ASYMPTOTIC L^4 NORM OF
POLYNOMIALS DERIVED FROM CHARACTERS

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Littlewood investigated polynomials with coefficients in $\{-1, 1\}$ (Littlewood polynomials), to see how small their ratio of norms $\|f\|_4/\|f\|_2$ on the unit circle can become as $\deg f \to \infty$. A small limit is equivalent to slow growth in the mean square autocorrelation of the associated binary sequences of coefficients of the polynomials. The autocorrelation problem for arrays and higher dimensional objects has also been studied; it is the natural generalization to multivariable polynomials. Here we find, for each $n > 1$, a family of n-variable Littlewood polynomials with lower asymptotic $\|f\|_4/\|f\|_2$ than any known hitherto. We discover these through a wide survey, infeasible with previous methods, of polynomials whose coefficients come from finite field characters. This is the first time that the lowest known asymptotic ratio of norms $\|f\|_4/\|f\|_2$ for multivariable polynomials $f(z_1, \ldots, z_n)$ is strictly less than what could be obtained by using products $f_1(z_1) \cdots f_n(z_n)$ of the best known univariate polynomials.

1. Introduction

1A. *History and main result.* Littlewood [1966; 1968] pioneered the study of the L^4 norm on the complex unit circle of polynomials whose coefficients lie in $\{-1, 1\}$, and in particular wanted to know how small their ratio of norms $\|f\|_4/\|f\|_2$ can become as $\deg f \to \infty$. He suspected, based on calculations of Swinnerton-Dyer, that this ratio could be made to approach 1 asymptotically, but the smallest limiting ratio he could find was $\sqrt[4]{4/3}$ for the Rudin–Shapiro polynomials [Littlewood 1968]. The L^4 norm is of particular interest since it serves as a lower bound for the L^∞ norm and is easier to calculate than most other L^r norms. Erdős [1957, Problem 22; 1962] had conjectured that $\|f\|_\infty/\|f\|_2$ is bounded away from 1 for nonconstant polynomials with complex coefficients of unit magnitude. This was disproved in [Kahane 1980], but the modified problem where we restrict to coefficients in $\{-1, 1\}$ remains open [Newman and Byrnes 1990], and would be solved if one could prove

that $\|f\|_4/\|f\|_2$ is bounded away from 1 as $\deg f \to \infty$. Polynomials in one or more variables with coefficients in $\{-1, 1\}$ and small $\|f\|_4/\|f\|_2$ are equivalent to binary sequences and arrays (that is, those that simply list the coefficients of the polynomials) with low mean square aperiodic autocorrelation. Such sequences and arrays are important in the theory of communications[1] [Golay 1977] and statistical physics [Bernasconi 1987]. Accordingly, we define a *Littlewood polynomial* in n variables to have the form

$$f(z_1, \ldots, z_n) = \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} f_{j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n},$$

with coefficient $f_{j_1,\ldots,j_n}$ in $\{-1, 1\}$, and our L^r norm for $f(z_1, \ldots, z_n)$ is

$$\|f\|_r = \left(\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| f\left(\exp(i\theta_1), \ldots, \exp(i\theta_n)\right) \right|^r d\theta_1 \cdots d\theta_n \right)^{1/r}.$$

Note that $\|f\|_2^2 = s_1 \cdots s_n$ for our Littlewood polynomial.

For univariate Littlewood polynomials, the lowest asymptotic ratio of norms $\|f\|_4/\|f\|_2$ found by Littlewood himself [1968] was $\sqrt[4]{4/3}$ for the Rudin–Shapiro polynomials. Two decades later, this was improved to $\sqrt[4]{7/6}$ by Høholdt and Jensen [1988], using modifications of Fekete polynomials. Over two decades later still, in [Jedwab et al. 2013b], another modification was shown to yield further improvement:

Theorem 1.1 (Jedwab, Katz, Schmidt). *There is a family of univariate Littlewood polynomials that*, *as* $\deg f \to \infty$, *has* $\|f\|_4/\|f\|_2 \to B_1$, *the largest real root of* $27x^{12} - 498x^8 + 1164x^4 - 722$, *which is less than* $\sqrt[4]{22/19}$.

Prior to this paper, for each n, the lowest known asymptotic $\|f\|_4/\|f\|_2$ for n-variable Littlewood polynomials

$$f(z_1, \ldots, z_n) \quad (\text{in the limit as } \deg_{z_1} f, \ldots, \deg_{z_n} f \to \infty)$$

was simply the n-th power of the lowest known ratio for univariate polynomials, based on the fact that if $f(z_1, \ldots, z_n) = f_1(z_1) \cdots f_n(z_n)$, then $\|f\|_r = \|f_1\|_r \cdots \|f_n\|_r$. For bivariate Littlewood polynomials, Schmidt [2011] obtained an asymptotic $\|f\|_4/\|f\|_2$ of $\sqrt{7/6}$ in this way (via Høholdt and Jensen's univariate polynomials mentioned above), and foresaw the possibility that the asymptotic $\|f\|_4/\|f\|_2$ could be lowered to B_1^2, contingent upon the conjecture that was later established as Theorem 1.1. In this paper, we show that one can do better than this product construction, even when based on the best univariate polynomials now known (those of Theorem 1.1).

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[1]In this milieu, results are expressed in terms of the *merit factor*, defined as $\|f\|_2^4/(\|f\|_4^4 - \|f\|_2^4)$.

Theorem 1.2. *For each $n > 1$, there is a family of n-variable Littlewood polynomials $f(z_1, \ldots, z_n)$ which, as $\deg_{z_1} f, \ldots, \deg_{z_n} f \to \infty$, has $\|f\|_4/\|f\|_2$ tending to a value strictly less than B_1^n.*

The lowest known asymptotic $\|f\|_4/\|f\|_2$ for n-variable Littlewood polynomials is an algebraic number depending on n, and is specified precisely in Section 1C after we define in Section 1B the polynomials that are involved.[2]

1B. *Character polynomials.* The polynomials used in [Høholdt and Jensen 1988], [Jedwab et al. 2013b], and the current paper to break previous records for the lowest known asymptotic $\|f\|_4/\|f\|_2$ are all character polynomials, that is, polynomials whose coefficients are given by characters of finite fields. The L^4 norms of character polynomials have already been studied extensively [Høholdt and Jensen 1988; Jensen and Høholdt 1989; Jensen et al. 1991; Bömer and Antweiler 1993; Borwein 2002; Borwein and Choi 2000; 2002; Jedwab 2005; Høholdt 2006; Jedwab and Schmidt 2010; Schmidt 2011], but it took the new methods of this paper to discover and verify the properties of the polynomials of our Theorem 1.2.

The interrelation between the additive and multiplicative structures of finite fields endow character polynomials with their remarkable qualities: the coefficients of an *additive character polynomial* are obtained by applying an additive character of a finite field to its nonzero elements arranged multiplicatively (listed as successive powers of a primitive element), while the coefficients of a *multiplicative character polynomial* are obtained by applying a multiplicative character of a finite field to its elements arranged additively (as $\mathbb{Z}$-linear combinations of the generators, arrayed in a box whose dimensionality equals the number of generators). Thus an additive character polynomial has the form

$$(1) \qquad f(z) = \sum_{j \in S} \psi(\alpha(j+t)) z^j,$$

where $\psi : \mathbb{F}_q \to \mathbb{C}$ is a nontrivial additive character, the *support* S is a set of the form $\{0, 1, \ldots, s-1\}$, the *translation* t is an element of $\mathbb{Z}$, and the *arrangement* α is a group epimorphism from $\mathbb{Z}$ to $\mathbb{F}_q^*$. A multiplicative character polynomial has the form

$$(2) \qquad f(z_1, \ldots, z_e) = \sum_{j=(j_1,\ldots,j_e) \in S} \chi(\alpha(j+t)) \, z_1^{j_1} \cdots z_e^{j_e},$$

where e is a positive integer, χ is a nontrivial complex-valued multiplicative character of $\mathbb{F}_q = \mathbb{F}_{p^e}$ with p prime, the *support* S is $S_1 \times \cdots \times S_e$ with each S_k a set of

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[2]Gulliver and Parker [2005] have also studied $\|f\|_4/\|f\|_2$ for multivariable Littlewood polynomials, but in a very different limit: they let the number of variables tend to infinity while keeping the degree in each variable less than or equal to one.

the form $\{0, 1, \ldots, s_k - 1\}$, while the *translation* $t = (t_1, \ldots, t_e)$ is in $\mathbb{Z}^e$, and the *arrangement* α is a group epimorphism from $\mathbb{Z}^e$ to $\mathbb{F}_q$. We always extend nontrivial multiplicative characters to take 0 to 0.

We now define the Fekete polynomials and their modifications used in [Høholdt and Jensen 1988] and [Jedwab et al. 2013b]. For an odd prime p, the p-th Fekete polynomial is a multiplicative character polynomial using the quadratic character (Legendre symbol) over the prime field $\mathbb{F}_p$, support $S = \{0, 1, \ldots, p-1\}$, translation $t = 0$, and arrangement $\alpha \colon \mathbb{Z} \to \mathbb{F}_p$ given by reduction modulo p. Fekete polynomials are themselves the subject of many fascinating studies linking number theory and analysis [Fekete and Pólya 1912; Pólya 1919; Montgomery 1980; Baker and Montgomery 1990; Conrey et al. 2000; Borwein et al. 2001; Borwein and Choi 2002].

The polynomials used in [Høholdt and Jensen 1988] to obtain asymptotic $\|f\|_4/\|f\|_2$ of $\sqrt[4]{7/6}$ have the same character, support, and arrangement, but the translations t are chosen such that $t/p \to 1/4$ as $p \to \infty$, and any coefficient of 0 (arising from the extended multiplicative character) is replaced with 1 to obtain Littlewood polynomials. To obtain asymptotic $\|f\|_4/\|f\|_2$ less than $\sqrt[4]{22/19}$, we used in [Jedwab et al. 2013b] a different limit for t/p, and allowed the support $S = \{0, 1, \ldots, s-1\}$ to be of size other than p, and in fact let s/p tend to a number slightly larger than 1 as $p \to \infty$.

