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**GEOMETRIC INEQUALITIES IN CARNOT GROUPS**

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## GEOMETRIC INEQUALITIES IN CARNOT GROUPS

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**Let  $\mathbb{G}$  be a subriemannian  $k$ -step Carnot group of homogeneous dimension  $Q$ . We prove several geometric inequalities concerning smooth hypersurfaces (submanifolds of codimension one) immersed in  $\mathbb{G}$ , endowed with the  $H$ -perimeter measure.**

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### 1. Introduction

During the last years there has been an increasing interest in studying analysis and geometric measure theory in metric spaces (see [Ambrosio 2001; Ambrosio and Kirchheim 2000a; 2000b; Cheeger and Kleiner 2010; David and Semmes 1997; Garofalo and Nhieu 1996; Varopoulos et al. 1992] and bibliographic references therein, but this list is far from being exhaustive). In this regard, important examples of highly noneuclidean geometries are represented by the so-called Carnot–Carathéodory (or subriemannian) geometries; see [Capogna et al. 1994; Gromov 1996; Montgomery 2002; Pansu 1982; 1989; 2005; Strichartz 1986; Vershik and Gershkovich 1994]. In this context, Carnot groups play the role of modeling the tangent space (in a suitable generalized sense, which is related to the Gromov–Hausdorff convergence) of a subriemannian manifold; see [Gromov 1996; Montgomery 2002]. For this and many other reasons, Carnot groups are an intriguing field of research; see [Ambrosio et al. 2006; Balogh 2003; Balogh et al. 2009; Capogna et al. 2010; Cheng et al. 2005; Danielli et al. 2007; 2010; Franchi

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et al. 2001; 2003a; 2003b; 2007; Hladky and Pauls 2008; Magnani 2002; Magnani and Vittone 2008; Montefalcone 2005; 2007a; Ritoré and Rosales 2008].

A  $k$ -step Carnot group  $(\mathbb{G}, \bullet)$  is an  $n$ -dimensional, connected, simply connected, nilpotent, stratified Lie group (with respect to the group multiplication  $\bullet$ ) whose Lie algebra  $\mathfrak{g} \cong \mathbb{R}^n$  satisfies

$$\mathfrak{g} = H_1 \oplus \cdots \oplus H_k, \quad [H_1, H_{i-1}] = H_i \quad (i = 2, \dots, k), \quad H_{k+1} = \{0\}.$$

We assume that  $h_i = \dim H_i$  ( $i = 1, \dots, k$ ) so that  $n = \sum_{i=1}^k h_i$ . Any Carnot group  $\mathbb{G}$  has a 1-parameter family of dilations, adapted to the stratification, that makes it a *homogeneous group*, in the sense of Stein's definition [1993]. We refer the reader to Section 1.1 for a more detailed introduction to Carnot groups.

In this paper, we shall prove some geometric inequalities concerning smooth hypersurfaces immersed in a subriemannian  $k$ -step Carnot group  $\mathbb{G}$  of homogeneous dimension  $Q := \sum_{i=1}^k i h_i$ . We have to stress that hypersurfaces will be endowed with the so-called  $H$ -perimeter measure  $\sigma_H^{n-1}$ , which is a natural substitute for the intrinsic  $(Q-1)$ -dimensional CC Hausdorff measure. In Section 1.2, we will discuss some preliminary notions concerning homogeneous measures and the horizontal geometry of hypersurfaces. Then we will recall some tools which will be important in the sequel, such as a coarea-type formula and the horizontal integration by parts theory; see Section 1.3.

In Section 2 we will extend to this setting some isoperimetric-type constants, introduced in [Cheeger 1970] for compact riemannian manifolds and later studied in [Yau 1975].

In particular, we shall prove the validity of some global inequalities for smooth compact hypersurfaces with (or without) boundary, immersed into  $\mathbb{G}$ . Here, we would like to remark that there is a strong relationship between these inequalities and some eigenvalue problems related to the second-order differential operator  $\mathcal{L}_{HS}$  (which is nothing but a horizontal version of the Laplace–Beltrami operator); see, more precisely, Definition 21 in Section 1.2.

Roughly speaking, after defining the isoperimetric constants (in purely geometric terms), we will show that they are equal to the infimum of some Rayleigh quotients. More precisely, let  $S \subset \mathbb{G}$  be a smooth hypersurface and assume  $\partial S \neq \emptyset$ . Furthermore, set

$$\text{Isop}(S) := \inf \frac{\sigma_H^{n-2}(N)}{\sigma_H^{n-1}(S_1)},$$

where  $N \subset S$  is a smooth hypersurface of  $S$  such that  $N \cap \partial S = \emptyset$  and  $S_1$  is the unique  $(n-2)$ -dimensional submanifold of  $S$  such that  $N = \partial S_1$ . We have to stress that  $\sigma_H^{n-1}$  and  $\sigma_H^{n-2}$  denote homogeneous measures on  $S_1$  and  $N$ , respectively. These measures can be thought of, respectively, as the  $(Q-1)$ -dimensional and

the  $(Q - 2)$ -dimensional CC Hausdorff measures on  $S_1$  and  $N$ ; see Section 1.2. Then, it will be shown that

$$\text{Isop}(S) = \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}},$$

where the infimum is taken over suitably smooth functions on  $S$  such that  $\psi|_{\partial S} = 0$ . As mentioned, this constant is related to the first nonzero eigenvalue  $\lambda_1$  of the following Dirichlet-type problem:

$$\begin{cases} -\mathcal{L}_{HS}\psi = \lambda\psi, \\ \psi|_{\partial S} = 0; \end{cases}$$

see Definition 21. Indeed, we shall see that

$$\lambda_1 \geq \frac{(\text{Isop}(S))^2}{4};$$

see Corollary 28. Some similar results concerning another isoperimetric constant will be proved; see Theorem 30 and Corollary 31. The proofs of these results follow the scheme of the riemannian case, for which we refer the reader to [Yau 1975]; see also [Cheeger 1970] and [Chavel 1984; 1993]. We also remark that the main technical tool in the original proofs is the coarea formula.

In Section 3 we shall prove two geometric inequalities involving volume,  $H$ -perimeter and the first eigenvalue of the operator  $\mathcal{L}_{HS}$  on  $S$ . These results generalize an inequality of Chavel [1978] and an inequality of Reilly [1977], respectively.

In Section 4 we will extend to the Carnot setting some classical differential-geometric results (such as linear isoperimetric inequalities); see, for instance, [Burago and Zalgaller 1988] and references therein. The starting point is an integral formula very similar to the euclidean Minkowski formula; see Corollary 20 for a precise statement. In particular, we will show that

$$(h - 1) \sigma_H^{n-1}(S) \leq R \left( \int_S (|\mathcal{K}_H| + |C_H \nu_H|) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right),$$

where  $S \subset \mathbb{G}$  is a compact hypersurface with boundary and  $R$  denotes the radius of a homogeneous  $\varrho$ -ball circumscribed about  $S$ . From this linear (isoperimetric) inequality, it is possible to infer some geometric consequences and, among them, we prove a weak monotonicity inequality for the  $H$ -perimeter; see Section 4.1, Proposition 38.

Section 5 contains a theorem about nonhorizontal graphs in 2-step Carnot groups. This generalizes a classical result of Heinz [1955]; see also [Chern 1965].

Let us describe this result in the simpler case of the Heisenberg group  $\mathbb{H}^1$ . So let  $S \subset \mathbb{H}^1$  be a  $T$ -graph associated with a function  $t = f(x, y)$  of class  $C^2$  over the

$xy$ -plane. If the horizontal mean curvature  $\mathcal{H}_H$  of  $S$  satisfies a bound  $|\mathcal{H}_H| \geq C > 0$ , then

$$C \mathcal{H}_{\text{Eu}}^2(\mathcal{P}_{xy}(\mathcal{U})) \leq \mathcal{H}_{\text{Eu}}^1(\mathcal{P}_{xy}(\partial\mathcal{U}))$$

for every  $C^1$ -smooth relatively compact open set  $\mathcal{U} \subset S$ , where  $\mathcal{H}_{\text{Eu}}^i$  ( $i = 1, 2$ ) is the usual  $i$ -dimensional euclidean Hausdorff measure and  $\mathcal{P}_{xy}$  is the orthogonal projection onto the  $xy$ -plane. Hence, taking  $\mathcal{U} := S \cap C_r(\mathcal{T})$ , where  $C_r(\mathcal{T})$  denotes a vertical cylinder of radius  $r$  around the  $T$ -axis of  $\mathbb{H}^1$ , yields

$$r \leq \frac{2}{C}$$

for every  $r > 0$ . It follows that any entire  $xy$ -graph of class  $C^2$ , having either constant or only bounded horizontal mean curvature  $\mathcal{H}_H$ , must be necessarily a  $H$ -minimal surface. An analogous result holds true in the framework of step 2 Carnot groups; see Theorem 42.

In Section 6 we shall study some (local) Poincaré-type inequalities, depending on the local geometry of the hypersurface  $S$  and, more precisely, on its characteristic set  $C_S$ ; see Theorems 44 and 45.

For instance, let  $S \subset \mathbb{G}$  be a  $C^2$ -smooth hypersurface with bounded horizontal mean curvature  $\mathcal{H}_H$ . Then, we shall prove that for every  $x \in S$  there exists  $R_0 \leq \text{dist}_\rho(x, \partial S)$  (which explicitly depends on  $C_S$ ) such that

$$\left( \int_{S_R} |\psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \leq C_p R \left( \int_{S_R} |\text{grad}_{HS} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}}, \quad p \in [1, +\infty[$$

for all  $\psi \in C_0^1(S_R)$  and all  $R \leq R_0$ , where  $S_R := S \cap B_\rho(x, R)$ .

These results are obtained by means of elementary “linear” estimates starting from the horizontal integration by parts formula, together with a simple analysis of the role played by the characteristic set. Finally, in Section 6.1 we will prove the validity of a Caccioppoli-type inequality for weak solutions of the operator  $\mathcal{L}_{HS}$ .

**1.1. Carnot groups.** A  $k$ -step Carnot group  $(\mathbb{G}, \bullet)$  is a finite-dimensional connected, simply connected, nilpotent and stratified Lie group with respect to a polynomial group law  $\bullet$ . The Lie algebra  $\mathfrak{g} \cong \mathbb{R}^n$  fulfills the conditions  $\mathfrak{g} = H_1 \oplus \dots \oplus H_k$ ,  $[H_1, H_{i-1}] = H_i$  for all  $i = 2, \dots, k+1$ ,  $H_{k+1} = \{0\}$ , where  $[\cdot, \cdot]$  denotes the Lie bracket and each  $H_i$  is a vector subspace of  $\mathfrak{g}$ . In particular, we denote by  $0$  the identity of  $\mathbb{G}$  and assume that  $\mathfrak{g} \cong T_0\mathbb{G}$ . We also use the notation  $H := H_1$  and  $V := H_2 \oplus \dots \oplus H_k$ . The subspaces  $H$  and  $V$  are smooth subbundles of  $T\mathbb{G}$  called *horizontal* and *vertical*, respectively.

**Notation 1.** Throughout this paper, we denote by  $\mathcal{P}_{H_i} : T\mathbb{G} \rightarrow H_i$  the orthogonal projection map from  $T\mathbb{G}$  onto  $H_i$  for any  $i = 1, \dots, k$ . In particular, we set

$\mathcal{P}_H := \mathcal{P}_{H_i}$ . Analogously, we set  $\mathcal{P}_V : T\mathbb{G} \rightarrow V$  to denote the orthogonal projection map from  $T\mathbb{G}$  onto  $V$ .

Let  $h_i := \dim H_i$  for any  $i = 1, \dots, k$ . Set  $n_0 := 0$  and  $n_i := \sum_{j=1}^i h_j$  for any  $i = 1, \dots, k$ . Note that  $n_1 = h_1, n_2 = h_1 + h_2, \dots$ , and  $n_k = n$ .

**Notation 2.** Throughout this paper, we set  $I_{H_i} := \{n_{i-1} + 1, \dots, n_i\}$  for any  $i = 1, \dots, k$ . We also set  $I_V := \{h_1 + 1, \dots, n\}$  and use Greek letters  $\alpha, \beta, \gamma, \dots$ , for indices in  $I_V$ . For the sake of simplicity, we set  $h := h_1$  and  $I_H := I_{H_1}$ .

The horizontal bundle  $H$  is generated by a frame  $\overline{X}_H := \{X_1, \dots, X_h\}$  of left-invariant vector fields. This frame can be completed to a global graded, left-invariant frame  $\underline{X} := \{X_1, \dots, X_n\}$  for  $T\mathbb{G}$ . Note that the standard basis  $\{e_i : i = 1, \dots, n\}$  of  $\mathbb{R}^n$  can be relabeled to be *graded* or *adapted to the stratification*. Any left-invariant vector field of the frame  $\underline{X}$  is given by  $X_i(x) = L_{x*}e_i$  ( $i = 1, \dots, n$ ), where  $L_{x*}$  denotes the differential of the left-translation  $L_x$ , defined by  $L_x y := x \bullet y$  for all  $y \in \mathbb{G}$ . We also fix a euclidean metric on  $\mathfrak{g} = T_0\mathbb{G}$  such that  $\{e_i : i = 1, \dots, n\}$  is an orthonormal basis. This metric  $g = \langle \cdot, \cdot \rangle$  extends to the whole tangent bundle by left-translations and makes  $\underline{X}$  an orthonormal left-invariant frame. Therefore  $(\mathbb{G}, g)$  is a riemannian manifold.

Let  $\exp : \mathfrak{g} \rightarrow \mathbb{G}$  be the exponential map. Hereafter, we will use *exponential coordinates of the first kind*; see [Varadarajan 1974, Chapter 2, p. 88].

As for the case of nilpotent Lie groups, the multiplication  $\bullet$  of  $\mathbb{G}$  is uniquely determined by the “structure” of the Lie algebra  $\mathfrak{g}$ . This is the content of the *Baker–Campbell–Hausdorff formula*; see [Corwin and Greenleaf 1990]. More precisely,

$$\exp(X) \bullet \exp(Y) = \exp(X \star Y) \quad \text{for all } X, Y \in \mathfrak{g},$$

where  $\star : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  denotes the *Baker–Campbell–Hausdorff product*, given by

$$(1) \quad X \star Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ + \text{brackets of length } \geq 3.$$

Using exponential coordinates and (1), the group multiplication  $\bullet$  turns out to be polynomial and explicitly computable; see [Corwin and Greenleaf 1990]. Moreover,  $0 = \exp(0, \dots, 0)$  and the inverse of  $x \in \mathbb{G}$  ( $x = \exp(x_1, \dots, x_n)$ ) is  $x^{-1} = \exp(-x_1, \dots, -x_n)$ .

