mathcal{C}$. The measure-theoretic entropy of μ is defined as

$$h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T, \alpha),$$

where

$$h_\mu(T, \alpha) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) = \inf_{n\ge1} H_\mu(\alpha_0^{n-1}).$$

For each $f \in C(X, \mathbb{R})$, the *measure-theoretic pressure of* μ is defined as

$$P_\mu(T, f) = h_\mu(T) + \int_X f\, d\mu.$$

For a given $\mathcal{U} \in \mathcal{C}_X$, set

$$H_\mu(\mathcal{U}) = \inf_{\beta\in\mathcal{P}_X,\, \beta\succeq\mathcal{U}} H_\mu(\beta) \quad \text{and} \quad H_\mu(\mathcal{U} \mid \mathcal{C}) = \inf_{\beta\in\mathcal{P}_X,\, \beta\succeq\mathcal{U}} H_\mu(\beta \mid \mathcal{C}).$$

When $\mu \in \mathcal{M}(X, T)$ and $\mathcal{C}$ is T-invariant (that is, $T^{-1}\mathcal{C} = \mathcal{C}$), $H_\mu(\mathcal{U}_0^{n-1} \mid \mathcal{C})$ is a nonnegative subadditive sequence for a given $\mathcal{U} \in \mathcal{U}$. Let

$$h_\mu(T, \mathcal{U} \mid \mathcal{C}) = \lim_{n\to+\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1} \mid \mathcal{C}) = \inf_{n\ge1} H_\mu(\mathcal{U}_0^{n-1} \mid \mathcal{C}).$$

For $\mathscr{C} = \{\varnothing, X\}$ (mod μ), we write $H_\mu(\mathscr{U} \mid \mathscr{C})$ and $h_\mu(T, \mathscr{U} \mid \mathscr{C})$ as $H_\mu(\mathscr{U})$ and $h_\mu(T, \mathscr{U})$, respectively. Romagnoli [2003] proved that

$$h_\mu(T) = \sup_{\mathscr{U} \in \mathscr{C}_X^o} h_\mu(T, \mathscr{U}).$$

It is well known that, for $\beta \in \mathscr{P}_X$, $h_\mu(T, \beta) = h_\mu(T, \beta \mid P_\mu(T)) \le H_\mu(\beta \mid P_\mu(T))$, where $P_\mu(T)$ is the Pinsker σ-algebra of $(X, \mathscr{B}_\mu, \mu, T)$.

Lemma 2.4 [Huang 2008, Lemma 2.1]. *Let (X, T) be a TDS, $\mu \in \mathscr{M}(X, T)$, and $\mathscr{U} \in \mathscr{C}_X$. Then*

$$h_\mu(T, \mathscr{U}) = h_\mu(T, \mathscr{U} \mid P_\mu(T)).$$

For $\mathscr{U} \in \mathscr{C}_X^o$, $\mu \in \mathscr{M}(X, T)$ and $f \in C(X, \mathbb{R})$, we define the *measure-theoretic pressure for T with respect to $\mathscr{U}$* as

$$P_\mu(T, f, \mathscr{U}) = h_\mu(T, \mathscr{U}) + \int_X f \, d\mu.$$

Obviously,

$$P_\mu(T, f) = h_\mu(T) + \int_X f \, d\mu = \sup_{\mathscr{U} \in \mathscr{C}_X^o} h_\mu(T, \mathscr{U}) + \int_X f \, d\mu = \sup_{\mathscr{U} \in \mathscr{C}_X^o} P_\mu(T, f, \mathscr{U}).$$

Let (X, T) be a TDS, $\mu \in \mathscr{M}(X, T)$, and $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under μ. Then $(X, \mathscr{B}_\mu, \mu, T)$ is a Lebesgue system. If $\{\alpha_i\}_{i \in I}$ is a countable family of finite partitions of X, the partition $\alpha = \bigvee_{i \in I} \alpha_i$ is called a *measurable partition*. The sets $A \in \mathscr{B}_\mu$, which are unions of atoms of α, form a sub-σ-algebra of $\mathscr{B}_\mu$ by $\hat{\alpha}$ or α if there is no ambiguity. Every sub-σ-algebra of $\mathscr{B}_\mu$ coincides with a σ-algebra constructed in this way (mod μ).

Given a measurable partition α, put $\alpha^- = \bigvee_{n=1}^{\infty} T^{-n}\alpha$ and $\alpha^T = \bigvee_{n=-\infty}^{+\infty} T^{-n}\alpha$. Define in the same way $\mathscr{F}^-$ and $\mathscr{F}^T$ if $\mathscr{F}$ is a sub-σ-algebra of $\mathscr{B}_\mu$. It is clear that for a measurable partition α of X, we have

$$\widehat{\alpha^-} = (\hat{\alpha})^- \quad \text{and} \quad \widehat{\alpha^T} = (\hat{\alpha})^T \quad (\text{mod } \mu).$$

Let $\mathscr{F}$ be a sub-σ-algebra of $\mathscr{B}_\mu$ and α be the measurable partition of X with $\alpha^- = \mathscr{F}$ (mod μ). μ can be disintegrated over $\mathscr{F}$ as $\mu = \int_X \mu_x \, d\mu(x)$, where $\mu_x \in \mathscr{M}(X)$ and $\mu_x(\alpha(x)) = 1$ for μ-a.e. $x \in X$. The disintegration is characterized by two properties:

(a) For every $f \in L^1(X, \mathscr{B}_X, \mu)$, $f \in L^1(X, \mathscr{B}_X, \mu_x)$ for μ-a.e. $x \in X$, and the map $x \mapsto \int_X f(y) \, d\mu_x(y)$ is in $L^1(X, \mathscr{F}, \mu)$.

(b) For every $f \in L^1(X, \mathscr{B}_X, \mu)$, $\mathbb{E}_\mu(f \mid \mathscr{F})(x) = \int_X f \, d\mu_x$ for μ a.e. $x \in X$.

Then, for any $f \in L^1(X, \mathscr{B}_X, \mu)$,

$$\int_X \left(\int_X f \, d\mu_x \right) d\mu(x) = \int_X f \, d\mu.$$

Lemma 2.5 [Huang 2008, Lemma 2.2]. *Let (X, T) be a TDS, $\mu \in \mathcal{M}(X, T)$, and $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{B}_\mu$. If $\mu = \int_X \mu_x \, d\mu(x)$ is the disintegration of μ over $\mathcal{F}$,*

(a) *for $\mathcal{V} \in \mathcal{C}_X$, $H_\mu(\mathcal{V} \mid \mathcal{F}) = \int_X H_{\mu_x}(\mathcal{V}) \, d\mu(x)$,*

(b) *for $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $H_\mu(\mathcal{U} \vee \mathcal{V} \mid \mathcal{F}) \le H_\mu(\mathcal{U} \mid \mathcal{F}) + H_\mu(\mathcal{V} \mid \mathcal{F})$.*

Let K be a nonempty closed subset of X. For $\epsilon > 0$, a subset of X is called an (n, ϵ)-*spanning set* of K, if for any $x \in K$ there exists $y \in F$ with $d_n(x, y) \le \epsilon$, where $d_n(x, y) = \max_{i=0}^{n-1} d(T^i x, T^i y)$; a subset E of K is called an (n, ϵ)-*separated set* of K, if $x, y \in E$, $x \ne y$ implies $d_n(x, y) > \epsilon$. Let $r_n(d, T, \epsilon, K)$ denote the smallest cardinality of any (n, ϵ)-spanning subset for K and $s_n(d, T, \epsilon, K)$ denote the largest cardinality of any (n, ϵ)-separated subset of K.

For each $\epsilon > 0$ and $f \in C(X, \mathbb{R})$, we define

$$P_n(T, f, \epsilon, K) = \sup\Big\{ \sum_{x \in E} \exp f_n(x) : E \text{ is an } (n, \epsilon)\text{-separated subset of } K \Big\}.$$

The topological pressure of T for the closed subset K is defined as

$$P(T, f, K) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(T, f, \epsilon, K).$$

Clearly, for $f \equiv 0$, we can write $P_n(T, 0, \epsilon, K) = s_n(d, T, \epsilon, K)$. It follows that $P(T, f, K) = h(T, K)$, where $h(T, K)$ is the Bowen entropy for the closed subset K defined in [Walters 1982]; see also [Huang 2008]. When $K = X$, $P(T, f, X) = P(T, f)$, where $P(T, f)$ is the standard notion of topological pressure defined in [Walters 1982]. Moreover, it is not hard to verify that $P(T, f, K) = \sup_{\mathcal{U} \in \mathcal{C}_X^o} P(T, f, \mathcal{U}, K)$.