The families of character polynomials used here are based on similar asymptotics: we say that a family $\{f_\iota\}_{\iota \in I}$ of additive character polynomials is *size-stable* to mean that if we write $\mathbb{F}_{q_\iota}$ and S_ι for the field and support of f_ι, then $\{q_\iota : \iota \in I\}$ is infinite and $|S_\iota|/(q_\iota - 1)$ tends to a positive real number σ (called the *limiting size*) as $q_\iota \to \infty$. Likewise, we say that a family of e-variable multiplicative character polynomials $\{f_\iota\}_{\iota \in I}$ is *size-stable* to mean that if we write $\mathbb{F}_{q_\iota} = \mathbb{F}_{p_\iota^e}$ and $S_\iota = S_{\iota,1} \times \cdots \times S_{\iota,e}$ for the field and support of f_ι, then the set of primes $\{p_\iota : \iota \in I\}$ is infinite and for each $k \in \{1, \ldots, e\}$, the ratio $|S_{\iota,k}|/p_\iota$ tends to a positive real number σ_k as $q_\iota \to \infty$. We call $\sigma_1, \ldots, \sigma_e$ the *limiting sizes*. And we say that a family of e-variable multiplicative character polynomials $\{f_\iota\}_{\iota \in I}$ is *translation-stable* to mean that if we write $\mathbb{F}_{q_\iota} = \mathbb{F}_{p_\iota^e}$ and $t_\iota = (t_{\iota,1}, \ldots, t_{\iota,e})$ for the field and translation of f_ι, then the set of primes $\{p_\iota : \iota \in I\}$ is infinite and for each $k \in \{1, \ldots, e\}$, the ratio $t_{\iota,k}/p_\iota$ tends to a real number τ_k as $q_\iota \to \infty$. We call $\tau_1, \ldots, \tau_e$ the *limiting translations*.

1C. *Subsidiary results.* We discovered the polynomials of Theorem 1.2 via a survey, enabled by the methods presented in this paper, of the asymptotic L^4 norms of both additive and multiplicative character polynomials. Quadratic multiplicative characters behave differently than nonquadratic ones, so we treat them separately: we have *quadratic families* in which every character is quadratic, and *nonquadratic families* in which none is. We then have three theorems: one for additive characters

and two for the different types of multiplicative characters, and we express our limiting norms in terms of the function

$$
(3) \qquad \Omega(x, y) = \sum_{n \in \mathbb{Z}} \max(0, 1 - |xn - y|)^2,
$$

which is defined and continuous on $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$.

Theorem 1.3. *Let $\{f_\iota\}_{\iota \in I}$ be a size-stable family, with limiting size σ, of additive character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$.*

(i) *As $q_\iota \to \infty$, we have*

$$
\frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \to -\tfrac{2}{3}\sigma + 2\,\Omega\Big(\frac{1}{\sigma}, 0\Big).
$$

(ii) *This limit is globally minimized if and only if σ is the unique root in $\left(1, 1 + \frac{9}{64}\right)$ of $x^3 - 12x + 12$.*

Theorem 1.4. *Let $\{f_\iota\}_{\iota \in I}$ be a size-stable family, with limiting sizes $\sigma_1, \dots, \sigma_e$, of e-variable nonquadratic multiplicative character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$.*

(i) *As $q_\iota \to \infty$, we have*

$$
\frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \to -\frac{2^e}{3^e} \prod_{j=1}^{e} \sigma_i + 2 \prod_{j=1}^{e} \Omega\Big(\frac{1}{\sigma_j}, 0\Big).
$$

(ii) *This limit is globally minimized if and only if $\sigma_1, \dots, \sigma_e$ all equal the unique root in $(1, 1 + 3^{e+1}/2^{2e+4})$ of*

$$
x^{3e} - \frac{3^e}{2^{e-3}}(x - 1)(3x^2 - 4x + 2)^{e-1}.
$$

Theorem 1.5. *Let $\{f_\iota\}_{\iota \in I}$ be a size- and translation-stable family, with limiting sizes $\sigma_1, \dots, \sigma_e$ and limiting translations $\tau_1, \dots, \tau_e$, of e-variable quadratic multiplicative character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$.*

(i) *As $q_\iota \to \infty$, we have*

$$
\frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \to -\frac{2^{e+1}}{3^e} \prod_{j=1}^{e} \sigma_i + 2 \prod_{j=1}^{e} \Omega\Big(\frac{1}{\sigma_j}, 0\Big) + \prod_{j=1}^{e} \Omega\Big(\frac{1}{\sigma_j}, 1 + \frac{2\tau_j}{\sigma_j}\Big).
$$

(ii) *This limit is globally minimized if and only if $\sigma_1, \dots, \sigma_e$ all equal the unique root in $(1, 1 + 3^{e+1}/2^{2e+3})$ of*

$$
x^{3e} - \frac{3^e}{2^{e-2}}(x - 1)(3x^2 - 4x + 2)^{e-1} - \frac{3^e}{2^{2e}}(2x - 1)^{2e-1},
$$

and $\tau_j \in \{\frac{1}{4}(1 - 2\sigma_j) + n/2 : n \in \mathbb{Z}\}$ for each $j \in \{1, \dots, e\}$.

These new theorems are much more general than the compositum of all previous results on the limiting ratio of L^4 to L^2 norm for character polynomials [Høholdt and Jensen 1988; Jensen and Høholdt 1989; Jensen et al. 1991; Borwein and Choi 2000; Schmidt 2011; Jedwab et al. 2013a; 2013b], and reveal for the first time the full functional form of the asymptotic ratio of norms as it depends on choice of character, limiting size, and limiting translation, thus enabling us to find multivariable Littlewood polynomials with lower asymptotic $\|f\|_4/\|f\|_2$ than any known hitherto.

For each $e \geq 1$, let A_e (resp., B_e) be the minimum asymptotic ratio of norms for a family of e-variable nonquadratic (resp., quadratic) character polynomials, as described in Theorem 1.4(ii) (for A_e) and Theorem 1.5(ii) (for B_e). Note that A_1 is also the minimum asymptotic ratio of norms achievable by a family of additive character polynomials as described in Theorem 1.3(ii). Rational approximations of $B_1^4, B_2^4, \ldots, B_5^4$ are obtained later in Lemma 7.1, and if desired, a computer may be used to obtain more accurate approximations of values of various A_e and B_e. For each $e \geq 1$, B_e is to date the lowest known asymptotic $\|f\|_4/\|f\|_2$ for a family of e-variable Littlewood[3] polynomials $f(z_1, \ldots, z_e)$ in the limit as $\deg_{z_1} f, \ldots, \deg_{z_e} f \to \infty$. For $e = 1$, this recapitulates Corollary 3.2 of [Jedwab et al. 2013b], while for $e > 1$, the ratio obtained here is strictly lower than any found to date. Until now, the smallest known asymptotic ratio has been whatever can be obtained from the best univariate polynomials and the product construction $\|f(z_1) \cdots f(z_e)\|_r = \|f(z)\|_r^e$, and so we are claiming that $B_e < B_1^e$ for every $e > 1$. This will give our main result, Theorem 1.2, but in fact we prove something more general: one always obtains a lower asymptotic ratio of norms with a single optimal family of quadratic character polynomials than one does using the product construction with two or more families of character polynomials (which could draw coefficients from $\{-1, 1\}$ or a larger set, depending on the characters involved).

Theorem 1.6. *For each $e \geq 1$, let A_e (resp., B_e) be the minimum asymptotic ratio of L^4 to L^2 norm achievable by families of e-variable nonquadratic (resp., quadratic) multiplicative character polynomials, as described in Theorem 1.4(ii) (resp., Theorem 1.5(ii)).*

Then $B_e < A_e$ for every $e \geq 1$ and $B_{e_1+e_2} < B_{e_1} B_{e_2}$ for every $e_1, e_2 \geq 1$.

1D. *Organization of this paper.* To prove Theorems 1.3–1.5, we first establish a general theorem for obtaining the L^4 norm of a polynomial from its Fourier interpolation in Section 3, after setting down notational conventions in Section 2. Our general theorem reduces the problem of computing L^4 norms of character

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[3]Quadratic character polynomials are not always Littlewood because the extended quadratic character χ has $\chi(0) = 0$, so we replace each coefficient of 0 thus produced with a 1, and Corollary A.3 shows that this has no effect on the asymptotic ratio of norms.

polynomials to a pair of calculations (one for additive and one for multiplicative characters) involving Gauss sums, which are presented in Section 4. We use these in Section 5 to prove Theorems 1.3(i), 1.4(i), and 1.5(i). These respectively imply Theorems 1.3(ii), 1.4(ii), and 1.5(ii), but showing this demands delicate arguments which are sketched in Section 6. We prove Theorem 1.6 in Section 7. Some technical results used in Sections 5 and 6 are collected and proved in the Appendix.

2. Notation and conventions

For the rest of this paper p is a prime, and $q = p^e$ with e a positive integer. For any group Γ, we use $\widehat{\Gamma}$ to denote the group of characters from Γ into $\mathbb{C}$: thus $\widehat{\mathbb{F}_q}$ is the group of additive characters from $\mathbb{F}_q$ to $\mathbb{C}$ and $\widehat{\mathbb{F}_q^*}$ the group of multiplicative characters from $\mathbb{F}_q^*$ to $\mathbb{C}$. We extend any nontrivial $\chi \in \widehat{\mathbb{F}_q^*}$ so that $\chi(0) = 0$.

To write the multiplicative character polynomial (2) compactly, we use the convention that if $j = (j_1, \ldots, j_e) \in \mathbb{Z}^e$, the notation z^j is a shorthand for $z_1^{j_1} \cdots z_e^{j_e}$. To make it easier to speak about supports of character polynomials (1) and (2), we call a finite set of consecutive integers a *segment*, and a finite Cartesian product of segments a *box*. If S is a subset of $\mathbb{Z}^n$ and $t \in \mathbb{Z}^n$, then $S + t$ is the translated subset $\{s + t : s \in S\}$.

3. L^4 norms via the Fourier transform

If Γ is a finite abelian group and $\{F_g\}_{g \in \Gamma}$ is a family of complex numbers, then for any $\eta \in \widehat{\Gamma}$, we have the Fourier transform

$$\hat{F}_\eta = \sum_{g \in \Gamma} F_g \eta(g),$$

with inverse

$$F_g = \frac{1}{|\Gamma|} \sum_{\eta \in \widehat{\Gamma}} \hat{F}_\eta \overline{\eta(g)}.$$

We express the L^4 norm in terms of the Fourier interpolation.