A *subriemannian metric*  $g_H$  is a symmetric positive bilinear form on the horizontal bundle  $H$ . The *CC-distance*  $d_{CC}(x, y)$  between  $x, y \in \mathbb{G}$  is given by

$$d_{CC}(x, y) := \inf \int \sqrt{g_H(\dot{\gamma}, \dot{\gamma})} dt,$$

where the infimum is taken over all piecewise-smooth horizontal paths  $\gamma$  joining  $x$  to  $y$ . Later, we shall choose  $g_H := g|_H$ .

Carnot groups are *homogeneous groups*; that is, they admit a 1-parameter group of automorphisms  $\delta_t : \mathbb{G} \rightarrow \mathbb{G}$  ( $t \geq 0$ ) defined by  $\delta_t x := \exp(\sum_{j, i_j} t^j x_{i_j} e_{i_j})$ , where  $x = \exp(\sum_{j, i_j} x_{i_j} e_{i_j}) \in \mathbb{G}$ . As already said, the *homogeneous dimension* of  $\mathbb{G}$  is the integer  $Q := \sum_{i=1}^k i h_i$  coinciding with the *Hausdorff dimension* of  $(\mathbb{G}, d_{CC})$  as a metric space; see [Montgomery 2002].

We recall that a continuous distance  $\varrho : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}_+ \cup \{0\}$  is a *homogeneous distance* if, and only if,

$$\varrho(x, y) = \varrho(z \bullet x, z \bullet y) \text{ for all } x, y, z \in \mathbb{G}; \quad \varrho(\delta_t x, \delta_t y) = t \varrho(x, y) \text{ for all } t \geq 0.$$

The *structural constants* of  $\mathfrak{g}$  (see [Chavel 1993]) associated with the frame  $\underline{X}$  are defined by  $C_{ij}^r := \langle [X_i, X_j], X_r \rangle$  for all  $i, j, r = 1, \dots, n$ . They are skew-symmetric and satisfy Jacobi's identity. The stratification of the Lie algebra  $\mathfrak{g}$  implies a fundamental "structural" property of Carnot groups: if  $X_i \in H_l, X_j \in H_m$ , then  $[X_i, X_j] \in H_{l+m}$ . Note that, if  $i \in I_{H_s}$  and  $j \in I_{H_r}$ , then

$$(2) \quad C_{ij}^m \neq 0 \implies m \in I_{H_{s+r}}.$$

Equivalently, if  $C_{ij}^r \neq 0$ , then  $\text{ord}(i) + \text{ord}(j) = \text{ord}(r)$ , where  $\text{ord} : \{1, \dots, n\} \rightarrow \{1, \dots, k\}$  is the function defined as  $\text{ord}(l) = i \iff l \in I_{H_i}$ .

**Notation 3.** Henceforth, we shall set

- $C_H^\alpha := [C_{ij}^\alpha]_{i, j=1, \dots, h} \in \mathcal{M}_{h \times h}(\mathbb{R})$  for all  $\alpha \in I_{H_2} = \{h + 1, \dots, h + h_2\}$ ;
- $C^\alpha := [C_{ij}^\alpha]_{i, j=1, \dots, n} \in \mathcal{M}_{n \times n}(\mathbb{R})$  for all  $\alpha \in I_V = \{h + 1, \dots, n\}$ .

**Remark 4.** It is important to observe that (2) immediately implies that the matrices just defined are the only ones which can be nonzero.

Let us define the left-invariant coframe  $\omega := \{\omega_i : i = 1, \dots, n\}$  dual to  $\underline{X}$ ; i.e.,  $\omega_i = X_i^*$  for every  $i = 1, \dots, n$ . The *left-invariant 1-forms*  $\omega_i$  for  $i = 1, \dots, n$  are uniquely determined by the condition  $\omega_i(X_j) = \langle X_i, X_j \rangle = \delta_i^j$  for all  $i, j = 1, \dots, n$ , where  $\delta_i^j$  denotes Kronecker delta.

**Definition 5.** We shall denote by  $\nabla$  the unique left-invariant Levi-Civita connection on  $\mathbb{G}$  associated with the left-invariant metric  $g = \langle \cdot, \cdot \rangle$ . Moreover, if  $X, Y \in \mathfrak{X}(H) := C^\infty(\mathbb{G}, H)$ , we shall set

$$\nabla_X^H Y := \mathcal{P}_H(\nabla_X Y).$$

Let  $\underline{X} = \{X_1, \dots, X_n\}$  be the global left-invariant frame on  $T\mathbb{G}$ . Then

$$(3) \quad \nabla_{X_i} X_j = \frac{1}{2} \sum_{r=1}^n (C_{ij}^r - C_{jr}^i + C_{ri}^j) X_r \quad \text{for all } i, j = 1, \dots, n;$$

see, for instance, [Milnor 1976, Section 5, pp. 310–311]. Furthermore, we stress

that  $\nabla^H$  is a partial connection, called *horizontal  $H$ -connection*; see [Ge 1992] or [Koiller et al. 2001]; see also [Montefalcone 2007a] and references therein. Using Definition 5 together with (3) and (2), it is not difficult to show the following:

- $\nabla^H$  is flat; i.e.,

$$\nabla_{X_i}^H X_j = 0 \quad \text{for all } i, j \in I_H;$$

- $\nabla^H$  is compatible with the subriemannian metric  $g_H$ ; i.e.,

$$X\langle Y, Z \rangle = \langle \nabla_X^H Y, Z \rangle + \langle Y, \nabla_X^H Z \rangle \quad \text{for all } X, Y, Z \in \mathfrak{X}(H);$$

- $\nabla^H$  is torsion-free; i.e.,

$$\nabla_X^H Y - \nabla_Y^H X - \mathcal{P}_H[X, Y] = 0 \quad \text{for all } X, Y \in \mathfrak{X}(H).$$

**Definition 6.** If  $\psi \in C^\infty(\mathbb{G})$  we define the horizontal gradient of  $\psi$  as the unique horizontal vector field  $\text{grad}_H \psi$  such that  $\langle \text{grad}_H \psi, X \rangle = d\psi(X) = X\psi$  for every  $X \in \mathfrak{X}(H)$ . The horizontal divergence of  $X \in \mathfrak{X}(H)$ ,  $\text{div}_H X$ , is defined, at each point  $x \in \mathbb{G}$ , by

$$\text{div}_H X(x) := \text{Trace}(Y \rightarrow \nabla_Y^H X)(x) \quad (Y \in H_x).$$

For any  $Y = \sum_{j \in I_H} y_j X_j \in \mathfrak{X}(H)$ , we denote by  $\mathcal{F}_H Y$  the horizontal Jacobian matrix of  $Y$ ; i.e.,

$$\mathcal{F}_H Y := [X_i(y_j)]_{j, i \in I_H}.$$

**Example 7** (Heisenberg group  $\mathbb{H}^n$  ( $n \geq 1$ )). The Lie algebra  $\mathfrak{h}_n \cong \mathbb{R}^{2n+1}$  of the  $n$ -th Heisenberg group  $\mathbb{H}^n$  can be described by means of a left-invariant frame  $\underline{Z} := \{X_1, Y_1, \dots, X_n, Y_n, T\}$ , where, at each  $p = \exp(x_1, y_1, x_2, y_2, \dots, x_n, y_n, t) \in \mathbb{H}^n$ , we have set  $X_i(p) := \partial/\partial x_i - \frac{1}{2}y_i \partial/\partial t$ ,  $Y_i(p) := \partial/\partial y_i + \frac{1}{2}x_i \partial/\partial t$  for every  $i = 1, \dots, n$ ;  $T(p) := \partial/\partial t$ . One has  $[X_i, Y_i] = T$  for every  $i = 1, \dots, n$ , and all other commutators vanish, so that  $T$  is the *center* of  $\mathfrak{h}_n$  and  $\mathfrak{h}_n$  turns out to be a nilpotent and stratified Lie algebra of step 2; i.e.,  $\mathfrak{h}_n = H \oplus H_2$ . The structural constants of  $\mathfrak{h}_n$  are described by the skew-symmetric  $(2n \times 2n)$ -matrix

$$C_H^{2n+1} := \begin{vmatrix} 0 & 1 & \cdot & 0 & 0 \\ -1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & -1 & 0 \end{vmatrix}.$$

**1.2. Hypersurfaces.** The (riemannian) left-invariant volume form of any Carnot group  $\mathbb{G}$  is defined as  $\sigma_R^n := \bigwedge_{i=1}^n \omega_i \in \bigwedge^n(T^*\mathbb{G})$ . By integration of the  $n$ -form  $\sigma_R^n$ , one obtains the Haar measure of  $\mathbb{G}$ , which equals the push-forward of the  $n$ -dimensional Lebesgue measure  $\mathcal{L}^n$  on  $\mathfrak{g} \cong \mathbb{R}^n$ . The symbols  $\mathcal{H}_{\text{CC}}^s$ ,  $\mathcal{H}_{\text{Eu}}^s$  will denote



the intrinsic CC  $s$ -dimensional Hausdorff measure and the euclidean  $s$ -dimensional Hausdorff measure, respectively. (Sometimes we will use the notation  $\sigma_R^n = \text{Vol}^n$ ). Let  $S \subset \mathbb{G}$  be a hypersurface (i.e., a codimension 1 submanifold of  $\mathbb{G}$ ) of class  $C^i$  ( $i \geq 1$ ). Let  $\nu$  denote the (riemannian) unit normal vector along  $S$ . Then  $x \in S$  is a *characteristic point* if and only if  $\dim H_x = \dim(H_x \cap T_x S)$ . The *characteristic set* of  $S$  is given by  $C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_x S)\}$ . In other words, a point  $x \in S$  is noncharacteristic (hereafter abbreviated as NC) if and only if  $H$  is transversal to  $S$  at  $x$ . Hence, one has  $C_S := \{x \in S : |\mathcal{P}_H \nu(x)| = 0\}$ , where  $\mathcal{P}_H$  denotes orthogonal projection onto  $H$ . It is of fundamental importance that the  $(Q-1)$ -dimensional CC Hausdorff measure of the characteristic set  $C_S$  vanishes; i.e.,  $\mathcal{H}_{\text{CC}}^{Q-1}(C_S) = 0$ ; see, for instance, Theorem 6.6.2 in [Magnani 2002]. We also stress that if  $S$  is a hypersurface of class  $C^2$ , then precise estimates of the riemannian Hausdorff dimension of  $C_S$  can be found in [Balogh et al. 2010]; see also [Balogh 2003] for the case of the Heisenberg group  $\mathbb{H}^n$  ( $n \geq 1$ ).

The  $(n-1)$ -dimensional riemannian measure along  $S$  is defined by integration of the  $(n-1)$ -differential form  $\sigma_R^{n-1} \lrcorner S := (\nu \lrcorner \sigma_R^n)|_S$ , where  $\lrcorner$  denotes the “contraction” operator on differential forms; see [Federer 1969]. We recall that  $\lrcorner : \bigwedge^k(T^*\mathbb{G}) \rightarrow \bigwedge^{k-1}(T^*\mathbb{G})$  is defined, for  $X \in T\mathbb{G}$  and  $\alpha \in \bigwedge^k(T^*\mathbb{G})$ , by setting  $(X \lrcorner \alpha)(Y_1, \dots, Y_{k-1}) := \alpha(X, Y_1, \dots, Y_{k-1})$ .

At each NC point  $x \in S \setminus C_S$  the *unit  $H$ -normal* is defined as

$$\nu_H := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}.$$

Similarly to the riemannian case, we define an  $(n-1)$ -differential form  $\sigma_H^{n-1} \in \bigwedge^{n-1}(T^*S)$  by setting

$$\sigma_H^{n-1} \lrcorner S := (\nu_H \lrcorner \sigma_R^n)|_S.$$

By integration of  $\sigma_H^{n-1} \lrcorner S$ , one gets a left-invariant and  $(Q-1)$ -homogeneous measure, which is called  *$H$ -perimeter measure*. This measure can be extended to the whole of  $S$  by setting  $\sigma_H^{n-1} \lrcorner C_S = 0$ . Note that  $\sigma_H^{n-1} \lrcorner S = |\mathcal{P}_H \nu| \sigma_R^{n-1} \lrcorner S$ . Furthermore, denoting by  $\mathcal{G}_{\text{CC}}^{Q-1}$  the  $(Q-1)$ -dimensional spherical intrinsic CC Hausdorff measure (i.e., associated with the CC-distance  $d_{\text{CC}}$ ), then

$$\sigma_H^{n-1}(S \cap B) = k(\nu_H) \mathcal{G}_{\text{CC}}^{Q-1} \lrcorner (S \cap B) \quad \text{for all } B \in \mathcal{B}(\mathbb{G}),$$

where the density-function  $k(\nu_H)$ , called *metric factor*, explicitly depends on  $\nu_H$  and  $d_{\text{CC}}$ ; see [Magnani 2002].

At each NC point  $x \in S \setminus C_S$ , the *horizontal tangent bundle*  $HS := H \cap TS \subset TS$  and the *horizontal normal bundle*  $\nu_H S \subset H$  split the horizontal bundle  $H$  into an orthogonal direct sum; i.e.,  $H = \nu_H \oplus HS$ . The stratification of  $\mathfrak{g}$  induces a

stratification of  $TS := \bigoplus_{i=1}^k H_i S$ , where we have set  $HS := H_1 S$ ; see [Gromov 1996]. Note that at any characteristic point  $x \in C_S$  one has  $H_x = H_x S$ , so that

$$\dim(H_x S) = \begin{cases} h - 1 & \text{if } x \in S \setminus C_S, \\ h & \text{if } x \in C_S. \end{cases}$$

**Notation 8.** Throughout this paper, we denote by  $\mathcal{P}_{HS} : TS \rightarrow HS$  the orthogonal projection map from  $TS$  onto  $HS$ .

Now let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$  and let  $\nabla^{TS}$  denote the induced connection on  $S$  from  $\nabla$ . The tangential connection  $\nabla^{TS}$  induces a partial connection on  $HS$  defined by

$$\nabla_X^{HS} Y := \mathcal{P}_{HS}(\nabla_X^{TS} Y) \quad \text{for all } X, Y \in \mathfrak{X}^1(HS) := C^1(S, HS).$$

It turns out that

$$\nabla_X^{HS} Y = \nabla_X^H Y - \langle \nabla_X^H Y, \nu_H \rangle \nu_H \quad \text{for every } X, Y \in \mathfrak{X}^1(HS);$$

see [Montefalcone 2007a].

