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## THIN $r$-NEIGHBORHOODS OF EMBEDDED GEODESICS WITH FINITE LENGTH AND NEGATIVE JACOBI OPERATOR ARE STRONGLY CONVEX


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# THIN $r$-NEIGHBORHOODS OF EMBEDDED GEODESICS WITH FINITE LENGTH AND NEGATIVE JACOBI OPERATOR ARE STRONGLY CONVEX

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**In a complete Riemannian manifold, an embedded geodesic $\gamma$ with finite length and negative Jacobi operator admits an $r$-neighborhood $N_r(\gamma)$ with radius $r > 0$ small enough such that each pair of points of $N_r(\gamma)$ can be joined by a unique geodesic contained in $N_r(\gamma)$ where it minimizes length among the piecewise $C^1$ paths joining its endpoints.**

## Introduction

Let $M$ be a connected complete Riemannian manifold; let $d$ denote its Riemannian distance function [do Carmo 1992]. A connected subset $S \subset M$ with nonempty interior $S^\circ$ is called strongly convex for a pair of points $(p, q) \in S \times S$ if there exists a unique geodesic path $t \in [0, 1] \to \gamma(t) \in M$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma(t) \in S^\circ$ for $t \in (0, 1)$, with $\gamma$ length-minimizing among piecewise $C^1$ paths from $p$ to $q$ in $\bar{S}$. The subset $S$ is just called *strongly convex* if it is so for each pair $(p, q) \in S \times S$.

**Definition 0.1.** Let $S \subset M$ be a strongly convex subset. For each pair $(p, q) \in S \times S$, the length of the geodesic path joining $p$ to $q$ with interior in $S^\circ$ is called the *inner distance* from $p$ to $q$ in $S$, denoted by $d_S(p, q)$.

It is quite natural to endow a strongly convex subset $S \subset M$ with its inner distance function $d_S$. The latter is nothing but the length metric associated with the metric space $(S, d|_S)$ [Gromov 1981].

Since Whitehead's landmark paper [1932], it has been known that small enough balls in $M$ are strongly convex. Moreover, if $B$ is such a ball, its inner distance function $d_B$ coincides with the restriction of $d$ to $B \times B$ [Kobayashi and Nomizu 1996; Cheeger and Ebin 2008; Aubin 1998; do Carmo 1992; Klingenberg 1995]. In the flat torus $\mathbb{R}^n/\mathbb{Z}^n$, if the radius of a ball $B$ belongs to the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$, the reader can check that $B$ remains strongly convex but $d_B$ no longer coincides

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with $d|_{B\times B}$. Here, we would like to construct a general family of examples of strongly convex subsets $S \subset M$ such that $d_S \not\equiv d|_{S\times S}$.

The notion of *extended* distance function used in [Figalli et al. 2012] is similar in spirit to that of inner metric; could it guide us toward an example? Let us recall its definition. If $t \in [0, 1] \to \gamma(t) \in M$ is an embedded geodesic without conjugate points, the map $\mathrm{Id} \times \exp : TM \to M \times M$ induces a diffeomorphism $\Psi_\gamma$ from a neighborhood $\mathcal{U}$ of $\big(\gamma(0), (d\gamma/dt)(0)\big)$ in $TM$ to a neighborhood $\mathcal{W}$ of $(\gamma(0), \gamma(1))$ in $M \times M$. The extended distance function $d_\gamma$ of [Figalli et al. 2012] is then defined in $\mathcal{W}$ by $d_\gamma(p, q) = |V|_p$ where $\Psi_\gamma(p, V) = (p, q)$. It is called so because, if $\gamma$ contains no cut point, shrinking $\mathcal{W}$ if necessary, it satisfies $d_\gamma(p, q) \equiv d(p, q)$. In this setting, we would like to know whether a thin enough tube about the geodesic $\gamma$ must be strongly convex. Anytime it is, one may identify $d_\gamma$ with the restriction to $\mathcal{W}$ of the inner distance function of the tube; in particular, the function $d_\gamma$ satisfies in effect the distance axioms.

By a *tube* about $\gamma$ is meant a closed subset of $M$ containing $\gamma([0, 1])$, with nonempty interior and each point of which admitting a unique *nearest point* in $\gamma([0, 1])$; moreover, if $p \mapsto p^\perp_\gamma$ denotes the nearest-point map, the geodesic from $p$ to $p^\perp_\gamma$ should meet $\gamma([0, 1])$ *orthogonally*. Finally, the lateral boundary of the tube is given by the equation $d(p, p^\perp_\gamma) = r$, where $r > 0$ is a small real number called the *radius* of the tube.

We are thus willing to study the question: *under which conditions must a tube about an embedded geodesic be strongly convex*?

First of all, indeed, we should restrict to geodesics without conjugate points (at least in their interior) since, by the Morse index theorem, they would not be minimizing otherwise [Milnor 1963]. To proceed further, let us take examples. In the domain of the unit sphere of $\mathbb{R}^3$ given by $0 \leqslant \text{longitude} < \pi$ and $-r \leqslant \text{latitude} \leqslant r$ with $r$ small, we see that the geodesic joining two points with equal latitude close enough to $r$ does not stay in that domain. But if we look at a similar domain about the *interior* equator of a torus of revolution in $\mathbb{R}^3$ and pick two points as above, the geodesic joining them does stay in the domain. So, a curvature assumption should be made along a geodesic before we can expect the strong convexity of a tube about it, and positive curvature rules out strong convexity.

Eventually, we will show that a tube $T_r(\gamma_0)$ with small enough radius $r$ about a geodesic $\gamma_0$ with negative Jacobi operator is essentially strongly convex. Specifically, we will prove the following result:

**Theorem 0.2.** *Let $\gamma_0 : s \in [0, \ell_0] \to \gamma_0(s) \in M$ be an embedded unit-speed geodesic with negative Jacobi operator. Given $\varsigma > 0$, there exists $\varrho > 0$ such that, if $r \in (0, \varrho)$, the tube $T_r(\gamma_0)$ is strongly convex for each pair $(p, q) \in T_r(\gamma_0) \times T_r(\gamma_0)$ of points satisfying either $|s(p^\perp_{\gamma_0}) - s(q^\perp_{\gamma_0})| \geqslant \varsigma$, or $s(p^\perp_{\gamma_0})$ and $s(q^\perp_{\gamma_0})$ belong to the subinterval*

$[\varsigma, \ell_0 - \varsigma]$. *Furthermore, if $M$ has dimension 2, the result holds with $\varsigma = 0$ provided we except the boundary pairs $(p, q)$ lying in the same end ($s = 0$ or $s = \ell_0$) of the tube.*

In this statement, we allow the geodesic $\gamma_0$ to contain cut points. For instance, if the image of $\gamma_0$ is contained in the curve $\{x^2 + y^2 = 1, z = 0\}$ viewed as the interior equator of a torus of revolution in $\mathbb{R}^3$, we allow its length $\ell_0$ to belong to the interval $[0, 2\pi)$. In this context, the inner distance function for which we are looking appears well approximated by the pseudometric defined in the tube by $\hat{d}(p, q) = |s(p^{\perp}_{\gamma_0}) - s(q^{\perp}_{\gamma_0})|$, at least for the pairs $(p, q) \in T_r(\gamma_0) \times T_r(\gamma_0)$ such that $\hat{d}(p, q) \gg r$. Accordingly, our proof will split in two parts; let us provide a rough outline of it.