3. ϵ-stable sets

Let (X, T) be a TDS with a compatible metric d. Given $\epsilon > 0$, the ϵ-*stable set* of x under T is the set of points whose forward orbit ϵ-shadows that of x:

$$W_\epsilon^s(x, T) = \{y \in X : d(T^n x, T^n y) \le \epsilon \text{ for all } n = 0, 1, \ldots\}.$$

Since the preimages of these sets can be nontrivial, we can consider the following function. For each $x \in X$, $f \in C(X, \mathbb{R})$, and $\epsilon > 0$, let

$$P_s(T, f, x, \epsilon) := \lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(T, f, \delta, T^{-n} W_\epsilon^s(x, T)).$$

$P_s(T, f, x, \epsilon)$ is called the *topological pressure of the preimages of the ϵ-stable sets of x*. For $f \equiv 0$, $P_s(T, 0, x, \epsilon) = h_s(T, x, \epsilon)$, where the latter is the dispersal rate function defined in [Fiebig et al. 2003]. It was proved in [Huang 2008] that $\sup_{x \in X} h_s(T, x, \epsilon) = h_{\text{top}}(T)$ for all $\epsilon > 0$. In the present section, we show that this is also true for the functions $P_s(T, f, x, \epsilon)$ and $P(T, f)$. By proving that, for any $\mu \in \mathcal{M}^e(X, T)$ with positive entropy, $\lim_{\epsilon\to 0} P_s(T, f, x, \epsilon) \geq P_\mu(T, f)$ for μ-a.e. $x \in X$, we can obtain the result. We need the following lemmas.

Lemma 3.1. *Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$, and $\{K_n\}$ be a sequence of nonempty closed subsets of X. Then*

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \delta, K_n) = \sup_{\mathcal{U}\in\mathcal{C}_X^0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, K_n).$$

Proof. For a fixed $\delta > 0$, choose $\mathcal{V} \in \mathcal{C}_X^o$ with $\operatorname{diam} \mathcal{V} < \delta$. For $n \in \mathbb{N}$ let A be an (n, δ)-separated set of K_n. Since $B \cap K_n$ contains at most one element of A for each B of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{V}$, for every $\mathcal{W} \in \mathcal{C}_X$ with $\mathcal{W} \succeq \mathcal{V}_0^{n-1}$, each element of $\mathcal{W}$ also contains at most one element of A. We get $\sum_{x\in A} \exp f_n(x) \leq P_n(T, f, \mathcal{V}, K_n)$. That is $P_n(T, f, \delta, K_n) \leq P_n(T, f, \mathcal{V}, K_n)$. Then

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \delta, K_n) &\leq \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{V}, K_n)\\ &\leq \sup_{\mathcal{U}\in\mathcal{C}_X^o} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, K_n).\end{aligned}$$

Letting $\delta \to 0$, we get

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \delta, K_n) \leq \sup_{\mathcal{U}\in\mathcal{C}_X^0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, K_n).$$

In the following, we show the converse inequality. For any fixed $\mathcal{U} \in \mathcal{C}_X^o$, let δ be the Lebesgue number of $\mathcal{U}$. For $n \in \mathbb{N}$, let E be an $(n, \delta/2)$-separated set of K_n with the largest cardinality. Then E is also an $(n, \delta/2)$-spanning set of K_n. From the definition of spanning sets, we know that

$$\bigcup_{x\in E} \bigcap_{i=0}^{n-1} T^{-i}\overline{B_{\delta/2}(T^i x)} \supset K_n, \quad \text{where } \overline{B_{\delta/2}(T^i x)} = \Big\{y \in X : d(T^i x, y) \leq \frac{\delta}{2}\Big\}.$$

Now, for each $x \in E$ and $0 \leq i \leq n-1$, $\overline{B_{\delta/2}(T^i x)}$ is contained in some element of $\mathcal{U}$ since δ is the Lebesgue number of the open cover $\mathcal{U}$. Hence, for each $x \in E$, the intersection $\bigcap_{i=0}^{n-1} T^{-i}\overline{B_{\delta/2}(T^i x)}$ is contained in some element of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$. Let $\mathcal{W} = \big\{\bigcap_{i=0}^{n-1} T^{-i}\overline{B_{\delta/2}(T^i x)} : x \in E\big\}$. Then $\mathcal{W} \in \mathcal{C}_X$ and $\mathcal{W} \succeq \mathcal{U}_0^{n-1}$. Let

$$Q_n(T, f, \mathcal{U}, K_n) = \inf\Big\{\sum_{V\in\mathcal{V}} \inf_{x\in V\cap K_n} \exp f_n(x) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1}\Big\}.$$

Then

$$Q_n(T, f, \mathcal{U}, K_n) \leq \sum_{x\in E} f_n(x) \leq P_n\Big(T, f, \frac{\delta}{2}, K_n\Big).$$

Let $\tau_{\mathcal{U}} = \sup\{| f(x) - f(y) | : d(x, y) \leq \operatorname{diam} \mathcal{U}\}$. Then

$$\exp(-n\tau_{\mathcal{U}}) P_n(T, f, \mathcal{U}, K_n) \leq Q_n(T, f, \mathcal{U}, K_n).$$

So

$$-\tau_{\mathcal{U}} + \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, K_n) \le \limsup_{n\to+\infty} \frac{1}{n} \log P_n\Big(T, f, \frac{\delta}{2}, K_n\Big)$$
$$\le \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n\Big(T, f, \frac{\delta}{2}, K_n\Big).$$

Since $\mathcal{U}$ is arbitrary, we get

$$\sup_{\mathcal{U}\in\mathscr{C}_X^0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, K_n) \le \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \delta, K_n). \quad \square$$

An immediate consequence of Lemma 3.1 is the following.

Lemma 3.2. *Let (X, T) be a TDS and $f \in C(X, \mathbb{R})$. Then, for each $x \in X$ and $\epsilon > 0$,*

$$P_s(T, f, x, \epsilon) = \sup_{\mathcal{U}\in\mathscr{C}_X^o} \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f, \mathcal{U}, T^{-n} W_\epsilon^s(x, T)).$$

Lemma 3.3 [Walters 1982, Lemma 9.9]. *Let $a_1, \dots, a_k$ be given real numbers. If $p_i \ge 0$, $i = 1, \dots, k$, and $\sum_{i=1}^k p_i = 1$,*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \le \log \sum_{i=1}^k e^{a_i},$$

and equality holds if and only if

$$p_i = \frac{e^{a_i}}{\sum_{i=1}^k e^{a_i}} \quad \text{for all } i = 1, \dots, k.$$

Let (X, T) be a TDS, $\mu \in \mathcal{M}(X, T)$, and $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under μ. The *Pinsker σ-algebra $P_\mu(T)$* is defined as the smallest sub-σ-algebra of $\mathscr{B}_\mu$ containing $\{\xi \in \mathscr{P}_X : h_\mu(T, \xi) = 0\}$. It is well known that $P_\mu(T) = P_\mu(T^{-1})$ and $P_\mu(T)$ is T-invariant, that is, $T^{-1}(P_\mu(T)) = P_\mu(T)$.

Lemma 3.4 [Huang 2008, Lemma 3.5]. *Let (X, T) be a TDS, $\mu \in \mathcal{M}(X, T)$, and $\delta > 0$. Then there exist $\{W_i\}_{i=1}^\infty \subset \mathscr{P}_X$ and $0 = k_1 < k_2 < \cdots$ such that*

(a) $\operatorname{diam} W_1 < \delta$ *and* $\lim_{i\to+\infty} \operatorname{diam} W_i = 0$,

(b) $\lim_{k\to+\infty} H_\mu(P_k \mid \mathscr{P}^-) = h_\mu(T)$, *where* $P_k = \bigvee_{i=1}^k T^{-k_i} W_i$ *and* $\mathscr{P} = \bigvee_{k=1}^\infty P_k$,

(c) $\bigcap_{n=0}^\infty \widehat{T^{-n}\mathscr{P}^-} = P_\mu(T)$.

Lemma 3.5. *Let (X, T) be a TDS, $\mathcal{U} \in \mathscr{C}_X^o$, $f \in C(X, \mathbb{R})$, and $K \in \mathscr{K}(X)$. Then, for each $n \in \mathbb{N}$,*

$$P_n(T, f, \mathcal{U}, T^{-n} K) = P_n(T, f \circ T^{-n}, T^n \mathcal{U}, K).$$

Proof. For each $\mathcal{V} \in \mathcal{C}_X$ and $\mathcal{V} \succeq \bigvee_{i=1}^{n} T^i\mathcal{U}$, obviously, $T^{-n}\mathcal{V} \in \mathcal{C}_x$ and $T^{-n}\mathcal{V} \succeq \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}$.

Since for each $V \in \mathcal{V}$,

$$\sup_{x \in T^{-n}V \cap T^{-n}K} \exp f_n(x) = \sup_{x \in V \cap K} \exp f_n(T^{-n}x),$$

it is easy to see that $P_n(T, f, \mathcal{U}, T^{-n}K) \le P_n(T, f \circ T^{-n}, T^n\mathcal{U}, K)$. From the homeomorphism of T, the inverse inequality holds. Then $P_n(T, f, \mathcal{U}, T^{-n}K) = P_n(T, f \circ T^{-n}, T^n\mathcal{U}, K)$. □

Recall that a set-valued map F from X to $\mathcal{K}(X)$ is said to be *measurable* if $\{x \in X : F(x) \cap A \ne \varnothing\} \in \mathcal{B}_X$ for every Borel (open or closed) subset A of X.