**Definition 9** (see [Montefalcone 2007a]). We call  $HS$ -gradient of  $\psi \in C^1(S)$  the unique horizontal tangent vector field  $\text{grad}_{HS} \psi$  such that

$$\langle \text{grad}_{HS} \psi, X \rangle = d\psi(X) = X\psi \quad \text{for all } X \in \mathfrak{X}^1(HS).$$

We denote by  $\text{div}_{HS}$  the  $HS$ -divergence; i.e., if  $X \in \mathfrak{X}^1(HS)$  and  $x \in S$ , then

$$\text{div}_{HS} X(x) := \text{Trace}(Y \rightarrow \nabla_Y^{HS} X)(x) \quad (Y \in H_x S).$$

The  $HS$ -Laplacian  $\Delta_{HS}$  is the second-order differential operator defined as

$$\Delta_{HS} \psi := \text{div}_{HS}(\text{grad}_{HS} \psi) \quad \text{for every } \psi \in C^2(S).$$

The horizontal second fundamental form of  $S \setminus C_S$  is the map given by

$$B_H(X, Y) := \langle \nabla_X^H Y, \nu_H \rangle \quad \text{for all } X, Y \in \mathfrak{X}^1(HS).$$

The horizontal mean curvature  $\mathcal{H}_H$  is the trace of  $B_H$ ; i.e.,  $\mathcal{H}_H := \text{Tr } B_H = -\text{div}_H \nu_H$ .

It is worth observing that the  $HS$ -connection admits, in general, a nonzero torsion because  $B_H$  is *not symmetric*; see [Montefalcone 2007a].

**Definition 10.** Let  $\mathcal{U} \subseteq S$  be an open set. We shall denote by  $C_{HS}^i(\mathcal{U})$  ( $i = 1, 2$ ) the space of functions whose  $HS$ -derivatives up to  $i$ -th order are continuous on  $\mathcal{U}$ .

We stress that the previous definitions concerning the horizontal second fundamental form  $B_H(\cdot, \cdot)$  and the  $HS$ -connection can also be reformulated by using the function space  $C_{HS}^i(\mathcal{U})$  ( $i = 1, 2$ ) and, more precisely, by replacing  $\mathfrak{X}^1(HS) = C^1(S, HS)$  with  $\mathfrak{X}_{HS}^1(HS) := C_{HS}^1(S, HS)$ .

Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^i$  ( $i \geq 1$ ) and let  $\nu$  be the outward-pointing unit normal vector field along  $S$ . We need to define some important geometric objects. To this end, we first note that  $\nu = \mathcal{P}_H \nu + \mathcal{P}_V \nu$ . By using the left-invariant frame  $\underline{X} = \{X_1, \dots, X_n\}$ , we see that  $\mathcal{P}_V \nu = \sum_{\alpha \in I_V} \nu_\alpha X_\alpha$ , where  $\nu_\alpha := \langle \nu, X_\alpha \rangle$ ; see Notation 2.

**Notation 11.** Hereafter we shall set

- $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|}$  for all  $\alpha \in I_V$ ;
- $\varpi := \sum_{\alpha \in I_V} \varpi_\alpha X_\alpha$ ;
- $C_H := \sum_{\alpha \in I_{H_2}} \varpi_\alpha C_H^\alpha$ ;

see, for instance, Notation 3 and Remark 4.

**1.3. Other tools.** Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^i$  ( $i \geq 1$ ). Let  $\partial S$  be an  $(n-2)$ -dimensional submanifold of  $S$  of class  $C^1$ , oriented by the outward pointing unit normal vector  $\eta \in TS \cap \text{Nor}(\partial S)$ . We shall denote by  $\sigma_R^{n-2}$  the riemannian measure on  $\partial S$ ; i.e.,  $\sigma_R^{n-2} \llcorner \partial S = (\eta \lrcorner \sigma_R^{n-1})|_{\partial S}$ . In particular, note that  $(X \lrcorner \sigma_H^{n-1})|_{\partial S} = \langle X, \eta \rangle |\mathcal{P}_H \nu| \sigma_R^{n-2} \llcorner \partial S$  for every  $X \in \mathfrak{X}^1(TS) := C^1(S, TS)$ . The *unit HS-normal* along  $\partial S$  is given by  $\eta_{HS} := \mathcal{P}_{HS} \eta / |\mathcal{P}_{HS} \eta|$ . In this way, we can define a homogeneous  $(n-2)$ -dimensional measure  $\sigma_H^{n-2} \in \bigwedge^{n-2}(T^* \partial S)$  by setting  $\sigma_H^{n-2} \llcorner \partial S := (\eta_{HS} \lrcorner \sigma_H^{n-1})|_{\partial S}$ . It follows that

$$\sigma_H^{n-2} \llcorner \partial S = |\mathcal{P}_H \nu| |\mathcal{P}_{HS} \eta| \sigma_R^{n-2} \llcorner \partial S$$

and that  $(X \lrcorner \sigma_H^{n-1})|_{\partial S} = \langle X, \eta_{HS} \rangle \sigma_H^{n-2} \llcorner \partial S$  for all  $X \in \mathfrak{X}^1(HS) := C^1(S, HS)$ .

Now let  $\nu \wedge \eta \in \Lambda^2(TS)$  be a unit 2-vector orienting  $\partial S$ , where  $\nu \in \text{Nor}(S)$  and  $\eta \in TS \cap \text{Nor}(\partial S)$ . Then, the *characteristic set* of  $\partial S$  is defined as

$$C_{\partial S} := \{p \in \partial S : |\mathcal{P}_H(\nu \wedge \eta)| = 0\},$$

where the orthogonal projection operator  $\mathcal{P}_H$  is extended to 2-vectors in the standard way.

**Proposition 12.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^1$  and let  $\phi \in C^1_{HS}(S)$ . Then*

$$(4) \quad \int_S |\text{grad}_{HS} \phi(x)| \sigma_H^{n-1}(x) = \int_{\mathbb{R}} \sigma_H^{n-2} \{\phi^{-1}[s] \cap S\} ds.$$

*Proof.* This formula follows from the riemannian coarea formula; see [Burago and Zalgaller 1988], [Chavel 2001] or [Montefalcone 2009]. We have

$$\int_S \phi(x) |\text{grad}_{TS} \phi(x)| \sigma_R^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\phi^{-1}[s] \cap S} \phi(y) \sigma_R^{n-2}(y)$$

for every  $\phi \in L^1(S, \sigma_R^{n-1})$ ; see [Burago and Zalgaller 1988; Chavel 2001]. Choosing

$$\phi = \frac{|\text{grad}_{HS} \phi|}{|\text{grad}_{TS} \phi|} |\mathcal{P}_H \nu|$$

yields

$$\int_S \phi |\text{grad}_{TS} \phi| \sigma_R^{n-1} = \int_S \frac{|\text{grad}_{HS} \phi|}{|\text{grad}_{TS} \phi|} |\text{grad}_{TS} \phi| \underbrace{|\mathcal{P}_H \nu| \sigma_R^{n-1}}_{=\sigma_H^{n-1}} = \int_S |\text{grad}_{HS} \phi| \sigma_H^{n-1}.$$

The (riemannian) unit normal  $\eta$  along  $\varphi^{-1}[s]$  is given by  $\eta = \text{grad}_{TS} \phi / |\text{grad}_{TS} \phi|$ . Hence  $|\mathcal{P}_{HS} \eta| = |\text{grad}_{HS} \phi| / |\text{grad}_{TS} \phi|$  and it turns out that

$$\begin{aligned} \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \sigma_R^{n-2} &= \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \frac{|\text{grad}_{HS} \phi|}{|\text{grad}_{TS} \phi|} |\mathcal{P}_H \nu| \sigma_R^{n-2} \\ &= \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \underbrace{|\mathcal{P}_{HS} \eta| |\mathcal{P}_H \nu| \sigma_R^{n-2}}_{=\sigma_H^{n-2}} \\ &= \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \sigma_H^{n-2}. \quad \square \end{aligned}$$

Below, we recall a basic integration by parts formula for horizontal vector fields; see [Montefalcone 2007a].

**Definition 13.** Let  $\mathcal{D}_{HS} : \mathfrak{X}_{HS}^1(HS) \rightarrow C(S)$  be the first-order differential operator given by

$$\mathcal{D}_{HS} X := \text{div}_{HS} X + \langle C_H \nu_H, X \rangle \quad \text{for all } X \in \mathfrak{X}_{HS}^1(HS) \quad (:= C_{HS}^1(S, HS)).$$

Furthermore, let  $\mathcal{L}_{HS} : C_{HS}^2(S) \rightarrow C(S)$  be the second-order differential operator given by

$$\mathcal{L}_{HS} \phi := \Delta_{HS} \phi + \langle C_H \nu_H, \text{grad}_{HS} \phi \rangle \quad \text{for all } \phi \in C_{HS}^2(S);$$

see Definition 9 and Notation 11.

The horizontal matrix  $C_H$  is a key object, related to the skew-symmetric part of the horizontal second fundamental form  $B_H$ . Note that  $\mathcal{D}_{HS}(\phi X) = \phi \mathcal{D}_{HS} X + \langle \text{grad}_{HS} \phi, X \rangle$  for every  $X \in \mathfrak{X}_{HS}^1(HS)$  and every  $\phi \in C_{HS}^1(S)$ . Moreover, one has  $\mathcal{L}_{HS} \phi = \mathcal{D}_{HS}(\text{grad}_{HS} \phi)$  for every  $\phi \in C_{HS}^2(S)$ . These definitions are motivated by Theorem 3.17, Corollary 3.18 and Corollary 3.19 in [Montefalcone 2007a].

**Theorem 14** (see [Montefalcone 2007a]). *Let  $S$  be a compact NC hypersurface of class  $C^2$  with boundary  $\partial S$  of class  $C^1$ . Then*

$$(5) \quad \int_S \mathcal{D}_{HS} X \sigma_H^{n-1} = - \int_S \mathcal{H}_H \langle X, \nu_H \rangle \sigma_H^{n-1} + \int_{\partial S} \langle X, \eta_{HS} \rangle \sigma_H^{n-2} \quad \text{for all } X \in \mathfrak{X}^1(H).$$

**Remark 15.** We note that, in general,  $\mathcal{H}_H \notin L^1_{\text{loc}}(S; \sigma_R^{n-1})$ ; see [Danielli et al. 2012]. However, it is always true that  $\mathcal{H}_H \in L^1_{\text{loc}}(S; \sigma_H^{n-1})$ ; see, for instance, [Montefalcone 2012].

**Remark 16.** Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$  and  $\nu$  the outward-pointing unit normal vector along  $S$ . For any  $X \in \mathfrak{X}(\mathbb{G})$  let us set  $X^\perp := \langle X, \nu \rangle \nu$  and  $X^\top := X - X^\perp$  to denote the riemannian normal and tangential components of  $X$  at any point of  $S$ . We would like to stress that formula (5) can be seen as a particular case of a general integral formula, the so-called first variation formula of the  $H$ -perimeter. More precisely, the first variation formula is given by

$$(6) \quad I_S(X, \sigma_H^{n-1}) = \int_S \left( -\mathcal{H}_H \langle X^\perp, \nu \rangle + \text{div}_{TS}(X^\top |\mathcal{P}_H \nu| - \langle X^\perp, \nu \rangle \nu_H^\top) \right) \sigma_R^{n-1},$$

where  $I_S(X, \sigma_H^{n-1})$  denotes the first derivative of the  $H$ -perimeter under a smooth variation of  $S$  with initial velocity  $X$ ; see [Montefalcone 2012, Theorem 4.6]. Formula (6) also holds if  $C_S \neq \emptyset$ , but in this case we need to assume  $\mathcal{H}_H \in L^1_{\text{loc}}(S; \sigma_R^{n-1})$ . We observe that, in the case of the first Heisenberg group  $\mathbb{H}^1$ , this formula coincides with that of Ritoré and Rosales [2008, Lemma 4.3, p. 14]. Note that, if  $X = X_H \in \mathfrak{X}(H)$ , then

$$\begin{aligned} X_H^\top |\mathcal{P}_H \nu| - \langle X_H^\perp, \nu \rangle \nu_H^\top &= (X_H - |\mathcal{P}_H \nu| \langle X_H, \nu_H \rangle \nu) |\mathcal{P}_H \nu| - |\mathcal{P}_H \nu| \langle X_H, \nu \rangle (\nu_H - |\mathcal{P}_H \nu| \nu) \\ &= (X_H - \langle X_H, \nu \rangle \nu_H) |\mathcal{P}_H \nu| = \mathcal{P}_{HS}(X_H) |\mathcal{P}_H \nu|, \end{aligned}$$

where we have used the fact that  $\nu = |\mathcal{P}_H \nu| \nu_H + \sum_{\alpha \in I_\nu} \nu_\alpha X_\alpha$  at each NC point. Finally, inserting this into (6), we obtain an equivalent form of (5). In particular, for any  $X \in \mathfrak{X}(H)$  the function  $\mathcal{D}_{HS} X$  turns out to be the Lie derivative of the differential  $(n-1)$ -form  $\sigma_H^{n-1} \lrcorner S$  with respect to the initial velocity  $X$  of a smooth variation of  $S$ . Roughly speaking, this can be rephrased by saying that the differential  $(n-1)$ -form  $(\mathcal{D}_{HS} X) \sigma_H^{n-1} \in \Lambda^{n-1}(T^*S)$  is the “infinitesimal” first variation of  $S$ .

Formula (5) holds true even if  $C_S \neq \emptyset$ , at least under suitable assumptions.

**Definition 17.** Let  $X \in C^1(S \setminus C_S, HS)$  and set  $\alpha_X := (X \lrcorner \sigma_H^{n-1})|_S$ . We say that  $X$  is *admissible (for the horizontal divergence formula)* if the differential forms  $\alpha_X$  and  $d\alpha_X$  are continuous on all of  $S$ , or, more generally, if  $\alpha, d\alpha \in L^\infty(S)$  and  $\iota_S^* \alpha \in L^\infty(\partial S)$ . We say that  $\phi \in C^2_{HS}(S \setminus C_S)$  is *admissible* if  $\text{grad}_{HS} \phi$  is admissible for the horizontal divergence formula.

