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SOME CHARACTERIZATIONS OF CAMPANATO SPACES VIA COMMUTATORS ON MORREY SPACES

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We give some creative characterizations of Campanato spaces via the boundedness of commutators associated with the Calderón–Zygmund singular integral operator by some new methods instead of the sharp maximal function theorem.

1. Introduction and main results

Let $-n/p \leq \beta < 1$ and $1 \leq p < \infty$. A locally integrable function f is said to belong to the Campanato spaces $C^{p,\beta}(\mathbb{R}^n)$ if

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)} = \sup_Q \|f\|_{C^{p,\beta}(Q)} := \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f - f_Q|^p \, dx \right)^{1/p} < \infty,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$, Q denotes any cube contained in $\mathbb{R}^n$ and $|Q|$ is the Lebesgue measure of Q.

Campanato spaces are useful tools in the regularity theory of PDEs as a result of their better structures, which allow us to give an integral characterization of the spaces of Hölder continuous functions. This leads to a generalization of the classical Sobolev embedding theorem (see, e.g., [Lemarié-Rieusset 2007; Lu 1995; 1998]). It is also well known that $C^{1,1/p-1}$ is the dual space of Hardy space $H^p(\mathbb{R}^n)$ when $0 < p < 1$ (see [Triebel 1992]). For a recent account of the theory on $C^{p,\beta}(\mathbb{R}^n)$, we refer the reader to [Duong et al. 2007; Lin et al. 2011; Nakai 2006; Yang et al. 2010].

It's obvious that $\beta = 0$ implies $C^{p,0}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$ with the norm

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| \, dx. \tag{1-1}$$

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When $0<\beta<1$ and $1\le p<\infty$, we have $C^{p,\beta}(\mathbb{R}^n)=\mathrm{Lip}_\beta(\mathbb{R}^n)$ (see [DeVore and Sharpley 1984; Janson et al. 1983]) with the equivalent norm

$$\|f\|_{\mathrm{Lip}_\beta(\mathbb{R}^n)} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f-f_Q| \approx \sup_Q \left(\frac{1}{|Q|^{1+q\beta/n}}\int_Q |f-f_Q|^q\right)^{1/q}, \tag{1-2}$$

where $1\le q\le\infty$ and $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is the Lipschitz functional space.

When $-n/p\le\beta<0$, there are several stages in the study of $C^{p,\beta}(\mathbb{R}^n)$. Let Ω be a connected open set of $\mathbb{R}^n$. Denote by $\overline{\Omega}$ the closure of Ω, and by $\operatorname{diam}\Omega$ the diameter of Ω. For any $x_0\in\mathbb{R}^n$ and $l\in(0,\infty)$, set $B(x_0,l)=\{x\in\mathbb{R}^n:|x-x_0|<l\}$ and $\Omega(x_0,l)=B(x_0,l)\cap\Omega$. The following space was first introduced by Morrey [1938] to investigate the local behavior of solutions to the second order elliptic PDE

$$\|f\|_{M^{p,\beta}(\Omega)} = \sup_{\substack{x_0\in\overline{\Omega}\\ l\in(0,\operatorname{diam}\Omega)}} \frac{1}{|\Omega(x_0,l)|^{\beta/n}}\left(\frac{1}{|\Omega(x_0,l)|}\int_{\Omega(x_0,l)}|f|^p\right)^{1/p},$$

where $f\in L^p_{\mathrm{loc}}(\Omega)$, $1\le p<\infty$ and $-n/p\le\beta<0$. When $\Omega=\mathbb{R}^n$, $M^{p,\beta}(\mathbb{R}^n)$ is the classical Morrey space, whose norm is defined by

$$\|f\|_{M^{p,\beta}(\mathbb{R}^n)} = \sup_Q \|f\|_{M^{p,\beta}(Q)} = \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}.$$

$M^{p,\beta}(\mathbb{R}^n)$ is an expansion of $L^p(\mathbb{R}^n)$ in the sense that $M^{p,-n/p}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$. Similarly, for $1\le p<\infty$, $-n/p\le\beta<0$, a function $f\in L^p_{\mathrm{loc}}(\Omega)$ is said to belong to the Campanato space $C^{p,\beta}(\Omega)$ if

$$\|f\|_{C^{p,\beta}(\Omega)} = \sup_{\substack{x_0\in\overline{\Omega}\\ l\in(0,\operatorname{diam}\Omega)}} \frac{1}{|\Omega(x_0,l)|^{\beta/n}}\left(\frac{1}{|\Omega(x_0,l)|}\int_{\Omega(x_0,l)}|f-f_{\Omega(x_0,l)}|^p\right)^{1/p}<\infty.$$