Lemma 3.6. *Let $G : X \to \mathcal{K}(X)$ be a measurable set-valued map, $f \in C(X, \mathbb{R}^+)$, and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$F : x \to \inf\Big\{\sum_{V \in \mathcal{V}} \sup_{y \in V \cap G(x)} f(y) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}\Big\}$$

is Borel-measurable, where $\sup_{y \in V \cap G(x)} f(y) = 0$ for $V \cap G(x) = \varnothing$.

Proof. By Lemma 2.1, for each $x \in X$, we have

$$\inf\Big\{\sum_{V \in \mathcal{V}} \sup_{y \in V \cap G(x)} f(y) : \mathcal{V} \in \mathcal{C}_X, \mathcal{V} \succeq \mathcal{U}\Big\} = \min\Big\{\sum_{V \in \mathcal{V}} \sup_{y \in V \cap G(x)} f(y) : \mathcal{V} \in \mathcal{P}^*(\mathcal{U})\Big\}.$$

It is sufficient to prove that, for each $V \in \mathcal{V}$, where $\mathcal{V} \in \mathcal{P}^*(\mathcal{U})$, the function $H_V : x \to \sup_{y \in V \cap G(x)} f(y)$ is Borel-measurable.

For each $r \in \mathbb{R}$, let $E_r = \{x : \sup_{y \in V \cap G(x)} f(y) > r\}$. Note that $U = f^{-1}(r, +\infty)$ is an open subset of X. For $r \ge 0$, if $V \cap U = \varnothing$, $E_r = \varnothing$. If $V \cap U \ne \varnothing$, then $E_r = \{x : V \cap G(x) \cap U \ne \varnothing\}$. Since $V \cap U \in \mathcal{B}(X)$, by the set-valued measurability of G, it is clear that E_r is a Borel subset of X. For $r < 0$, $E_r = E_0 \cup F$, where $F = \{x : \sup_{y \in V \cap G(x)} f(y) = 0\}$. Since

$$F = \{x : V \cap G(x) = \varnothing\} = X \backslash \{x : V \cap G(x) \ne \varnothing\}$$

is Borel-measurable, E_r is also a Borel subset of X; thus H_V is Borel-measurable. □

The next theorem clearly implies the main result of this paper.

Theorem 3.7. *Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$, and $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$. Then, for μ-a.e. $x \in X$, $\lim_{\epsilon \to 0} P_s(T, f, x, \epsilon) \ge P_\mu(T, f)$.*

Proof. It suffices to prove that, for a given $\epsilon > 0$, $P_s(T, f, x, \epsilon) \ge P_\mu(T, f)$ for μ-a.e. $x \in X$.

Fix $\epsilon > 0$. Since T is a homeomorphism on X, there exists $\delta \in (0, \epsilon)$ such that $d(T^{-1}x, T^{-1}y) < \epsilon$ when $d(x, y) < \delta$. By Lemma 3.4, there exists $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}_X$ satisfying diam $P_1 \le \delta$, $\bigcap_{n=0}^{\infty} \widehat{T^{-n}\mathcal{P}^-} = P_\mu(T)$, and $H_\mu(P_k \mid \mathcal{P}^-) \to h_\mu(T)$ when

$k \to +\infty$, where $\mathcal{P} = \bigvee_{i=1}^{\infty} P_i$. Since diam $P_1 \le \delta$, it is clear that $\mathcal{P}^-(x) \subseteq W_\epsilon^s(x, T)$ for each $x \in X$.

Let $\mu = \int_X \mu_x \, d\mu(x)$ be the disintegration of μ over $\mathcal{P}^-$. Then

$$\operatorname{supp}(\mu_x) \subseteq \overline{\mathcal{P}^-(x)} \subseteq W_\epsilon^s(x, T) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Let $k \in \mathbb{N}$. By inequality (3.3) in [Huang 2008], we know that there exists $\mathcal{U}_k \in \mathcal{C}_X^o$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} H_\mu\Bigl(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}_k \mid T^{-n}\mathcal{P}^-\Bigr) \ge H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k}. \tag{2}$$

For $n \in \mathbb{N}$, let $F_n(x) = (1/n) \log P_n(T, f \circ T^{-n}, T^n\mathcal{U}_k, W_\epsilon^s(x, T))$. Noting that the map $x \to W_\epsilon^s(x, T)$ is upper semicontinuous, it follows from Lemma 3.6 that F_n is a Borel-measurable function. Let $F(x) = \limsup_{n\to+\infty} F_n(x)$ for $x \in X$. Then F is also Borel-measurable. Since $T W_\epsilon^s(x, T) \subseteq W_\epsilon^s(Tx, T)$ for each $x \in X$, we have

$$\begin{aligned}
&P_n(T, f \circ T^{-n}, T^n\mathcal{U}_k, W_\epsilon^s(x, T)) \\
&\quad \le \inf\Bigl\{\sum_{V\in\mathcal{V}} \sup_{y\in V\cap T W_\epsilon^s(x,T)} \exp f_n \circ T^{-(n+1)}(y) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \bigvee_{i=2}^{n+1} T^i\mathcal{U}_k\Bigr\} \\
&\quad \le \inf\Bigl\{\sum_{V\in\mathcal{V}} \sup_{y\in V\cap W_\epsilon^s(Tx,T)} \exp f_n \circ T^{-(n+1)}(y) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \bigvee_{i=1}^{n+1} T^i\mathcal{U}_k\Bigr\} \\
&\quad = P_{n+1}(T, f \circ T^{-(n+1)}, T^{n+1}\mathcal{U}_k, W_\epsilon^s(Tx, T)).
\end{aligned}$$

Then

$$\begin{aligned}
F(x) &= \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T, f \circ T^{-n}, T^n\mathcal{U}_k, W_\epsilon^s(x, T)) \\
&\le \limsup_{n\to+\infty} \frac{n+1}{n} \cdot \frac{1}{n+1} \log P_{n+1}(T, f \circ T^{-(n+1)}, T^{n+1}\mathcal{U}_k, W_\epsilon^s(Tx, T)) \\
&= F(Tx).
\end{aligned}$$

Thus, $F(x) \le F(Tx)$ for each $x \in X$. Since $\mu \in \mathcal{M}(X, T)$, $\int_X F(Tx)\, d\mu(x) = \int_X F(x)\, d\mu(x)$, we have, $F(Tx) = F(x)$ for μ-a.e. $x \in X$. Moreover, $F(x) \equiv a_k$ for μ-a.e. $x \in X$ as μ is ergodic, where $a_k \ge 0$ is a constant.

From Lemma 2.1, there exists a finite partition

$$\beta \in \mathcal{P}^*\Bigl(\bigvee_{i=1}^{n} T^i\mathcal{U}_k\Bigr)$$

such that

$$P_n(T, f \circ T^{-n}, T^n\mathcal{U}_k, W_\epsilon^s(x, T)) = \sum_{B\in\beta} \sup_{x\in B\cap W_\epsilon^s(x,T)} \exp f_n \circ T^{-n}(x).$$

It follows from Lemma 3.3 that

$$
\begin{aligned}
&\log P_n(T, f \circ T^{-n}, T^n \mathcal{U}_k, W^s_\epsilon(x,T)) \\
&\quad = \log \sum_{B \in \beta} \sup_{x \in B \cap W^s_\epsilon(x,T)} \exp f_n \circ T^{-n}(x) \\
&\quad \geq \sum_{B \in \beta} \mu_x(B \cap W^s_\epsilon(x,T)) \big(\sup_{x \in B \cap W^s_\epsilon(x,T)} \exp f_n \circ T^{-n}(x) - \log \mu_x(B \cap W^s_\epsilon(x,T)) \big) \\
&\quad = H_{\mu_x}(\beta) + \sum_{B \in \beta} \sup_{x \in B \cap W^s_\epsilon(x,T)} f_n \circ T^{-n}(x) \cdot \mu_x(B) \quad (\mathrm{supp}(\mu_x) \subseteq W^s_\epsilon(x,T) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \mu\text{-a.e. } x \in X) \\
&\quad \geq H_{\mu_x}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \Big) + \int_X f_n \circ T^{-n} \, d\mu_x
\end{aligned}
$$