We stress that, if the differential forms  $\alpha_X$  and  $d\alpha_X$  are continuous on all of  $S$  (or, more generally, if  $\alpha, d\alpha \in L^\infty(S)$  and  $\iota_S^* \alpha \in L^\infty(\partial S)$ , where  $\iota_S : \partial M \rightarrow \bar{M}$  is the natural inclusion), then Stokes’ formula holds true; see, for instance, [Taylor 2006]. This fact motivates the following:

**Corollary 18.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  with boundary  $\partial S$  of class  $C^1$ . Then:*

- (i)  $\int_S \mathfrak{D}_{HS} X \sigma_H^{n-1} = \int_{\partial S} \langle X, \eta_{HS} \rangle \sigma_H^{n-2}$  for every admissible  $X \in C^1(S \setminus C_S, HS)$ .
- (ii)  $\int_S \mathcal{L}_{HS} \phi \sigma_H^{n-1} = \int_{\partial S} \langle \text{grad}_{HS} \phi, \eta_{HS} \rangle \sigma_H^{n-2}$  for every admissible  $\phi \in C^2_{HS}(S \setminus C_S)$ .
- (iii) If  $\partial S = \emptyset$ , then

$$-\int_S \phi \mathcal{L}_{HS} \phi \sigma_H^{n-1} = \int_S |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1}$$

for every  $\phi \in C^2_{HS}(S \setminus C_S)$  such that  $\phi^2$  is admissible.

The last formula holds even if  $\partial S \neq \emptyset$ , but for compactly supported functions. One can show that  $\phi^2$  is admissible if and only if  $\phi \in C^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma_H^{n-1})$ , where we have set  $W^{1,2}_{HS}(S, \sigma_H^{n-1}) := \{\phi \in L^2(S, \sigma_H^{n-1}) : |\text{grad}_{HS} \phi| \in L^2(S, \sigma_H^{n-1})\}$ . We also remark that any vector field  $X \in C^1(S, HS)$  turns out to be admissible. Analogously, any  $\phi \in C^2_{HS}(S)$  is admissible.

**Lemma 19.** *Let  $x_H := \sum_{i \in I_H} x_i X_i$  be the ‘‘horizontal position vector’’ and let  $g_H$  denote its component along the  $H$ -normal  $\nu_H$ ; i.e.,  $g_H := \langle x_H, \nu_H \rangle$ . In the sequel, the function  $g_H$  will be called ‘‘horizontal support function’’ of  $x_H$ . Then, we have:*

- (i)  $\text{div}_H x_H = h$ ;
- (ii)  $\mathfrak{D}_{HS}(x_{HS}) = (h - 1) + g_H \mathfrak{H}_H + \langle C_H \nu_H, x_{HS} \rangle$  at each NC point  $x \in S \setminus C_S$ , where  $x_{HS} := x_H - g_H \nu_H$ .

*Proof.* We have  $\text{div}_H x_H = \sum_{i=1}^h \langle \nabla_{X_i} x_H, X_i \rangle = \sum_{i,j=1}^h (X_i(x_j) + \langle \nabla_{X_i} X_j, X_i \rangle) = \sum_{i,j=1}^h \delta_i^j = h$ , where  $\delta_i^j$  denotes Kronecker’s delta; here we have used  $\mathcal{F}_h(x_H) = \text{Id}_h$  and  $\langle \nabla_{X_i} X_j, X_i \rangle = 0$  for all  $i, j \in I_H$ ; see Definition 6 and formula (6). Furthermore, by definition, one has  $\text{div}_{HS} x_H = \text{div}_H x_H - \langle \nabla_{\nu_H} x_H, \nu_H \rangle$ . Hence  $\text{div}_{HS} x_H = h - \langle \nu_H, \nu_H \rangle = h - 1$ . Furthermore, by definition, we have

$$(7) \quad \text{div}_{HS} x_{HS} = \sum_{i=2}^h \langle \nabla_{\tau_i} (x_H - g_H \nu_H), \tau_i \rangle,$$

where we have used an orthonormal horizontal frame  $\underline{\tau}_H := \{\tau_1, \dots, \tau_h\}$  in an open neighborhood  $U \subset \mathbb{G}$  of  $S$  such that  $\tau_1(x) = \nu_H(x)$  at any  $x \in S \setminus C_S$ ; see, for instance, Definition 3.4 in [Montefalcone 2007a]. Starting from (7), we compute

$$\text{div}_{HS} x_{HS} = \sum_{i=2}^h (\langle \tau_i, \tau_i \rangle - g_H \langle \nabla_{\tau_i}^H \nu_H, \tau_i \rangle) = (h-1) - g_H \text{div}_H \nu_H = (h-1) + g_H \mathfrak{H}_H$$

for every  $x \in S \setminus C_S$ . The thesis easily follows from the definition of  $\mathfrak{D}_{HS}$ .  $\square$

A simple consequence of Corollary 18 and Lemma 19 is given by the following:

**Corollary 20** (Minkowski-type formula). *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$ . Let  $x_H = \sum_{i \in I_H} x_i X_i$  be the horizontal position vector. Furthermore, set  $g_H = \langle x_H, \nu_H \rangle$  and  $x_{HS} = x - g_H \nu_H$  for every  $x \in S \setminus C_S$ . Then*

$$\int_S ((h - 1) + g_H \mathcal{K}_H + \langle C_H \nu_H, x_{HS} \rangle) \sigma_H^{n-1} = 0.$$

*Proof.* It is enough to apply Corollary 18 to the horizontal tangent vector field  $x_{HS} \in C^1(S \setminus C_S, HS)$ . Using Remark 15 and Lemma 19 the thesis easily follows.  $\square$

**Definition 21** (eigenvalue problems for  $\mathcal{L}_{HS}$ ). *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  without boundary. Then we look for solutions of class  $C^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma_H^{n-1})$  to the problem*

$$(P_1) \quad \begin{cases} -\mathcal{L}_{HS}\psi = \lambda\psi; \\ \int_S \psi \sigma_H^{n-1} = 0. \end{cases}$$

If  $\partial S \neq \emptyset$ , we look for solutions of class  $C^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma_H^{n-1})$  to the problems

$$(P_2) \quad \begin{cases} -\mathcal{L}_{HS}\psi = \lambda\psi; \\ \psi|_{\partial S} = 0; \end{cases} \quad (P_3) \quad \begin{cases} -\mathcal{L}_{HS}\psi = \lambda\psi; \\ \frac{\partial \psi}{\partial \eta_{HS}}|_{\partial S} = 0. \end{cases}$$

We explicitly remark that  $\partial\psi/\partial\eta_{HS} = \langle \text{grad}_{HS}\psi, \eta_{HS} \rangle$ .

The problems  $(P_1)$ ,  $(P_2)$  and  $(P_3)$  generalize to our context the classical *closed, Dirichlet and Neumann eigenvalue problems* for the Laplace–Beltrami operator on riemannian manifolds; see [Chavel 1984; 1993].

Finally, we recall a recent general result about the size of horizontal tangencies to noninvolutive distributions, which applies to our Carnot setting; see Theorem 4.5 in [Balogh et al. 2010].

**Theorem 22** (generalized Derridj’s theorem). *Let  $\mathbb{G}$  be a  $k$ -step Carnot group.*

- (i) *If  $S \subset \mathbb{G}$  is a hypersurface of class  $C^2$ , the euclidean-Hausdorff dimension of the characteristic set  $C_S$  of  $S$  satisfies  $\dim_{\text{Eu-Hau}}(C_S) \leq n - 2$ .*
- (ii) *If  $V = H^\perp \subset T\mathbb{G}$  satisfies  $\dim V \geq 2$  and  $N \subset \mathbb{G}$  is an  $(n - 2)$ -dimensional submanifold of class  $C^2$ , then the euclidean-Hausdorff dimension of the characteristic set  $C_N$  of  $N$  satisfies  $\dim_{\text{Eu-Hau}}(C_N) \leq n - 3$ .*

**Remark 23.** Let  $N \subset \mathbb{G}$  be an  $(n - 2)$ -dimensional submanifold of class  $C^2$ . This smoothness condition is sharp; see [Balogh et al. 2010]. Moreover, we stress that  $\dim V = 1$  just for Heisenberg groups and 2-step Carnot groups having 1-dimensional center. For Heisenberg groups  $\mathbb{H}^n$ ,  $n > 1$ , using Frobenius’ theorem yields  $\dim_{\text{Eu-Hau}}(C_N) \leq n$ , where  $n = \frac{1}{2} \dim H$ ; see also [Balogh et al. 2010]. On the contrary, 1-dimensional curves in  $\mathbb{H}^1$  can be horizontal or transversal to  $H$ . For

2-step groups having 1-dimensional center (or, equivalently, horizontal bundle  $H$  of codimension 1) a simple analysis shows that  $\dim_{\text{Eu-Hau}}(C_N) = n - 2$  if, and only if,  $\mathbb{G}$  reduces to the direct product of  $\mathbb{H}^1$  and of a euclidean space  $\mathbb{R}^{h-2}$ .

**2. Isoperimetric constants and the first eigenvalue of  $\mathcal{L}_{HS}$  on compact hypersurfaces**

As a consequence of the coarea formula (4) we may generalize to the Carnot groups setting some results about isoperimetric constants and global Poincaré inequalities for which we refer the reader to [Chavel 1984; 1993]; see also [Cheeger 1970; Yau 1975].

Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  with (or without) boundary. Similarly as in the riemannian setting (see [Cheeger 1970; Yau 1975]), we may give the following:

**Definition 24.** The *isoperimetric constant*  $\text{Isop}(S)$  of  $S$  is defined as follows:

- If  $\partial S = \emptyset$ , we set

$$\text{Isop}(S) := \inf \frac{\sigma_H^{n-2}(N)}{\min\{\sigma_H^{n-1}(S_1), \sigma_H^{n-1}(S_2)\}},$$

where the infimum is taken over all  $C^2$ -smooth  $(n - 2)$ -dimensional submanifolds  $N$  of  $S$  which divide  $S$  into two hypersurfaces  $S_1, S_2$  with common boundary  $N = \partial S_1 = \partial S_2$ .

- If  $\partial S \neq \emptyset$ , we set

$$\text{Isop}(S) := \inf \frac{\sigma_H^{n-2}(N)}{\sigma_H^{n-1}(S_1)},$$

where  $N \subset S$  is a smooth hypersurface of  $S$  such that  $N \cap \partial S = \emptyset$  and  $S_1$  is the unique  $C^2$ -smooth  $(n - 2)$ -dimensional submanifold of  $S$  such that  $N = \partial S_1$ .

Here  $\partial S, S_1, S_2$  and  $N = \partial S_i$  ( $i = 1, 2$ ) are not assumed to be connected.

This definition requires some comments. As recalled in the introduction, in the riemannian setting analogous isoperimetric constants were introduced in [Cheeger 1970], in order to give a geometric lower bound for the smallest eigenvalue of the Laplace–Beltrami operator on smooth compact riemannian manifolds. This definition was somewhat motivated by an example of Calabi, the so-called *dumbbell* manifold, homeomorphic to  $\mathbb{S}^2$ . Actually, an analysis of this example shows that, in order to bound  $\lambda$  from below, the diameter and the volume are not enough.

We also have to recall that these isoperimetric constants turn out to be strictly positive. Although this claim turns out to be (more or less) elementary in dimension



$n = 2$ , it becomes a bit more difficult when  $n > 2$ ; see [Cheeger 1970]. Some years after Cheeger’s result, Yau [1975] reconsidered the isoperimetric constants and demonstrated that  $\lambda$  has a bound in terms of volume, diameter and (of a lower bound of the) Ricci curvature. See the survey [Li 1982] for a glimpse on this topic.

Below we shall generalize some of the results of [Yau 1975]. Our results will follow the original scheme, which is based mainly on a suitable use of the coarea formula for smooth functions. Note also that, instead of  $C^\infty$ -smooth hypersurfaces, here we are considering hypersurfaces of class  $C^2$ . We have to observe that all the results could also be stated for  $C^1$  hypersurfaces. But the delicate matter here is that in our setting, new difficulties come from the presence of characteristic points and, in the  $C^1$  case, it is not simple to prove that isoperimetric constants are strictly positive. Actually, the following further hypothesis seems to be unavoidable in order to have nonzero isoperimetric constants:

**(H)** *Every  $C^2$ -smooth  $(n - 2)$ -dimensional submanifold  $N \subset S$  satisfies*

$$\dim C_N < n - 2.$$

This assumption can be overcome by using the generalized Derridj’s theorem, (Theorem 22); see also Remark 23. As a consequence, the results of this section are “meaningful” (in the sense that the isoperimetric constants do not vanish) at least for any Carnot group  $\mathbb{G}$  such that  $\dim V \geq 2$  and for all Heisenberg groups  $\mathbb{H}^n$ , with  $n > 1$ .

**Theorem 25.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$ .*

(i) *If  $\partial S = \emptyset$ , then*

$$\text{Isop}(S) = \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}},$$

*where the infimum is taken over all  $C^2$ -smooth functions on  $S$  such that  $\int_S \psi \sigma_H^{n-1} = 0$ .*

(ii) *If  $\partial S \neq \emptyset$ , then*

$$\text{Isop}(S) = \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}},$$

*where the infimum is taken over all  $C^2$ -smooth functions on  $S$  such that  $\psi|_{\partial S} = 0$ .*

**Warning 26.** The definition of  $\text{Isop}(S)$  can be weakened. For instance, part (i) of Definition 24 can be given by assuming  $S$  of class  $C^1$  and then by taking the infimum over all  $(n - 2)$ -dimensional submanifolds  $N$  of  $S$  of class  $C^1$  which divide  $S$  into two hypersurfaces  $S_1, S_2$  with common boundary  $N = \partial S_1 = \partial S_2$ .

In this case, Theorem 25(i) holds, without modifications, by taking the infimum over  $C^1_{HS}$ -smooth functions. If  $\partial S \neq \emptyset$  an analogous claim holds, for the other isoperimetric constant. Furthermore, equivalent remarks can be given for all the results of this section. Nevertheless, as already said, this weaker formulation seems to be less meaningful because of the presence of characteristic points.

**Warning 27.** Throughout this section, we shall fix a homogeneous distance  $\varrho$  on  $\mathbb{G}$  of class  $C^1$  outside the diagonal of  $\mathbb{G}$ .