Campanato [1963] proved that, if $\operatorname{diam}\Omega<\infty$ and there exists a positive constant C such that

$$|\Omega(x_0,l)|\ge Cl^n, \tag{1-3}$$

for every $x_0\in\overline{\Omega}$ and $l\in(0,\operatorname{diam}\Omega)$, then

$$M^{p,\beta}(\Omega)=C^{p,\beta}(\Omega). \tag{1-4}$$

(For more accounts about (1-4), see [Rupflin 2008], for example.) Throughout this paper, the letter C stands for a positive constant which may vary from line to line. When $\operatorname{diam}\Omega=\infty$ (i.e., Ω is unbounded, as when $\Omega=\mathbb{R}^n$, for example), Sakamoto and Yabuta [1999] pointed out that when $1\le p<\infty$ and $\beta\in[-n/p,0)$, $C^{p,\beta}(\mathbb{R}^n)$

is equivalent to $M^{p,\beta}(\mathbb{R}^n)$. But Lin [2009] gave a counterexample to verify that when $1 \le p < \infty$ and $\beta \in [-n/p, 0)$, we have

$$M^{p,\beta}(\mathbb{R}^n) \subsetneq C^{p,\beta}(\mathbb{R}^n), \tag{1-5}$$

which implies that the statement in [Sakamoto and Yabuta 1999] may be inaccurate. More precisely, on account of the remark above, we have

$$C^{p,\beta}(\mathbb{R}^n) \begin{cases} = BMO(\mathbb{R}^n) & \text{for } \beta = 0, \\ = \mathrm{Lip}_\beta(\mathbb{R}^n) & \text{for } 0 < \beta < 1, \\ \supset M^{p,\beta}(\mathbb{R}^n) & \text{for } -n/p < \beta < 0. \end{cases}$$

Let T be a linear operator and b a suitable function. For a proper function f, the commutator T_b is defined by

$$T_b(f) := bTf - T(bf).$$

In this paper, we give some characterizations of $C^{p,\beta}(\mathbb{R}^n)$ in terms of the boundedness of T_b, where T is the Calderón–Zygmund singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy;$$

here $K \in C^\infty(S^{n-1})$ is a Calderón–Zygmund kernel that satisfies

$$K(x) = K(x/|x|)/|x|^n \quad \text{for } |x| \ne 0 \tag{1-6}$$

and

$$\int_{S^{n-1}} K = 0. \tag{1-7}$$

For more on the theory of the Calderón–Zygmund singular integral operator T, see [Grafakos 2004; Janson 1978; Lu 2011; Lu et al. 2007; Stein 1970], for example.

There are many classical works about the characterizations of Campanato spaces by the boundedness of T_b on Lebesgue spaces. Coifman, Rochberg and Weiss [Coifman et al. 1976] gave a characterization of $BMO(\mathbb{R}^n)$ in terms of the commutator T_b:

$$b \in BMO(\mathbb{R}^n) \iff T_b : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \text{ if } 1 < p < \infty.$$

Janson [1978] gave a characterization of $\mathrm{Lip}_\beta(\mathbb{R}^n)$ by the (L^p, L^q)-boundedness of the commutator T_b: If $0 < \beta < 1$, then

$$b \in \mathrm{Lip}_\beta(\mathbb{R}^n) \iff T_b : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \text{ if } 1 < p < q < \infty \text{ and } 1/q = 1/p - \beta/n.$$

Paluszyński [1995] gave a new characterization of $\mathrm{Lip}_\beta(\mathbb{R}^n)$ by the $(L^p, \dot{F}^{\beta}_{p,\infty})$-boundedness of the commutator T_b: If $0 < \beta < 1$, then

$$b \in \mathrm{Lip}_\beta(\mathbb{R}^n) \iff T_b : L^p(\mathbb{R}^n) \to \dot{F}^\beta_{p,\infty}(\mathbb{R}^n) \text{ if } 1 < p < \infty,$$

where $\dot{F}^\beta_{p,\infty}(\mathbb{R}^n)$ is the homogeneous Triebel–Lizorkin space with the equivalent norm

$$\|f\|_{\dot{F}^\beta_{p,\infty}(\mathbb{R}^n)} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b - b_Q| \right\|_{L^p}.$$

As a natural extension of Lebesgue space, it is interesting to know whether Campanato spaces can be characterized by the boundedness of T_b on Morrey spaces.