Then

$$
\begin{aligned}
a_k &= \int_X F(x)\, d\mu = \int_X \limsup_{n\to+\infty} F_n(x)\, d\mu \geq \limsup_{n\to+\infty} \int_X F_n(x)\, d\mu \\
&\geq \limsup_{n\to+\infty} \int_X \frac{1}{n} \Big(H_{\mu_x}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \Big) + \int f_n \circ T^{-n}\, d\mu_x \Big) d\mu(x) \\
&= \limsup_{n\to+\infty} \Big(\int_X \frac{1}{n} H_{\mu_x}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \Big) d\mu(x) + \frac{1}{n} \int_X \int f_n \circ T^{-n}\, d\mu_x d\mu(x) \Big) \\
&= \limsup_{n\to+\infty} \Big(\int_X \frac{1}{n} H_{\mu_x}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \Big) d\mu(x) + \frac{1}{n} \int_X f_n \circ T^{-n}\, d\mu(x) \Big) \\
&= \limsup_{n\to+\infty} \int_X \frac{1}{n} H_{\mu_x}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \Big) d\mu(x) + \int_X f\, d\mu(x) \quad (\text{since } \mu \in \mathcal{M}(X,T)) \\
&= \limsup_{n\to+\infty} \frac{1}{n} H_{\mu}\Big(\bigvee_{i=1}^{n} T^i \mathcal{U}_k \mid \mathcal{P}^- \Big) + \int_X f\, d\mu(x) \quad (\text{by Lemma 2.5(a)}) \\
&= \limsup_{n\to+\infty} \frac{1}{n} H_{\mu}\Big(\bigvee_{i=1}^{n-1} T^{-i} \mathcal{U}_k \mid T^{-n}\mathcal{P}^- \Big) + \int_X f\, d\mu(x) \\
&\geq H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k} + \int_X f\, d\mu(x) \quad (\text{by inequality (2)}).
\end{aligned}
$$

Since $P_s(T, f, x, \epsilon) \geq F(x)$ for each $x \in X$, we have

$$
\begin{aligned}
P_s(T, f, x, \epsilon) &\geq \lim_{k\to+\infty} \Big(H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k} + \int_X f\, d\mu(x) \Big) \\
&= h_\mu(T) + \int_X f\, d\mu(x) = P_\mu(T, f)
\end{aligned}
$$

for μ-a.e. $x \in X$. $\square$

We introduce the ϵ-pressure point and pressure point for a TDS. Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$. For $\epsilon > 0$, we call $x \in X$ an *ϵ-pressure point* for T if $P_s(T, f, x, \epsilon) = P(T, f)$, and we call it a *pressure point* if $\lim_{\epsilon \to o} P_s(T, f, x, \epsilon) = P(T, f)$. The function $P_s(T, f, x, \epsilon)$ is decreasing in ϵ. It follows that every pressure point is also an ϵ-pressure point for each $\epsilon > 0$. Note that, while the notion of an ϵ-pressure point depends on the choice of the metric, that of pressure point does not. Denote by $\mathcal{P}(T, f)$ the set of all pressure points of (X, T) for $f \in C(X, \mathbb{R})$. For $f \equiv 0$, the ϵ-pressure point and pressure point are the ϵ-entropy point and entropy point, respectively, which are introduced in [Fiebig et al. 2003]. Moreover, $\mathcal{P}(T, 0) = \mathcal{E}(T)$, where $\mathcal{E}$ is the set of all entropy points of (X, T).

Remark 3.8. Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$. If there exists $\mu \in \mathcal{M}^e(X, T)$ such that $P(T, f) = P_\mu(T, f)$, $\mathcal{P}(T, f) \neq \varnothing$.

4. Stable sets

The main results of the present section are Theorems 4.1 and 4.5. Recall that, for a TDS (X, T) and $x \in X$,

$$W^s(x, T) = \{y \in X : \lim_{n \to +\infty} d(T^n x, T^n y) = 0\},$$
$$W^u(x, T) = \{y \in X : \lim_{n \to +\infty} d(T^{-n} x, T^{-n} y) = 0\}.$$

$W^s(x, T)$ is called the *stable set* of x for T, and $W^u(x, T)$ is called the *unstable set* of x for T. Obviously, $W^s(x, T) = W^u(x, T^{-1})$ and $W^u(x, T) = W^s(x, T^{-1})$.

Theorem 4.1. *Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$, and $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$. Then, for μ-a.e. $x \in X$,*

(a) *there exists a closed subset $A(x) \subseteq W^s(x, T)$ such that*

$$\lim_{n \to +\infty} \operatorname{diam} T^n A(x) = 0 \quad \textit{and} \quad P(T^{-1}, f, A(x)) \geq P_\mu(T, f);$$

(b) *there exists a closed subset $B(x) \subseteq W^u(x, T)$ such that*

$$\lim_{n \to +\infty} \operatorname{diam} T^{-n} B(x) = 0 \quad \textit{and} \quad P(T, f, B(x)) \geq P_\mu(T, f).$$

Proof. Since $\mu \in \mathcal{M}^e(X, T)$, $P_\mu(T^{-1}, f) = P_\mu(T, f)$, and $W^s(x, T^{-1}) = W^u(x, T)$, (a) implies (b). It remains to prove (a).

By Lemma 3.4, there exist $\{W_i\}_{i=1}^\infty \subset \mathcal{P}_X$ and $0 = k_1 < k_2 < \cdots$ satisfying

(a) $\operatorname{diam} W_1 < \delta$ and $\lim_{i \to +\infty} \operatorname{diam} W_i = 0$,

(b) $\lim_{k \to +\infty} H_\mu(P_k \mid \mathcal{P}^-) = h_\mu(T)$, where $P_k = \bigvee_{i=1}^k T^{-k_i} W_i$ and $\mathcal{P} = \bigvee_{k=1}^\infty P_k$,

(c) $\bigcap_{n=0}^\infty \widehat{T^{-n}\mathcal{P}^-} = P_\mu(T)$.

Let $Q_i = \bigvee_{j=1}^{i} T^{-j}(P_1 \vee P_2 \vee \cdots \vee P_i)$ for $i \in \mathbb{N}$. Then $Q_i \in \mathcal{P}_X$, $Q_1 \preceq Q_2 \preceq \cdots$, and $\bigvee_{i=1}^{\infty} Q_i = \mathcal{P}^-$.

For $x \in X$, let $A(x) = \bigcap_{i=1}^{\infty} \overline{Q_i(x)}$. Then $A(x)$ is a closed set and $A(x) \supseteq \overline{\mathcal{P}^-(x)}$. The set $A(x)$ also has the properties $\lim_{n\to+\infty} \operatorname{diam} T^n A(x) = 0$ and $A(x) \subseteq W^s(x, T)$ (see the proof of [Huang 2008, Theorem 4.2] for details).

Moreover, the set-valued map $A : x \to A(x)$ is measurable. In fact, for each open set U of X,

$$\left\{x : \bigcap_{n=1}^{\infty} \overline{Q_i(x)} \subseteq U\right\} = \bigcup_{n\geq 1} \bigcap_{k\geq n} \bigcap \{A \in Q_k : \bar{A} \subseteq U\}$$

is a Borel set of X. Then, for each closed set V of X, $\{x : \overline{Q_i(x)} \subseteq X\backslash V\}$ is a Borel set. It follows that $\{x : \overline{Q_i(x)} \cap V \neq \varnothing\}$ is Borel and then $A : x \to A(x)$ is set-valued measurable.

Let $\mu = \int_X \mu_x \, d\mu(x)$ be the disintegration of μ over $\mathcal{P}^-$. Then

$$\operatorname{supp}(\mu_x) \subseteq \overline{\mathcal{P}^-(x)} \subseteq A(x) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{3}$$

We now prove that, for μ-a.e. $x \in X$, $P(T^{-1}, f, A(x)) \geq P_\mu(T, f)$. Since $\lim_{k\to+\infty} H_\mu(P_k \mid \mathcal{P}^-) = h_\mu(T)$, it is sufficient to prove that, for each $k \in \mathbb{N}$, $P(T^{-1}, f, A(x)) \geq H_\mu(P_k \mid \mathcal{P}^-) - 1/k + \int_X f \, d\mu(x)$ for μ-a.e. $x \in X$.

For a given $k \in \mathbb{N}$, there exists $\mathcal{U}_k \in \mathcal{C}_X^o$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}_k \mid T^{-n}\mathcal{P}^-\right) \geq H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k} \quad \text{for each } n \in \mathbb{N} \tag{4}$$

(see [Huang 2008] for details).

Let $F_n(x) = (1/n) \log P_n(T^{-1}, f, \mathcal{U}_k, A(x))$, where

$$P_n(T^{-1}, f, \mathcal{U}_k, A(x)) = \inf\left\{\sum_{V\in\mathcal{V}} \sup_{y\in V\cap A(x)} \exp f_n \circ T^{-(n-1)}(y) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \bigvee_{i=0}^{n-1} T^i\mathcal{U}_k\right\},$$

and $f_n(z) = \sum_{i=0}^{n-1} f(T^i z)$. By Lemma 3.6, F_n is a Borel-measurable function. Let $F(x) = \limsup_{n\to+\infty} F_n(x)$ for each $x \in X$. Then F is also a Borel-measurable function on X.