<u>Case 1</u>: For $\hat{d}(p, q)$ less than a suitable positive constant $c$ independent of $r$ as $r \downarrow 0$, there exists a unique minimizing geodesic $t \in [0, 1] \to \gamma(t) \in M$ from $p$ to $q$, so we only have to prove the inclusion $\gamma((0, 1)) \subset (T_r(\gamma_0))^{\circ}$. We do it using a one-parameter family of geodesics $\lambda \in [0, 1] \to c_\lambda$ interpolating between $c_0$ given by $t \in [0, 1] \to \gamma_0\big(ts(q^{\perp}_{\gamma_0}) + (1 - t)s(p^{\perp}_{\gamma_0})\big)$ and $c_1 = \gamma$. For $\lambda$ small, we certainly have $c_\lambda((0, 1)) \subset (T_r(\gamma_0))^{\circ}$. We must rule out the possibility that $c_\lambda(t)$ first touches the boundary of $T_r(\gamma_0)$ for some $t \in (0, 1)$. If $n = 2$, it could happen but on the *lateral* part of $\partial T_r(\gamma_0)$ because the ends of $T_r(\gamma_0)$ are *totally geodesic*. If $n > 2$, the pinching $s[(c_\lambda(t))^{\perp}_{\gamma_0}] \in (0, \ell_0)$ is obtained relying on the assumption (ignored elsewhere in the proof) that $\hat{d}(p, q) \geqslant \varsigma$ or $s(p^{\perp}_{\gamma_0})$ and $s(q^{\perp}_{\gamma_0})$ lie in $[\varsigma, \ell_0 - \varsigma]$. As for the lateral part of $T_r(\gamma_0)$, the estimate $d\big(c_\lambda(t), (c_\lambda(t))^{\perp}_{\gamma_0}\big) < r$ (unless $p = q$) follows from a *maximum principle* for geodesics shown to hold in $T_r(\gamma_0)$ due to our curvature assumption.

<u>Case 2</u>: $\hat{d}(p, q) \geqslant c$. Here, we must work harder, shrink $r > 0$ and show that, if $t \in [0, 1] \to \gamma(t) \in M$ is a geodesic from $p$ to $q$ ranging in $T_r(\gamma_0)$, its Jacobi operator should stay, like the one of $\gamma_0$, *negative*. Moreover, we infer from the latter property that $\gamma$ must be minimizing and unique. We are thus left with proving the very *existence* of $\gamma$. It will be done by a tricky connectedness argument, fixing $p$, letting $q$ vary in the tube and using the parameter $z = \hat{d}(p, q) \in [c, \ell_0]$ itself. The openness part of that argument is based on the invertibility of $d(\exp_p)(\dot{\gamma}(0))$, which holds due to the curvature property of $\gamma$; the closedness part relies on the aforementioned maximum principle.

Can one find a quicker proof? We did not. With Theorem 0.2 and its proof at hand, it becomes easy to obtain a full strong convexity result if, instead of the tube $T_r(\gamma_0)$, we consider the closure of the $r$-neighborhood of $\gamma_0$, that is, the subset $N_r(\gamma_0) = \{m \in M, d(\gamma_0([0, \ell_0]), m) \leqslant r\}$. In this way, we get the main result of the paper, namely:

**Corollary 0.3** (main result). *Let $\gamma_0 : s \in [0, \ell_0] \to \gamma_0(s) \in M$ be an embedded unit-speed geodesic with negative Jacobi operator. There exists $\varrho > 0$ such that the subset $N_r(\gamma_0) \subset M$ is strongly convex for $r \in (0, \varrho)$.*

The paper is organized as follows: the next two sections are devoted to preliminary tools for the proof, general properties of thin tubes are recorded in Section 1 and further ones under our curvature assumption in Section 2, the proof of Theorem 0.2 itself is given in Section 3, and that of Corollary 0.3, in Section 4.

## 1. Properties of a thin tube about an embedded geodesic

Throughout this section, we use the setting of Theorem 0.2 but drop the assumption made on the Jacobi operator of the geodesic $\gamma_0$.

**1A.** ***Fermi map, cylinders and Gauss lemma.*** Let us recall how the tube $T_r(\gamma_0)$ can be precisely defined [Aubin 1998; Gray 2004]. The geodesic $\gamma_0$ extends uniquely as a geodesic embedding of an interval $I = (-\epsilon, \ell_0 + \epsilon)$ with $\epsilon$ small. We consider the map

$$(V, s) \in V_0^\perp \times I \to E_0(V, s) = \exp^\perp_{\gamma_0(s)}(\|^{\gamma_0}(V)) \in M,$$

where we have denoted by $V_0^\perp$ the subspace of $T_{\gamma_0(0)}M$ orthogonal to the velocity vector $V_0 = (d\gamma_0/ds)(0)$, by $\|^{\gamma_0}(V)$ the vector field along $\gamma_0$ obtained by parallel transport of the vector $V$ and by $\exp^\perp_{\gamma_0(s)}$ the restriction of the exponential map to $\|^{\gamma_0}(V_0)(s)^\perp$. The differential of $E_0$ at $(0, s)$ is given by

$$(\delta V, \delta s) \in V_0^\perp \times \mathbb{R} \to dE_0(0, s)(\delta V, \delta s) = \frac{d\gamma_0}{ds}(s)\delta s + \|^{\gamma_0}(\delta V)(s) \in T_{\gamma_0(s)}M;$$

it is an isomorphism since orthogonality is preserved by parallel transport. From the inverse function theorem [Lang 2002] and the compactness of $[0, \ell_0]$ (or bounded length of $\gamma_0$), we infer[1] the existence of a real $R > 0$ such that, setting $|V|$ for the norm of a vector $V$ and $\overline{B}^\perp(0, R) = \{V \in V_0^\perp, |V| \leqslant R\}$, the map $E_0$ induces a *diffeomorphism* from a neighborhood of $\overline{B}^\perp(0, R) \times [0, \ell_0]$ onto a neighborhood of its image. Let us fix such a radius $R$ once for all. For $r \leqslant R$, we denote by $T_r(\gamma_0)$ the image by $E_0$ of $\overline{B}^\perp(0, r) \times [0, \ell_0]$ and call it the *tube* about $\gamma_0$ with radius $r$ [Gray 2004]. We set $p \mapsto F_0(p) = (v_0^\perp(p), z(p))$ for the *inverse* of the mapping $E_0$ and refer to it as the *Fermi map along* $\gamma_0$. We call $z(p)$ the *height* of the point $p$ relative to $\gamma_0$ and the subsets $E_R^{\text{top}}(\gamma_0) = \{p \in T_R(\gamma_0), z(p) = \ell_0\}$ and $E_R^{\text{bot}}(\gamma_0) = \{p \in T_R(\gamma_0), z(p) = 0\}$, respectively, the *top and bottom ends* of the

[1] Full details are given in Section 1D for a construction encompassing the present one.

tube. If $p \in T_R(\gamma_0)$, the unit-speed geodesic

$$s \in [0, |v_0^\perp(p)|] \to E_0\left(s \frac{v_0^\perp(p)}{|v_0^\perp(p)|}, z(p)\right)$$

is the unique minimizing geodesic from $\gamma_0$ to $p$; its length $\mathfrak{r}_{\gamma_0}(p) = |v_0^\perp(p)|$ is thus equal to $d(\gamma_0, p)$. For short, that geodesic will be denoted by $s \mapsto [\gamma_0, p](s) \in T_R(\gamma_0)$, and the function $\mathfrak{r}_{\gamma_0}$ itself simply by $\mathfrak{r}$ unless a confusion may occur. We let $N_{\gamma_0}(p)$, or just $N(p)$ if no confusion, denote the velocity vector $d[\gamma_0, p]/ds$ evaluated at $s = d(\gamma_0, p)$. The unit vector field $p \mapsto N(p)$ is defined in the open subset of the tube $T_R(\gamma_0)$ where $\mathfrak{r}(p) > 0$, that is, outside the geodesic $\gamma_0$; moreover, it is readily seen to satisfy $dz(N) = 0$, $d\mathfrak{r}(N) = 1$ and $\nabla_N N = 0$, with $\nabla$ the Levi-Civita connection. If $r \in (0, R]$, we set $C_r(\gamma_0) = \{p \in T_R(\gamma_0), \mathfrak{r}(p) = r\}$ for the *cylinder* of radius $r$ about $\gamma_0$, sometimes called the *lateral* part of the boundary of the tube $T_r(\gamma_0)$. The outward unit *normal* to that cylinder at $p \in C_r(\gamma_0)$ is nothing but $N(p)$ due to the *generalized Gauss lemma* according to which the gradient of the function $\mathfrak{r}$ and the vector field $N$ coincide [Gray 2004, pp. 26–28]. The identity $N = \operatorname{grad} \mathfrak{r}$ will be central for us. It yields the following identity, recorded here for later use, valid at each $p \in T_R(\gamma_0)$ such that $\mathfrak{r}(p) > 0$:

$$(1)\quad (g - d\mathfrak{r}^2)(V, W) = (g - d\mathfrak{r}^2)(\Pi_N^\perp(V), \Pi_N^\perp(W)) \quad \text{for all } (V, W) \in T_pM \times T_pM,$$

where we have set $\Pi_N^\perp(V) = V - g(V, N)N$ for the orthogonal projection of $T_pM$ onto $N(p)^\perp$; in other words, if we write $TM = \mathbb{R}N \oplus N^\perp$ on $\{\mathfrak{r} > 0\}$, the generalized Gauss lemma implies that the metric $g$ splits into the sum of $d\mathfrak{r}^2$ along $\mathbb{R}N$ and $(g - d\mathfrak{r}^2)$ along $N^\perp$.