Ding [1997] characterized $BMO(\mathbb{R}^n)$ by the $(M^{p,\beta}(\mathbb{R}^n), M^{p,\beta}(\mathbb{R}^n))$-boundedness of T_b:

$$b \in BMO(\mathbb{R}^n) \iff T_b : M^{p,\beta}(\mathbb{R}^n) \to M^{p,\beta}(\mathbb{R}^n) \text{ if } 1 < p < \infty,\ -n/p \le \beta < 0.$$

In the rest of this paper, we shall establish the characterizations of other cases of Campanato spaces — namely, $\mathrm{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta < 1$ and $M^{p,\beta}(\mathbb{R}^n)$ for $-n/p \le \beta < 0$ — using certain boundedness properties of T_b on Morrey spaces.

Now, we formulate our first result as follows:

Theorem 1.1. *Let* $1 < p < \infty$, $0 < \alpha < 1$, $-n/p \le \beta < 0$, $1 + p\beta/n < p/q$, $1/q = 1/p - \alpha/n$ *and* $\tilde{\beta} = (q-p)/p + q\beta/n$. *The following statements are equivalent*:

(1) $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

(2) T_b *is a bounded operator from* $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\tilde{\beta}}(\mathbb{R}^n)$.

We say that a nonnegative function f belongs to the reverse Hölder class RH^r if for any $Q \subset \mathbb{R}^n$ and $1 < r < \infty$ we have

$$\left(\frac{1}{|Q|} \int_Q |f|^r \, dx \right)^{1/r} \le \frac{C}{|Q|} \int_Q |f| \, dx.$$

When $r = \infty$, we say that $f \in RH^\infty$ if $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists a constant C such that

$$\|f\|_{L^\infty(Q)} := \sup_{Q \ni x} |f(x)| \le \frac{C}{|Q|} \int_Q |f| \, dx. \tag{1-8}$$

For $1 < r < \infty$, it is easy to see that $RH^\infty = \bigcup_{r>1} RH^r$. Reverse Hölder classes contain many kinds of functions. For example, if $P(x)$ is a polynomial and $\gamma > 0$, then $f(x) = |P(x)|^\gamma \in RH^\infty$(see [Fefferman 1983]). (For more theories about RH^r, see [Cruz-Uribe and Neugebauer 1995; Harboure et al. 1998], for example.)

Theorem 1.2. *Assume* $\max\{1, n/(1-\beta)\} < p < \infty$, $-n/p \le \beta < 0$, $1 < p_i < \infty$ $(i = 1, 2)$, $p_1 \in \mathbb{N}$ *even*, $-n/p_i \le \beta_i < 0$, $1/p = 1/p_1 + 1/p_2$ *and* $\beta = \beta_1 + \beta_2$. *If* Ω *satisfies* (1-3) *and* $\operatorname{diam} \Omega < \infty$, *the following statements are equivalent*:

(1) $b \in M^{p_1,\beta_1}(\Omega)$.

(2) *If* $b \in RH^\infty$, T_b *is a bounded operator from* $M^{p_2,\beta_2}(\Omega)$ *to* $M^{p,\beta}(\Omega)$.

The advantage of using the assumption $\operatorname{diam}\Omega < \infty$ lies in the fact that the equivalent norm of (1-4) is used in the proof of Theorem 1.2. If $\Omega = \mathbb{R}^n$, we can obtain the following characterizations of Campanato spaces:

Theorem 1.3. *Assume* $\max\{1, n/(1-\beta)\} < p < \infty$, $-n/p \le \beta < 0$, $1 < p_i < \infty$ $(i = 1, 2)$, $p_1 \in \mathbb{N}$ *even*, $-n/p_i \le \beta_i < 0$, $1/p = 1/p_1 + 1/p_2$ *and* $\beta = \beta_1 + \beta_2$. *The following statements are equivalent*:

(1) $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$.

(2) T_b *is a bounded operator from* $M^{p_2,\beta_2}(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$ *if* b *further satisfies that there exists a constant* $C > 0$ *such that for any* $Q \subset \mathbb{R}^n$,

$$\sup_Q |b - b_Q| \le \frac{C}{|Q|} \int_Q |b - b_Q|. \tag{1-9}$$

Remark 1. Inequalities (1-8) and (1-9) can be thought of as a form of mean value equality. Besides polynomial functions, mean value equalities also characterize harmonic functions (see [Gilbarg and Trudinger 1983]).