For each $\mathcal{V} \succeq \bigvee_{i=0}^{n-1} T^i\mathcal{U}_k$, $T^{-1}\mathcal{V} \succeq \bigvee_{i=0}^{n-1} T^i\mathcal{U}_k$. Since $T(A(x)) \subseteq A(T(x))$ (see the proof of [Huang 2008, Theorem 4.2]), for each $V \in \mathcal{V}$,

$$\begin{aligned}\sup_{y\in T^{-1}V\cap A(x)} \sum_{i=0}^{n-1} f(T^{-i}y) &\leq \sup_{y\in T^{-1}(V\cap A(Tx))} \sum_{i=0}^{n-1} f(T^{-i}y) \\ &= \sup_{y\in V\cap A(Tx)} \sum_{i=1}^{n} f(T^{-i}y) \leq \sup_{y\in V\cap A(Tx)} \sum_{i=0}^{n} f(T^{-i}y),\end{aligned}$$

it is not hard to see that $P_n(T^{-1}, f, \mathcal{U}_k, A(x)) \le P_{n+1}(T^{-1}, f, \mathcal{U}_k, A(Tx))$. Hence

$$\begin{aligned} F(x) &= \limsup_{n\to+\infty} \frac{1}{n} \log P_n(T^{-1}, f, \mathcal{U}_k, A(x)) \\ &\le \limsup_{n\to+\infty} \frac{n+1}{n} \cdot \frac{1}{n+1} \log P_n(T^{-1}, f, \mathcal{U}_k, A(Tx)) = F(Tx). \end{aligned}$$

Thus $F(x) \le F(Tx)$ for each $x \in X$. Since $\mu \in \mathcal{M}(X, T)$, we have

$$\int_X (f(Tx) - f(x))\, d\mu(x) = 0.$$

Then $F(Tx) = F(x)$ for μ-a.e. $x \in X$. From the ergodicity of μ, there exists a constant $a_k \ge 0$ such that $F(x) \equiv a_k$ for μ-a.e. $x \in X$.

By Lemma 2.1, there exists a partition $\beta \in \mathcal{P}^*(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k)$ such that, for μ-a.e. $x \in X$,

$$\begin{aligned} &\log P_n(T^{-1}, f, \mathcal{U}_k, A(x)) \\ &= \log \sum_{B\in\beta} \sup_{y\in B\cap A(x)} \exp \sum_{i=0}^{n-1} f(T^{-i}y) \\ &\ge \sum_{B\in\beta} \mu_x(B) \Big(\sup_{y\in B\cap A(x)} \exp \sum_{i=0}^{n-1} f(T^{-i}y) - \log \mu_x(B) \Big) \quad \text{(by (3) and Lemma 3.3)} \\ &= H_{\mu_x}(\beta) + \sum_{B\in\beta} \sup_{y\in B\cap A(x)} \exp \sum_{i=0}^{n-1} f(T^{-i}y) \cdot \mu_x(B) \\ &\ge H_{\mu_x}\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \Big) + \int_X f_n \circ T^{-(n-1)}\, d\mu_X. \end{aligned}$$

Then

$$\begin{aligned} a_k &= \int_X F(x)\, d\mu = \int_X \limsup_{n\to+\infty} F_n(x)\, d\mu(x) \ge \limsup_{n\to+\infty} \int_X F_n(x)\, d\mu(x) \\ &\ge \limsup_{n\to+\infty} \frac{1}{n} \int_X \Big(H_{\mu_x}\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \Big) + \int_X f_n \circ T^{-(n-1)}\, d\mu_x \Big) d\mu(x) \\ &= \limsup_{n\to+\infty} \frac{1}{n} \Big(\int_X H_{\mu_x}\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \Big) d\mu(x) + \int_X f_n \circ T^{-(n-1)}\, d\mu(x) \Big) \\ &= \limsup_{n\to+\infty} \frac{1}{n} \int_X H_{\mu_x}\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \Big) d\mu(x) + \int_X f\, d\mu(x) \quad (\text{since } \mu \in \mathcal{M}(X, T)) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to+\infty} \frac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \mid \mathcal{P}^-\Big) + \int_X f\, d\mu(x) \quad \text{(by Lemma 2.5(a))}\\
&= \limsup_{n\to+\infty} \frac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1} T^i \mathcal{U}_k \mid T^{-(n-1)}\mathcal{P}^-\Big) + \int_X f\, d\mu(x)\\
&\ge H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k} + \int_X f\, d\mu(x) \quad \text{(by (4))}.
\end{aligned}
$$

Therefore, for μ-a.e. $x \in X$,

$$P(T^{-1}, f, A(x)) \ge P(T^{-1}, f, \mathcal{U}_k, A(x)) = F(x) \ge H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k} + \int_X f\, d\mu(x)$$

for each $k \in \mathbb{N}$.

Then

$$
\begin{aligned}
P(T^{-1}, f, A(x)) &\ge \lim_{n\to+\infty}\Big(H_\mu(P_k \mid \mathcal{P}^-) - \frac{1}{k}\Big) + \int_X f\, d\mu(x)\\
&= H_\mu(T) + \int_X f\, d\mu(x) = P_\mu(T, f). \qquad \square
\end{aligned}
$$

This completes the proof of Theorem 4.1.

A direct consequence of Theorem 4.1 is the following.

Corollary 4.2. *Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$. If there exists $\mu \in \mathcal{M}^e(X, T)$ with $P_\mu(T, f) = P(T, f)$, there exists $x \in X$, a closed subset $A(x) \subseteq W^s(x, T)$, and a closed subset $B(x) \subseteq W^u(x, T)$ such that*

(a) $\lim_{n\to+\infty} \operatorname{diam} T^n A(x) = 0$ *and* $P(T^{-1}, f, A(x)) = P(T, f)$;

(b) $\lim_{n\to+\infty} \operatorname{diam} T^{-n} B(x) = 0$ *and* $P(T, f, B(x)) = P(T, f)$.

A TDS (X, T) is transitive if, for each pair of nonempty open subsets U and V of X, there exists $n \ge 0$ such that $U \cap T^{-n} V \neq \varnothing$; and it is weakly mixing if $(X \times X, T \times T)$ is transitive. These notions describe the global properties of the whole TDS. Blanchard and Huang [2008] give a new criterion to picture “a certain amount of weakly mixing” in some consistent sense. The notion of a weakly mixing set was introduced as follows.

If X, Y are topological spaces, denote by $\mathscr{C}(X, Y)$ the set of all continuous maps from X to Y.

Definition 4.3. Let (X, T) be a TDS and $A \in 2^X$. The set A is said to be weakly mixing for T if there exists $B \subset A$ satisfying

(a) B is the union of countably many Cantor sets;

(b) the closure of B equals A;

(c) for any $C \in B$ and $g \in \mathscr{C}(C, A)$, there exists an increasing sequence of natural numbers $\{n_i\} \subset \mathbb{N}$ such that $\lim_{i\to+\infty} T^{n_i} x = g(x)$ for any $x \in C$.

Denote by $WM_s(X, T)$ the family of weakly mixing subsets of (X, T). The system (X, T) itself is called partially mixing when it contains a weakly mixing set. The whole space X is a weakly mixing set if and only if TDS (X, T) is weakly mixing [Xiong and Yang 1991]. The following result (See [Blanchard and Huang 2008, Proposition 4.2]) gives an equivalent characterization of the weakly mixing set in another way.

Proposition 4.4. *Let (X, T) be a TDS and A be a nonsingleton closed subset of X. Then A is a weakly mixing subset of X if and only if, for any $k \in \mathbb{N}$ and any choice of nonempty open subsets $V_1, \dots, V_k$ of A and nonempty open subsets $U_1, \dots, U_k$ of X with $A \cap U_i \neq \varnothing$, $i = 1, 2, \dots, k$, there exists $m \in \mathbb{N}$ such that $T^m V_i \cap U_i \neq \varnothing$ for each $1 \le i \le k$.*

Now we prove the following theorem. Part (a) of Theorem 4.5 was already proved in [Huang 2008]. For completeness, we state it in the theorem.

Theorem 4.5. *Let (X, T) be a TDS and $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$. Then, for μ-a.e. $x \in X$, there exists a closed subset*

$$E(x) \subseteq \overline{W^s(x, T)} \cap \overline{W^u(x, T)}$$

such that

(a) $E(x) \in WM_s(X, T) \cap WM_s(X, T^{-1})$, *i.e., $E(x)$ is weakly mixing for T, T^{-1};*

(b) $P(T, f, E(x)) \ge P_\mu(T, f)$ *and* $P(T^{-1}, f, E(x)) \ge P_\mu(T, f)$.