Finally, $\mathfrak{i} \in (0, \infty]$ will stand for the injectivity radius of $T_R(\gamma_0)$, that is, for the minimum of the distance from a point $p$ to its cut locus as $p$ varies in $T_R(\gamma_0)$ [do Carmo 1992, pp. 267–273]. For each $r \in (0, R]$, the injectivity radius of $T_r(\gamma_0)$ will thus be *at least equal* to $\mathfrak{i}$. If $M$ is compact, $\mathfrak{i}$ is finite, but $\mathfrak{i} = \infty$ if $M$ is the hyperbolic space, for instance.

**1B.** ***Fermi charts and related notions.*** Let $n = \dim M$. Given an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_{\gamma_0(0)}M$ with $e_n = (d\gamma_0/ds)(0)$, let us assign to each $p \in T_R(\gamma_0)$ the $n$-tuple $x = (\tilde{x}, x^n) \in \bar{B}^{n-1}(0, R) \times [0, \ell_0]$, where $\bar{B}^{n-1}(0, R)$ denotes the closure of the ball of radius $R$ in the Euclidean space $\mathbb{R}^{n-1}$, given by $x(p) = (x^1, \dots, x^{n-1}, x^n)$ if and only if $v_0^\perp(p) = \sum_{\alpha=1}^{n-1} x^\alpha e_\alpha$ and $z(p) = x^n$. The map $x : T_R(\gamma_0) \to \bar{B}^{n-1}(0, R) \times [0, \ell_0]$ so defined is called a *Fermi chart* along the embedded geodesic $\gamma_0$. (In 1922, while a PhD student at the Scuola Normale Superiore in Pisa, motivated by the study of the equivalence principle in general relativity, Enrico Fermi was the first to consider such local coordinates, which he used along timelike paths; see [Gray and Vanhecke 1982, p. 217] and references therein.)

We see from this construction that $y = (\tilde{y}, y^n)$ is another such chart if and only if $y^n = x^n$ and there exists an orthogonal transformation $\mathfrak{R} \in \mathrm{O}(n-1)$ such that $\tilde{y} = \mathfrak{R}\tilde{x}$. The calculations which we will perform in the tube $T_R(\gamma_0)$ will be *invariant* (or tensorial) with respect to change of Fermi charts. We will freely use the local Euclidean metric $\mathfrak{e}_{\gamma_0} = \sum_{i=1}^n (dx^i)^2$ (just denoted by $\mathfrak{e}$, unless confusing) and the *affine structure* inherited from its (flat) Levi-Civita connection $D_{\gamma_0} = D$. The latter will be convenient to identify distinct tangent spaces and hence view vectors tangent to $T_R(\gamma_0)$ at distinct points as belonging to *the same* vector space. We will also view the Christoffel symbols $\Gamma_{ij}^k(x)$ of our original (global) connection $\nabla$ as the components in the chart $x$ of the local *tensorial* difference $(\nabla - D)$.

In the Fermi chart $x$, the components of the metric tensor $g$ satisfy $g_{ij}(0, x^n) = \delta_{ij}$, $dg_{ij}(0, x^n) = 0$, so the Christoffel symbols vanish at $(0, x^n)$, meaning that $g$ is osculating to $\mathfrak{e}$ along $\gamma_0$. We set $\|\cdot\|$ for the norm associated to the Euclidean metric $\mathfrak{e}$ and $\theta_0 = \min \|U\| \leqslant 1 \leqslant \Theta_0 = \max \|U\|$, where $U$ runs over all unit[2] tangent vectors at points of $T_R(\gamma_0)$. For each $p \in T_R(\gamma_0)$, setting $\rho(x) = \sqrt{\sum_{\alpha=1}^{n-1} (x^\alpha)^2}$, we have $\mathfrak{r}(p) = \rho(x(p))$. The geodesic ray $t \in [0, 1] \to E_0(tv_0^\perp(p), z(p)) \in M$ reads $t \mapsto \mathcal{R}(t) = (tx^1, \dots, tx^{n-1}, x^n)$ with $x = x(p)$; being constant, its speed is equal to $\rho(x)$, so the unit vector field $N$ reads $N(p) = \nu(x(p))$ with $\nu(x) = (1/\rho)(x) \sum_{\alpha=1}^{n-1} x^\alpha \, \partial/\partial x^\alpha$.

If $W = \sum_{i=1}^n W^i \, \partial/\partial x^i \in T_pM$, we may view $W$ as a *constant* vector field in $T_R(\gamma_0)$, in other words, extend it to $T_R(\gamma_0)$ by $D_{\gamma_0}$ parallelism, a notion well defined in any Fermi chart along $\gamma_0$. Following [Gray 2004, p. 21], let us call any such vector field a *Fermi field* (here, with respect to $\gamma_0$). Given a point $p \in T_R(\gamma_0)$ and vector field $Z$ on $T_R(\gamma_0)$, we may similarly consider the Fermi field $Z(p)$, thinking of it as $Z$ frozen at $p$. Among Fermi fields, one may distinguish those with $W^n = 0$ from those writing $Z = Z^n \, \partial/\partial x^n$ (sometimes called *axial*). For later use, we record the brackets identities

$$\left[\nu, \frac{\partial}{\partial x^n}\right] = 0 \quad \text{and} \quad \left[\nu, \rho \frac{\partial}{\partial x^\alpha}\right] = \frac{\partial \rho}{\partial x^\alpha} \nu \quad \text{for all } \alpha < n. \tag{2}$$

Finally, it will be convenient to consider on $T_R(\gamma_0)$ the field of projections $\Pi_0 = \sum_{\alpha=1}^{n-1} dx^\alpha \otimes \partial/\partial x^\alpha$, which is the constant (or Fermi) extension of the orthogonal projection of $T_{\gamma_0(0)}M$ onto $V_0^\perp$.

**1C. *Estimates for geodesics in a thin tube.*** Beforehand, let us recall a classical result, namely: there exists a continuous function $p \in M \to \chi(p) \in (0, \infty]$ called the *convexity radius*, which is smaller than the injectivity radius, such that, for each $\varrho \in (0, \chi(p))$, the Riemannian ball $B(p, \varrho)$ is strongly convex [Cheeger and Ebin 2008, pp. 103–105; Klingenberg 1995, pp. 84–85; Whitehead 1932]. For $r > 0$

[2]Here and below, to be understood for the metric $g$, unless otherwise specified.

small, we may thus consider the function $r \mapsto \chi_{\gamma_0}(r) = \min\{\chi(p), p \in T_r(\gamma_0)\}$, which is nonincreasing. We set $\mathfrak{c} = \chi_{\gamma_0}(R)$ and stress that $\mathfrak{c} \leqslant \mathfrak{i}$. Our first estimate is an upper bound on the length of the geodesics contained in the tube $T_{R_0}(\gamma_0)$ with $R_0 = \min(R, \mathfrak{c}/3)$.