Theorem 3.1. *Let Γ be a finite abelian group, $\{F_g\}_{g \in \Gamma}$ a family of complex numbers, n a positive integer, and $\pi \in \mathrm{Hom}(\mathbb{Z}^n, \Gamma)$. For any $\eta \in \widehat{\Gamma}$, let $\eta' \in \widehat{\mathbb{Z}^n}$ be $\eta \circ \pi$. If U is a finite subset of $\mathbb{Z}^n$ and $F(z) = \sum_{u \in U} F_{\pi(u)} z^u \in \mathbb{C}[z_1, \ldots, z_n]$, then*

$$\|F\|_4^4 = \frac{1}{|\Gamma|^5} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\Gamma}} \overline{\kappa'(a)\lambda'(b)} \mu'(c)\nu'(d) H(\kappa, \lambda, \mu, \nu),$$

where

$$H(\kappa, \lambda, \mu, \nu) = \sum_{\xi \in \widehat{\Gamma}} \hat{F}_{\xi\kappa} \hat{F}_{\xi\lambda} \overline{\hat{F}_{\xi\mu} \hat{F}_{\xi\nu}}.$$

Proof. By the definition of the L^4 norm, we have

$$\|F\|_4^4 = \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} F_{\pi(a)} F_{\pi(b)} \overline{F_{\pi(c)} F_{\pi(d)}},$$

and thus, using the inverse Fourier transform,

$$\|F\|_4^4 = \frac{1}{|\Gamma|^4} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\Gamma}} \hat{F}_\kappa \hat{F}_\lambda \overline{\hat{F}_\mu \hat{F}_\nu} \kappa'(a) \lambda'(b) \mu'(c) \nu'(d).$$

Since we are summing κ over all $\widehat{\Gamma}$, we can replace κ by $\xi\kappa$ for any given $\xi \in \widehat{\Gamma}$, and also do likewise with λ, μ, ν to obtain

$$\|F\|_4^4 = \frac{1}{|\Gamma|^4} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\Gamma}} \hat{F}_{\xi\kappa} \hat{F}_{\xi\lambda} \overline{\hat{F}_{\xi\mu} \hat{F}_{\xi\nu}} \kappa'(a) \lambda'(b) \mu'(c) \nu'(d)$$

where we have omitted mention of the resulting factor of $\overline{\xi'(a)\xi'(b)}\xi'(c)\xi'(d)$, which equals 1 in view of the constraint in the first summation. Now sum ξ over $\widehat{\Gamma}$ and divide by $|\Gamma| = |\widehat{\Gamma}|$ to finish. $\qquad\square$

We apply this general theorem to additive and multiplicative character polynomials in two corollaries below. Such polynomials have Gauss sums as their Fourier coefficients, so for any $\psi \in \widehat{\mathbb{F}}_q$ and $\chi \in \widehat{\mathbb{F}}_q^*$, we define the *Gauss sum* associated with ψ and χ to be

$$G(\psi, \chi) = \sum_{a \in \mathbb{F}_q^*} \psi(a)\chi(a).$$

Corollary 3.2. *If $f(z)$ is an additive character polynomial with character $\psi \in \widehat{\mathbb{F}}_q$, support S, translation t, and arrangement α, then*

$$\|f\|_4^4 = \frac{1}{(q-1)^5} \sum_{\substack{a,b,c,d \in S+t \\ a+b=c+d}} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\mathbb{F}}_q^*} \overline{\kappa'(a)\lambda'(b)}\mu'(c)\nu'(d) H(\kappa, \lambda, \mu, \nu)$$

where for any $\eta \in \widehat{\mathbb{F}}_q^*$, we let $\eta' \in \widehat{\mathbb{Z}}$ be $\eta \circ \alpha$, and

$$H(\kappa, \lambda, \mu, \nu) = \sum_{\xi \in \widehat{\mathbb{F}}_q^*} G(\psi, \xi\kappa) G(\psi, \xi\lambda) \overline{G(\psi, \xi\mu) G(\psi, \xi\nu)}.$$

Proof. Our additive character polynomial $f(z) = \sum_{s \in S} \psi(\alpha(s+t)) z^s$ has the same L^r norms as $F(z) = z^t f(z) = \sum_{u \in S+t} \psi(\alpha(u)) z^u$, so take $\Gamma = \mathbb{F}_q^*$, $F_g = \psi(g)$, $n = 1$, $\pi = \alpha$, and $U = S+t$ in [Theorem 3.1], and note that for $\eta \in \widehat{\mathbb{F}}_q^*$ we have $\hat{F}_\eta = G(\psi, \eta)$. $\qquad\square$

Corollary 3.3. *If $f(z)$ is a multiplicative character polynomial with character $\chi \in \widehat{\mathbb{F}_q^*}$, support S, translation t, and arrangement α, then*

$$\|f\|_4^4 = \frac{1}{q^5} \sum_{\substack{a,b,c,d \in S+t \\ a+b=c+d}} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\mathbb{F}}_q} \overline{\kappa'(a)\lambda'(b)}\mu'(c)\nu'(d) H(\kappa, \lambda, \mu, \nu)$$

where for any $\eta \in \widehat{\mathbb{F}}_q$, we let $\eta' \in \widehat{\mathbb{Z}^e}$ be $\eta \circ \alpha$, and

$$H(\kappa, \lambda, \mu, \nu) = \sum_{\xi \in \mathbb{F}_q} G(\xi\kappa, \chi)G(\xi\lambda, \chi)\overline{G(\xi\mu, \chi)G(\xi\nu, \chi)}.$$

Proof. Our multiplicative character polynomial $f(z) = \sum_{s \in S} \chi(\alpha(s+t))z^s$ has the same L^r norms as $F(z) = z^t f(z) = \sum_{u \in S+t} \chi(\alpha(u))z^u$, so take $\Gamma = \mathbb{F}_q$, $F_g = \chi(g)$, $n = e$, $\pi = \alpha$, and $U = S+t$ in Theorem 3.1, and note that for $\eta \in \widehat{\mathbb{F}}_q$ we have $\hat{F}_\eta = G(\eta, \chi)$. $\qquad\square$

The key to L^4 norms is then the evaluation of the sums $H(\kappa, \lambda, \mu, \nu)$ in the above two corollaries, which we take up in the next section.

4. Two propositions on summations of Gauss sums

Here we estimate the values of the summations H that appear in Corollaries 3.2 and 3.3. We begin with some basic facts about Gauss sums, which are proved in Theorems 5.11 and 5.12 of [Lidl and Niederreiter 1997].

Lemma 4.1. *If $\psi \in \widehat{\mathbb{F}}_q$ and $\chi \in \widehat{\mathbb{F}_q^*}$, then*

(i) $G(\psi, \chi) = q - 1$ *if both characters are trivial,*

(ii) $G(\psi, \chi) = 0$ *if ψ is trivial and χ is not,*

(iii) $G(\psi, \chi) = -1$ *if χ is trivial and ψ is not,*

(iv) $|G(\psi, \chi)| = \sqrt{q}$ *if both characters are nontrivial, and*

(v) $\sum_{a \in \mathbb{F}_q^*} \psi(ba)\chi(a) = \overline{\chi}(b)G(\psi, \chi)$ *for any $b \in \mathbb{F}_q^*$.*

We first estimate the summation H appearing in Corollary 3.2.

Proposition 4.2. *Let ψ be a nontrivial character in $\widehat{\mathbb{F}}_q$, and $\kappa, \lambda, \mu, \nu \in \widehat{\mathbb{F}_q^*}$. If*

$$H = \sum_{\xi \in \widehat{\mathbb{F}_q^*}} G(\psi, \xi\kappa)G(\psi, \xi\lambda)\overline{G(\psi, \xi\mu)G(\psi, \xi\nu)},$$

$$M = \begin{cases} (q-1)^3 & \text{if } \{\kappa, \lambda\} = \{\mu, \nu\}, \\ 0 & \text{otherwise}, \end{cases}$$

then $|H - M| \leq (q-1)q\sqrt{q}$.

Proof. First we consider the case where $\{\kappa, \lambda\} = \{\mu, \nu\}$, wherein

$$H = \sum_{\xi \in \widehat{\mathbb{F}_q^*}} |G(\psi, \xi\kappa)|^2 |G(\psi, \xi\lambda)|^2.$$

One can work out from parts (iii) and (iv) of Lemma 4.1 that $H = (q-2)q^2 + 1$ if $\kappa = \lambda = \mu = \nu$ and $H = (q-3)q^2 + 2q$ otherwise. Thus $H - M = (q-1)(q-2)$ or $1 - q$.

Now we consider the case where $\{\kappa, \lambda\} \neq \{\mu, \nu\}$, wherein

$$H = \sum_{\xi \in \widehat{\mathbb{F}_q^*}} \sum_{w,x,y,z \in \mathbb{F}_q^*} \psi(w + x - y - z)\xi(wxy^{-1}z^{-1})\kappa(w)\lambda(x)\overline{\mu(y)\nu(z)}$$

$$= (q-1) \sum_{\substack{w,x,y,z \in \mathbb{F}_q^* \\ wx = yz}} \psi(w + x - y - z)\kappa(w)\lambda(x)\overline{\mu(y)\nu(z)}.$$

Now reparametrize the sum with $w = uy$ and $z = ux$ to obtain

$$H = (q-1) \sum_{u,x,y \in \mathbb{F}_q^*} \overline{\psi}((u-1)x)\psi((u-1)y)\kappa\overline{\nu}(u)\lambda\overline{\nu}(x)\kappa\overline{\mu}(y),$$

and since $\{\kappa, \lambda\} \neq \{\mu, \nu\}$, we can restrict to $u \neq 1$ without changing the value of the summation. Then Lemma 4.1(v) tells us that when we sum over x and y, we obtain

$$H = (q-1)G(\psi, \kappa\overline{\mu})G(\overline{\psi}, \lambda\overline{\nu}) \sum_{u \neq 0,1} \overline{\kappa\lambda\mu\nu}(u-1)\kappa\overline{\nu}(u).$$

Now $\overline{\kappa\lambda\mu\nu}$ and $\kappa\overline{\nu}$ cannot both be the trivial character since $\{\kappa, \lambda\} \neq \{\mu, \nu\}$. If $\overline{\kappa\lambda\mu\nu}$ is trivial, then the sum over u is -1; if $\kappa\overline{\nu}$ is trivial, the sum is $-\overline{\kappa\lambda\mu\nu}(-1)$; otherwise, let ω be a generator of $\widehat{\mathbb{F}_q^*}$ and we can write the sum over u as

$$\sum_{u \neq 0,1} \omega((u-1)^a u^b)$$

for some nonzero $a, b \in \mathbb{Z}/(q-1)\mathbb{Z}$, and use the Weil bound [Weil 1948; Lidl and Niederreiter 1997, Theorem 5.41] to see that this sum is bounded in magnitude by $\sqrt{q}$. We can use this fact, along with Lemma 4.1(iii), (iv), to see that $|H| \leq (q-1)q\sqrt{q}$. $\qquad\square$

Similarly, we estimate the summation H appearing in Corollary 3.3.