*Proof of Theorem 25.* The proof repeats almost verbatim the arguments of Theorem 1 in [Yau 1975]. We just prove the theorem for  $\partial S = \emptyset$  since the other case is analogous. First, let us prove the inequality

$$\text{Isop}(S) \leq \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}},$$

where  $\psi \in C^2(S)$  and  $\int_S \psi \sigma_H^{n-1} = 0$ . To prove this inequality let us consider the auxiliary functions  $\psi^+ = \max\{0, \psi\}$ ,  $\psi^- = \max\{0, -\psi\}$ . By applying the coarea formula (4) and the definition of  $\text{Isop}(S)$  we get that

$$\int_S |\text{grad}_{HS} \psi^\pm| \sigma_H^{n-1} = \int_0^{+\infty} \sigma_H^{n-2} \{x \in S : \psi^\pm = t\} dt \geq \text{Isop}(S) \int_S |\psi^\pm| \sigma_H^{n-1}.$$

Now we shall prove the reversed inequality. So let us assume that  $\sigma_H^{n-1}(S_1) \leq \sigma_H^{n-1}(S_2)$  and let  $\epsilon > 0$ . By making use of the fixed homogeneous distance  $\varrho$  on  $\mathbb{G}$ , we now define a function  $\psi_\epsilon : S \rightarrow \mathbb{R}$  by setting

$$(8) \quad \begin{aligned} \psi_\epsilon(x)|_{S_1} &:= \begin{cases} \frac{\varrho(x, N)}{\epsilon} & \text{if } \varrho(x, N) \leq \epsilon, \\ 1 & \text{if } \varrho(x, N) > \epsilon, \end{cases} \\ \psi_\epsilon(x)|_{S_2} &:= \begin{cases} -\alpha \frac{\varrho(x, N)}{\epsilon} & \text{if } \varrho(x, N) \leq \epsilon, \\ -\alpha & \text{if } \varrho(x, N) > \epsilon, \end{cases} \end{aligned}$$

where the constant  $\alpha$  depends on  $\epsilon$  and is chosen in a way that  $\int_S \psi_\epsilon \sigma_H^{n-1} = 0$ . Obviously

$$\lim_{\epsilon \rightarrow 0} \alpha = \frac{\sigma_H^{n-1}(S_1)}{\sigma_H^{n-1}(S_2)}.$$

Since

$$\begin{aligned} \int_S |\text{grad}_{HS} \psi_\epsilon| \sigma_H^{n-1} &= \frac{1 + \alpha}{\epsilon} \int_{N_\epsilon := \{x \in S : \varrho(x, N) \leq \epsilon\}} |\text{grad}_{HS} \varrho(x, N)| \sigma_H^{n-1} \\ &= \frac{1 + \alpha}{\epsilon} \int_0^\epsilon \sigma_H^{n-2} \{x \in N_\epsilon : \varrho(x, N) = t\} dt, \end{aligned}$$

one gets

$$\lim_{\epsilon \rightarrow 0} \int_S |\text{grad}_{HS} \psi_\epsilon| \sigma_H^{n-1} = (1 + \alpha) \sigma_H^{n-2}(N).$$

Moreover  $\lim_{\epsilon \rightarrow 0} \int_S |\psi_\epsilon| \sigma_H^{n-1} = \sigma_H^{n-1}(S_1) + \alpha \sigma_H^{n-1}(S_2)$ . Putting it all together we get

$$\inf_{\psi} \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}} \leq \lim_{\epsilon \rightarrow 0} \frac{\int_S |\text{grad}_{HS} \psi_\epsilon| \sigma_H^{n-1}}{\int_S |\psi_\epsilon| \sigma_H^{n-1}} \leq \frac{\sigma_H^{n-1}(N)}{\sigma_H^{n-2}(S_1)}.$$

If we take the infimum over  $N$  and  $S_1$ , the inequality follows. □

**Corollary 28.** *Let  $\lambda_1$  be the first nonzero eigenvalue of either the closed eigenvalue problem or the Dirichlet eigenvalue problem (see Definition 21). Then we have  $\lambda_1 \geq \frac{1}{4}(\text{Isop}(S))^2$ .*

*Proof.* We just prove the first claim, as the second claim is similar. Let  $\psi$  be an eigenfunction of  $\mathcal{L}_{HS}$  corresponding to  $\lambda_1$ . Then

$$\begin{aligned} \lambda_1 &= -\frac{\int_S \psi \mathcal{L}_{HS} \psi \sigma_H^{n-1}}{\int_S |\psi|^2 \sigma_H^{n-1}} = \frac{\int_S |\text{grad}_{HS} \psi|^2 \sigma_H^{n-1}}{\int_S |\psi|^2 \sigma_H^{n-1}} = \frac{\int_S |\text{grad}_{HS} \psi|^2 \sigma_H^{n-1}}{(\int_S |\psi|^2 \sigma_H^{n-1})^2} \int_S |\psi|^2 \sigma_H^{n-1} \\ &\geq \frac{(\int_S |\psi| |\text{grad}_{HS} \psi| \sigma_H^{n-1})^2}{(\int_S |\psi|^2 \sigma_H^{n-1})^2} = \frac{1}{4} \frac{(\int_S |\text{grad}_{HS} \psi|^2 \sigma_H^{n-1})^2}{(\int_S \psi^2 \sigma_H^{n-1})^2} \geq \frac{(\text{Isop}(S))^2}{4}, \end{aligned}$$

where we have used Theorem 25 together with the Cauchy–Schwarz inequality. □

We now extend, to Carnot groups, another isoperimetric constant and some related facts which, in the riemannian case, were studied in [Yau 1975].

**Definition 29.** The *isoperimetric constant*  $\text{Isop}_0(S)$  of any  $C^2$ -smooth compact hypersurface  $S \subset \mathbb{G}$  with boundary  $\partial S$  is given by

$$\text{Isop}_0(S) := \inf \left\{ \frac{\sigma_H^{n-2}(\partial S_1 \cap \partial S_2)}{\min\{\sigma_H^{n-1}(S_1), \sigma_H^{n-1}(S_2)\}} \right\},$$

where the infimum is taken over all decompositions  $S = S_1 \cup S_2$  such that  $\sigma_H^{n-1}(S_1 \cap S_2) = 0$ .

**Theorem 30.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  with boundary. Then*

$$\text{Isop}_0(S) = \inf \left\{ \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\inf_{\beta \in \mathbb{R}} \int_S |\psi - \beta| \sigma_H^{n-1}} \right\},$$

where the infimum is taken over all  $C^2$ -functions defined on  $S$ .

*Proof.* The proof is analogous to that of Theorem 6 in [Yau 1975]. First, let us prove the inequality

$$\text{Isop}(S) \leq \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_H^{n-1}}{\int_S |\psi| \sigma_H^{n-1}}.$$

To this end, let us define the functions  $\psi^+ := \max\{0, \psi - k\}$ ,  $\psi^- := -\min\{0, \psi - k\}$ , where  $k \in \mathbb{R}$  is any constant such that

$$\sigma_H^{n-1}\{x \in S : \psi^+ > 0\} \leq \frac{1}{2}\sigma_H^{n-1}(S), \quad \sigma_H^{n-1}\{x \in S : \psi^- > 0\} \leq \frac{1}{2}\sigma_H^{n-1}(S).$$

By using again the coarea formula (4) together with the definition of  $\text{Isop}_0(S)$  we get that

$$\int_S |\text{grad}_{HS} \psi^\pm| \sigma_H^{n-1} = \int_0^{+\infty} \sigma_H^{n-2}\{x \in S : \psi^\pm = t\} dt \geq \text{Isop}(S) \int_S |\psi^\pm| \sigma_H^{n-1}.$$

We prove the other inequality. Assuming  $\sigma_H^{n-1}(S_1) \leq \sigma_H^{n-1}(S_2)$  and  $\epsilon > 0$ , we define the function

$$(9) \quad \psi_\epsilon(x)|_{S_1} := 1, \quad \psi_\epsilon(x)|_{S_2} := \begin{cases} 1 - \frac{\varrho(x, \partial S_1 \cap \partial S_2)}{\epsilon} & \text{if } \varrho(x, \partial S_1 \cap \partial S_2) \leq \epsilon, \\ 0 & \text{if } \varrho(x, \partial S_1 \cap \partial S_2) > \epsilon. \end{cases}$$

Furthermore, one can find a constant  $k(\epsilon)$  satisfying

$$\int_S |\psi_\epsilon - k(\epsilon)| \sigma_H^{n-1} = \inf_{\beta \in \mathbb{R}} \int_S |\psi_\epsilon - \beta| \sigma_H^{n-1}$$

and such that  $k(\epsilon) \rightarrow 0$  for  $\epsilon \rightarrow 0^+$ . Hence

$$\lim_{\epsilon \rightarrow 0} \left\{ \frac{\int_S |\text{grad}_{HS} \psi_\epsilon| \sigma_H^{n-1}}{\inf_{\beta \in \mathbb{R}} \int_S |\psi_\epsilon - \beta| \sigma_H^{n-1}} \right\} \leq \frac{\sigma_H^{n-2}(\partial S_1 \cap \partial S_2)}{\min\{\sigma_H^{n-1}(S_1), \sigma_H^{n-1}(S_2)\}}. \quad \square$$

**Corollary 31.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$ . Then*

$$(10) \quad \int_S |\psi - k|^2 \sigma_H^{n-1} \leq \frac{4}{(\text{Isop}_0(S))^2} \int_S |\text{grad}_{HS} \psi|^2 \sigma_H^{n-1}$$

for every  $\psi \in C^2(S)$  and every  $k \in \mathbb{R}$  such that

$$\sigma_H^{n-1}\{x \in S : \psi \geq k\} \geq \frac{1}{2}\sigma_H^{n-1}(S), \quad \sigma_H^{n-1}\{x \in S : \psi \leq k\} \geq \frac{1}{2}\sigma_H^{n-1}(S).$$

Furthermore, if  $\psi \in C^2(S)$  and  $\int_S \psi \sigma_H^{n-1} = 0$ , then

$$(11) \quad \int_S |\psi|^2 \sigma_H^{n-1} \leq \frac{4}{(\text{Isop}_0(S))^2} \int_S |\text{grad}_{HS} \psi|^2 \sigma_H^{n-1}.$$

*Proof.* One has  $\int_S (\psi^+ \cdot \psi^-) \sigma_H^{n-1} = 0$ , where the functions  $\psi^\pm$  are defined as in the proof of Theorem 30. Moreover, by using once more coarea formula, we get

$$\begin{aligned} \int_S |\psi - k|^2 \sigma_H^{n-1} &= \int_S |\psi^+ + \psi^-|^2 \sigma_H^{n-1} \leq \int_S |\psi^+|^2 \sigma_H^{n-1} + \int_S |\psi^-|^2 \sigma_H^{n-1} \\ &\leq \frac{1}{\text{Isop}_0(S)} \left( \int_S |\text{grad}_{HS}(\psi^+)|^2 \sigma_H^{n-1} + \int_S |\text{grad}_{HS}(\psi^-)|^2 \sigma_H^{n-1} \right) \\ &\leq \frac{2}{\text{Isop}_0(S)} \int_S (\psi^+ + \psi^-) |\text{grad}_{HS} \psi| \sigma_H^{n-1} \\ &\leq \frac{2}{\text{Isop}_0(S)} \|\psi^+ + \psi^-\|_{L^2(S; \sigma_H^{n-1})} \|\text{grad}_{HS} \psi\|_{L^2(S; \sigma_H^{n-1})}. \end{aligned}$$

This proves (10). In order to prove (11) we note that the hypothesis  $\int_S \psi \sigma_H^{n-1} = 0$  actually implies that

$$\int_S \psi^2 \sigma_H^{n-1} = \inf_{k \in \mathbb{R}} \int_S (\psi - k)^2 \sigma_H^{n-1},$$

which, together with (10), implies the thesis of the theorem.  $\square$

### 3. Two upper bounds on $\lambda_1$

Below we shall extend two (nowadays classical) inequalities obtained, respectively, by Chavel and Reilly in the euclidean/riemannian setting. An important feature of these results is that they give explicit upper bounds for the first nontrivial eigenvalue (of the Laplacian) of a compact submanifold of  $\mathbb{R}^n$ . For further details we refer to [Chavel 1978] and [Reilly 1977]; see also [Heintze 1988]. To begin with, let  $\Omega \subsetneq \mathbb{G}$  be a bounded domain and assume that  $S := \partial\Omega$  is a connected hypersurface of class  $C^2$ , with orientation given by the outward normal vector  $\nu$ . Moreover, let  $x_H$  be the horizontal position vector field and let us apply the usual divergence formula. We also set  $\sigma_R^n = \mathcal{V}ol^n$ . We have

$$h \mathcal{V}ol^n(\Omega) = \int_\Omega \text{div}_H x_H \sigma_R^n = \int_{\partial\Omega} \langle x_H, \nu \rangle \sigma_R^{n-1} = \int_S \langle x_H, \nu_H \rangle \sigma_H^{n-1},$$

where we have used Lemma 19(i). Furthermore, we may further assume that the ‘‘center of mass’’ of  $\partial\Omega$  (with respect to the  $H$ -perimeter) is placed at the identity  $0 \in \mathbb{G}$ . In other words, let us assume that  $\int_S x_i \sigma_H^{n-1} = 0$  for every  $i \in I_H = \{1, \dots, h\}$ , where  $x_H \equiv (x_1, \dots, x_i, \dots, x_h)$  is the horizontal position vector; see Lemma 19.