**Proposition 1.1.** *If $\gamma : [0, \ell] \to \gamma(s) \in T_{R_0}(\gamma_0)$ is a unit-speed geodesic,*[3] *its length $\ell$ is bounded above by $L_0$, with $L_0 = \ell_0 + 2R$ if $\mathfrak{i} = \infty$, and $L_0 = 2(\ell_0 + \mathfrak{c})$ if $\mathfrak{c} < \infty$.*

*Proof.* If $\mathfrak{i} = \infty$, the geodesic $\gamma$ is minimizing and unique in $M$. But we can join its endpoints $p = \gamma(0)$, $q = \gamma(\ell)$ by a geodesic path broken twice, namely, first by going along the geodesic ray from $p$ to $\gamma_0(z(p))$, next by going from $\gamma_0(z(p))$ to $\gamma_0(z(q))$ along $\gamma_0$, then by going along the geodesic ray from $\gamma_0(z(q))$ to $q$. The total length of that broken path must be larger than $\ell$ and it is, indeed, at most equal to $L_0 = \ell_0 + 2R$.

If $\mathfrak{c} < \infty$, for each $\epsilon > 0$ small enough, the triangle inequality satisfied by the Riemannian distance on $M$ shows that we can cover the tube $T_{R_0}(\gamma_0)$ by $N$ open balls of radius $r = \mathfrak{c} - \epsilon$, successively centered at the points $\gamma_0(0), \gamma_0(r), \gamma_0(2r), \dots, \gamma_0((N-1)r), \gamma_0(\ell_0)$, with $N = [\ell_0/\mathfrak{c}] + 1$. Now, the length of the restriction of the geodesic $\gamma$ to each ball is bounded above by $2r$ and, letting $\epsilon \downarrow 0$, we obtain $\ell \leqslant 2N\mathfrak{c}$. $\square$

Using a Fermi chart along $\gamma_0$, setting $R_1 = \frac{9}{10} R_0$, we can readily find a positive constant $c_1$ such that, for each $p \in T_{R_1}(\gamma_0)$, the following estimates hold at $x = x(p)$:

$$\|g - \mathfrak{e}\| \leqslant c_1 \rho^2(x), \quad \|\nabla - D\| \leqslant c_1 \rho(x). \tag{3}$$

The purpose of our next proposition is twofold. On the one hand, it provides a radius under which the geodesics contained in a tube about $\gamma_0$ and longer than a given length $\delta > 0$ keep moving axially in a single direction; in particular, they must be embedded, like $\gamma_0$. On the other hand, it provides an estimate describing how $C^0$-close to $\gamma_0$ a geodesic should be in order to get $C^1$-close to it.

**Proposition 1.2.** *Fixing $\delta \in (0, L_0)$, let $r_1 > 0$ be given by*

$$r_1^2 \left( c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left( \frac{4}{\delta} + c_1 L_0 \Theta_0^2 \right)^2 \right) = 1.$$

*For each $r \in (0, \min(R_1, r_1))$ and each unit-speed geodesic $s \in [0, \ell] \to \gamma(s) \in T_r(\gamma_0)$ with length $\ell \geqslant \delta$, the axial component $d\gamma^n/ds$ of the velocity cannot vanish. Moreover, the following estimate holds*:

$$\left\| \varepsilon \frac{d\gamma}{ds} - \frac{\partial}{\partial x^n} \right\| \leqslant \left( \frac{4}{\ell} + c_1 \ell \Theta_0^2 \right) \rho_\gamma + \left( c_1 \Theta_0^2 + \frac{1}{\theta_0^2} \left( \frac{4}{\ell} + c_1 \ell \Theta_0^2 \right)^2 \right) \rho_\gamma^2, \tag{4}$$

[3]Throughout the paper, $\ell$ denotes the length of $\gamma$ which may vary; it should be written $\ell(\gamma)$, of course, but we will stick to the short notation $\ell$ instead.

*where* $\rho_\gamma$ *stands for* $\max_{\sigma\in[0,\ell]}\rho(\gamma(\sigma))$ *and* $\varepsilon=\pm 1$, *according to the sign of* $d\gamma^n/ds$.

*Proof.* Before proving the first assertion we require an estimate; namely, letting $s\in[0,\ell]\to\gamma(s)\in T_{R_1}(\gamma_0)$ be a unit-speed geodesic, we have

$$\left\|\Pi_0\frac{d\gamma}{ds}(s)\right\|\leqslant\left(\frac{4}{\ell}+c_1\ell\Theta_0^2\right)\rho_\gamma\quad\text{for all } s\in[0,\ell]. \tag{5}$$

Indeed, if $s\in\left[0,\frac{\ell}{2}\right]$, we write, for all $\alpha\in\{1,\dots,n-1\}$,

$$(\ell-s)\frac{d\gamma^\alpha}{ds}(s)=\gamma^\alpha(\ell)-\gamma^\alpha(s)-\int_s^\ell\int_s^S\frac{d^2\gamma^\alpha}{d\sigma^2}(\sigma)\,d\sigma\,dS,$$

while if $s\in\left[\frac{\ell}{2},\ell\right]$, we write instead

$$s\frac{d\gamma^\alpha}{ds}(s)=\gamma^\alpha(s)-\gamma^\alpha(0)-\int_0^s\int_s^S\frac{d^2\gamma^\alpha}{d\sigma^2}(\sigma)\,d\sigma\,dS.$$

In either case, transforming the last term of the right-hand side by means of the geodesic equation, recalling (3) and using the triangle and Schwarz inequalities, we readily infer (5). Writing

$$\left|\frac{d\gamma^n}{ds}\right|=\left\|\frac{d\gamma}{ds}\right\|\sqrt{1-\frac{\|\Pi_0\,d\gamma/ds\|^2}{\|d\gamma/ds\|^2}}\quad\text{and}\quad\left\|\frac{d\gamma}{ds}\right\|=\sqrt{1-(g-\mathfrak{e})\left(\frac{d\gamma}{ds},\frac{d\gamma}{ds}\right)},$$

the latter to be combined with (3), we get

$$\left|\frac{d\gamma^n}{ds}\right|\geqslant 1-c_1\rho_\gamma^2\Theta_0^2-\frac{1}{\theta_0^2}\left\|\Pi_0\frac{d\gamma}{ds}\right\|^2;$$

hence, using (5), we obtain the important lower bound

$$\left|\frac{d\gamma^n}{ds}(s)\right|\geqslant 1-\left(c_1\Theta_0^2+\frac{1}{\theta_0^2}\left(\frac{4}{\ell}+c_1\ell\Theta_0^2\right)^2\right)\rho_\gamma^2\quad\text{for all } s\in[0,\ell]. \tag{6}$$

Recalling Proposition 1.1 and the assumption $\ell\geqslant\delta$, this shows that $d\gamma^n/ds$ cannot vanish provided the radius $r$ of the tube in which the geodesic ranges satisfies

$$r^2\left(c_1\Theta_0^2+\frac{1}{\theta_0^2}\left(\frac{4}{\delta}+c_1L_0\Theta_0^2\right)^2\right)<1,$$

or else $r\in(0,\min(R_1,r_1))$, as we assumed. The first part of Proposition 1.2 is thus proved.