Proposition 4.3. *Let χ be a nontrivial character in $\widehat{\mathbb{F}_q^*}$, and $\kappa, \lambda, \mu, \nu \in \widehat{\mathbb{F}_q}$. If*

$$H = \sum_{\xi \in \widehat{\mathbb{F}_q}} G(\xi\kappa, \chi)G(\xi\lambda, \chi)\overline{G(\xi\mu, \chi)G(\xi\nu, \chi)}$$

and

$$M = \begin{cases} q^3 & \text{if } \{\kappa, \lambda\} = \{\mu, \nu\}, \\ q^3 & \text{if } \kappa = \lambda, \mu = \nu, \text{ and } \chi \text{ is the quadratic character,} \\ 0 & \text{otherwise,} \end{cases}$$

then $|H - M| \leq 3q^2 \sqrt{q}$.

Proof. Let ϵ be the canonical additive character over $\mathbb{F}_q$. Then for any $\eta \in \widehat{\mathbb{F}_q}$, there is a unique $y \in \mathbb{F}_q$ such that $\eta(z) = \epsilon(yz)$ for all $z \in \mathbb{F}_q$. Let a, b, c, d be chosen so that $\kappa(z) = \epsilon(az)$, $\lambda(z) = \epsilon(bz)$, $\mu(z) = \epsilon(cz)$, and $\nu(z) = \epsilon(dz)$ for all $z \in \mathbb{F}_q$. Furthermore, we shall parametrize the sum of ξ over $\widehat{\mathbb{F}_q}$ in the definition of H by a sum over $x \in \mathbb{F}_q$, and replace $\xi(z)$ with $\epsilon(xz)$ wherever it occurs. Thus, in view Lemma 4.1(v) and (ii), we have

$$H = |G(\epsilon, \chi)|^4 \sum_{x \in \mathbb{F}_q} \overline{\chi}((x + a)(x + b))\chi((x + c)(x + d)),$$

and $|G(\epsilon, \chi)| = \sqrt{q}$ by Lemma 4.1(iv). Let m be the order of χ. Then

$$H = q^2 \sum_{x \in \mathbb{F}_q} \chi((x + a)^{m-1}(x + b)^{m-1}(x + c)(x + d)).$$

The magnitude of the Weil sum over x is bounded by $3\sqrt{q}$ unless the polynomial $(x+a)^{m-1}(x+b)^{m-1}(x+c)(x+d)$ is an m-th power in $\mathbb{F}_q[x]$. (See [Weil 1948; Lidl and Niederreiter 1997, Theorem 5.41].) It is an m-th power only if $\{a, b\} = \{c, d\}$ or if $m = 2$, $a = b$, and $c = d$, in which cases the Weil sum is either $q - 1$ (if $a = b = c = d$) or $q - 2$ (if there are two distinct roots). $\qquad \square$

5. Asymptotic L^4 norm

We prove Theorems 1.3(i), 1.4(i), and 1.5(i) in this section, by using the propositions from the previous section with Corollaries 3.2 and 3.3.

Proof of Theorems 1.4(i) and 1.5(i). Let χ be a nontrivial character in $\widehat{\mathbb{F}_q^*}$, let α be an epimorphism from $\mathbb{Z}^e$ to $\mathbb{F}_q$, let $t \in \mathbb{Z}^e$, and let $S = S_1 \times \cdots \times S_e$ be a box where each S_j is a nonempty segment of the form $\{0, 1, \ldots, s_j - 1\}$. Recall from Section 2 our notational convention that z^s is shorthand for $z_1^{s_1} \cdots z_e^{s_e}$ when $s = (s_1, \ldots, s_e) \in \mathbb{Z}^e$. Let $f(z)$ be the multiplicative character polynomial $\sum_{s \in S} \chi(\alpha(s + t))z^s$. We shall calculate $\|f\|_4^4$ first, and then investigate what happens asymptotically to this quantity in the limits considered in Theorems 1.4(i) and 1.5(i). By Corollary 3.3, we have

$$(4) \qquad \|f\|_4^4 = \frac{1}{q^5} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa, \lambda, \mu, \nu \in \widehat{\mathbb{F}_q}} \overline{\kappa'(a)\lambda'(b)}\mu'(c)\nu'(d)H(\kappa, \lambda, \mu, \nu),$$

where we let $U = S + t$, and for any $\eta \in \widehat{\mathbb{F}}_q$, we let $\eta' = \eta \circ \alpha$, and

$$H(\kappa, \lambda, \mu, \nu) = \sum_{\xi \in \widehat{\mathbb{F}}_q} G(\xi\kappa, \chi) G(\xi\lambda, \chi) \overline{G(\xi\mu, \chi)} \overline{G(\xi\nu, \chi)}.$$

By Proposition 4.3, we can write $H(\kappa, \lambda, \mu, \nu) = M(\kappa, \lambda, \mu, \nu) + N(\kappa, \lambda, \mu, \nu)$ with

$$M(\kappa, \lambda, \mu, \nu) = \begin{cases} q^3 & \text{if } \{\kappa, \lambda\} = \{\mu, \nu\}, \\ q^3 & \text{if } \kappa = \lambda, \mu = \nu, \text{ and } \chi \text{ is the quadratic character}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$(5) \qquad\qquad |N(\kappa, \lambda, \mu, \nu)| \le 3q^2 \sqrt{q},$$

for all $\kappa, \lambda, \mu, \nu \in \widehat{\mathbb{F}}_q^*$.

If χ is nonquadratic, when we write out separately the contributions from M and N to (4), we get $\|f\|_4^4 = A + B - D + E$, where

$$A = \frac{1}{q^2} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa,\lambda \in \widehat{\mathbb{F}}_q} \kappa'(c-a)\lambda'(d-b),$$

$$B = \frac{1}{q^2} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa,\lambda \in \widehat{\mathbb{F}}_q} \kappa'(d-a)\lambda'(c-b),$$

$$D = \frac{1}{q^2} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa \in \widehat{\mathbb{F}}_q} 1,$$

$$E = \frac{1}{q^5} \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\mathbb{F}}_q} N(\kappa, \lambda, \mu, \nu) \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \overline{\kappa'(a)\lambda'(b)}\mu'(c)\nu'(d).$$

Here A accounts for the value of M when $(\kappa, \lambda) = (\mu, \nu)$, and B accounts for the value of M when $(\kappa, \lambda) = (\nu, \mu)$, while D corrects for the double counting by A and B of the case $\kappa = \lambda = \mu = \nu$.

Note that $A = B$, and that A counts the number of $(a, b, c, d) \in U^4$ with $c - a = b - d \in \ker\alpha$. If we write $a = (a_1, \ldots, a_e)$, $b = (b_1, \ldots, b_e)$, $c = (c_1, \ldots, c_e)$, and $d = (d_1, \ldots, d_e)$, then $c - a \in \ker\alpha$ is equivalent to $c_1 - a_1 \equiv \cdots \equiv c_e - a_e \equiv 0$ (mod p), because α is an epimorphism from $\mathbb{Z}^e$ to $\mathbb{F}_q = \mathbb{F}_{p^e}$ and so factors as $\alpha = \gamma \circ \beta$, with $\beta \colon \mathbb{Z}^e \to (\mathbb{Z}/p\mathbb{Z})^e$ coordinate-wise reduction modulo p and $\gamma \colon (\mathbb{Z}/p\mathbb{Z})^e \to \mathbb{F}_q$ a group isomorphism. Now $U = U_1 \times \cdots \times U_e$ with each $U_j = \{t_j, t_j + 1, \ldots, t_j + |S_j| - 1\}$, so for each $n \in \mathbb{Z}$, there are $\max(0, |S_j| - p|n|)$ ways for $c_j - a_j$ to equal pn and the same number of ways for $b_j - d_j$ to equal pn.

So

$$A = B = \prod_{j=1}^{e} \sum_{n_j \in \mathbb{Z}} \max(0, |S_j| - p|n_j|)^2.$$

On the other hand, qD counts the number of $(a, b, c, d) \in U^4$ with $c - a = b - d$, so by the same argument we just used (with modulus 1 instead of p),

$$qD = \prod_{j=1}^{e} \sum_{n_j \in \mathbb{Z}} \max(0, |S_j| - |n_j|)^2,$$

from which we can compute

$$D = \prod_{j=1}^{e} \left(\frac{2|S_j|^3 + |S_j|}{3p} \right).$$

Now we bound E via two bounds: our bound (5) on N, and a technical result, Lemma A.1 in the Appendix, bounding the inner sum of E. We satisfy the condition on α demanded by this lemma, since $\alpha = \gamma \circ \beta$ with $\beta \colon \mathbb{Z}^e \to (\mathbb{Z}/p\mathbb{Z})^e$ coordinate-wise reduction modulo p and $\gamma \colon (\mathbb{Z}/p\mathbb{Z})^e \to \mathbb{F}_q$ a group isomorphism. With these two bounds, we obtain

$$(6) \qquad |E| \leq 3 \cdot 64^e q \sqrt{q} \prod_{j=1}^{e} \max\left(1, \frac{|S_j|}{p}\right)^3 \prod_{j=1}^{e} (1 + \log p)^3.$$

Now we divide $\|f\|_4^4 = A + B - D + E$ by $\|f\|_2^4$ and consider the limit where each $|S_j|/p \to \sigma_j$ as $q \to \infty$, that is, consider what happens in a size-stable family of polynomials. Another technical result, Lemma A.2, shows that we can replace the denominator $\|f\|_2^4$ with $|S|^2$ without changing the limit. Then recall the definition (3) of Ω, and note that $A/|S|^2$ and $B/|S|^2$ tend to $\prod_{j=1}^{e} \Omega(1/\sigma_j, 0)$, that $D/|S|^2$ tends to $\left(\frac{2}{3}\right)^e \prod_{j=1}^{e} \sigma_j$, and that $|E|/|S|^2$ tends to 0 in this limit.

If χ is quadratic, the proof is done in the same manner, except that there is now a contribution from M in the case where $\kappa = \lambda$ and $\mu = \nu$, and so we get $\|f\|_4^4 = A + B + C - 2D + E$, where A, B, D, and E are as defined above, and

$$C = \frac{1}{q^2} \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \sum_{\kappa, \mu \in \widehat{\mathbb{F}}_q} \kappa'(-a-b)\mu'(c+d).$$

Note that we subtract D twice now because A, B, and C count the case where $\kappa = \lambda = \mu = \nu$ three times. C counts the number of $(a, b, c, d) \in U^4$ with $a + b = c + d \in \ker \alpha$. Following the method we used to determine A, write $a = (a_1, \ldots, a_e)$, $b = (b_1, \ldots, b_e)$, $c = (c_1, \ldots, c_e)$, and $d = (d_1, \ldots, d_e)$, and note that $a + b \in \ker \alpha$ is equivalent to $a_1 + b_1 \equiv \cdots \equiv a_e + b_e \equiv 0 \pmod{p}$.