Remark 2. Solutions to a large class of elliptic second order PDEs satisfy the mean value inequality. Therefore, Theorem 1.2 and Theorem 1.3 can give characterizations of the space of solutions to some second order elliptic PDEs. Take Laplace's equation, for example. If b is a solution to the equation

$$\Delta u = 0, \tag{1-10}$$

where Δ is the Laplace operator and u is a function defined on the bounded domain $\Omega \subset \mathbb{R}^n$, then b satisfies (1-9); see [Gilbarg and Trudinger 1983, Theorem 2.1]. Therefore, if the commutator T_b associated to b is bounded from $M^{p_2,\beta_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$, then the space of solutions to (1-10) is the Campanato space $C^{p_1,\beta_1}(\mathbb{R}^n)$.

Remark 3. We emphasize that the methods in dealing with $C^{p,\beta}$ when $\beta < 0$ are quite different from that of $\beta \ge 0$, and there are essential difficulties in establishing the characterizations of Campanato spaces on Morrey spaces when $\beta < 0$. Therefore, we set up Theorem 1.3 under the condition that the symbol of the commutator satisfies the mean value inequality. Condition (1-9) in Theorem 1.3 was intrinsic to the proof of the converse characterizations of $C^{p_1,\beta_1}(\mathbb{R}^n)$. Of course, there are essential differences between the ideas in the proof of Theorem 1.2 and Theorem 1.3 and that of [Janson 1978] and [Paluszyński 1995], where the sharp maximal function theorem were used.

Our theorems provide natural and intrinsic characterizations of Campanato spaces on Morrey spaces. It is also worth pointing out that our paper is the first work on the problem of commutators whose symbol belongs to Morrey spaces. Our viewpoints will shed some new lights on characterizations of Campanato spaces via commutators formed by other operators on Morrey spaces, such as fractional integrals, oscillatory integral operators and Hardy–Littlewood–Paley operators. Besides Euclidean space, characterizations of Campanato spaces on other spaces can similarly be considered, such as on homogeneous groups. Partly inspired by [Janson 1978] and [Paluszyński 1995], we prove Theorems 1.1–1.3 in Section 2.

2. Proof of the main results

For the proofs we need some lemmas about the estimates of operators on Morrey spaces.

Lemma 2.1 [Chiarenza and Frasca 1987]. *Let* $1 < p < n/\alpha$, $0 < \alpha < n$, $1/q = 1/p - \alpha/n$, $0 < 1 + p\beta/n < p/q$, $-n/p \le \beta < 0$ *and* $\tilde{\beta} = (q-p)/p + q\beta/n$. *Then the fractional integral operator*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

is bounded from $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\tilde{\beta}}(\mathbb{R}^n)$.

Lemma 2.2 [Komori and Shirai 2009]. *Let* $1 < p < \infty$ *and* $-n/p \le \beta < 0$. *Then* T *is bounded from* $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{p,\beta}(\mathbb{R}^n)$.

Proof of Theorem 1.1. (1) ⇒ (2). Together, (1-2) and (1-6) imply

$$\begin{aligned} |T_b f(x)| &\le \int_{\mathbb{R}^n} |b(x)-b(y)|\,|K(x-y)|\,|f(y)|\, dy \\ &\le \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le I_\alpha(|f(x)|). \end{aligned}$$

Therefore, T_b is bounded from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{q,\tilde{\beta}}(\mathbb{R}^n)$ by Lemma 2.1.

(2) ⇒ (1). The proof consists of the construction of a proper commutator. We follow [Janson 1978] in choosing $z_0 \ne 0$ and $\delta > 0$ such that $1/K(z)$ can be expressed in the neighborhood $|z - z_0| < \sqrt{n}\delta$ as the absolute convergent Fourier series

$$\frac{1}{K(z)} = \sum a_n e^{i v_n \cdot z},$$

where the exact form of the vectors v_n is irrelevant. Set $z_1 = \delta^{-1} z_0$. If $|z - z_1| < \sqrt{n}$, it follows from (1-6) that

$$\frac{1}{K(z)} = \frac{\delta^{-n}}{K(\delta z)} = \delta^{-n} \sum a_n e^{i v_n \cdot \delta z}. \tag{2-1}$$