The last assumption is justified by the following:

**Lemma 32.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^i$  ( $i \geq 1$ ). We can always choose a system of exponential coordinates  $x = \exp(x_1, \dots, x_n)$  on  $\mathbb{G}$  such that  $\int_S x_i \sigma_H^{n-1}(x) = 0$  for every  $i \in I_H = \{1, \dots, h\}$ .*

*Proof.* Let

$$a_i := \frac{\int_S x_i \sigma_H^{n-1}(x)}{\sigma_H^{n-1}(S)} \quad \text{for all } i \in I_H = \{1, \dots, h\}$$

and  $a_H \equiv (a_1, \dots, a_i, \dots, a_h)$ . Set  $a := \exp(a_H, 0_V)$ , where the symbol  $0_V$  denotes the origin of  $V \subset \mathfrak{g}$ . Consider the change of variables  $y := \Phi(x) = a^{-1} \bullet x$  ( $x \in \mathbb{G}$ ). Equivalently, we have  $\Phi(x) = L_{a^{-1}}(x)$ , where  $L_{a^{-1}}$  is the left-translation by  $a^{-1} = -a$ ; see Section 1.1. The usual change of variables formula together with standard properties of the pull-back imply the following chain of equalities:

$$\begin{aligned} (12) \quad \int_{\Phi(S)} f(y) \sigma_H^{n-1}(y) &= \int_S f(\Phi(x)) \mathcal{J}\text{ac}(\Phi)(x) \sigma_H^{n-1}(x) \\ &= \int_S \Phi^*(f \sigma_H^{n-1}) = \int_S (f \circ \Phi)(\Phi^* \sigma_H^{n-1}) \end{aligned}$$

for every smooth function  $f : S \rightarrow \mathbb{R}$ ; see, for instance, [Lee 2003, Lemma 9.11, p. 214]. Using the left-invariance of the  $H$ -perimeter yields  $\mathcal{J}\text{ac}(\Phi) = 1$ , or equivalently,  $\Phi^* \sigma_H^{n-1} = \sigma_H^{n-1}$ . Now, let us assume that  $f(y) := y_i$  for any  $i \in I_H$ . Equivalently, let  $f$  be the  $i$ -th exponential coordinate of the variable  $y \in \mathbb{G}$ . Note also that  $(f \circ \Phi)(x) = \Phi_i(x) = -a_i + x_i$  for any  $i \in I_H$ . Actually, this follows from the fact that the group law  $\bullet$  acts linearly on the horizontal layer; see (1). Then, using (12) yields

$$\int_{\Phi(S)} y_i \sigma_H^{n-1}(y) = \int_S (-a_i + x_i) \sigma_H^{n-1}(x) = 0 \quad \text{for all } i \in I_H,$$

which achieves the proof. □

We therefore get that

$$\begin{aligned} h \mathcal{V}\text{ol}^n(\Omega) &= \int_S \langle x_H, \nu_H \rangle \sigma_H^{n-1} \leq \int_S |x_H| \sigma_H^{n-1} \leq \sqrt{\sigma_H^{n-1}(S)} \sqrt{\int_S |x_H|^2 \sigma_H^{n-1}} \\ &= \sqrt{\sigma_H^{n-1}(S)} \sqrt{\int_S \sum_{i \in I_H} x_i^2 \sigma_H^{n-1}} \leq \sqrt{\frac{\sigma_H^{n-1}(S)}{\lambda_1}} \sqrt{\int_S \sum_{i \in I_H} |\text{grad}_{HS} x_i|^2 \sigma_H^{n-1}}, \end{aligned}$$

where the last identity follows from Lord Rayleigh’s characterization of the first nontrivial eigenvalue  $\lambda_1$  of the operator  $\mathcal{L}_{HS}$  on  $S$ . Now a direct computation gives the pointwise identity  $\sum_{i \in I_H} |\text{grad}_{HS} x_i|^2 = h - 1$ . Hence, putting it all together, we have shown the following:

**Theorem 33.** *Let  $\Omega \subsetneq \mathbb{G}$  be a bounded domain with  $C^2$  boundary  $S = \partial D$ . Let  $\lambda_1$  be the first (nontrivial) eigenvalue of the operator  $\mathcal{L}_{HS}$  on  $S$ . Then*

$$\sqrt{\lambda_1} \frac{\mathcal{V}\text{ol}^n(\Omega)}{\sigma_H^{n-1}(S)} \leq \frac{\sqrt{h-1}}{h}.$$

We now discuss another geometric inequality, which looks very similar to the last one. More precisely, let  $S$  be a  $C^2$ -smooth compact hypersurface without boundary. So let us make use of Rayleigh’s principle:

$$\lambda_1 \int_S \varphi^2 \sigma_H^{n-1} \leq \int_S |\text{grad}_{HS} \varphi|^2 \sigma_H^{n-1}$$

for any function  $\varphi \in C^2(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma_H^{n-1})$  satisfying  $\int_S \varphi \sigma_H^{n-1} = 0$ . Again, we assume that the center of mass of  $S = \partial\Omega$  is placed at  $0 \in \mathbb{G}$  so that  $\int_S x_i \sigma_H^{n-1} = 0$  for every  $i \in I_H$ . Hence, similarly as above, we get that

$$\lambda_1 \int_S |x_H|^2 \sigma_H^{n-1} = \lambda_1 \sum_{i \in I_H} \int_S x_i^2 \sigma_H^{n-1} \leq \lambda_1 \sum_{i \in I_H} \int_S |\text{grad}_{HS} x_i|^2 \sigma_H^{n-1} = (h-1) \sigma_H^{n-1}(S).$$

At this point, we reformulate Corollary 20 as follows:

$$\int_S ((h-1) + \langle (\mathcal{H}_H \nu_H + C_H \nu_H), x_H \rangle) \sigma_H^{n-1} = 0.$$

From this identity and the Cauchy–Schwarz inequality, we easily get that

$$\begin{aligned} (h-1) \sigma_H^{n-1}(S) &\leq \sqrt{\int_S |x_H|^2 \sigma_H^{n-1}} \sqrt{\int_S |\mathcal{H}_H \nu_H + C_H \nu_H|^2 \sigma_H^{n-1}} \\ &\leq \sqrt{\int_S |x_H|^2 \sigma_H^{n-1}} \sqrt{\int_S (\mathcal{H}_H^2 + |C_H \nu_H|^2) \sigma_H^{n-1}}. \end{aligned}$$

Therefore

$$\frac{((h-1) \sigma_H^{n-1}(S))^2}{\int_S (\mathcal{H}_H^2 + |C_H \nu_H|^2) \sigma_H^{n-1}} \leq \int_S |x_H|^2 \sigma_H^{n-1}$$

and hence

$$\lambda_1 \frac{((h-1) \sigma_H^{n-1}(S))^2}{\int_S (\mathcal{H}_H^2 + |C_H \nu_H|^2) \sigma_H^{n-1}} \leq (h-1) \sigma_H^{n-1}(S),$$

which proves the following:

**Theorem 34.** *Let  $\Omega \subsetneq \mathbb{G}$  be a bounded domain with  $C^2$  boundary  $S = \partial D$  and  $\nu$  the outward-pointing unit normal vector along  $S$ . Let  $\lambda_1$  be the first eigenvalue of the operator  $\mathcal{L}_{HS}$  on  $S$ . The following upper bound for  $\lambda_1$  holds:*

$$\lambda_1 \leq \frac{\int_S (\mathcal{H}_H^2 + |C_H \nu_H|^2) \sigma_H^{n-1}}{(h-1) \sigma_H^{n-1}(S)} = \frac{f_S (\mathcal{H}_H^2 + |C_H \nu_H|^2) \sigma_H^{n-1}}{h-1}.$$

#### 4. Horizontal linear isoperimetric inequalities

Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  with (or without) boundary. Let  $x_H$  be the horizontal position vector of  $S$  and set  $x_{HS} := x_H - g_H \nu_H$  where  $g_H = \langle x_H, \nu_H \rangle$  is the horizontal support function of  $S$ ; see Lemma 19. We recall that

$$(13) \quad \int_S ((h-1) + g_H \mathcal{K}_H + \langle C_H \nu_H, x_{HS} \rangle) \sigma_H^{n-1} = \int_{\partial S} \langle x_H, \eta_{HS} \rangle \sigma_H^{n-2};$$

see Corollary 20. Note that, if  $\partial S = \emptyset$ , then the boundary integral vanishes. From this we easily get that

$$(14) \quad (h-1) \sigma_H^{n-1}(S) \leq \int_S (|g_H| |\mathcal{K}_H| + |\langle C_H \nu_H, x_{HS} \rangle|) \sigma_H^{n-1} + \int_{\partial S} |\langle x_H, \eta_{HS} \rangle| \sigma_H^{n-2}.$$

**Remark 35** (assumptions on  $\varrho$ ). Let  $\varrho(x) = \varrho(0, x) = \|x\|_\varrho$  be a homogeneous norm on  $\mathbb{G}$  and let  $\varrho(x, y) = \|y^{-1} \cdot x\|_\varrho$  be the associated (homogeneous) distance on  $\mathbb{G}$ . In this section we assume the following:

- (i)  $\varrho$  is piecewise  $C^1$  outside the diagonal of  $\mathbb{G}$ ;
- (ii)  $|\text{grad}_H \varrho| \leq 1$  at each regular point of  $\varrho$ ;
- (iii)  $|x_H| \leq \varrho(x, 0)$  for all  $x \in \mathbb{G}$ .

**Example 36.** On the Heisenberg group  $\mathbb{H}^n$ , the CC-distance  $d_{CC}$  satisfies these assumptions. Another example is the distance associated with the Koranyi norm defined by  $\|x\|_\varrho := \varrho(x) = \sqrt[4]{|x_H|^4 + 16t^2}$  for  $x = \exp(x_H, t) \in \mathbb{H}^n$ . This norm is homogeneous and  $C^\infty$ -smooth out of  $0 \in \mathbb{H}^n$  and satisfies conditions (ii) and (iii). This example can easily be generalized to any Carnot group having step 2 and satisfying  $C_H^\alpha C_{H_2}^\beta = -\mathbf{1}_{H_1} \delta_\alpha^\beta$ , ( $\alpha, \beta \in I_{H_2}$ ). Actually, in this case, one can show that the homogeneous norm  $\|\cdot\|_\varrho$ , defined by  $\|x\|_\varrho := \sqrt[4]{|x_H|^4 + 16|x_{H_2}|^2}$  for all  $x = \exp(x_H, x_{H_2})$  satisfies all the conditions in Remark 35.

Let  $R$  be the radius of the  $\varrho$ -ball  $B_\varrho(0, R)$ , centered at the identity  $0$  of the group  $\mathbb{G}$  and circumscribed about  $S$ . It is important to remark that, because of the left-invariance of the  $H$ -perimeter, we may replace  $0$  with any  $x \in \mathbb{G}$ . Below, we shall estimate (by the Cauchy–Schwarz inequality) the right-hand side of (14). To this aim, note that  $g_H \leq |x_H| \leq \|x\|_\varrho$ . So we have

$$(15) \quad (h-1) \sigma_H^{n-1}(S) \leq R \left( \int_S (|\mathcal{K}_H| + |C_H \nu_H|) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right),$$

which is a linear inequality. Obviously, if  $S$  is  $H$ -minimal, i.e.,  $\mathcal{K}_H = 0$ , we have

$$(16) \quad (h-1) \sigma_H^{n-1}(S) \leq R \left( \int_S |C_H \nu_H| \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right).$$



Furthermore, if  $\mathcal{H}_H^0 := \max\{\mathcal{H}_H(x) | x \in S\}$ , one gets

$$(17) \quad \sigma_H^{n-1}(S)((h-1) - R\mathcal{H}_H^0) \leq R \left( \int_S |C_H \nu_H| \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right).$$

Equivalently, we have

$$(18) \quad R \geq \frac{(h-1) \sigma_H^{n-1}(S)}{\mathcal{H}_H^0 \sigma_H^{n-1}(S) + \left( \int_S |C_H \nu_H| \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right)},$$

and, by assuming  $R\mathcal{H}_H^0 < h-1$ , we also get

$$(19) \quad \sigma_H^{n-1}(S) \leq \frac{R \left( \int_S |C_H \nu_H| \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right)}{(h-1) - R\mathcal{H}_H^0}.$$

Here, we just remark that there are no closed compact  $H$ -minimal hypersurfaces immersed in Carnot groups. This fact can be proved by using the first variation formula of the  $H$ -perimeter; see [Montefalcone 2012]. The previous formulae have been proved for hypersurfaces with boundary, but they hold even if  $\partial S = \emptyset$ . More precisely:

**Proposition 37.** *Let  $S \subset \mathbb{G}$  be a compact hypersurface of class  $C^2$  without boundary. Let  $R$  be the radius of the  $\varrho$ -ball  $B_\varrho(0, R)$ , centered at the identity  $0$  of the group  $\mathbb{G}$  and circumscribed about  $S$ . Then*

$$(20) \quad (h-1) \sigma_H^{n-1}(S) \leq R \int_{\mathfrak{a}_U} (|\mathcal{H}_H| + |C_H \nu_H|) \sigma_H^{n-1};$$

$$(21) \quad R \geq \frac{(h-1) \sigma_H^{n-1}(S)}{\mathcal{H}_H^0 \sigma_H^{n-1}(S) + \int_S |C_H \nu_H| \sigma_H^{n-1}};$$

$$(22) \quad \sigma_H^{n-1}(S) \leq \frac{R \int_S |C_H \nu_H| \sigma_H^{n-1}}{(h-1) - R\mathcal{H}_H^0}.$$

**4.1. Application: a weak monotonicity formula.** In the sequel, we shall set  $S_t = S \cap B_\varrho(x, t)$ . The “natural” monotonicity formula which can be deduced from the inequality (15) is contained in:

**Proposition 38.** *The following inequality holds for  $\mathcal{L}^1$ -a.e.  $t > 0$ :*

$$(23) \quad -\frac{d}{dt} \frac{\sigma_H^{n-1}(S_t)}{t^{h-1}} \leq \frac{1}{t^{h-1}} \left( \int_{S_t} (|\mathcal{H}_H| + |C_H \nu_H|) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S \cap B_\varrho(x, t)) \right).$$

*Proof.* Since we are assuming that the homogeneous distance  $\varrho$  is smooth (at least piecewise  $C^1$ ), by applying the classical Sard’s theorem we get that  $S_t$  is a  $C^2$ -smooth manifold with boundary for  $\mathcal{L}^1$ -a.e.  $t > 0$  (or, equivalently, this claim

follows by intersecting  $S$  with the boundary of a  $\varrho$ -ball  $B_\varrho(x, t)$  centered at  $x$  and of radius  $t$ . So let us apply formula (13) for the set  $S_t$ . We have

$$(h-1) \sigma_H^{n-1}(S_t) \leq t \left( \int_{S_t} (|\mathcal{H}_H| + |C_H \nu_H|) \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S_t) \right),$$

where  $t$  is the radius of a  $\varrho$ -ball centered at  $x$  and intersecting  $S$ . Since

$$\partial S_t = \{\partial S \cap B_\varrho(x, t)\} \cup \{\partial B_\varrho(x, t) \cap S\},$$

we get

$$(24) \quad (h-1) \sigma_H^{n-1}(S_t) \leq t \left( \underbrace{\int_{S_t} (|\mathcal{H}_H| + |C_H \nu_H|) \sigma_H^{n-1}}_{=: \mathcal{A}(t)} + \underbrace{\sigma_H^{n-2}(\partial S \cap B_\varrho(x, t))}_{=: \mathcal{B}(t)} + \sigma_H^{n-2}(\partial B_\varrho(x, t) \cap S) \right).$$

Now let us consider the function  $\psi(y) := \|y^{-1} \bullet x\|_\varrho$  for all  $y \in S$ . By hypothesis,  $\psi$  is a  $C^1$ -smooth function — at least piecewise — satisfying  $|\text{grad}_H \psi| \leq 1$ ; see Remark 35. So we may apply the coarea formula to this function. Since  $|\text{grad}_{HS} \psi| \leq |\text{grad}_H \psi|$ , we easily get that

$$\begin{aligned} \sigma_H^{n-1}(S_{t_1}) - \sigma_H^{n-1}(S_t) &\geq \int_{S_{t_1} \setminus S_t} |\text{grad}_{HS} \psi| \sigma_H^{n-1} = \int_t^{t_1} \sigma_H^{n-2} \{\psi^{-1}[s] \cap S\} ds \\ &= \int_t^{t_1} \sigma_H^{n-2}(\partial B_\varrho(x, s) \cap S) ds. \end{aligned}$$

From the last inequality we infer that

$$\frac{d}{dt} \sigma_H^{n-1}(S_t) \geq \sigma_H^{n-2}(\partial B_\varrho(x, t) \cap S)$$

for  $\mathcal{L}^1$ -a.e.  $t > 0$ . Hence, from this inequality and (24), we obtain

$$(h-1) \sigma_H^{n-1}(S_t) \leq t \left( \mathcal{A}(t) + \mathcal{B}(t) + \frac{d}{dt} \sigma_H^{n-1}(S_t) \right),$$

which is an equivalent form of (23).  $\square$

We have to notice however that, in order to prove an “intrinsic” isoperimetric inequality, the number  $(h-1)$  in the previous differential inequality *is not the correct one*, which is  $(Q-1)$ . This fact motivates a further study, made by the author in [Montefalcone 2009; 2010].