Since $U = U_1 \times \cdots \times U_e$ with each $U_j = \{t_j, t_j + 1, \ldots, t_j + |S_j| - 1\}$, there are $\max(0, |S_j| - |np - (2t_j + |S_j| - 1)|)$ ways to obtain $a_j + b_j = np$ with $(a_j, b_j) \in U_j^2$, and the same number of ways to obtain $c_j + d_j = np$ with $(c_j, d_j) \in U_j^2$, so

$$C = \prod_{j=1}^{e} \sum_{n_j \in \mathbb{Z}} \max(0, |S_j| - |pn_j - |S_j| - 2t_j + 1|)^2,$$

and if we have both size- and translation-stability, then $|S_j|/p \to \sigma_j$ and $t_j/p \to \tau_j$ as $q \to \infty$, so that $C/|S|^2 \to \prod_{j=1}^{e} \Omega(1/\sigma_j, 1 + 2\tau_j/\sigma_j)$.   □

Proof of Theorem 1.3(i). The proof is the same, *mutatis mutandis*, as for the $e = 1$ case of Theorem 1.4(i), with the roles of $\mathbb{F}_q$ and $\mathbb{F}_q^*$ exchanged. Corollary 3.2 and Proposition 4.2 replace Corollary 3.3 and Proposition 4.3, and Lemma A.2 becomes unnecessary as $\|f\|_2^2$ for an additive character polynomial f is always precisely equal to the cardinality of the support of f. These, and other attendant minor changes resulting from the exchange of $\mathbb{F}_q$ and $\mathbb{F}_q^*$, cause (5) to become $|N(\kappa, \lambda, \mu, \nu)| \le (q-1)q\sqrt{q}$, and (6) to become

$$|E| \le 64q\sqrt{q} \max(1, |S|/(q-1))^3 (1 + \log(q-1))^3,$$

and any other printed instance of p or q should be replaced with $q - 1$.   □

6. Minimizing the asymptotic ratio of L^4 to L^2 norm

Here we prove Theorems 1.3(ii), 1.4(ii), and 1.5(ii) by finding the limiting sizes and (for quadratic multiplicative character polynomials) the limiting translations that globally minimize the ratio of the L^4 to L^2 norm.

Proof of Theorem 1.4(ii). In view of Theorem 1.4(i), we are trying to minimize the limiting ratio of norms, given by the function

$$K(x_1, \ldots, x_e) = -\frac{2^e}{3^e} \prod_{j=1}^{e} x_i + 2 \prod_{j=1}^{e} \Phi(x_j),$$

for $x_1, \ldots, x_e$ positive real numbers (the limiting sizes), where for positive x, we define

$$(7) \qquad \Phi(x) = \Omega\left(\frac{1}{x}, 0\right) = \sum_{n \in \mathbb{Z}} \max\left(0, 1 - \frac{|n|}{x}\right)^2,$$

which is differentiable for $x \ne 0$ and is C^∞ for $x \notin \mathbb{Z}$.

Step 1. We can assume that each $x_j > 1$ because otherwise the partial derivative of K with respect to x_j would be negative.

Step 2. We can assume that $(x_1, \ldots, x_e) \in (1, 3)^e$: indeed, Lemma A.6 shows that $K(x_1, \ldots, x_e) \geq \Phi(x_1) \cdots \Phi(x_e)$, and note that $\Phi(x)$ is increasing for $x > 1$, that $\Phi(1) = 1$, $\Phi(3) = \frac{19}{9} > 2$, and $K(1, \ldots, 1) = 2 - \left(\frac{2}{3}\right)^e < 2$. This proves that a global minimum exists and lies in $(1, 3)^e$: the closure of $(1, 3)^e$ is compact and K is continuous thereupon.

Step 3. Suppose $(\sigma_1, \ldots, \sigma_e)$ to be global minimizer of K. Then the partial derivatives of K must vanish there, whence for each $k \in \{1, \ldots, e\}$, we have $2u(\sigma_k) \prod_{j=1}^{e} U(\sigma_j) = 1$, where $u(x) = x\Phi'(x)/\Phi(x)$ and $U(x) = 3\Phi(x)/(2x)$ for $x > 0$.

Step 4. Then one can show that $U(x)$ is strictly decreasing on $[1, 3]$, with $U(3) = \frac{19}{18}$. Thus we must have $u(\sigma_k) \leq \frac{1}{2} \cdot \left(\frac{18}{19}\right)^e < \frac{1}{2}$ for all $k \in \{1, \ldots, e\}$. Then examination of $u(x)$ shows that $u(x)$ strictly increases from 0 to $\frac{1}{2}$ for $x \in [1, 2 - \sqrt{2/3}]$, and then $u(x) > \frac{1}{2}$ for $x \in (2 - \sqrt{2/3}, 3)$. This then forces $\sigma_1 = \cdots = \sigma_e < 2 - \sqrt{2/3} < \frac{6}{5}$.

Step 5. Now $U(\sigma_1) > U\left(\frac{6}{5}\right) > \frac{9}{7}$, so this forces $u(\sigma_1) < \frac{1}{2} \cdot \left(\frac{7}{9}\right)^e \leq \frac{7}{18}$, which in turn forces $\sigma_1 < \frac{8}{7}$. Then $U(\sigma_1) > U\left(\frac{8}{7}\right) > \frac{4}{3}$, so this forces $u(\sigma_1) < \frac{1}{2} \cdot \left(\frac{3}{4}\right)^e$. Since $u(x) \geq 8(x-1)/3$ for $x \in \left[1, \frac{8}{7}\right]$, this forces $\sigma_1 < 1 + 3^{e+1}/2^{2e+4}$.

Step 6. Now our problem is reduced to the single-variable minimization of

$$\Theta(x) = K(x, \ldots, x) = -\left(\frac{2x}{3}\right)^e + 2\Phi(x)^e$$

on the interval $(1, 1 + 3^{e+1}/2^{2e+4})$. It is not hard to see that $d\Theta/dx$ vanishes if and only if $x^{3e} - (3^e/2^{e-3})(x-1)(3x^2 - 4x + 2)^{e-1}$ vanishes. Meanwhile $d^2\Theta/dx^2 > 0$ on our interval: by computing its value and then dropping a nonnegative term, we can see that $d^2\Theta/dx^2$ is at least

$$-e(e-1)\frac{2^e x^{e-2}}{3^e} + 8e\frac{3-2x}{x^4}\Phi(x)^{e-1} \geq -e(e-1)\frac{19^e}{24^e} + 2e > 0.$$

This proves that there is a unique minimum: the unique root a_e of

$$x^{3e} - \frac{3^e}{2^{e-3}}(x-1)(3x^2 - 4x + 2)^{e-1}$$

lying in $(1, 1 + 3^{e+1}/2^{2e+4})$. ☐

Proof of Theorem 1.3(ii). This is accomplished exactly as the $e = 1$ case of the proof of Theorem 1.4(ii) above, save that Lemma A.5 replaces Lemma A.6. ☐

Proof of Theorem 1.5(ii). In view of Theorem 1.5 (i), we are trying to minimize the limiting ratio of norms, given by the function

$$(8) \qquad -\frac{2^{e+1}}{3^e}\prod_{j=1}^{e} x_j + 2\prod_{j=1}^{e}\Omega\left(\frac{1}{x_j}, 0\right) + \prod_{j=1}^{e}\Omega\left(\frac{1}{x_j}, 1 + \frac{2y_j}{x_j}\right)$$

for $x_1, \ldots, x_e$ positive real numbers (the limiting sizes) and $y_1, \ldots, y_e$ arbitrary real numbers (the limiting translations).

Step 1. We invoke Lemma A.4(i) to see that we can confine our search to $x_1, \ldots, x_e$ greater than or equal to $\frac{1}{2}$. For as long as $x_j \leq \frac{1}{2}$, the lemma shows that we can always arrange for y_j to be such that $\Omega(x_j^{-1}, 1 + 2x_j^{-1} y_j) = 0$, and we note that $\Omega(x_j^{-1}, 0) = 1$ for all $x_j \in (0, \frac{1}{2}]$. Thus we can increase x_j to $\frac{1}{2}$ to lower (8) through the term $-(2^{e+1}/3^e) \prod_{j=1}^e x_j$ while keeping the other terms constant.

Step 2. Now we invoke Lemma A.4(ii) to see that for fixed $x_1, \ldots, x_e$, we minimize the last term of (8) if and only if we arrange that

$$y_j \in \left\{ \frac{1 - 2x_j}{4} + \frac{m}{2} : m \in \mathbb{Z} \right\}$$

for each $j \in \{1, \ldots, e\}$. The problem is thus reduced to the minimization of

$$\Lambda(x_1, \ldots, x_e) = -\frac{2^{e+1}}{3^e} \prod_{j=1}^e x_j + 2 \prod_{j=1}^e \Phi(x_j) + \prod_{j=1}^e \Psi(x_j)$$

for $x_1, \ldots, x_e$ positive real numbers, where $\Phi(x)$ is as defined in (7), and

$$\Psi(x) = \sum_{n \in \mathbb{Z}} \max\left(0, 1 - \frac{|2n+1|}{2x}\right)^2$$

for $x > 0$. Note that Ψ is differentiable for $x \neq 0$ and is C^∞ for $x \notin \mathbb{Z} + \frac{1}{2}$.

Step 3. We can assume that each $x_j > \frac{1}{2}$ because otherwise the partial derivative of Λ with respect to x_j would be negative.

Step 4. We can assume that $x_1, \ldots, x_e \in (\frac{1}{2}, 3)$: indeed, Lemma A.6 shows that $\Lambda(x_1, \ldots, x_e) \geq \Phi(x_1) \cdots \Phi(x_e)$, and note that $\Phi(x)$ is nondecreasing for $x > \frac{1}{2}$, that $\Phi(\frac{1}{2}) = 1$, $\Phi(3) = \frac{19}{9} > 2$, and $\Lambda(1, \ldots, 1) = 2 - 2(\frac{2}{3})^e + (\frac{1}{2})^e < 2$. This proves that a global minimum exists and lies in $(\frac{1}{2}, 3)^e$: the closure of $(\frac{1}{2}, 3)^e$ is compact and Λ is continuous thereupon.