Choose now any cube $Q = Q(x_0, r)$. Set $y_0 = x_0 - r z_1$ and $Q' = Q(y_0, r)$. Thus, if $x \in Q$ and $y \in Q'$,

$$\left|\frac{x-y}{r} - z_1\right| \le \left|\frac{x-x_0}{r} - \frac{y-y_0}{r}\right| \le \sqrt{n}.$$

Denoting $s(x) = \operatorname{sgn}(b(x) - b_{Q'})$, by (2-1) we have

$$\begin{aligned}\int_Q |b(x) - b_{Q'}|\,dx &= \int_Q (b(x) - b_{Q'})s(x)\,dx\\ &= \frac{1}{|Q'|}\int_Q\int_{Q'}(b(x)-b(y))s(x)\,dy\,dx\\ &= \frac{1}{r^n}\iint_{\mathbb{R}^n}(b(x)-b(y))s(x)\frac{r^n K(x-y)}{K((x-y)/r)}\chi_Q(x)\chi_{Q'}(y)\,dy\,dx\\ &= C\sum a_n \iint_{\mathbb{R}^n}(b(x)-b(y))K(x-y)e^{i(\delta/r)v_n\cdot x}\\ &\qquad\qquad \times s(x)\chi_Q(x)e^{-i(\delta/r)v_n\cdot y}\chi_{Q'}(y)\,dy\,dx.\end{aligned}$$

Taking

$$g_n(y) = e^{-i(\delta/r)v_n\cdot y}\chi_{Q'}(y) \quad \text{and} \quad h_n(x) = e^{i(\delta/r)v_n\cdot x}s(x)\chi_Q(x),$$

we obtain

$$\begin{aligned}\int_Q |b(x) - b_{Q'}|\,dx &= C\sum a_n \iint_{\mathbb{R}^n}(b(x)-b(y))K(x-y)g_n(y)h_n(x)\,dy\,dx\\ &= C\sum a_n \int_{\mathbb{R}^n} T_b g_n(x)h_n(x)\,dx\\ &\le C\sum |a_n| \int_{\mathbb{R}^n} |T_b g_n(x)|\,|h_n(x)|\,dx\\ &\le C\sum |a_n| \int_{\mathbb{R}^n} |T_b g_n(x)|\,dx.\end{aligned}$$

Applying the Hölder inequality to $\int_{\mathbb{R}^n} |T_b g_n(x)|\,dx$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |T_b g_n(x)|\,dx &\le |Q|^{1+(\alpha+\beta)/n}\|T_b g_n\|_{M^{q,\tilde\beta}(\mathbb{R}^n)}\\ &\le C|Q|^{1+(\alpha+\beta)/n}\|g_n\|_{M^{p,\beta}(\mathbb{R}^n)}\\ &\le C|Q|^{1+\alpha/n},\end{aligned}$$

since $\|g_n\|_{M^{p,\beta}(\mathbb{R}^n)} = |Q|^{-\beta/n}$. Thus we have obtained

$$\frac{1}{|Q|^{1+\alpha/n}}\int_Q |b(x) - b_{Q'}|\,dx \le C,$$

which completes the proof of Theorem 1.1 by (1-2). □

Theorem 1.2 is a restatement of Theorem 1.3 when $C^{p,\beta}$ spaces over domains with finite volume, so we give the proof of Theorem 1.3 first. Again, we begin with some lemmas that are essential to our analysis.

Lemma 2.3. *Let p, p_1, p_2, β, β_1, β_2 and b be the same as in Theorem 1.3. Then*

$$\|(b-b_Q)f\chi_Q\|_{L^p(\mathbb{R}^n)} \le |Q|^{1/p+\beta/n}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.$$

This follows from Hölder's inequality:

$$\begin{aligned}\|(b-b_Q)f\chi_Q\|_{L^p(\mathbb{R}^n)} &\le \left(\int_Q |b-b_Q|^{p_1}\right)^{1/p_1}\left(\int_Q |f|^{p_2}\right)^{1/p_2}\\ &\le |Q|^{1/p(1+p\beta/n)}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.\end{aligned}$$

Lemma 2.4. *Suppose that $Q_* \subset Q$ and $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$ with $1 < p_1 < \infty$ and $-n/p_1 \le \beta_1 < 0$. Then the following estimate holds*:

$$|b_{Q_*}-b_Q| \le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}|Q_*|^{\beta_1/n}. \tag{2-2}$$

We divide the proof into two cases.