**5. A theorem about nonhorizontal graphs in 2-step Carnot groups**

We begin by describing our result in the simpler setting of the first Heisenberg group  $\mathbb{H}^1$ ; see also [Montefalcone 2007b]. For the notation, see Example 7.

**Theorem 39** (Heinz’s estimate for  $T$ -graphs). *Let  $S = \{p = \exp(x, y, t) \in \mathbb{H}^1 : t = f(x, y) \text{ for all } (x, y) \in \mathbb{R}^2\}$  be a  $T$ -graph of class  $C^2$  over the  $xy$ -plane. If  $|\mathcal{H}_H| \geq C > 0$ , then*

$$C \mathcal{H}_{\text{Eu}}^2(\mathcal{P}_{xy}(\mathcal{U})) \leq \mathcal{H}_{\text{Eu}}^1(\mathcal{P}_{xy}(\partial\mathcal{U}))$$

for every  $C^1$ -smooth relatively compact open set  $\mathcal{U} \subset S$ . Hence, taking  $\mathcal{U} := S \cap C_r(\mathcal{T})$ , where  $C_r(\mathcal{T})$  denotes a vertical cylinder of radius  $r$  around the  $T$ -axis  $\mathcal{T} := \{p = \exp(0, 0, t) \in \mathbb{H}^1, t \in \mathbb{R}\}$ , yields, for every  $r > 0$ ,

$$r \leq \frac{2}{C}.$$

It follows that any entire  $xy$ -graph of class  $C^2$  having constant (or just bounded) horizontal mean curvature  $\mathcal{H}_H$  must be necessarily a  $H$ -minimal surface. To see this fact, it is enough to send  $r \rightarrow +\infty$ . The proof of the previous theorem is elementary. More precisely, one uses the following identity:

$$-\int_{\mathcal{U}} \mathcal{H}_H \varpi \sigma_H^2 = \int_{\partial\mathcal{U}} \nu_H \lrcorner d\theta,$$

where  $\theta = T^* = dt + \frac{ydx - xdy}{2}$  denotes the dual 1-form to the vertical direction  $T$ . We also have to remark that  $\varpi \sigma_H^2 = -d\theta = dx \wedge dy$ . The previous theorem is a generalization to our context of a classical result obtained in [Heinz 1955]. This was generalized in [Chern 1965] and then by other authors in a number of different directions.

Below, we shall restrict ourselves to consider only 2-step Carnot groups.

**Definition 40** (nonhorizontal graphs in 2-step Carnot groups). Let  $\mathbb{G}$  be a 2-step Carnot group and let  $Z = \sum_{\alpha \in I_V} z_\alpha X_\alpha \in V$  be a constant vertical vector. In this case, for the sake of simplicity, we reorder the variables in  $\mathfrak{g}$  as  $x \equiv (x_{Z^\perp}, x_Z)$ , where  $x_Z := \langle x, Z \rangle \in \mathbb{R}$  and  $x_{Z^\perp} := x - x_Z Z \in Z^\perp$ . Then, we say that  $S \subset \mathbb{G}$  is a  $Z$ -graph (over the hyperplane  $Z^\perp$ ) if there exists a function  $\psi : Z^\perp \rightarrow \mathbb{R}$  such that  $S = \{p = \exp(x_{Z^\perp}, \psi(x_{Z^\perp})) \in \mathbb{G}, x_{Z^\perp} \in Z^\perp\}$ .

Let us fix a constant vertical vector  $Z \in V$  and let  $S = \{p = \exp(x_{Z^\perp}, \psi(x_{Z^\perp})) \in \mathbb{G}, x_{Z^\perp} \in Z^\perp\}$  be a  $Z$ -graph of class  $C^2$  over the  $Z^\perp$ -hyperplane. For the sake of simplicity and without loss of generality, we may take  $Z = X_\alpha$  for a fixed index  $\alpha \in I_V = \{h + 1, \dots, n\}$ .

Now let us define a differential  $(n-2)$ -form on  $S \subset \mathbb{G}$  by setting

$$\xi^\alpha := (\nu_H \lrcorner X_\alpha \lrcorner \sigma_R^n)|_{S \setminus C_S} \in \Lambda^2(T^*S).$$

This differential  $(n-2)$ -form  $\xi^\alpha$  is well-defined out of  $C_S$  and we have to compute its exterior derivative. Below we will briefly sketch a proof, which can also be found in [Montefalcone 2007a]; see Claim 3.22.

**Lemma 41.** *At each NC point,*

$$d\xi^\alpha|_{S \setminus C_S} = -\mathcal{H}_H \varpi_\alpha \sigma_H^{n-1}|_{S \setminus C_S}.$$

*Proof.* Let us set  $\zeta_j := (X_\alpha \lrcorner X_j \lrcorner \sigma_R^n)|_S$  for any  $\alpha \in I_V$  and  $j \in I_H$  and compute  $d\zeta_j := d(X_\alpha \lrcorner X_j \lrcorner \sigma_R^n)|_S$ . Let  $\mathbb{G}$  be a  $k$ -step Carnot group. We claim that

$$(25) \quad d\zeta_j|_{S \setminus C_S} = \sum_{k=\alpha+1}^n C_{\alpha j}^k (X_k \lrcorner \sigma_R^n)|_{S \setminus C_S} = \sum_{k=\alpha+1}^n C_{\alpha j}^k \nu_k \sigma_R^{n-1}|_{S \setminus C_S}.$$

The proof of this claim is just a long, but elementary, calculation. Since we are assuming that  $\mathbb{G}$  has step 2, using the properties of the Carnot structural constants yields  $C_{\alpha j}^k = 0$  whenever  $j, k \in I_H$  and  $\alpha \in I_V$ . Hence  $d\zeta_j = 0$  for every  $j \in I_H$ . By linearity  $\xi^\alpha = -\sum_{j \in I_H} \nu_H^j \zeta_j$ , where  $\nu_H^j = \langle \nu_H, X_j \rangle$  for any  $j \in I_H$ . It follows easily that  $d\xi^\alpha = -\mathcal{H}_H \varpi_\alpha \sigma_H^{n-1}$ , as wished.  $\square$

**Theorem 42** (Heinz's estimate for nonhorizontal graphs in 2-step Carnot groups). *Let  $\mathbb{G}$  be a 2-step Carnot group and let  $Z \in V$  be a constant vertical vector. Furthermore, let  $S$  be a  $Z$ -graph of class  $C^2$  over the  $Z^\perp$ -hyperplane. If  $|\mathcal{H}_H| \geq C > 0$ , then*

$$(26) \quad C \mathcal{H}_{\text{Eu}}^{n-1}(\mathcal{P}_{Z^\perp}(\mathcal{U})) \leq \mathcal{H}_{\text{Eu}}^{n-2}(\mathcal{P}_{Z^\perp}(\partial\mathcal{U}))$$

for every  $C^1$ -smooth relatively compact open set  $\mathcal{U} \subset S$ . Hence, taking  $\mathcal{U} := S \cap C_r(\mathcal{X})$ , where  $C_r(\mathcal{X})$  denotes a euclidean cylinder of radius  $r$  around the  $Z$ -axis given by  $\mathcal{X} := \{p = \exp(0_{Z^\perp}, t) \in \mathbb{G}, t \in \mathbb{R}\}$ , yields, for every  $r > 0$ ,

$$(27) \quad r \leq \frac{n-1}{C}.$$

*Proof.* Without loss of generality, we may assume  $-\mathcal{H}_H \geq C > 0$  and take  $Z = X_\alpha$  for some fixed index  $\alpha \in I_V$ . In this case, one has

$$\varpi_\alpha \sigma_H^{n-1}|_S = \nu_\alpha \sigma_R^{n-1}|_S = (X_\alpha \lrcorner \sigma_R^n)|_S = d\mathcal{H}_{\text{Eu}}^{n-1} \lrcorner X_\alpha^\perp,$$

where the last identity follows from our assumption that  $S$  is a  $X_\alpha$ -graph. By using Lemma 41 and Stokes' formula, we obtain the integral identity

$$-\int_{\mathcal{U}} \mathcal{H}_H \varpi_\alpha \sigma_H^{n-1} = \int_{\partial\mathcal{U}} \nu_H \lrcorner X_\alpha \lrcorner \sigma_R^n.$$

Furthermore, we have

$$\begin{aligned}
 - \int_{\mathcal{U}} \mathcal{H}_H \varpi_\alpha \sigma_H^{n-1} &= - \int_{\mathcal{P}_{X_\alpha^\perp}(\mathcal{U})} \mathcal{H}_H d\mathcal{H}_{\text{Eu}}^{n-1}, \\
 \int (v_H \lrcorner d\mathcal{H}_{\text{Eu}}^{n-1})|_{\mathcal{P}_{X_\alpha^\perp}(\partial\mathcal{U})} &= \int \langle v_H, \eta \rangle d\mathcal{H}_{\text{Eu}}^{n-2} \llcorner \mathcal{P}_{X_\alpha^\perp}(\partial\mathcal{U}).
 \end{aligned}$$

Putting it all together, we get  $C\mathcal{H}_{\text{Eu}}^{n-1}(\mathcal{P}_{X_\alpha^\perp}(\mathcal{U})) \leq \mathcal{H}_{\text{Eu}}^{n-2}(\mathcal{P}_{X_\alpha^\perp}(\partial\mathcal{U}))$ , which proves (26) when  $Z = X_\alpha$ . The thesis follows by linearity. Finally, (27) follows from (26) and the elementary calculation

$$\frac{\mathcal{H}_{\text{Eu}}^{n-2}(\partial B_{\text{Eu}}^{n-1})}{\mathcal{H}_{\text{Eu}}^{n-1}(B_{\text{Eu}}^{n-1})} = n - 1,$$

where  $B_{\text{Eu}}^{n-1}$  denotes a euclidean unit ball in  $Z^\perp \cong \mathbb{R}^{n-1}$ . □

It follows that an entire  $Z$ -graph of class  $C^2$  over the  $Z^\perp$ -hyperplane having constant (or bounded) horizontal mean curvature  $\mathcal{H}_H$  must be necessarily a  $H$ -minimal hypersurface.

### 6. Local Poincaré-type inequality

By using an elementary technique, somehow analogous to the one used in Section 4, we will state a local Poincaré-type inequality for smooth compactly supported functions on NC domains. First we need the following:

**Definition 43.** Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$  and let  $\mathcal{U} \subseteq S$  be an open domain. We say that  $\mathcal{U}$  is *uniformly noncharacteristic* (abbreviated UNC) if

$$\sup_{x \in \mathcal{U}} |\varpi(x)| = \sup_{x \in \mathcal{U}} \frac{|\mathcal{P}_V v(x)|}{|\mathcal{P}_H v(x)|} < +\infty.$$

We stress that

$$(28) \quad |C_H v_H| = \left| \sum_{\alpha \in I_V} \omega_\alpha C_H^\alpha v_H \right| \leq \sum_{\alpha \in I_V} |\omega_\alpha| \|C_H^\alpha\|_{\text{Gr}} \leq \frac{C}{|\mathcal{P}_H v|},$$

where  $C := \sum_{\alpha \in I_V} \|C_H^\alpha\|_{\text{Gr}}$  only depends on the structural constants of  $\mathfrak{g}$ . Set

$$R_{\mathcal{U}} := \frac{1}{2[\|\mathcal{H}_H\|_{L^\infty(\mathcal{U})} + C\|\varpi\|_{L^\infty(\mathcal{U})}]}.$$

From (28), we have  $|C_H v_H| \leq C \max_{\alpha \in I_V} |\varpi_\alpha|$ . Moreover,  $\int_B |\varpi_\alpha| \sigma_H^{n-1} = \int_B |v_\alpha| \sigma_R^{n-1} \leq \sigma_R^{n-1}(B)$  for every Borel set  $B \subseteq S$ .

**Theorem 44.** *Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$ . Let  $\mathcal{U} \subset S$  be a uniformly NC open domain. Then, for all  $x \in \mathcal{U}$  and for all  $R \leq \min\{\text{dist}_\mathfrak{g}(x, \partial\mathcal{U}), R_{\mathcal{U}}\}$ ,*

$$(29) \quad \left( \int_{\mathfrak{U}_R} |\psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \leq C_p R \left( \int_{\mathfrak{U}_R} |\text{grad}_{HS} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}}, \quad p \in [1, +\infty[,$$

for every  $\psi \in C_{HS}^1(\mathfrak{U}_R) \cap C_0(\mathfrak{U}_R)$ . More generally, let  $\tilde{\mathfrak{U}} \subset \mathfrak{U}$  be a bounded open subset of  $\mathfrak{U}$  with smooth boundary such that  $\text{diam}_\rho(\tilde{\mathfrak{U}}) \leq 2 \min\{\text{dist}_\rho(x, \partial\mathfrak{U}), R_{\mathfrak{U}}\}$ . Then

$$(30) \quad \left( \int_{\tilde{\mathfrak{U}}} |\psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \leq C_p \text{diam}_\rho(\tilde{\mathfrak{U}}) \left( \int_{\tilde{\mathfrak{U}}} |\text{grad}_{HS} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}}, \quad p \in [1, +\infty[,$$

for every  $\psi \in C_{HS}^1(\tilde{\mathfrak{U}}) \cap C_0(\tilde{\mathfrak{U}})$ .