Moreover, letting now $\varepsilon$ stand for the *sign* of $d\gamma^n/ds$, we have

$$|d\gamma^n/ds(s)|\equiv\varepsilon\,d\gamma^n/ds,$$

so we readily get from (6) and the obvious inequality $|d\gamma^n/ds| \leqslant \|d\gamma/ds\|$, the pinching

$$-\frac{1}{2}c_1\Theta_0^2\rho_\gamma^2 \leqslant 1-\varepsilon\frac{d\gamma^n}{ds} \leqslant \left(c_1\Theta_0^2+\frac{1}{\theta_0^2}\left(\frac{4}{\ell}+c_1\ell\Theta_0^2\right)^2\right)\rho_\gamma^2.$$

Combined with (5) this yields (4), since

$$\left\|\varepsilon\frac{d\gamma}{ds}(s)-\frac{\partial}{\partial x^n}\right\| \leqslant \left\|\Pi_0\frac{d\gamma}{ds}(s)\right\| + \left|\varepsilon\frac{d\gamma^n}{ds}(s)-1\right|. \qquad \square$$

Writing $UM$ for the unit tangent bundle and $\mathrm{End}_s(TM)$ for the bundle of symmetric[4] endomorphisms of $TM$, let us consider the map

$$(p,U)\in UM \to \mathbb{J}(p,U)=R_p(\,\cdot\,,U)U \in \mathrm{End}_s(TM),$$

where $R_p$ stands for the Riemann curvature tensor at the point $p\in M$. It satisfies $g(V,\mathbb{J}(p,U)W)\equiv S_p(V,U,W,U)$ where $S_p$ stands for the sectional (or covariant Riemann) curvature tensor of the metric $g$ at the point $p$; it is thus, indeed, *symmetric*. We denote by $\kappa^1(p,U)\leqslant\cdots\leqslant\kappa^{n-1}(p,U)$ the eigenvalues (each repeated with its multiplicity) of the nontrivial part of $\mathbb{J}(p,U)$, namely of its restriction to $U^\perp$. For each $\alpha\in\{1,\dots,n-1\}$, the map $(p,U)\in UM\to\kappa^\alpha(p,U)\in\mathbb{R}$ is $C^1_{\mathrm{loc}}$ [Kato 1995, pp. 122–123], hence uniformly Lipschitz for $p\in T_{R_0}(\gamma_0)$. So there exists a constant $k_0$ such that, for each pair $((p,U),(p',U'))\in UM^2$ with $\max(\mathfrak{r}_{\gamma_0}(p),\mathfrak{r}_{\gamma_0}(p'))\leqslant R_0$ and each $\alpha\in\{1,\dots,n-1\}$, the following uniform estimate holds:

$$|\kappa^\alpha(p,U)-\kappa^\alpha(p',U')| \leqslant k_0\big(d(p,p')+\|U-U'\|\big). \tag{7}$$

For each unit-speed geodesic $\sigma\in[0,\ell]\to\gamma(\sigma)\in M$, we write $s\mapsto\mathfrak{J}_\gamma(s)$ for the pullback to $[0,\ell]$ of the map $\mathbb{J}$ by the section

$$t\mapsto\left(\gamma(s),\frac{d\gamma}{d\sigma}(s)\right)\in UM$$

and call $\mathfrak{J}_\gamma(s)$ the *Jacobi operator* along the geodesic $\gamma$ at $s$. We further write $\kappa_\gamma^1(s)\leqslant\cdots\leqslant\kappa_\gamma^{n-1}(s)$ for the eigenvalues of the restriction of $\mathfrak{J}_\gamma(s)$ to $\frac{d\gamma}{d\sigma}(s)^\perp$ and call them the *Jacobi curvatures* along $\gamma$ at $s$.

**Corollary 1.3.** *Given $\delta$ and $r$ as in Proposition 1.2, set*

$$k=k_0\left(1+\frac{4}{\delta}+c_1L_0\Theta_0^2+\left(c_1\Theta_0^2+\frac{1}{\theta_0^2}\left(\frac{4}{\delta}+c_1L_0\Theta_0^2\right)^2\right)r\right).$$

*For each unit-speed geodesic $\sigma\in[0,\ell]\to\gamma(\sigma)\in T_r(\gamma_0)$ with length $\ell\geqslant\delta$ and*

[4]Here, "unit" and "symmetric" refer to the Riemannian metric $g$, of course.

*each* $s \in [0, \ell]$, *the following estimate holds*:

$$\left|\kappa_\gamma^\alpha(s) - \kappa_0^\alpha(\gamma^n(s))\right| \leqslant k\rho_\gamma \quad \textit{for all } \alpha \in \{1, \dots, n-1\},$$

*where* $\kappa_0^1 \leqslant \cdots \leqslant \kappa_0^{n-1}$ *stand for the Jacobi curvatures along* $\gamma_0$.

*Proof.* Fixing $\gamma$ as stated, we may apply Proposition 1.2 to it. This yields an estimate for $\|(d\gamma/d\sigma)(s) - \partial/\partial x^n\|$ which, combined with the estimate (7) read at $(p, U) = \big(\gamma(s), (d\gamma/d\sigma)(s)\big)$ and $(p', U') = \big(\gamma_0(\gamma^n(s)), \partial/\partial x^n\big)$, yields the desired result. □

Corollary 1.3 shows in particular that, if the Jacobi operator along $\gamma_0$ stays *definite*, it must stay so (with the same signature) along geodesics longer than a given length and contained in a tube about $\gamma_0$ of small enough radius.

**1D. *Family of Fermi maps near $\gamma_0$*.** For each unit-speed geodesic $s \in [0, \ell] \to \gamma(s) \in T_{R_1}(\gamma_0)$, let $I_{\gamma_0}(\gamma) \subset [0, \ell_0]$ denote the axial image interval $\gamma^n([0, \ell])$ and $T(\gamma_0, \gamma)$, the shortest piece of tube about $\gamma_0$ containing $\gamma$, equal to $\{m \in T_{\rho_\gamma}(\gamma_0),\ x^n(m) \in I_{\gamma_0}(\gamma)\}$. If such a geodesic $\gamma$ is an embedding, when is it possible to construct a Fermi map along it such that a point $m \in T(\gamma_0, \gamma)$ may stay *outside* the corresponding tube about $\gamma$ if and only if its height $z_\gamma(m)$ relative to $\gamma$ satisfies either $z_\gamma(m) < 0$ or $z_\gamma(m) > \ell$? When such a possibility occurs, we call $(\gamma_0, \gamma)$-*exceptional* the latter points and $(\gamma_0, \gamma)$-*accessible* all other points of $T(\gamma_0, \gamma)$. Sticking to the notations of Proposition 1.2, we will prove the following:

**Proposition 1.4.** *For each* $\delta \in (0, \ell_0)$, *there exists* $r_2 \in (0, \min(R_1, r_1))$ *such that, for each unit-speed geodesic* $\gamma$ *longer than* $\delta$ *and contained in* $T_{r_2}(\gamma_0)$, *a Fermi map can be constructed along* $\gamma$ *with corresponding tube about* $\gamma$ *containing the whole of* $T(\gamma_0, \gamma)$ *but its* $(\gamma_0, \gamma)$*-exceptional points.*

We call *family of Fermi maps near* $\gamma_0$ the map which assigns, to each unit-speed geodesic $\gamma$ as stated and each $(\gamma_0, \gamma)$-accessible point $m \in T_{r_2}(\gamma_0)$, the image of $m$ by the Fermi map along $\gamma$.

*Proof.* The idea is to use a suitable implicit function theorem argument along $\gamma_0$. Since it is absent from the literature, we will present it carefully. Let us fix $\delta \in (0, \ell_0)$ and a unit-speed geodesic $\sigma \in [0, \ell^*] \to \gamma^*(\sigma) \in T_{r_2}(\gamma_0)$, with $\ell^* \geqslant \delta$ and $r_2 \in (0, \min(R_1, r_1))$ to be chosen later. From Proposition 1.2, we know that $\gamma^*$ is an embedding. We can thus construct a tube $T_\varrho(\gamma^*)$ about $\gamma^*$, for some radius $\varrho > 0$, as done for $\gamma_0$ in Section 1A. We want $\rho_{\gamma^*} \leqslant r_2$ small enough compared to $\varrho$ such that the tube $T_\varrho(\gamma^*)$ contains $T(\gamma_0, \gamma^*)$ but its exceptional points. Can we choose the radius $r_2$ such that this property holds for every such geodesic $\gamma^*$?

First, we observe that the required property holds for $\gamma^*$ if and only if it holds for the reversed geodesic $\gamma^*_{\text{rev}}$, given by $\sigma \in [0, \ell^*] \to \gamma^*_{\text{rev}}(\sigma) = \gamma^*(\ell^* - \sigma)$. Therefore, applying Proposition 1.2 to $\gamma^*$, we may assume with no loss of generality that $d\gamma^{*n}/d\sigma$ is positive.