Case 1: Suppose $Q_* \subset Q \subseteq 2Q_*$. Hölder's inequality yields

$$\begin{aligned}|b_{Q_*}-b_Q| &\le \frac{1}{|Q_*|}\int_{Q_*}|b-b_Q| + \frac{1}{|Q|}\int_Q |b-b_Q|\\ &\le C\frac{1}{|Q|}\int_Q |b-b_Q| \le C\left(\int_Q |b-b_Q|^{p_1}\right)^{1/p_1}|Q|^{-1/p_1}\\ &\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}|Q_*|^{\beta_1/n}.\end{aligned}$$

Case 2: Suppose $2Q_* \subset Q$. Choose a sequence of nested cubes

$$Q_* =: Q_1 \subset Q_2 \subset \cdots Q =: Q_{m+1},$$

with $|Q_{i+1}| = 2^n|Q_i|$ for $1 \le i \le m$. By the results of Case 1, we have

$$\begin{aligned}|b_{Q_*}-b_Q| &= |b_{Q_1}-b_{Q_2}+b_{Q_2}-\cdots+b_{Q_m}-b_{Q_{m+1}}|\\ &\le \sum_{i=1}^m |b_{Q_i}-b_{Q_{i+1}}| \le \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\sum_{i=1}^m 2^{(i+1)\beta_1}|Q_*|^{\beta_1/n}\\ &\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}|Q_*|^{\beta_1/n},\end{aligned}$$

which is (2-2).

The following imbedding theorem for L^p spaces over domains with finite volume is very useful in the analysis of inequality, which you can find in any book about Sobolev spaces (see [Adams and Fournier 2003], for example).

Lemma 2.5. *Suppose that* $|\Omega| = \int_\Omega 1\,dx < \infty$ *and* $1 \le p \le q \le \infty$. *If* $f \in L^q(\Omega)$, *then* $f \in L^p(\Omega)$ *and*

$$\|f\|_{L^p(\Omega)} \le C|\Omega|^{1/p-1/q}\|f\|_{L^q(\Omega)}.$$

Proof of Theorem 1.3. (1) $\Rightarrow$ (2). For a cube $Q = Q(x_Q, r) \subset \mathbb{R}^n$ and $y \in Q$, take $f \in M^{p_2,\beta_2}(\mathbb{R}^n)$ and set $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. After noticing that

$$T_b f = T_{(b-b_Q)} f,$$

we have

$$\begin{aligned}
&\left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q |T_b f - (T_b f)_Q|^p\right)^{1/p} \\
&\qquad = \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q |T_{(b-b_Q)} f - (T_{(b-b_Q)} f)_Q|^p\right)^{1/p} \\
&\qquad \le \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q |T_{(b-b_Q)} f - (b-b_Q) f_2(x_Q)|^p\right)^{1/p} \\
&\qquad \le I + II + III,
\end{aligned}$$

where

$$\begin{aligned}
I &:= \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q |(b-b_Q)Tf|^p\right)^{1/p}, \\
II &:= \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q |T(b-b_Q)f_1|^p\right)^{1/p}, \\
III &:= \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q \left|(T(b-b_Q)f_2)(y) - (T(b-b_Q)f_2)(x_Q)\right|^p\right)^{1/p}.
\end{aligned}$$

Hölder's inequality and Lemma 2.2 imply

$$\begin{aligned}
I &= \frac{1}{|Q|^{1/p+\beta/n}}\left(\int_Q |(b-b_Q)Tf|^p\right)^{1/p} \\
&\le \frac{1}{|Q|^{1/p+\beta/n}}\left(\int_Q |b-b_Q|^{p_1}\right)^{1/p_1}\left(\int_Q |Tf|^{p_2}\right)^{1/p_2} \\
&\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|Tf\|_{M^{p_2,\beta_2}(\mathbb{R}^n)} \\
&\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.
\end{aligned}$$

From Lemma 2.3, it follows that

$$II \le \frac{1}{|Q|^{1/p+\beta/n}}\|(b-b_Q)f_1\|_{L^p} \le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.$$