Proof. Let $\mathcal{B}_\mu$ be the completion of $\mathcal{B}_X$ under μ. Then $(X, \mathcal{B}_\mu, \mu, T)$ is a Lebesgue system. Let $P_\mu(T)$ be the Pinsker σ-algebra of $(X, \mathcal{B}_\mu, \mu, T)$. Let $\mu = \int_X \mu_x d\mu(x)$ be the disintegration of μ over $P_\mu(T)$. Then, for μ-a.e. $x \in X$,

$$\operatorname{supp}(\mu_x) \subseteq \overline{W^s(x, T)} \cap \overline{W^u(x, T)}$$

and

$$\operatorname{supp}(\mu_x) \in WM_s(X, T) \cap WM_s(X, T^{-1})$$

(see [Huang 2008, Theorem 4.6] for details).

We now prove that, for μ-a.e. $x \in X$,

$$P(T, f, \operatorname{supp}(\mu_x)) \ge P_\mu(T, f) \quad \text{and} \quad P(T^{-1}, f, \operatorname{supp}(\mu_x)) \ge P_\mu(T, f).$$

By the symmetry of T and T^{-1}, $P_\mu(T, f) = P_\mu(T^{-1}, f)$. It remains to prove that, for μ-a.e. $x \in X$, $P(T, f, \operatorname{supp}(\mu_x)) \ge P_\mu(T, f)$. Since $P_\mu(T)$ is T-invariant, $T\mu_x = \mu_{Tx}$ for μ-a.e. $x \in X$. Therefore, there exists a T-invariant measurable set $X_0 \subset X$ with $\mu(X_0) = 1$ and $T\mu_x = \mu_{Tx}$ for $x \in X_0$.

For each $\mathcal{U} \in \mathcal{C}_X^o$, $x \in X_0$, and $n \in \mathbb{N}$, by Lemma 2.1, there exists a $\beta \in \mathcal{P}^*(\mathcal{U}_0^{n-1})$ such that

$$
\begin{aligned}
(5)\quad & \log P_n(T, f, \mathcal{U}, \operatorname{supp}(\mu_x)) \\
&= \log \inf\Big\{ \sum_{V \in \mathcal{V}} \sup_{y \in V \cap \operatorname{supp}(\mu_x)} \exp f_n(x) : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_0^{n-1} \Big\} \\
&= \log \sum_{B \in \beta} \sup_{y \in B \cap \operatorname{supp}(\mu_x)} \exp f_n(x) \\
&\geq \sum_{B \in \beta} \mu_x(B) \Big(\sup_{y \in B \cap \operatorname{supp}(\mu_x)} f_n(x) - \log \mu_x(B) \Big) \quad \text{(by Lemma 3.3)} \\
&= H_{\mu_x}(\beta) + \sum_{B \in \beta} \mu_x(B) \sup_{y \in B \cap \operatorname{supp}(\mu_x)} f_n(x) \\
&\geq H_{\mu_x}(\mathcal{U}_0^{n-1}) + \int_X f_n \, d\mu_x
\end{aligned}
$$

Fix $\mathcal{U} \in \mathcal{C}_X^o$ and $n \in \mathbb{N}$. Denote $F_n(x) = H_{\mu_x}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) + \int_X f_n \, d\mu_x$ for each $x \in X_0$. Then

$$
\begin{aligned}
F_{n+m}(x) &= H_{\mu_x}\Big(\bigvee_{i=0}^{n+m-1} T^{-i}\mathcal{U} \Big) + \int_X f_{n+m} \, d\mu_x \\
&\leq H_{\mu_x}\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U} \Big) + H_{\mu_x}\Big(T^{-n} \bigvee_{i=0}^{m-1} T^{-i}\mathcal{U} \Big) + \int_X f_n \, d\mu_x + \int_X f_m \circ T^n \, d\mu_x \\
&\leq F_n(x) + H_{T^n\mu_x}\Big(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U} \Big) + \int_X f_m \circ T^n \, d\mu_x \\
&= F_n(x) + H_{T^n\mu_x}\Big(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U} \Big) + \int_X f_m dT^n \mu_x \\
&= F_n(x) + H_{\mu_{T^n x}}\Big(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U} \Big) + \int_X f_m \, d\mu_{T^n x} \\
&= F_n(x) + F_m(T^n x),
\end{aligned}
$$

that is, $\{F_n\}$ is subadditive. Since the map $x \to \mu_x(A)$ for each $A \in \mathcal{B}$ is measurable on X_0, it follows that $F_n(x)$ is measurable on X_0. By Kingman's subadditive ergodic theorem, $\lim_{n\to\infty}(1/n)F_n(x) \equiv a_{\mathcal{U}}$ for μ-a.e. $x \in X$, where $a_{\mathcal{U}}$ is a constant. Then, by (5),

$$
P(T, f, \mathcal{U}, \operatorname{supp}(\mu_x)) \geq a_{\mathcal{U}}
$$

for each $\mathcal{U} \in \mathcal{C}_X^0$ and μ-a.e. $x \in X$. Therefore

$$\begin{aligned} a_{\mathcal{U}} &= \int_X \lim_{n\to\infty} \frac{1}{n} F_n(x)\, d\mu = \lim_{n\to\infty} \frac{1}{n} \int_X F_n(x)\, d\mu \\ &= \lim_{n\to\infty} \frac{1}{n} \int_X \left(H_{\mu_x}(\mathcal{U}_0^{n-1}) + \int_X f_n \, d\mu_x \right) d\mu(x) \\ &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1} \mid P_\mu(T)) + \int_X f \, d\mu \\ &= h_\mu(T, \mathcal{U} \mid P_\mu(T)) + \int_X f\, d\mu \\ &= P_\mu(T, f, \mathcal{U}) \qquad \text{(by Lemma 2.4)}. \end{aligned}$$

It follows that

$$P(T, f, \mathcal{U}, \operatorname{supp}(\mu_x)) \geq P_\mu(T, f, \mathcal{U})$$

for each $\mathcal{U} \in \mathcal{C}_X^o$ and μ-a.e. $x \in X$.

Choose a sequence of open covers $\{\mathcal{U}_m\}_{m=1}^\infty$ with $\lim \operatorname{diam}\{\mathcal{U}_m\} = 0$. Then

$$\begin{aligned} \lim_{n\to\infty} P_\mu(T, f, \mathcal{U}_m) &= \lim_{n\to\infty} \left(h_\mu(T, \mathcal{U}_m) + \int_X f\, d\mu \right) \\ &= h_\mu(T) + \int_X f\, d\mu = P_\mu(T, f). \end{aligned}$$

Since for each $m \in \mathbb{N}$ and μ-a.e. $x \in X$, $P(T, f, \mathcal{U}_m, \operatorname{supp}(\mu_x)) \geq P_\mu(T, f, \mathcal{U}_m)$, we have

$$P(T, f, \operatorname{supp}(\mu_x)) = \sup_{m\in\mathbb{N}} P(T, f, \mathcal{U}_m, \operatorname{supp}(\mu_x)) \geq \sup_{m\geq 1} P_\mu(T, f, \mathcal{U}_m) = P_\mu(T, f)$$

for each μ-a.e. $x \in X$. □

It is not hard to see that the following corollary holds.

Corollary 4.6. *Let (X, T) be a TDS and $f \in C(X, \mathbb{R})$. Then*

(a) $\sup_{x\in X} P(T, f, \overline{W^s(x, T)} \cap \overline{W^u(x, T)}) = P(T, f)$;

(b) *if there exists $\mu \in \mathcal{M}^e(X, T)$ with $P_\mu(T, f) = P(T, f)$, then, for μ-a.e. $x \in X$, there exists a closed subsets $E(x) \subseteq \overline{W^s(x, T)} \cap \overline{W^u(x, T)}$ such that*

  (i) $E(x) \in WM_s(X, T) \cap WM_s(X, T^{-1})$,

  (ii) $P(T, f, E(x)) = P(T^{-1}, f, E(x)) = P(T, f)$.

5. Noninvertible case

In this section, we generalize the results in Sections 3 and 4 to the noninvertible case. Let (X, T) be a noninvertible TDS, that is, X is a compact metric space, and $T : X \to X$ is a surjective continuous map but not one-to-one.

Set $\widetilde{X} = \{(x_1, x_2, \dots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$. It is clear that $\widetilde{X}$ is a subspace of the product space $\Pi_{i=1}^{\infty} X$ with the metric d_T defined by

$$d_T((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

Let $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ be the shift homeomorphism, that is,

$$\widetilde{T}(x_1, x_2, \dots) = (T(x_1), x_1, x_2, \dots).$$

We refer to the TDS $(\widetilde{X}, \widetilde{T})$ as the inverse limit of (X, T). Let $\pi_i : \widetilde{X} \to X$ be the natural projection onto the i-th coordinate. Then $\pi_i : (\widetilde{X}, \widetilde{T}) \to (X, T)$ is a factor map.

Lemma 5.1. *Let (X, T) be a noninvertible TDS, $f \in C(X, \mathbb{R})$. Then, for each $\mathcal{U} \in \mathscr{C}_X^o$ and $K \in \mathscr{K}(X)$,*

$$P_{n+m}(T, f, \mathcal{U}, K) \le P_m(T, f, \mathcal{U}, K) \cdot P_n(T, f \circ T^m, T^{-m}\mathcal{U}, K)$$

for each $n, m \in \mathbb{N}$.