Step 5. Suppose that $(\sigma_1, \ldots, \sigma_e)$ is a global minimizer of Λ. Then the partial derivatives of Λ must vanish there, whence

$$(9) \qquad u(\sigma_k) \prod_{j=1}^e U(\sigma_j) + \tfrac{1}{2} v(\sigma_k) \prod_{j=1}^e V(\sigma_j) = 1,$$

where $u(x) = \dfrac{x \Phi'(x)}{\Phi(x)}$, $U(x) = \dfrac{3\Phi(x)}{2x}$, $v(x) = \dfrac{x \Psi'(x)}{\Psi(x)}$, and $V(x) = \dfrac{3\Psi(x)}{2x}$ for $x > \frac{1}{2}$.

Step 6. We can assume $\sigma_1, \ldots, \sigma_e \in (1, 3)$: see (9) and note that $u(x) = 0$ for $x \in (\frac{1}{2}, 1]$, $\frac{1}{2} v(x) V(x) < 1$ for $x \in (\frac{1}{2}, 1]$, and $V(x) < 1$ for $x \in (\frac{1}{2}, 3)$.

Step 7. It is not difficult to show that $U(x)$ strictly decreases and $V(x)$ strictly increases on $[1, 3]$ with $U(3) = \frac{19}{18}$ and $V(1) = \frac{3}{4}$, and that $0 \leq u(x) < 1 \leq v(x)$ for $x \in [1, 3]$. Thus (9) shows that we must have $u(\sigma_k) < \left(\frac{18}{19}\right)^e (1 - 3^e/2^{2e+1})$ for all k. This forces $u(\sigma_k) < \frac{7}{10}$ for all k, and examination of the function u shows that $u(x) \geq \frac{7}{10}$ for $x \in \left[\frac{5}{2}, 3\right]$, and so we must have $\sigma_k < \frac{5}{2}$ for all k.

Now one can repeat the argument on the interval $\left[1, \frac{5}{2}\right]$ to show that every $\sigma_k < 2$, then repeat it again on $[1, 2]$ to show $\sigma_k < \frac{5}{4}$. Further repetitions give $\sigma_k < \frac{6}{5}$, $\sigma_k < \frac{13}{11}$, and $\sigma_k < \frac{7}{6}$. Since $U(x) > \frac{4}{3}$ while $v(x)$, $V(x) \geq 0$ for $x \in \left(1, \frac{7}{6}\right)$, we have $u(\sigma_k) < \left(\frac{3}{4}\right)^e$ for all k, and since $u(x) \geq 8(x-1)/3$ for $x \in \left[1, \frac{7}{6}\right]$, this means that $\sigma_k < 1 + 3^{e+1}/2^{2e+3}$ for all k.

Step 8. So we have $1 < \sigma_k < \min\left(\frac{7}{6}, 1+3^{e+1}/2^{2e+3}\right)$ for all k. Consider the products in (9): since each $\sigma_j \in \left(1, \frac{7}{6}\right)$, we have

$$\left(\tfrac{4}{3}\right)^e < \prod_{j=1}^{e} U(\sigma_j) < \left(\tfrac{3}{2}\right)^e, \quad \text{while} \quad \left(\tfrac{3}{4}\right)^e < \prod_{j=1}^{e} V(\sigma_j) < \left(\tfrac{7}{8}\right)^e.$$

We now claim that for a given $A \in \left[\left(\frac{4}{3}\right)^e, \left(\frac{3}{2}\right)^e\right]$ and $B \in \left[\left(\frac{3}{4}\right)^e, \left(\frac{7}{8}\right)^e\right]$, there is at most one solution $x \in \left(1, \frac{7}{6}\right)$ to $Au(x) + \frac{1}{2}Bv(x) = 1$, which will force $\sigma_1 = \cdots = \sigma_e$. For if we set $w(x) = Au(x) + \frac{1}{2}Bv(x)$, we can show that $w'(x) > 0$ for $x \in \left(1, \frac{7}{6}\right)$: on this interval, we have $u(x) = 4(x-1)/(3x^2 - 4x + 2)$ $v(x) = 2/(2x-1)$, and it is not difficult to show that $u'(x) > \frac{3}{2}$ and $v'(x) > -4$, so that

$$w'(x) > \tfrac{3}{2}A - 2B \geq \left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)^e - 2\left(\tfrac{7}{8}\right)^e > 0.$$

Step 9. Now our problem is reduced to the single-variable minimization of

$$T(x) = \Lambda(x, \ldots, x) = -2(2x/3)^e + 2\Phi(x)^e + \Psi(x)^e$$

for $x \in (1, 1 + 3^{e+1}/2^{2e+3})$. It is not hard to see that dT/dx vanishes if and only if

$$x^{3e} - \frac{3^e}{2^{e-2}}(x-1)(3x^2 - 4x + 2)^{e-1} - \frac{3^e}{2^{2e}}(2x-1)^{2e-1}$$

does. Meanwhile we claim that the second derivative of T is strictly positive on our interval: by dropping some nonnegative terms we see that

$$\frac{d^2 T}{dx^2}(x) \geq -e(e-1)\frac{2^{e+1}x^{e-2}}{3^e} + e\frac{8(3-2x)}{x^4}\Phi(x)^{e-1} + e\frac{3-4x}{x^4}\Psi(x)^{e-1}.$$

Thus for $e = 1$, the second derivative is at least $(27 - 20x)/x^4$, which is strictly positive on our interval $\left(1, 1 + \frac{9}{32}\right)$. For $e = 2$, we can use the fact that

$$0 \leq \Psi(x) \leq 1 \leq \Phi(x)$$

on our interval $\left(1, 1 + \frac{27}{128}\right)$ to show that the second derivative is at least

$$-\frac{16}{9} + \frac{54 - 40x}{x^4},$$

which is strictly positive on our interval. Finally, if $e \geq 3$, we have

$$1 + \frac{3^{e+1}}{2^{2e+3}} < \frac{7}{6},$$

and so on our interval $(1, 1 + 3^{e+1}/2^{2e+3})$ we have $8(3 - 2x)/x^4 > \frac{20}{7}$, $\Phi(x) \geq 1$, $(3 - 4x)/x^4 \geq -1$, and $0 \leq \Psi(x) \leq 1$, so that

$$\frac{d^2 T}{dx^2}(x) \geq -\frac{8}{9}\left(\frac{7}{9}\right)^{e-2} e(e-1) + \frac{13}{7} e > 0.$$

This proves that there is a unique global minimum for this single-variable problem: the unique root b_e of

$$x^{3e} - \frac{3^e}{2^{e-2}}(x-1)(3x^2 - 4x + 2)^{e-1} - \frac{3^e}{2^{2e}}(2x-1)^{2e-1}$$

lying in $(1, 1 + 3^{e+1}/2^{2e+3})$. Thus we have found that global minima are obtained precisely when $\sigma_1 = \cdots = \sigma_e = b_e$ and $\tau_j \in \{\frac{1}{4}(1 - 2b_e) + m/2 : m \in \mathbb{Z}\}$ for each $j \in \{1, \ldots, e\}$. $\qquad\square$

7. Proof of Theorem 1.6

For A_e and B_e as defined in Theorem 1.6, we first show that $B_e < A_e$ for each $e \geq 1$. Given the minimizing conditions described in Theorems 1.4(ii) and 1.5(ii), it suffices to show that

$$-\frac{2^{e+1}}{3^e} x^e + 2\Omega\left(\frac{1}{x}, 0\right)^e + \Omega\left(\frac{1}{x}, \frac{1}{2x}\right)^e < -\frac{2^e}{3^e} x^e + 2\Omega\left(\frac{1}{x}, 0\right)^e.$$

for $x \in \left[1, \frac{3}{2}\right]$. This follows if $\Omega(1/x, 1/2x) < 2x/3$, or using the definition (3) of Ω, if $4x^3 - 12x^2 + 12x - 3 > 0$ for $x \in \left[1, \frac{3}{2}\right]$, which is routine to show.

Now we show that $B_{e_1+e_2} < B_{e_1} B_{e_2}$ for any $e_1, e_2 \geq 1$. We use a technical Lemma 7.1 below, which provides bounds on the B_e. It shows that if $e_1 \geq 5$ or $e_2 \geq 5$, then $B_{e_1} B_{e_2} > \sqrt[4]{2} > B_{e_1+e_2}$. So we may confine ourselves to the case where $1 \leq e_1 \leq e_2 \leq 4$. If we define $B_0 = 1$, then the bounds in Lemma 7.1 also show us that

$$\frac{B_1}{B_0} > \frac{B_2}{B_1} > \frac{B_3}{B_2} > \frac{B_4}{B_3} > \frac{B_5}{B_4}.$$

Thus we note that $B_{e_1} B_{e_2} > B_{e_1-1} B_{e_2+1}$. If $e_1 + e_2 > 5$, we can repeat this argument to show that $B_{e_1} B_{e_2}$ is greater than $B_{e_1+e_2-5} B_5$, which we have already shown to exceed $B_{e_1+e_2}$. On the other hand, if $e_1 + e_2 \leq 5$, repetition of the same argument produces $B_{e_1} B_{e_2} > B_0 B_{e_1+e_2} = B_{e_1+e_2}$.

Lemma 7.1. *For each $e \geq 1$, let B_e be the minimum asymptotic ratio of L^4 to L^2 norm achievable by a family of e-variable quadratic multiplicative character polynomials as described in* Theorem 1.5(ii). *Then*

(i) $\sqrt[4]{103/89} < B_1 < \sqrt[4]{22/19}$,

(ii) $\sqrt[4]{86/65} < B_2 < \sqrt[4]{75/56}$,

(iii) $\sqrt[4]{142/95} < B_3 < \sqrt[4]{116/77}$,

(iv) $\sqrt[4]{100/61} < B_4 < \sqrt[4]{107/65}$,

(v) $\sqrt[4]{7/4} < B_5 < \sqrt[4]{128/73}$, *and*

(vi) $\sqrt[4]{7/4} < B_e < \sqrt[4]{2}$ *for all $e \geq 6$.*

Proof. By Section 6 of the proof of Theorem 1.5(ii) (see page 389), for each $e \geq 1$ the quantity B_e^4 is the minimum of the function

$$T(x) = -\frac{2^{e+1}}{3^e}x^e + 2\Omega\left(\frac{1}{x}, 0\right)^e + \Omega\left(\frac{1}{x}, \frac{1}{2x}\right)^e$$

on the interval $(1, 1+3^{e+1}/2^{2e+3})$, upon which the second derivative of T is shown to be positive. Thus if we can find x_1, x_2, x_3 in this interval with $x_1 < x_2 < x_3$ and $T(x_2) < \min(T(x_1), T(x_3))$, then we will have shown that the minimizing value of x lies in the interval (x_1, x_3). Then we can use $B_e^4 \leq T(x_2)$ for our upper bound and, by the monotonicity of $\Omega(1/x, 0)$ and $\Omega(1/x, 1/(2x))$, we can use

$$B_e^4 > -\frac{2^{e+1}}{3^e}x_3^e + 2\Omega\left(\frac{1}{x_1}, 0\right)^e + \Omega\left(\frac{1}{x_1}, \frac{1}{2x_1}\right)^e$$

as a lower bound. We use this technique to prove bounds in (i)–(v). The calculations done by hand are tedious, so here we simply state choices of the triple (x_1, x_2, x_3) that establish stricter bounds than the ones we claim above:

for B_1 use $\left(\frac{55}{52}, \frac{128}{121}, \frac{73}{69}\right)$,   for B_2 use $\left(\frac{18}{17}, \frac{17}{16}, \frac{16}{15}\right)$,   for B_3 use $\left(\frac{21}{20}, \frac{20}{19}, \frac{19}{18}\right)$,
for B_4 use $\left(\frac{26}{25}, \frac{25}{24}, \frac{24}{23}\right)$,   and for B_5 use $\left(\frac{36}{35}, \frac{35}{34}, \frac{34}{33}\right)$.