We now turn to the estimate for the term III. From (1-6), it may be concluded that

$$\begin{aligned}
&\left|(T(b-b_Q)f_2)(y)-T((b-b_Q)f_2)(x_Q)\right| \\
&\qquad = \left|\int_{R^n}(K(y-z)-K(x_Q-z))(b(z)-b_Q)f_2(z)\,dz\right| \\
&\qquad \le C\int_{(2Q)^c}\frac{|y-x_Q|}{|z-x_Q|^{n+1}}|b(z)-b_Q||f(z)|\,dz \\
&\qquad \le C\sum_{k=2}^{\infty}\int_{2^kQ\setminus 2^{k-1}Q}\frac{1}{2^k|2^kQ|}(|b(z)-b_{2^kQ}|+|b_Q-b_{2^kQ}|)|f(z)|\,dz \\
&\qquad \le C\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|}\int_{2^kQ}(|b(z)-b_{2^kQ}|+|b_Q-b_{2^kQ}|)|f(z)|\,dz,
\end{aligned}$$

which yields

$$\begin{aligned}
III &\le \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q\left|\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|}\int_{2^kQ}|b(z)-b_{2^kQ}||f(z)|\,dz\right|^p\right)^{1/p} \\
&\quad + \left(\frac{1}{|Q|^{1+p\beta/n}}\int_Q\left|\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|}\int_{2^kQ}|b_Q-b_{2^kQ}||f(z)|\,dz\right|^p\right)^{1/p} \\
&=: III_1+III_2.
\end{aligned}$$

With repeated application of Lemma 2.3 and the L^p-boundedness of the Hardy–Littlewood maximal operator M, we can obtain

$$\begin{aligned}
III_1 &\le \frac{1}{|Q|^{1/p+\beta/n}}\left\|\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|}\int_{2^kQ}|b(z)-b_{2^kQ}||f(z)|\,dz\right\|_{L^p} \\
&\le \frac{1}{|Q|^{1/p+\beta/n}}\sum_{k=2}^{\infty}\frac{1}{2^k}\left\|\frac{1}{|2^kQ|}\int_{2^kQ}|b(z)-b_{2^kQ}||f(z)|\,dz\right\|_{L^p} \\
&\le \sum_{k=2}^{\infty}\frac{1}{2^k}\frac{1}{|Q|^{1/p+\beta/n}}\left\|M(|b-b_{2^kQ}||f|)\right\|_{L^p} \\
&\le \sum_{k=2}^{\infty}\frac{1}{2^k}\frac{1}{|Q|^{1/p+\beta/n}}\left\||b-b_{2^kQ}||f|\right\|_{L^p} \\
&\le \sum_{k=2}^{\infty}\frac{1}{2^{k(1-n/p-\beta)}}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)} \\
&\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}.
\end{aligned}$$

Applying Lemma 2.4 and Lemma 2.5 to III_2, we have

$$\begin{aligned} III_2 &\le \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\frac{1}{|Q|^{1+p\beta/n}} \int_Q \left| \frac{1}{|2^k Q|} \int_{2^k Q} |b_Q - b_{2^k Q}| |f(z)| \, dz \right|^p dx \right)^{1/p} \\ &\le \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \left(\frac{|Q|^{\beta_1/n}}{|Q|^{1+p\beta/n}} \int_Q \left| \frac{1}{|2^k Q|} \int_{2^k Q} |f(z)| \, dz \right|^p dx \right)^{1/p} \\ &\le \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \frac{1}{|Q|^{1/p+\beta_2/n}} \|M(|f|)\|_{L^p} \\ &\le \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \frac{1}{|Q|^{1/p+\beta_2/n}} \|f\|_{L^p} \\ &\le \sum_{k=2}^{\infty} \frac{1}{2^{k(1-n/p-\beta_2)}} \|b\|_{C^{p_1,\beta_1}} \|f\|_{M^{p_2,\beta_2}} \\ &\le C \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we have obtained $III \le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \|f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}$. Our proof ends with the definition of $\|\cdot\|_{C^{p,\beta}(\mathbb{R}^n)}$.