Proof. Since for each $\mathcal{V}_1 \succeq \mathcal{U}_0^{m-1}$ and $\mathcal{V}_2 \succeq \mathcal{U}_0^{n-1}$ we have $\mathcal{V}_1 \vee T^{-m}\mathcal{V}_2 \succeq \mathcal{U}_0^{n+m-1}$, it follows that

$$\begin{aligned} P_{n+m}(T, f, \mathcal{U}, K) &\le \sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \in \mathcal{V}_2} \sup_{x \in V_1 \cap T^{-m} V_2 \cap K} \exp f_{n+m}(x) \\ &= \sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \in \mathcal{V}_2} \sup_{x \in V_1 \cap T^{-m} V_2 \cap K} \exp(f_m(x) + f_n(T^m x)) \\ &\le \sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \in \mathcal{V}_2} \sup_{x \in V_1 \cap K} \exp f_m(x) \cdot \sup_{x \in T^{-m} V_2 \cap K} \exp f_n(T^m x) \\ &= \sum_{V_1 \in \mathcal{V}_1} \sup_{x \in V_1 \cap K} \exp f_m(x) \cdot \sum_{V_2 \in \mathcal{V}_2} \sup_{x \in T^{-m} V_2 \cap K} \exp(f \circ T^m)_n(x). \end{aligned}$$

By the arbitrariness of $\mathcal{V}_1$ and $\mathcal{V}_2$, we have

$$P_{n+m}(T, f, \mathcal{U}, K) \le P_m(T, f, \mathcal{U}, K) \cdot P_n(T, f \circ T^m, T^{-m}\mathcal{U}, K). \qquad \square$$

Lemma 5.2. *Let (X, T) be a noninvertible TDS, $f \in C(X, \mathbb{R})$. Then, for each $\mathcal{U} \in \mathcal{C}_X^o$ and $K \in \mathcal{K}(X)$,*

$$P_n(T, f \circ T^m, T^{-m}\mathcal{U}, T^{-m}K) = P_n(T, f, \mathcal{U}, K) \quad \textit{for each } n, m \in \mathbb{N}.$$

Proof. Fix $n, m \in \mathbb{N}$. For each $\mathcal{V} \succeq (T^{-m}\mathcal{U})_0^{n-1}$,

$$\begin{aligned}\sum_{V \in \mathcal{V}} \sup_{x \in V \cap T^{-m}K} \exp(f \circ T^m)_n(x) &= \sum_{V \in \mathcal{V}} \sup_{x \in V \cap T^{-m}K} \exp f_n(T^m x) \\ &= \sum_{V \in \mathcal{V}} \sup_{x \in T^m V \cap K} \exp f_n(x).\end{aligned}$$

Since $T^m\mathcal{V} \succeq \mathcal{U}_0^{n-1}$,

$$P_n(T, f \circ T^m, T^{-m}\mathcal{U}, T^{-m}K) \le P_n(T, f, \mathcal{U}, K).$$

Conversely, for each $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$, $T^{-m}\mathcal{V} \succeq (T^{-m}\mathcal{U})_0^{n-1}$ and

$$\begin{aligned}\sum_{V \in \mathcal{V}} \sup_{x \in V \cap K} \exp f_n(x) &= \sum_{V \in \mathcal{V}} \sup_{x \in T^{-m}(V \cap K)} \exp f_n(T^m x) \\ &= \sum_{V \in \mathcal{V}} \sup_{x \in T^{-m}V \cap T^{-m}K} \exp(f \circ T^m)_n(x).\end{aligned}$$

Then

$$P_n(T, f \circ T^m, T^{-m}\mathcal{U}, T^{-m}K) \ge P_n(T, f, \mathcal{U}, K),$$

which completes the proof. □

Lemma 5.3. *Let $(\widetilde{X}, \widetilde{T})$ be the inverse limit of a noninvertible TDS (X, T). Let $f \in C(X, \mathbb{R})$ and let $\pi_1 : \widetilde{X} \to X$ be the projection to the first coordinate. Then, for any sequence of nonempty closed subsets K_n of $\widetilde{X}$,*

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(\widetilde{T}, f \circ \pi_1, \delta, K_n) = \lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(T, f, \delta, \pi_1(K_n)).$$

Proof. Let $\mathcal{U} \in \mathcal{C}_X^o$. For each $\mathcal{V} \in \mathcal{C}_X$ with $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$ and $x \in V \cap \pi_1(K_n)$, obviously, $\pi_1^{-1}\mathcal{V} \succeq (\pi_i^{-1}\mathcal{U})_0^{n-1}$ and

$$(f \circ \pi_1)_n(\tilde{x}) = \sum_{j=0}^{n-1} (f \circ \pi_1)(\widetilde{T}^j(\tilde{x})) = \sum_{j=0}^{n-1} f \circ T^j(\pi_1 \tilde{x}) = f_n(\pi_1 \tilde{x}) = f_n(x),$$

where $x = \pi_1 \tilde{x}$. Then

$$\sum_{V \in \mathcal{V}} \sup_{\tilde{x} \in \pi_1^{-1} V \cap K_n} \exp(f \circ \pi_1)_n(\tilde{x}) = \sum_{V \in \mathcal{V}} \sup_{x \in V \cap \pi_1(K_n)} \exp f_n(x).$$

It follows that

$$P_n(\widetilde{T}, f \circ \pi_1, \pi_1^{-1}\mathcal{U}, K_n) \le P_n(T, f, \mathcal{U}, \pi_1(K_n)). \tag{6}$$

On the other hand, for each $\widetilde{\mathscr{V}} \in \mathscr{C}_X^{\mathbb{N}}$ with $\widetilde{\mathscr{V}} \succeq (\pi_1^{-1}\mathscr{U})_0^{n-1}$, $\tilde{x} \in \widetilde{V} \cap K_n$, $\pi_1 \widetilde{V} \succeq \mathscr{U}_0^{n-1}$, and

$$\begin{aligned}\sum_{\widetilde{V}\in\widetilde{\mathscr{V}}} \sup_{\tilde{x}\in\widetilde{V}\cap K_n} \exp(f\circ\pi_1)_n(\tilde{x}) &= \sum_{\widetilde{V}\in\widetilde{\mathscr{V}}} \sup_{x\in\pi_1(\widetilde{V}\cap K_n)} \exp f_n(x)\\ &= \sum_{V\in\pi_1\widetilde{V}} \sup_{x\in\pi_1\widetilde{V}\cap\pi_1 K_n} \exp f_n(x),\end{aligned}$$

where $x = \pi_1\tilde{x}$. Then we get the opposite part of the inequality of (6), and consequently

$$P_n(\widetilde{T}, f\circ\pi_1, \pi_1^{-1}\mathscr{U}, K_n) = P_n(T, f, \mathscr{U}, \pi_i(K_n)). \tag{7}$$

Now we have

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(\widetilde{T}, f\circ\pi_1, \pi_1^{-1}\mathscr{U}, K_n) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(T, f, \mathscr{U}, \pi_1(K_n)).$$

From Lemma 3.1, we get

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(\widetilde{T}, f\circ\pi_1, \delta, K_n) \geq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(T, f, \delta, \pi_1(K_n)).$$

Conversely, let $\pi_i : \widetilde{X} \to X$ be the projection to the i[th] coordinate and $\widetilde{\mathscr{U}} \in \mathscr{C}_{\widetilde{X}}^o$. By the definition of $\widetilde{X}$, it is easy to see that there exists some $\mathscr{U} \in \mathscr{C}_X^o$ such that $\pi_i^{-1}(\mathscr{U}) \succeq \widetilde{U}$. Since for any two closed subsets C and D of X, $P_n(T, f, \mathscr{U}, C) \leq P_n(T, f, \mathscr{U}, D)$ and $\pi_i(K_n) \succeq T^{-(i-1)}\pi_i(K_n)$, by (7), we have

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \log P_n(\widetilde{T}, f\circ\pi_1, \widetilde{\mathscr{U}}, K_n)\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n} \log P_n(\widetilde{T}, f\circ\pi_1, \pi_i^{-1}\mathscr{U}, K_n)\\
&\quad= \limsup_{n\to\infty} \frac{1}{n} \log P_n(T, f, \mathscr{U}, \pi_i(K_n))\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n} \log P_n(T, f, \mathscr{U}, T^{-(i-1)}\pi_i(K_n))\\
&\quad= \limsup_{n\to\infty} \frac{1}{n+i-1} \log P_{n+i-1}(T, f, \mathscr{U}, T^{-(i-1)}\pi_i(K_n))\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n+i-1} \log\big(P_{i-1}(T, f, \mathscr{U}, T^{-(i-1)}\pi_i(K_n))\\
&\qquad\qquad \cdot P_n(T, f\circ T^{i-1}, T^{-(i-1)}\mathscr{U}, T^{-(i-1)}\pi_i(K_n))\big) \quad \text{(by Lemma 5.1)}\\
&\quad= \limsup_{n\to\infty} \frac{1}{n} P_n(T, f, \mathscr{U}, \pi_1(K_n)) \qquad \text{(by Lemma 5.2)}\\
&\quad\leq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} P_n(T, f, \delta, \pi_1(K_n)).\end{aligned}$$

By Lemma 3.1, we get

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(\widetilde{T}, f\circ\pi_1, \delta, K_n) \le \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(T, f, \delta, \pi_1(K_n)). \quad \square$$

Now we can prove the following theorem.