Next, we note that the geodesic $\gamma^*$ is given by its Cauchy data

$$(p^*, u^*) = \left(\gamma^*(0), \frac{d\gamma^*}{d\sigma}(0)\right) \in UM$$

and its length $\ell^* \in [\delta, L_0]$, while the generic point $m^*$ of the tube $T_\varrho(\gamma^*)$ is determined by its Fermi map image $F_{\gamma^*}(m^*)$, namely by its height $\sigma^* = z_{\gamma^*}(m^*) \in [0, \ell^*]$ and by the vector $V^* = v^\perp_{\gamma^*}(m^*) \in (u^*)^\perp$ such that $|V^*| \leqslant \varrho$ and $E_{\gamma^*}(V^*, \sigma^*) = m^*$. Here, we have denoted by $E_{\gamma^*} : (u^*)^\perp \times (-\epsilon, \ell^* + \epsilon) \to M$ (respectively, by $v^\perp_{\gamma^*}$) the analogue for $\gamma^*$ of the map $E_0$ (respectively, of the component $v^\perp_0$) defined for $\gamma_0$ at the beginning of Section 1A.

The resulting point $(p^*, u^*, V^*)$, the amalgam of the Cauchy data of $\gamma^*$ with the Fermi component $V^* = v^\perp_{\gamma^*}(m^*) \in (u^*)^\perp$ of $m^*$, lies in the vector bundle $\ker T\pi \to UM$, the kernel of the tangent map to the natural projection $\pi : UM \to M$. Sticking to the Fermi chart $x$ along $\gamma_0$, we use it to build a chart of $\ker T\pi$ near $(p^*, u^*, V^*)$ by assigning to each neighboring point $(p, u, V)$ the $(3n-2)$-tuple $(x^1, \dots, x^n, u_0^1, \dots, u_0^{n-1}, V_0^1, \dots, V_0^{n-1})$ with $x^i = x^i(p)$ and $u_0^\alpha$, $V_0^\alpha$ defined as follows. Firstly, for each tangent vector $W \in T_pM$, let $\overline{W}_0 \in T_{p_0^\perp}M$, with $p_0^\perp = p^\perp_{\gamma_0} \equiv \gamma_0(x^n(p))$, denote its (backward) parallel transport[5] along the geodesic ray $[\gamma_0, p]$, and $W_0 \in T_{\gamma_0(0)}M$, similarly from the latter now along $\gamma_0$. We pause to record a lemma (the proof of which is left as an easy exercise):

**Lemma 1.5.** *If $U$ is a unit tangent vector at $p \in T_{R_1}(\gamma_0)$ and $\overline{U}_0$ stands for its parallel transport to the point $\gamma_0(x^n(p))$ along the geodesic ray $[\gamma_0, p]$, the following estimate holds*:

$$\|U - \overline{U}_0\| \leqslant c_1 \Theta_0 \mathfrak{r}^2(p).$$

Applying this lemma, combined with Proposition 1.2 and the triangle inequality, to the vector $u^* \in T_{p^*}M$, and recalling that $\|\cdot\| \equiv |\cdot|$ along $\gamma_0$, we infer the estimate

$$|u_0^* - e_n| \leqslant k_1 r_2, \tag{8}$$

with

$$k_1 = \frac{4}{\delta} + c_1 L_0 \Theta_0^2 + \left(c_1\Theta_0 + c_1\Theta_0^2 + \frac{1}{\theta_0^2}\left(\frac{4}{\delta} + c_1 L_0 \Theta_0^2\right)^2\right) r_1.$$

Here, we used the positivity assumption made above on $(u^*)^n$. Taking $r_2 < 1/k_1$, this estimate implies the positivity of $(u_0^*)^n$. Back to the definition of the chart of

[5] Henceforth, with respect to the Levi-Civita connection $\nabla$, unless otherwise specified.

$\ker T\pi$ under elaboration, we take $(p, u, V)$ close enough to $(p^*, u^*, V^*)$ for $u_0^n$ to be still positive, and we define the $u_0^\alpha$'s and $V_0^\alpha$'s by

$$\sum_{\alpha=1}^{n-1} u_0^\alpha e_\alpha = \Pi_0 u_0, \quad \sum_{\alpha=1}^{n-1} V_0^\alpha e_\alpha = \Pi_0 V_0.$$

We recover the full parallel transported vectors $u_0$, $V_0$, by setting

$$u_0^n = \sqrt{1 - \sum_{\alpha=1}^{n-1} (u_0^\alpha)^2},$$

since $|u_0| = 1$ and $u_0^n > 0$, and

$$V_0^n = -\frac{1}{u_0^n} \sum_{\alpha=1}^{n-1} u_0^\alpha V_0^\alpha,$$

since $V_0 \perp u_0$. So $(x^i, u_0^\alpha, V_0^\alpha)$ is, indeed, a local chart of $\ker T\pi$. Although heavier, let us denote it rather by $(x^{*i}, u_0^{*\alpha}, V_0^{*\alpha})$ since we are now willing to move around the geodesic $\gamma^*$ and the point $m^* \in T_\varrho(\gamma^*)$, hence to let the point $(p^*, u^*, V^*)$ itself vary in $\ker T\pi$ near $(p_0, u_0, V_0) = \big(\gamma_0(s_0), (d\gamma_0/ds)(s_0), 0\big)$ with $s_0 \in [0, \ell_0]$. Deferring the completion of the present proof, we pause to set up an appropriate implicit function theorem.

*Implicit function theorem argument.* In this section, the requirement that the geodesic $\gamma^*$ be longer than $\delta$ will be unnecessary, thus ignored provisionally. Given $s_0 \in [0, \ell_0]$ and $\sigma_0 \in [0, \ell_0 - s_0]$, let the point $(p^*, u^*, V^*) \in \ker T\pi$ be close to $(p_0, u_0, V_0)$ and the real $\sigma^* \in \mathbb{R}^+$ be close to $\sigma_0$; let a further point $m$ belong to $T_{r_2}(\gamma_0)$. Setting $\gamma^*(\sigma) = \exp_{p^*}(\sigma u^*)$ and $m^* = E_{\gamma^*}(V^*, \sigma^*)$, consider the map

$$\Psi(p^*, u^*, V^*, \sigma^*, m) = x(m^*) - x(m) \in \mathbb{R}^n.$$

Using the chart $(x^{*i}, u_0^{*\alpha}, V_0^{*\alpha})$ for $(p^*, u^*, V^*)$ and the chart $x^i$ for $m$, let us denote the local expression of $\Psi$ (respectively, $x \circ E_{\gamma^*}$) by

$$\Psi^i(x^{*j}, u_0^{*\alpha}, V_0^{*\alpha}, \sigma^*, x^j) = E^i(x^{*j}, u_0^{*\alpha}, V_0^{*\alpha}, \sigma^*) - x^i.$$

At the point given by[6] $x^{*\alpha} = 0$, $x^{*n} = s_0$, $u_0^{*\alpha} = 0$, $V_0^{*\alpha} = 0$, $\sigma^* = \sigma_0$, $x^\alpha = 0$, $x^n = s_0 + \sigma_0$, we have

$$\Psi^i\big((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0, (\vec{0}, s_0 + \sigma_0)\big) = 0 \quad \text{for all } i \in \{1, \dots, n\}$$

and

$$\det\left(\frac{\partial \Psi^j}{\partial(V_0^{*\alpha}, \sigma^*)}\big((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0, (\vec{0}, s_0 + \sigma_0)\big)\right) \neq 0,$$

[6]Throughout with $\alpha$ ranging in $\{1, \dots, n-1\}$.