In this theorem one can take  $C_p := \frac{2p}{2h-3}$ .

*Proof.* Let us set  $\psi_\varepsilon := \sqrt{\varepsilon^2 + \psi^2}$  ( $\varepsilon \geq 0$ ). By applying Theorem 14 with  $X = \psi_\varepsilon x_H$  we get

$$\begin{aligned} \int_{\mathfrak{U}_R} \{ \psi_\varepsilon ((h-1) + g_H \mathcal{H}_H + \langle C_H \nu_H, x_{HS} \rangle) + \langle \text{grad}_{HS} \psi_\varepsilon, x_H \rangle \} \sigma_H^{n-1} \\ = \int_{\partial\mathfrak{U}_R} \psi_\varepsilon \langle x_H, \eta_{HS} \rangle \sigma_H^{n-2}, \end{aligned}$$

and so

$$\begin{aligned} (h-1) \int_{\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-1} &\leq R \left( \int_{\mathfrak{U}_R} [\psi_\varepsilon (|\mathcal{H}_H| + |C_H \nu_H|) + |\text{grad}_{HS} \psi_\varepsilon|] \sigma_H^{n-1} + \int_{\partial\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-2} \right) \\ &\leq R (\|\mathcal{H}_H\|_{L^\infty(\mathfrak{U}_R)} + C \|\varpi\|_{L^\infty(\mathfrak{U}_R)}) \int_{\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-1} \\ &\quad + R \left( \int_{\mathfrak{U}_R} |\text{grad}_{HS} \psi_\varepsilon| \sigma_H^{n-1} + \int_{\partial\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-2} \right). \end{aligned}$$

By using Fatou's lemma and the estimate  $R \leq R_{\mathfrak{U}}$  we get that

$$\begin{aligned} (h-1) \int_{\mathfrak{U}_R} |\psi| \sigma_H^{n-1} \\ \leq (h-1) \liminf_{\varepsilon \rightarrow 0^+} \int_{\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-1} \\ \leq \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-1} + R \lim_{\varepsilon \rightarrow 0^+} \left( \int_{\mathfrak{U}_R} |\text{grad}_{HS} \psi_\varepsilon| \sigma_H^{n-1} + \int_{\partial\mathfrak{U}_R} \psi_\varepsilon \sigma_H^{n-2} \right). \end{aligned}$$

Obviously,  $\psi_\varepsilon \rightarrow |\psi|$  and  $|\text{grad}_{HS} \psi_\varepsilon| \rightarrow |\text{grad}_{HS} \psi|$  as long as  $\varepsilon \rightarrow 0$ ; moreover  $|\psi| = 0$  along  $\partial\mathfrak{U}_R$ . Now since, as it is well-known,  $|\text{grad}_{HS} |\psi|| \leq |\text{grad}_{HS} \psi|$ , we easily get the claim by Lebesgue's dominated convergence theorem. So we have shown that

$$\int_{\mathfrak{U}_R} |\psi| \sigma_H^{n-1} \leq \frac{2R}{2h-3} \int_{\mathfrak{U}_R} |\text{grad}_{HS} \psi| \sigma_H^{n-1}$$

for every  $\psi \in C_{HS}^1(\mathcal{U}_R) \cap C_0(\mathcal{U}_R)$ . Finally, the general case follows by Hölder's inequality. More precisely, let us use the last inequality with  $|\psi|$  replaced by  $|\psi|^p$ . This implies

$$\begin{aligned} \int_{\mathcal{U}_R} |\psi|^p \sigma_H^{n-1} &\leq \frac{2R}{(2h-3)} \int_{\mathcal{U}_R} p |\psi|^{p-1} |\text{grad}_{HS} \psi| \sigma_H^{n-1} \\ &\leq \frac{2pR}{(2h-3)} \left( \int_{\mathcal{U}_R} |\psi|^{(p-1)q} \sigma_H^{n-1} \right)^{\frac{1}{q}} \left( \int_{\mathcal{U}_R} |\text{grad}_{HS} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}}, \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . This achieves the proof of (29). Finally, (30) can be proved by repeating the same arguments as above, just by replacing  $R$  with  $\text{diam}(\tilde{\mathcal{U}})$ .  $\square$

With some extra hypotheses one can show that (29) still holds up to the characteristic set.

**Theorem 45.** *Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$  with (or without) boundary  $\partial S$ . We assume that  $S$  has bounded horizontal mean curvature  $\mathcal{H}_H$  and that  $\dim C_S < n - 2$ . Furthermore, let  $\mathcal{U}_\epsilon$  ( $\epsilon > 0$ ) be a family of open subsets of  $S$  with  $C^1$  boundaries, such that:*

- (i)  $C_S \subset \mathcal{U}_\epsilon$  for every  $\epsilon > 0$ ;
- (ii)  $\sigma_R^{n-1}(\mathcal{U}_\epsilon) \rightarrow 0$  for  $\epsilon \rightarrow 0^+$ ;
- (iii)  $\int_{\mathcal{U}_\epsilon} |\mathcal{P}_H \nu| \sigma_R^{n-2} \rightarrow 0$  for  $\epsilon \rightarrow 0^+$ .

Then, for every  $x \in S$  and every (small enough)  $\epsilon > 0$  there exists  $R_0 := R_0(x, \epsilon) \leq \text{dist}_\varrho(x, \partial S)$  such that

$$(31) \quad \left( \int_{S_R} |\psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \leq C_p R \left( \int_{S_R} |\text{grad}_{HS} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}}, \quad p \in [1, +\infty[$$

holds for every  $\psi \in C_{HS}^1(S_R) \cap C_0(S_R)$  and every  $R \leq R_0$ , where

$$R_0 := \min \left\{ \text{dist}_\varrho(x, \partial S), \frac{1}{2[C(1 + \|\varpi\|_{L^\infty(S_R \setminus \mathcal{U}_\epsilon)}) + \|\mathcal{H}_H\|_{L^\infty(S_R)}]} \right\}.$$

*Proof.* Set  $\psi_\epsilon := \sqrt{\epsilon^2 + \psi^2}$  ( $0 \leq \epsilon < 1$ ). We shall prove the theorem for  $p = 1$ . The general case will follow by using Hölder's inequality. Let  $\mathcal{U}_\epsilon$  ( $\epsilon > 0$ ) be as above. Fix  $\epsilon_0 > 0$ . For every  $\epsilon \leq \epsilon_0$  one has

$$\int_{\mathcal{U}_\epsilon} \psi_\epsilon |C_H \nu_H| \sigma_H^{n-1} \leq 2C \|\psi\|_{L^\infty(\mathcal{U}_{\epsilon_0})} \sigma_R^{n-1}(\mathcal{U}_\epsilon),$$

where we have put  $C := \sum_{\alpha \in I_V} \|C_H^\alpha\|_{\text{Gr}}$ . Furthermore (ii) implies that for every  $\delta > 0$  there exists  $\epsilon_\delta > 0$  such that  $\sigma_R^{n-1}(\mathcal{U}_\epsilon) < \delta$  whenever  $\epsilon < \epsilon_\delta$ . Taking

$$\tilde{\delta} \leq \frac{\int_{S_R} \psi_\varepsilon \sigma_H^{n-1}}{2\|\psi\|_{L^\infty(\mathcal{U}_{\varepsilon_0})}}, \text{ one gets}$$

$$\int_{\mathcal{U}_{\varepsilon}} \psi_\varepsilon |C_H \nu_H| \sigma_H^{n-1} \leq C \int_{S_R} \psi_\varepsilon \sigma_H^{n-1}$$

for every  $\varepsilon \leq \min\{\varepsilon_{\tilde{\delta}}, \varepsilon_0\}$ . Moreover, for any  $\varepsilon \in ]0, \min\{\varepsilon_{\tilde{\delta}}, \varepsilon_0\}[$ , one has

$$\int_{S_R \setminus \mathcal{U}_{\varepsilon}} \psi_\varepsilon |C_H \nu_H| \sigma_H^{n-1} \leq C \|\varpi\|_{L^\infty(S_R \setminus \mathcal{U}_{\varepsilon})} \int_{S_R} \psi_\varepsilon \sigma_H^{n-1}.$$

It follows that

$$\int_{S_R} \psi_\varepsilon |C_H \nu_H| \sigma_H^{n-1} \leq C(1 + \|\varpi\|_{L^\infty(S_R \setminus \mathcal{U}_{\varepsilon})}) \int_{S_R} \psi_\varepsilon \sigma_H^{n-1}.$$

Since, by hypothesis, the horizontal mean curvature is bounded, we clearly have

$$\int_{S_R} \psi_\varepsilon |\mathcal{H}_H| \sigma_H^{n-1} \leq \|\mathcal{H}_H\|_{L^\infty(S_R)} \int_{S_R} \psi_\varepsilon \sigma_H^{n-1}.$$

Applying Theorem 14 with  $X = \psi_\varepsilon x_H$  (and arguing as in the proof of Theorem 44) yields

$$(h-1) \int_{S_R} \psi_\varepsilon \sigma_H^{n-1} \leq R \left( \int_{S_R} \{ \psi_\varepsilon (|\mathcal{H}_H| + |C_H \nu_H|) + |\text{grad}_{HS} \psi_\varepsilon| \} \sigma_H^{n-1} + \int_{\partial S_R} \psi_\varepsilon \sigma_H^{n-2} \right)$$

$$\leq R [ C(1 + \|\varpi\|_{L^\infty(S_R \setminus \mathcal{U}_{\varepsilon})}) + \|\mathcal{H}_H\|_{L^\infty(S_R)} ] \int_{S_R} \psi_\varepsilon \sigma_H^{n-1}$$

$$+ R \left( \int_{S_R} |\text{grad}_{HS} \psi_\varepsilon| \sigma_H^{n-1} + \int_{\partial S_R} \psi_\varepsilon \sigma_H^{n-2} \right).$$

So if  $R \leq R_0$ , one gets

$$\int_{S_R} \psi_\varepsilon \sigma_H^{n-1} \leq \frac{2R}{2h-3} \left( \int_{S_R} |\text{grad}_{HS} \psi_\varepsilon| \sigma_H^{n-1} + \int_{\partial S_R} \psi_\varepsilon \sigma_H^{n-2} \right).$$

We have  $\psi_\varepsilon \rightarrow |\psi|$  and  $|\text{grad}_{HS} \psi_\varepsilon| \rightarrow |\text{grad}_{HS} \psi|$  as long as  $\varepsilon \rightarrow 0$  and  $|\psi| = 0$  along  $\partial S_R$ . Since  $|\text{grad}_{HS} |\psi|| \leq |\text{grad}_{HS} \psi|$ , the thesis follows from Fatou's lemma and Lebesgue's dominated convergence theorem.  $\square$

**6.1. A Caccioppoli-type inequality.** Our final result is a generalization of the classical *Caccioppoli inequality* (see, for instance, [Ambrosio 1997]) for the operator  $\mathcal{L}_{HS}$  on smooth hypersurfaces.

Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$  and set  $S_R := S \cap B_\varrho(x, R)$  for any  $x \in \mathbb{G}$ . We are going to consider the functions satisfying, in the distributional sense,

$$(32) \quad -\mathcal{L}_{HS} \phi = \psi \quad \text{on } S_R,$$

whenever  $\psi \in L^2(S_R, \sigma_H^{n-1})$ .



So let us take a function  $\zeta \in C_{HS}^1(S_R) \cap C_0(S_R)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta = 1$  on  $S_{R/2} = S \cap B_\rho(0, R/2)$  and  $|\text{grad}_{HS} \zeta| \leq C_0/R$ . Inserting into the above equation the function  $\varphi = \zeta^2(\phi - \phi_0)$ , where  $\phi_0 \in \mathbb{R}$  is a fixed constant, and then integrating over  $S_R$ , yields

$$(33) \quad \underbrace{\int_{S_R} \zeta^2 |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1}}_{=: I_1} + 2 \underbrace{\int_{S_R} \zeta(\phi - \phi_0) \langle \text{grad}_{HS} \zeta, \text{grad}_{HS} \phi \rangle \sigma_H^{n-1}}_{=: I_2} = \underbrace{\int_{S_R} \psi \zeta^2(\phi - \phi_0) \sigma_H^{n-1}}_{=: I_3}.$$

We have

$$I_2 \leq \frac{1}{2} \int_{S_R} |\zeta|^2 |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1} + 2 \underbrace{\int_{S_R} |\phi - \phi_0|^2 |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1}}_{=: I_4}.$$

Moreover  $I_4 \leq 2C_0^2/R^2 \|\phi - \phi_0\|_{L^2(S_R)}$ . Now let us estimate the third integral  $I_3$ :

$$\begin{aligned} \int_{S_R} \psi \zeta^2(\phi - \phi_0) \sigma_H^{n-1} &= \int_{S_R} 2 \left( (2R\psi) \frac{\zeta^2(\phi - \phi_0)}{4R} \right) \sigma_H^{n-1} \\ &\leq 4R^2 \int_{S_R} \psi^2 \sigma_H^{n-1} + \frac{1}{16R^2} \int_{S_R} \zeta^4 |\phi - \phi_0|^2 \sigma_H^{n-1} \\ &\leq 4R^2 \int_{S_R} 2\psi^2 \sigma_H^{n-1} + \frac{1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_H^{n-1}. \end{aligned}$$

Since  $\zeta = 1$  on  $S_{R/2}$ , using the previous estimates yields

$$\int_{S_{R/2}} |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1} \leq \frac{2C_0^2 + 1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_H^{n-1} + 4R^2 \int_{S_R} \psi^2 \sigma_H^{n-1}.$$

We summarize these calculations, as follows:

**Theorem 46.** *Let  $S \subset \mathbb{G}$  be a hypersurface of class  $C^2$ ; let  $\phi_0 \in \mathbb{R}$  and let  $\phi$  be a distributional solution to the equation  $-\mathcal{L}_{HS} \phi = \psi$  on  $S_R$ , where  $\psi \in L^2(S_R, \sigma_H^{n-1})$ . Then, there exists a positive constant  $C > 0$  such that the following ‘‘Caccioppoli-type’’ inequality holds:*

$$\int_{S_{R/2}} |\text{grad}_{HS} \phi|^2 \sigma_H^{n-1} \leq C \left( \frac{1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_H^{n-1} + R^2 \int_{S_R} \psi^2 \sigma_H^{n-1} \right)$$

for every (small enough)  $R > 0$ , where  $S_R := S \cap B_\rho(x, R)$ , for any  $x \in S$ .

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
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