Theorem 5.4. *Let (X, T) be a noninvertible TDS, $f \in C(X, \mathbb{R})$, and $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$. Then, for μ-a.e. $x \in X$, $\lim_{\epsilon\to 0} P_s(T, f, x, \epsilon) \ge P_\mu(T, f)$.*

Proof. Let $(\widetilde{X}, \widetilde{T})$ be the inverse limit of (X, T). For $\epsilon > 0$, $n \in \mathbb{N}$, and $\tilde{x} \in \widetilde{X}$, denote $K_n = \widetilde{T}^{-n} W^s_{\epsilon/2}(\tilde{x}, \widetilde{T})$. Then, from the definitions of d_T and $\widetilde{X}$, it is easy to see that $\pi_1(K_n) \subseteq T^{-n} W^s_\epsilon(x, T)$, where $x = \pi_1(\tilde{x})$. By Lemma 5.3, we have

$$\begin{aligned} P_s(T, f, x, \epsilon) &= \lim_{\delta\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log P_n(T, f, \delta, T^{-n}W^s_\epsilon(x, T)) \\ &\ge \lim_{\delta\to 0}\limsup_{n\to+\infty}\log P_n(T, f, \delta, \pi_1(K_n)) \\ &= \lim_{\delta\to 0}\limsup_{n\to+\infty}\log P_n(\widetilde{T}, f\circ\pi_1, \delta, K_n) \\ &= P_s\Big(\widetilde{T}, f\circ\pi_1, \tilde{x}, \frac{\epsilon}{2}\Big). \end{aligned}$$

It follows that, for each $\tilde{x} \in \widetilde{X}$,

$$\lim_{\epsilon\to 0} P_s(T, f, \pi_1(\tilde{x}), \epsilon) \ge \lim_{\epsilon\to 0} P_s\Big(\widetilde{T}, f\circ\pi_1, \tilde{x}, \frac{\epsilon}{2}\Big). \tag{8}$$

Let $\tilde{\mu} \in \mathcal{M}^e(\widetilde{X}, \widetilde{T})$ with $\pi_1(\tilde{\mu}) = \mu$. Then, by Theorem 3.7, there exists a Borel subset $\widetilde{X}_0 \subseteq \widetilde{X}$ with $\tilde{\mu}(\widetilde{X}_0) = 1$ such that, for any $\tilde{x} \in \widetilde{X}_0$,

$$\begin{aligned} \lim_{\epsilon\to 0} P_s\Big(\widetilde{T}, f\circ\pi_1, \tilde{x}, \frac{\epsilon}{2}\Big) &\ge P_{\tilde{\mu}}(\widetilde{T}, f\circ\pi_1) = h_{\tilde{\mu}}(\widetilde{T}) + \int_{\widetilde{X}} f\circ\pi_1 d\tilde{\mu} \\ &\ge h_\mu(T) + \int_X f\,d\mu = P_\mu(T, f). \end{aligned} \tag{9}$$

Let $X_0 = \pi_1(\widetilde{X}_0)$. Then $X_0 \in \mathcal{B}_\mu$ and $\mu(X_0) = 1$. By the inequality (8) and (9), we have

$$\lim_{\epsilon\to 0} P_s(T, f, x, \epsilon) \ge P_\mu(T, f) \quad \text{for each } x \in X_0, \qquad \square$$

Theorem 5.4 immediately leads to the following corollary.

Corollary 5.5. *Let (X, T) be a noninvertible TDS and $f \in C(X, \mathbb{R})$. If there exists a $\mu \in \mathcal{M}^e(X, T)$ such that $P_\mu(T, f) = P(T, f)$, $\mathcal{P}(T, f) \ne \emptyset$.*

Lemma 5.6. *Let $(\widetilde{X}, \widetilde{T})$ be the inverse limit of a noninvertible TDS (X, T). If $A \subseteq \widetilde{E}$ is weak mixing, so is $\pi_1(A)$ and $P(\widetilde{T}, f\circ\pi_1, A) = P(T, f, \pi_1(A))$.*

Proof. The fact that $\pi_1(A)$ is weak mixing follows from Lemma 4.8 in [Blanchard and Huang 2008]. The latter follows from Lemmas 5.3 and 3.1. □

The following theorem shows that Theorem 4.5 also holds for noninvertible TDS.

Theorem 5.7. *Let (X, T) be a noninvertible TDS and $\mu \in \mathcal{M}^e(X, T)$ with $h_\mu(T) > 0$. Then, for μ-a.e. $x \in X$, there exists a closed subset $E(x) \subseteq \overline{W^s(x, T)}$ such that $P(T, f, E(x)) \geq P_\mu(T, f)$ and $E(x) \in WM_s(X, T)$.*

Proof. Let $(\widetilde{X}, \widetilde{T})$ be the inverse limit of (X, T). Then there exists $\tilde{\mu} \in \mathcal{M}^e(\widetilde{X}, \widetilde{T})$ with $\pi_1(\tilde{\mu}) = \mu$, where π_1 is the projection to the first coordinate. Obviously,

$$P_{\tilde{\mu}}(\widetilde{T}, f \circ \pi_1) = h_{\tilde{\mu}}(\widetilde{T}) + \int_{\widetilde{X}} f \circ \pi_1 \, d\tilde{\mu} \geq h_\mu(T) + \int_X f \, d\mu = P(T, f).$$

By Theorem 4.5, there exists a Borel set $\widetilde{X}_0 \subseteq \widetilde{X}$ with $\tilde{\mu}(\widetilde{X}_0) = 1$ such that, for each $\tilde{x} \in \widetilde{X}_0$, there exists a closed subset $E(\tilde{x}) \subseteq \overline{W^s(\tilde{x}, \widetilde{T})}$ such that

$$P(\widetilde{T}, f \circ \pi_1, E(\tilde{x})) \geq P_{\tilde{\mu}}(\widetilde{T}, f \circ \pi_1) \quad \text{and} \quad E(\tilde{x}) \in WM_s(\widetilde{X}, \widetilde{T}).$$

Let $(X_0) = \pi_1(\widetilde{X}_0)$. Then $X_0 \in \mathcal{B}_\mu$ and $\mu(X_0) = 1$. For each $x \in X_0$ let $E(x) = \pi_1(E(\tilde{x}))$, where $x = \pi_1(\tilde{x})$. Then $E(x) \subseteq \pi_1(\overline{W^s(\tilde{x}, \widetilde{T})}) \subseteq \overline{W^s(x, T)}$. By Lemma 5.6, we have

$$P(T, f, E(x)) = P(\widetilde{T}, f \circ, E(\tilde{x})) \geq P_{\tilde{\mu}}(\widetilde{T}, f \circ \pi_1) \geq P_\mu(T, f)$$

and $E(x) \in WM_s(X, T)$. □

The following result is immediate.

Corollary 5.8. *Let (X, T) be a noninvertible TDS. Then*

(a) $\sup_{x \in X} P(T, f, \overline{W^s(x, T)}) = P(T, f)$;

(b) *if there exists $\mu \in \mathcal{M}^e(X, T)$ with $P_\mu(T, f) = P(T, f)$, then, for μ-a.e. $x \in X$, there exists a closed subset $E(x) \subseteq \overline{W^s(x, T)}$ such that $E(x) \in WM_s(X, T)$ and $P(T, f, E(x)) = P(T, f)$.*

Remark 5.9. From the proof of Theorem 4.5, we know that $E(x) = \operatorname{supp}(\mu_x)$, where μ_x is a probability measure determined by the disintegration of $\mu \in \mathcal{M}^e(X, T)$ over the Pinsker σ-algebra $P_\mu(T)$.

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XIANFENG MA
DEPARTMENT OF MATHEMATICS
EAST CHINA UNIVERSITY OF SCIENCE AND TECHNOLOGY
SHANGHAI 200237
CHINA

xianfengma@gmail.com

ERCAI CHEN
SCHOOL OF MATHEMATICAL SCIENCES AND INSTITUTE OF MATHEMATICS
NANJING NORMAL UNIVERSITY
NANJING 210023
CHINA

and

CENTER OF NONLINEAR SCIENCE
NANJING UNIVERSITY
NANJING, 210093
CHINA

ecchen@njnu.edu.cn

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