Henceforth assume that $e \geq 6$, and let b_e be the unique value in the interval $(1, 1 + 3^{e+1}/2^{2e+3})$ such that $T(b_e) = B_e^4$. Now note that $\Omega(1/x, 0) \geq 1$ and $\Omega(1/x, 1/(2x)) \geq \frac{1}{2}$ for $x \geq 1$, and that $2b_e/3 < (2^{15}+3^7)/(3 \cdot 2^{14}) < 1$, so that

$$B_6^4 > -2\left(\frac{2^{15}+3^7}{3\cdot 2^{14}}\right)^6 + 2 + \frac{1}{2^6} > \frac{7}{4},$$

and if $e \geq 7$, then

$$B_e^4 > -2\left(\frac{2^{15}+3^7}{3\cdot 2^{14}}\right)^7 + 2 > \frac{7}{4}.$$

This proves our lower bound on B_e when $e \geq 6$.

To prove our upper bound on B_e when $e \geq 6$, write $b_e = 1 + c_e$ with $0 < c_e < 3^{e+1}/2^{2e+3}$, and use the definition (3) of Ω and the fact that $1/(1+c_e) > 1 - c_e$ to estimate $B_e = T(b_e)$ as

$$
\begin{aligned}
B_e^4 &< -\frac{2^{e+1}}{3^e} + 2(1 + 2c_e^2)^e + \frac{1}{2^e}(1 + c_e)^{2e} \\
&\leq -\frac{2^{e+1}}{3^e} + 2 + 4ec_e^2 + 2^{e+3}c_e^4 + \frac{1}{2^e}(1 + c_e)^{2e},
\end{aligned}
$$

where the second inequality follows from a crude approximation with the binomial expansion. Now note that

$$
2^{e+3}c_e^4 < \frac{3^{4e+4}}{2^{7e+9}} < \frac{2^e}{6 \cdot 3^e} \quad \text{and} \quad 4ec_e^2 < e\frac{3^{2e+2}}{2^{4e+4}} < \frac{5 \cdot 2^e}{4 \cdot 3^e},
$$

so that

$$
B_e^4 < 2 - \frac{7}{12} \cdot \frac{2^e}{3^e} + \frac{1}{2^e}(1 + c_e)^{2e},
$$

which will imply $B_e^4 < 2$ if we can show that $(3(1 + c_e)^2/4)^e \leq \frac{7}{12}$. Given that $c_e \leq 3^7/2^{15} < \frac{1}{10}$, the quantity being raised to the e-th power on the left hand side is less than 1, so it suffices to show that $(3(\frac{11}{10})^2/4)^6 \leq \frac{7}{12}$. $\square$

Appendix A: Proofs of auxiliary results

Here we collect, for the sake of completeness, technical results used in our proofs. The first is a bound on a character sum used in the proofs of Theorems 1.3(i), 1.4(i), and 1.5(i) in Section 5.

Lemma A.1. *Let Γ be a finite abelian group, n a positive integer, and*

$$
\pi_1, \ldots, \pi_n \in \mathrm{Hom}(\mathbb{Z}, \Gamma)
$$

such that $\mathrm{im}\,\pi_1 + \cdots + \mathrm{im}\,\pi_n$ is the internal direct sum of $\mathrm{im}\,\pi_1, \ldots, \mathrm{im}\,\pi_n$ in Γ. Let $\pi \in \mathrm{Hom}(\mathbb{Z}^n, \Gamma)$ with $\pi(u_1, \ldots, u_n) = \pi_1(u_1) + \cdots + \pi_n(u_n)$ for all $(u_1, \ldots, u_n) \in \mathbb{Z}^n$, and let $U = U_1 \times \cdots \times U_n$ be a box in $\mathbb{Z}^n$. Then

$$
T = \sum_{\kappa,\lambda,\mu,\nu \in \widehat{\Gamma}} \left| \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \overline{\kappa(\pi(a))\lambda(\pi(b))}\mu(\pi(c))\nu(\pi(d)) \right|
$$

is no greater than

$$
64^n |\Gamma|^4 \prod_{j=1}^{n} \max\left(1, \frac{|U_j|}{|\mathrm{im}\,\pi_j|}\right)^3 \prod_{j=1}^{n}(1 + \log|\mathrm{im}\,\pi_j|)^3.
$$

Proof. Write $K = \bigoplus_{j=1}^{n} \mathrm{im}\,\pi_j$, so that $\widehat{K} = \prod_{j=1}^{n} \widehat{\mathrm{im}\,\pi_j}$. Each character of K extends to $[\Gamma : K]$ characters of Γ, and for any $\eta \in \widehat{\mathrm{im}\,\pi_j}$, let $\eta' \in \widehat{\mathbb{Z}}$ be $\eta \circ \pi_j$,

so that

$$T = [\Gamma : K]^4 \prod_{j=1}^{n} \sum_{\kappa_j, \lambda_j, \mu_j, \nu_j \in \widehat{\mathrm{im}\,\pi_j}} \left| \sum_{\substack{a_j, b_j, c_j, d_j \in U_j \\ a_j + b_j = c_j + d_j}} \overline{\kappa'_j(a_j) \lambda'_j(b_j)} \mu'_j(c_j) \nu'_j(d_j) \right|,$$

and so it suffices to prove the bound when $n = 1$ and π is surjective. In this case Γ is a finite cyclic group, which we identify with $\mathbb{Z}/m\mathbb{Z}$ by identifying $\pi(1)$ with $1 \in \mathbb{Z}/m\mathbb{Z}$. Then we set $\epsilon_a = \exp(2\pi i a/m)$ for every $a \in \mathbb{Z}/m\mathbb{Z}$, and note that $\widehat{\Gamma}$ is the set of maps $a \mapsto \epsilon_{xa}$ with $x \in \mathbb{Z}/m\mathbb{Z}$. Thus,

$$T = \sum_{w,x,y,z \in \mathbb{Z}/m\mathbb{Z}} \left| \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \epsilon_{-wa-xb+yc+zd} \right|.$$

U is a set of consecutive integers in $\mathbb{Z}$, and note that translation of U does not influence the magnitude of the inner sum in T, so without loss of generality, we assume that $U = \{0, 1, \ldots, |U| - 1\}$. Then reparametrize the outer sum in T with $x' = w - x$, $y' = y - w$, $z' = z - w$, and w to obtain

$$T = m \sum_{x',y',z' \in \mathbb{Z}/m\mathbb{Z}} \left| \sum_{\substack{a,b,c,d \in U \\ a+b=c+d}} \epsilon_{-x'b+y'c+z'd} \right|,$$

which is not more than $64m \max(m, |U|)^3 (1 + \log m)^3$ by [Jedwab et al. 2013b, Lemma 2.2]. $\square$

The next result is used in the proofs of Section 5 to understand the asymptotic behavior of the L^2 norm for multiplicative character polynomials.

Lemma A.2. *Let $\{f_\iota\}_{\iota \in I}$ be a size-stable family of e-variable multiplicative character polynomials with $\mathbb{F}_{q_\iota}$, S_ι, t_ι, and α_ι the field, support, translation, and arrangement, respectively, of f_ι for each $\iota \in I$. Then there is a Q and an N such that for all $\iota \in I$ with $q_\iota \geq Q$, we have $|S_\iota \cap (\ker \alpha_\iota - t_\iota)| \leq N$. Thus $|S_\iota \cap \ker(\alpha_\iota - t_\iota)|/|S_\iota| \to 0$ and $\|f_\iota\|_2^2/|S_\iota| \to 1$ as $q_\iota \to \infty$.*

Proof. Suppose that the limiting sizes for our size-stable family $\{f_\iota\}_{\iota \in I}$ of multiplicative character polynomials are $\sigma_1, \ldots, \sigma_e$. For each $\iota \in I$, let χ_ι be the character of f_ι, so that $f_\iota(z) = \sum_{s \in S_\iota} \chi_\iota(\alpha_\iota(s + t_\iota)) z^s$, and let $p_\iota = q_\iota^{1/e}$, which is the characteristic of the field $\mathbb{F}_{q_\iota}$ of f_ι. Since α_ι is an epimorphism, its restriction to each $p_\iota \times \cdots \times p_\iota$ cubical box in $\mathbb{Z}^e$ is a bijection to $\mathbb{F}_{q_\iota}$, and by the definition of size-stability, there is some Q such that for every $q_\iota \geq Q$, the support $S_\iota = S_{\iota,1} \times \cdots \times S_{\iota,e}$ can be covered with $N = \prod_{j=1}^{e} \left(\lfloor \sigma_j \rfloor + 1 \right)$ such cubes, each of which contains one point of $\ker \alpha_\iota - t_\iota$, so $|S_\iota \cap (\ker \alpha_\iota - t_\iota)| \leq N$. Since the family is size-stable, $|S_\iota| \to \infty$ as $q_\iota \to \infty$. The squared L^2 norm of a polynomial is the sum of the squared magnitudes of its coefficients, and $\chi_\iota(\alpha_\iota(s + t_\iota)) = 0$ for $s \in S_\iota \cap (\ker \alpha_\iota - t_\iota)$ while

$|\chi_\iota(\alpha_\iota(s+t_\iota))| = 1$ for all other $s \in S_\iota$. Thus $\|f_\iota\|_2^2 = |S_\iota \smallsetminus (S_\iota \cap (\ker \alpha_\iota - t_\iota))|$, and so $\|f_\iota\|_2^2/|S_\iota| \to 1$ as $q_\iota \to \infty$. $\qquad\square$

Recall from footnote 3 of Section 1C that we sometimes wish to obtain a Littlewood polynomial from a quadratic character polynomial $f(z) = \sum_{s \in S} \chi(\alpha(s+t))z^s$, but f may have some coefficients equal to 0 because an extended nontrivial multiplicative character χ has $\chi(0) = 0$. More generally, if χ is a nontrivial multiplicative character (not necessarily quadratic), we may wish to obtain from f a polynomial with coefficients of unit magnitude. So we replace the zero coefficient for each z^s such that $s \in S \cap (\ker \alpha - t)$ with a coefficient of unit magnitude. We may choose each replacement coefficient independently of the others, and any polynomial g resulting from such replacements is called a *unimodularization* of f. The following corollary to Lemma A.2 shows that unimodularizing all the polynomials in a size-stable family of multiplicative character polynomials does not affect asymptotic ratio $\|f\|_4/\|f\|_2$.