where $\vec{0}$ stands for the zero vector of $\mathbb{R}^{n-1}$. The latter equation holds since

$$\frac{\partial \Psi^j}{\partial(V_0^{*\alpha}, \sigma^*)} \equiv \frac{\partial E^j}{\partial(V_0^{*\alpha}, \sigma^*)}$$

and $dE^j((\vec{0}, s_0), \vec{0}, \vec{0}, \sigma_0) \equiv dx^j \circ dE_0(0, s_0 + \sigma_0)$, where $dE_0(0, s_0 + \sigma_0)$ is an *isomorphism* as seen in Section 1A. We are thus in position to apply the implicit function theorem [Lang 2002]. There exists a real $\epsilon > 0$ and a unique map $(x^{*j}, u_0^{*\alpha}, x^j) \to F^* = (\mathcal{V}_0^{*1}, \dots, \mathcal{V}_0^{*n-1}, \varsigma^*)$ such that, if

(9) $\rho(x^*) \leqslant \epsilon, \quad |x^{*n} - s_0| \leqslant \epsilon, \quad |\Pi_0 u_0^*| \leqslant \epsilon, \quad \rho(x) \leqslant \epsilon, \quad |x^n - (s_0 + \sigma_0)| \leqslant \epsilon,$

the identities

$$\Psi^i\big(x^{*j}, u_0^{*\alpha}, \mathcal{V}_0^{*\alpha}(x^{*k}, u_0^{*\alpha}, x^k), \varsigma^*(x^{*k}, u_0^{*\alpha}, x^k), x^j\big) \equiv 0 \quad \text{for all } i \in \{1, \dots, n\}$$

are satisfied with $\sum_{\alpha=1}^{n-1} (\mathcal{V}_0^{*\alpha}(x^{*k}, u_0^{*\alpha}, x^k))^2$ and $\left|\varsigma^*(x^{*k}, u_0^{*\alpha}, x^k) - \sigma_0\right|$ small. By construction, these identities imply $m = m^*$; in other words, the map

$$x^j \to F^{*i}(x^{*j}, u_0^{*\alpha}, x^j)$$

is nothing but the expression of the Fermi map $F_{\gamma^*}$ along the geodesic $\gamma^*(\sigma) = \exp_{p^*}(\sigma u^*)$ read in the Fermi chart $x$ along $\gamma_0$. Finally, let us stress that the real $\epsilon > 0$ occurring in (9) may be chosen so small that it becomes independent of the pair of parameters $(s_0, \sigma_0)$, because the latter lies in a compact subset of $\mathbb{R}^2$, namely in the triangle of the positive quadrant given by $s_0 + \sigma_0 \leqslant \ell_0$. Henceforth, we fix $\epsilon > 0$ so.

*Completion of the proof of Proposition 1.4.* Back to the case of our previous geodesic $\gamma^*$, supposed longer than $\delta$ and with positive axial component, we are now in position to choose the radius $r_2$ of the tube about $\gamma_0$ in which $\gamma^*$ should lie. First of all, we fix a point $m \in T(\gamma_0, \gamma^*)$. So far, we have required $r_2 \in (0, \min(R_1, r_1, 1/k_1))$. Redoing the preceding implicit function theorem argument now with $p^* = \gamma^*(0)$, $s_0 = x^n(p^*)$, $s_0 + \sigma_0 = x^n(m)$, the first and fourth inequalities of (9) prompt us to take $r_2 \leqslant \epsilon$. Besides, we must further shrink $r_2 > 0$ in order to keep $\gamma^*$ nearly vertical so that the third inequality of (9) holds as well. From (8), we can do it by taking $r_2 \leqslant \epsilon/k_1$, as easily verified. Altogether, if the geodesic $\gamma^*$ is longer than $\delta \in (0, \ell_0)$ with $d\gamma^{*n}/d\sigma > 0$ and if it is contained in the tube $T_{r_2}(\gamma_0)$ with $r_2 \in (0, \min(R_1, r_1, \epsilon/k_1))$, the triple

$$\left(x^{*i} = x^{*i}(\gamma^*(0)),\ u_0^{*\alpha} = u_0^{*\alpha}\left(\frac{d\gamma^*}{d\sigma}(0)\right),\ x^i = x^i(m)\right)$$

satisfies the bounds (9). So we may consider its image by the local map $F^*$ precedingly constructed. In particular, it follows that the point $m$ lies in a tube about the embedded geodesic $\gamma^*$ if and only if its height $z_{\gamma^*}(m) = \varsigma^*(x^{*i}, u_0^{*\alpha}, x^i)$

lies in the interval $[0, \ell^*]$. Since the point $m$ was arbitrarily fixed in $T(\gamma_0, \gamma^*)$, we are done. $\square$

**1E. *Second fundamental form of a cylinder.*** If $n > 2$, sticking to the notations of Section 1A, let us study the second fundamental form of a cylinder $C_r(\gamma_0)$ of small radius $r$ about $\gamma_0$.

**Proposition 1.6.** *Given $r \in (0, \min(1, R))$, a point $p \in C_r(\gamma_0)$ and a pair of vectors $(V, W) \in T_pC_r(\gamma_0) \times T_pC_r(\gamma_0)$, let us denote by $\mathrm{II}_p(V, W)$ the second fundamental form of the cylinder $C_r(\gamma_0)$ calculated at $p$ on $(V, W)$. If we extend the vectors $V$, $W$ and $N(p)$ as Fermi fields on $T_R(\gamma_0)$ and set $p^\perp = \gamma_0(z(p))$, the following asymptotic expansion holds*:

$$\mathrm{II}_p(V, W) = -\frac{1}{r} g(\Pi_0 V, \Pi_0 W)(p^\perp) + r\big(S(V, N(p), W, N(p))(p^\perp) \\ - \tfrac{1}{3} S(\Pi_0 V, N(p), \Pi_0 W, N(p))(p^\perp)\big) + O(r^2),$$

*where, again, $S$ stands for the sectional curvature tensor.*

*Proof.* By definition [Gray 2004, p. 33; do Carmo 1992, p. 128], we have $\mathrm{II}_p(V, W) = g(-\nabla_V N, W)(p)$ and, here, one may allow the vectors $V$, $W$ to be arbitrary in $T_pM$ since $N$ is a vector field defined outside $C_r(\gamma_0)$. Covariant differentiation of the generalized Gauss lemma identity $g(N, \cdot) = d\mathfrak{r}$ on $\{\mathfrak{r} > 0\} \subset T_R(\gamma_0)$ yields

$$\mathrm{II}_p(V, W) = -\nabla d\mathfrak{r}(V, W)(p). \tag{10}$$

More generally, for each pair of vector fields $(A, B)$, we find $\nabla d\mathfrak{r}(A, B) = g(A, \nabla_B N) = g(B, \nabla_A N)$; hence also, using Lie brackets,

$$2\nabla d\mathfrak{r}(A, B) = N.g(A, B) + g(A, [B, N]) + g(B, [A, N]), \tag{11}$$

since $\nabla$ is torsionless. Taking a Fermi chart $x$ along $\gamma_0$ such that

$$x(p) = (r, \underbrace{0, \dots, 0}_{n-2}, x^n(p)),$$

let us calculate $\nabla d\rho(r, 0, x^n)$ using (11) with $A$ and $B$ equal to the $\partial/\partial x^i$. Note that $\nu(r, 0, x^n) = \partial/\partial x^1$ and $d\rho(r, 0, x^n) = dx^1$. From (1), we get $g_{1i}(r, 0, x^n) = \delta_{1i}$ and $N \cdot g(\partial/\partial x^1, \partial/\partial x^i)(r, 0, x^n) = 0$. From (2), we find $[\partial/\partial x^n, \nu](r, 0, x^n) = 0$ and

$$\left[\frac{\partial}{\partial x^\alpha}, \nu\right](r, 0, x^n) = \frac{1}{r}\left(\frac{\partial}{\partial x^\alpha} - \delta_{1\alpha}\frac{\partial}{\partial x^1}\right) \quad \text{for all } \alpha < n;$$