(2) ⇒ (1). We first claim that for fixed $Q \subset \mathbb{R}^n$, $b \in C^{p_1,\beta_1}(Q)$ and $f \in M^{p_2,\beta_2}(Q)$ with $\|f\|_{M^{p_2,\beta_2}(Q)} = |Q|^{-\beta_2/n}$, we have

$$\|T_b^m f\|_{C^{p,\beta}(Q)} \le C|Q|^{\beta_1(m-1)/n} \|f\|_{M^{p_2,\beta_2}(Q)} \le C|Q|^{(\beta_1 m-\beta)/n}, \tag{2-3}$$

where T_b^m is the m-th ($m \in Z^+$) commutator defined by

$$T_b^m f(x) = \text{p.v.} \int (b(x) - b(y))^m K(x-y) f(y) \, dy.$$

We shall prove (2-3) by induction. The case $m = 1$ is trivial. We now assume that for any $b \in C^{p_1,\beta_1}(Q)$, we have

$$\|T_b^{m-1} f\|_{C^{p,\beta}(Q)} \le C|Q|^{\beta_1(m-2)/n} \|f\|_{M^{p_2,\beta_2}(Q)} \le C|Q|^{(\beta_1(m-1)-\beta)/n}. \tag{2-4}$$

Next, we show the case m. We now observe that

$$\begin{aligned} \left|T_b^m f(x)\right| &= \left| \int (b(x)-b(y))^{m-1} K(x-y) f(y) (b(x)-b(y)) \, dy \right| \\ &\le \left| \int (b(x)-b(y))^{m-1} K(x-y) f(y) (b(x)-b_Q) \, dy \right| \\ &\quad + \left| \int (b(x)-b(y))^{m-1} K(x-y) f(y) (b(y)-b_Q) \, dy \right| \\ &\le (|b-b_Q| |T_b^{m-1} f|)(x) + |T_b^{m-1}((b-b_Q) f)(x)| =: J_1 + J_2. \end{aligned}$$

Equation (2-4) enables us to estimate J_1 as

$$\begin{aligned}\|J_1\|_{C^{p,\beta}(Q)} &\le \||b-b_Q||T_b^{m-1}f|\|_{C^{p,\beta}(Q)}\\ &\le \|b-b_Q\|_{L^\infty}\|T_b^{m-1}f\|_{C^{p,\beta}(Q)}\\ &\le \frac{1}{|Q|}\int_Q |b-b_Q|\,dx\,|Q|^{\beta_1(m-2)/n}\|f\|_{M^{p_2,\beta_2}(Q)}\\ &\le C|Q|^{\beta_1(m-1)/n}\|f\|_{M^{p_2,\beta_2}(Q)} \le C|Q|^{(\beta_1 m-\beta)/n}.\end{aligned}$$

With repeated application of (2-4), we have

$$\begin{aligned}\|J_2\|_{C^{p,\beta}(Q)} &= \|T_b^{m-1}((b-b_Q)f)\|_{C^{p,\beta}(Q)}\\ &\le C|Q|^{\beta_1(m-2)/n}\|(b-b_Q)f\|_{M^{p_2,\beta_2}(Q)}\\ &\le C|Q|^{\beta_1(m-2)/n}\|b-b_Q\|_{L^\infty}\|f\|_{M^{p_2,\beta_2}(Q)}\\ &\le C|Q|^{\beta_1(m-1)/n}\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_2,\beta_2}(Q)}.\\ &\le C|Q|^{(\beta_1 m-\beta)/n}.\end{aligned}$$

We come back to the proof of (2) $\Rightarrow$ (1). The rest of the proof proceeds similarly to that of Theorem 1.1. We apply the same argument again with $s(x)$ replaced by $\operatorname{sgn}(b(x)-b_Q)^{p_1}$ to obtain

$$\int_Q |b-b_Q|^{p_1}\,dx \le C\sum a_n\int_Q |T_b^{p_1}(g_n)|\,dx. \tag{2-5}$$

Combining (2-3) and observing $g_n \in M^{p_2,\beta_2}(Q)$ with the norm $\|g_n\|_{M^{p_2,\beta_2}(Q)} = |Q|^{-\beta_2/n}$, we estimate (2-5) as

$$\begin{aligned}\sum a_n\int_Q |T_b^{p_1}(g_n)|\,dx &\le C|Q|^{1+\beta/n}\|T_b^{p_1}(g_n)\|_{C^{p,\beta}(Q)}\\ &\le C|Q|^{1+p_1\beta_1/n+\beta_2/n}\|g_n\|_{M^{p_2,\beta_2}(Q)}\\ &\le C|Q|^{1+p_1\beta_1/n}.\end{aligned}$$

and then take the supremum of Q, completing the proof of Theorem 1.3. □

Proof of Theorem 1.2. This proof can be handled in much the same way as that of Theorem 1.3, using (1-5) and replacing $b-b_Q$ by b in the proof of (2-3). □

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