Corollary A.3. *Let $\{f_\iota\}_{\iota \in I}$ be a size-stable family of multiplicative character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$, and let g_ι be a unimodularization of f_ι for each $\iota \in I$. If r is a real number with $r \geq 2$ or if $r = \infty$, then $\|f_\iota\|_r/\|g_\iota\|_r \to 1$ as $q_\iota \to \infty$.*

Proof. If $u \in \mathbb{C}$ with $|u| = 1$ and if $s = (s_1, \ldots, s_e) \in \mathbb{Z}^e$, then L^r norm of $uz^s = uz_1^{s_1} \cdots z_e^{s_e}$ is 1. By Lemma A.2 there is an N and a Q such that whenever $q_\iota \geq Q$, the two polynomials f_ι and g_ι differ by the sum of N or fewer such monomials, and so by the L^r triangle inequality, the difference between $\|f_\iota\|_r$ and $\|g_\iota\|_r$ cannot be greater in magnitude than N. Now $\|g_\iota\|_r \geq \|g_\iota\|_2 = \sqrt{|S_\iota|}$ by monotonicity of L^r norms and the fact that the squared L^2 norm of a polynomial is the sum of the squared magnitudes of its coefficients, and $|S_\iota| \to \infty$ as $q_\iota \to \infty$ for a size-stable family. $\qquad\square$

The next result is used in Section 6 to find the limiting translations that globally minimize the asymptotic ratio of L^4 to L^2 norm for quadratic character polynomials.

Lemma A.4. *Let x be a fixed nonzero value in $\mathbb{R}$ and let y vary over $\mathbb{R}$.*

(i) *If $|x| \geq 2$, the function $\Omega(x, y)$, considered as a function of y, achieves a global minimum value of 0 for $y \in \bigcup_{m \in \mathbb{Z}}[m|x|+1, (m+1)|x|-1]$ and for no other value of y.*

(ii) *If $0 < |x| \leq 2$, the function $\Omega(x, y)$, considered as a function of y, achieves a global minimum value of*

$$\Omega\left(x, \frac{x}{2}\right) = \sum_{n \in \mathbb{Z}} \max\left(0, 1 - \left|\left(n + \tfrac{1}{2}\right)x\right|\right)^2$$

for $y \in \left\{x\left(m + \tfrac{1}{2}\right) : m \in \mathbb{Z}\right\}$ and for no other value of y.

Proof. For part (i), note that all the terms of $\Omega(x, y)$ are nonnegative. Since $\Omega(x, y) = \Omega(-x, y)$, we may assume without loss of generality that $x \geq 2$, and then the term $\max(0, 1-|xn-y|)^2$ is nonvanishing if and only if $(y-1)/x < n < (y+1)/x$. Thus we can obtain a global minimum value of 0 if we can arrange that no integer lie in the interval $((y-1)/x, (y+1)/x)$. If m is the greatest integer lying below this interval (so that $y \geq mx + 1$), then for the next integer $m + 1$ to lie above the interval, it is necessary and sufficient that $y \leq (m + 1)x - 1$.

For part (ii), it is clear from the definition (3) of Ω that $\Omega(-x, y) = \Omega(x, y)$, $\Omega(x, -y) = \Omega(x, y)$, and $\Omega(x, y) = \Omega(x, y+x)$. So without loss of generality we may restrict our attention to the case where $0 < x \leq 2$ and $0 \leq y \leq x/2$. In this case

$$\Omega(x, y) = \sum_{\lceil (y-1)/x \rceil \leq n \leq 0} (y - 1 - nx)^2 + \sum_{0 < n \leq \lfloor (y+1)/x \rfloor} (y + 1 - nx)^2,$$

and we reparametrize the sums to obtain

$$\Omega(x, y) = \sum_{0 \leq n \leq \lfloor (1-y)/x \rfloor} (y - 1 + nx)^2 + \sum_{0 \leq n \leq \lfloor (1+y-x)/x \rfloor} (y - x + 1 - nx)^2,$$

and calculate

$$\frac{\partial}{\partial y} \Omega(x, y) = \sum_{0 \leq n \leq \lfloor (1-y)/x \rfloor} 2(y - 1 + nx) + \sum_{0 \leq n \leq \lfloor (1+y-x)/x \rfloor} 2(y - x + 1 - nx)$$

$$= 2 \left\lfloor \frac{y+1}{x} \right\rfloor (2y - x) + \sum_{\lfloor (1+y-x)/x \rfloor < n \leq \lfloor (1-y)/x \rfloor} 2(y - 1 + nx),$$

because $(1-y)/x$ is greater than or equal to $(1+y-x)/x$, and note for the remainder of this proof that their difference is at most 1. Since $0 \leq y \leq x/2 \leq 1$, we can see that both terms in the last expression for our partial derivative are nonpositive, with the summation over n strictly negative if $y < x - 1$, and the other term is strictly negative if $x - 1 \leq y < x/2$. Thus our partial derivative is strictly negative for $0 \leq y < x/2$, and so for our ranges of x and y values, the unique minimum is obtained when $y = x/2$. $\square$

The last two results are used in Section 6 to show that a large limiting size will make the ratio of L^4 to L^2 norm large.

Lemma A.5. *If $\{f_\iota\}_{\iota \in I}$ is a size-stable family, with limiting size σ, of additive character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$, then*

$$\liminf_{q_\iota \to \infty} \frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \geq \Omega\left(\frac{1}{\sigma}, 0\right).$$

Proof. For each $\iota \in I$, let f_ι have character ψ_ι, support S_ι, translation t_i, and arrangement α_ι, so that $f_\iota(z) = \sum_{s \in S_\iota} \psi_\iota(\alpha_\iota(s + t_i))z^s$. When we confine the values

of z to the complex unit circle, we have

$$\overline{f_\iota(z)} = \sum_{s \in S_\iota} \overline{\psi_\iota(\alpha_\iota(s + t_i))} z^{-s}.$$

We can consider $f_\iota(z)$ and $\overline{f_\iota(z)}$ formally as elements of $\mathbb{C}[z, z^{-1}]$, and view $\|f_\iota\|_4^4$ as the sum of the squared magnitudes of the coefficients of $f_\iota(z)\overline{f_\iota(z)}$. The coefficient of z^s in $f_\iota(z)\overline{f_\iota(z)}$ is

$$\sum_{\substack{u, v \in S_\iota \\ u - v = s}} \psi_\iota(\alpha_\iota(u + t_i)) \overline{\psi_\iota(\alpha_\iota(v + t_i))}.$$

Since α_ι is an epimorphism from $\mathbb{Z}$ to $\mathbb{F}_{q_\iota}^*$, we see that $\psi_\iota(\alpha_\iota(u+t_i)) = \psi_\iota(\alpha_\iota(v+t_i))$ whenever $u \equiv v \pmod{q_\iota - 1}$. Thus if $s \equiv 0 \pmod{q_\iota - 1}$, the coefficient of z^s in $f_\iota(z)\overline{f_\iota(z)}$ is equal to $|S_\iota \cap (s + S_\iota)|$. Since $\|f_\iota\|_4^4$ is the sum of the squared magnitudes of the coefficients of $f_\iota(z)\overline{f_\iota(z)}$ while $\|f_\iota\|_2^2$ is the coefficient of z^0 of the same, we have

$$\frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \geq \frac{\sum_{n \in \mathbb{Z}} |S_\iota \cap (n(q_\iota - 1) + S_\iota)|^2}{|S_\iota|^2},$$

and then we note that $|S_\iota \cap (n(q_\iota - 1) + S_\iota)| = \max(0, |S_\iota| - |n(q_\iota - 1)|)$ and apply the size-stability limit $|S_\iota|/(q_\iota - 1) \to \sigma$ as $q_\iota \to \infty$. $\qquad\square$

Lemma A.6. *If $\{f_\iota\}_{\iota \in I}$ is a size-stable family, with limiting sizes $\sigma_1, \ldots, \sigma_e$, of e-variable multiplicative character polynomials over fields $\{\mathbb{F}_{q_\iota}\}_{\iota \in I}$, then*

$$\liminf_{q_\iota \to \infty} \frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \geq \prod_{j=1}^e \Omega\left(\frac{1}{\sigma_j}, 0\right).$$

Proof. For each $\iota \in I$, let f_ι have character χ_ι, support $S_\iota \subseteq \mathbb{Z}^e$, translation $t_i \in \mathbb{Z}^e$, and arrangement α_ι, so that $f_\iota(z_1, \ldots, z_e) = \sum_{s \in S_\iota} \chi_\iota(\alpha_\iota(s + t_i)) z^s$, where we write z^s for $z_1^{s_1} \cdots z_e^{s_e}$ when $s = (s_1, \ldots, s_e)$. Our proof runs the same as that of the previous lemma for additive character polynomials once we replace ψ_ι with χ_ι, but we must take care of the fact that $\chi_\iota(\alpha_\iota(s + t_i)) = 0$ when $s \in -t_i + \ker \alpha_\iota$; otherwise, the coefficients are of unit magnitude. And of course the polynomials are in e variables and the coefficients have periodicity p in each direction. Thus if we define $V_\iota = S_\iota \smallsetminus (-t_i + \ker \alpha_\iota)$ we have

$$\frac{\|f_\iota\|_4^4}{\|f_\iota\|_2^4} \geq \frac{\sum_{n \in \mathbb{Z}^e} |V_\iota \cap (n p_\iota + V_\iota)|^2}{|V_\iota|^2},$$

but Lemma A.2 can be used to show that the ratio $|V_\iota \cap (n p_\iota + V_\iota)|/|V_\iota|$ has the same limit as $|S_\iota \cap (n p_\iota + S_\iota)|/|S_\iota|$ as $q_\iota \to \infty$. $\qquad\square$

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