in particular, $[\partial/\partial x^1, \nu](r, 0, x^n) = 0$. Besides, for $i, j \in \{2, \dots, n\}$, we can derive the local expressions of $N.g(\partial/\partial x^i, \partial/\partial x^j)(r, 0, x^n) = (\partial g_{ij}/\partial x^1)(r, 0, x^n)$ from the following Riemann-type formulas extended to the Fermi setting [Spivak 1979; Delanoë and Ge 2010, Lemma 2]:

$$g_{ab}(x^1, \underbrace{0, \dots, 0}_{n-2}, x^n) = \delta_{ab} - \tfrac{1}{3}(x^1)^2 R_{a1b1}(\underbrace{0, \dots, 0}_{n-1}, x^n) + O((x^1)^3),$$

with $a, b \in \{2, \dots, n-1\}$, and

$$\begin{aligned} g_{an}(x^1, 0, x^n) &= -\tfrac{2}{3}(x^1)^2 R_{a1n1}(0, x^n) + O((x^1)^3), \\ g_{nn}(x^1, 0, x^n) &= 1 - (x^1)^2 R_{n1n1}(0, x^n) + O((x^1)^3), \end{aligned} \tag{12}$$

where $x^1$ stands for a small real parameter and $R_{ijkl}$ for the components of the sectional curvature tensor. Doing so, we obtain the expression

$$\begin{aligned} \nabla d\rho(r, 0, x^n) = \sum_{a=2}^{n-1}\sum_{b=2}^{n-1}\Big(\frac{1}{r}\delta_{ab} - \tfrac{2}{3} r R_{a1b1}(0, x^n) + O(r^2)\Big)\, dx^a \otimes dx^b \\ + \sum_{a=2}^{n-1}\big(-r R_{a1n1}(0, x^n) + O(r^2)\big)(dx^a \otimes dx^n + dx^n \otimes dx^a) \\ + \big(-r R_{n1n1}(0, x^n) + O(r^2)\big)\, dx^n \otimes dx^n. \end{aligned} \tag{13}$$

The latter combined with (10) yields the proposition. □

**Remark 1.7.** For later use, we record here that, if $n = 2$, recalling (1), the expansion of the metric in the Fermi chart $x$ becomes simply

$$g(x^1, x^2) = dx^1 \otimes dx^1 + \big(1 - (x^1)^2 K(0, x^2) + O((x^1)^3)\big)\, dx^2 \otimes dx^2,$$

where $K$ stands for the Gauss curvature of $M$. Accordingly, still from (11), the Hessian formula (13) becomes

$$\nabla d\rho(r, x^2) = \big(-r K(0, x^2) + O(r^2)\big)\, dx^2 \otimes dx^2.$$

## 2. Further properties when the Jacobi operator is negative

From the properties established is the preceding section for a thin tube about the geodesic $\gamma_0$, we will now derive stronger ones by assuming that the operator $\mathcal{J}_{\gamma_0}$ is negative, as done in Theorem 0.2. Specifically, using the notations of Corollary 1.3 and setting $\bar\kappa_0 = \max_{s \in [0, \ell_0]} \kappa_0^{n-1}(s)$, our assumption means that $\bar\kappa_0 < 0$; henceforth, it is implicitly made.

**Proposition 2.1** (the second fundamental form stays definite). *One can find a small real $r_3 > 0$ such that, for each $p \in T_{r_3}(\gamma_0)$ with $r = \mathfrak{r}(p) \neq 0$, the second fundamental form of $C_r(\gamma_0)$ at the point $p$ is negative definite.*

*Proof.* Let us take a Fermi chart $x$ at the point $p$ like the one used in the proof of Proposition 1.6 and write with it the expression of $\mathrm{II}_p(V, W)$ found in that proposition, with $V = W = \sum_{i=2}^{n} V^i \partial/\partial x^i \in T_p C_r(\gamma_0)$. We find

$$
\begin{aligned}
(14)\quad & \mathrm{II}_p(V, V) \\
&= -\frac{1}{2r}\sum_{a=2}^{n-1}(V^a)^2 + \frac{r}{2}R_{n1n1}(0, x^n)(V^n)^2 \\
&\quad - \frac{1}{4r}\left(\sum_{a=2}^{n-1}(V^a)^2 - 8r^2\sum_{a=2}^{n-1} R_{a1n1}(0, x^n)V^a V^n - 2r^2 R_{n1n1}(0, x^n)(V^n)^2\right) \\
&\quad - \frac{1}{4r}\sum_{a=2}^{n-1}\sum_{b=2}^{n-1} V^a V^b\big(\delta_{ab} - \tfrac{8}{3}r^2 R_{a1b1}(0, x^n)\big) + O(r^2)
\end{aligned}
$$

and the result readily follows from $R_{n1n1}(0, x^n) \leqslant \bar{\kappa}_0 < 0$, provided $r$ is taken small enough. □

**Proposition 2.2** (geodesics obey a maximum principle). *One can find a small real $r_4 > 0$ such that, for each geodesic path $t \in [0, 1] \to \gamma(t) \in T_{r_4}(\gamma_0)$, the following inequality holds*:

$$\max_{t\in[0,1]} \mathfrak{r}(\gamma(t)) \leqslant \max\big(\mathfrak{r}(\gamma(0)), \mathfrak{r}(\gamma(1))\big).$$

*Moreover, if $\mathfrak{r}(\gamma(\vartheta)) = \max\big(\mathfrak{r}(\gamma(0)), \mathfrak{r}(\gamma(1))\big)$ for some $\vartheta \in (0, 1)$, the path $\gamma$ must be constant.*

*Proof.* Anytime $t \in [0, 1] \to \gamma(t) \in T_R(\gamma_0)$ is a geodesic, at each $t \in [0, 1]$ such that $\mathfrak{r}(\gamma(t)) \neq 0$, we have

$$\frac{d^2}{dt^2}\big(\mathfrak{r}(\gamma(t))\big) = \nabla d\mathfrak{r}(\gamma(t))\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right).$$

If $n > 2$, combining (13) with (14) written with $V = \frac{d\gamma}{dt}$, we infer that the second derivative of the auxiliary real function $t \in [0, 1] \to \mathfrak{r}(\gamma(t))$ is *nonnegative* on $[0, 1]$ provided $\mathfrak{r}(\gamma(t)) \leqslant r_4 = r_3$. If $n = 2$, the same conclusion holds with $r_4$ small enough, due to Remark 1.7 read with $K(0, x^2) \leqslant \bar{\kappa}_0 < 0$. In any case, the maximum principle [Protter and Weinberger 1967] implies the first part of the proposition. Moreover, it yields $\mathfrak{r} \circ \gamma \equiv \mathfrak{r}(\gamma(\vartheta)) =: r_\vartheta > 0$; hence $(d\gamma/dt)(t) \in T_{\gamma(t)} C_{r_\vartheta}(\gamma_0)$ for each $t \in [0, 1]$. From (10) and Proposition 2.1 combined with

$$\frac{d^2}{dt^2}\big(\mathfrak{r}(\gamma(t))\big) \leqslant 0,$$

we infer that $d\gamma/dt \equiv 0$, so $\gamma$ must indeed be constant. □

Before moving on to the next property, we state a lemma of independent interest. (A reader parachuting to this point should understand it preceded by: "*Let $\gamma_0$ be an embedded unit-speed geodesic with negative Jacobi operator.*")

**Lemma 2.3.** *One can find a small real $r_5 > 0$ such that the inequality $g \geqslant d\mathfrak{r}_{\gamma_0}^2 + dz_{\gamma_0}^2$ between quadratic forms holds at each point of $\{p \in T_{r_5}(\gamma_0), \mathfrak{r}(p) > 0\}$.*

*Proof.* Take a point $p$ as stated and a Fermi chart $x$ along $\gamma_0$ such that $x(p) = (r, 0, \ldots, 0, x^n)$. From Remark 1.7 read with $K(0, x^2) \leqslant \bar{\kappa}_0 < 0$, the lemma appears straightforward if $n = 2$. In higher dimensions, from (1) and the expansion of $g_{ij}(x^1, 0, x^n)$ in (12), we infer that, for each vector $V = \sum_{i=1}^n V^i\, \partial/\partial x^i \in T_pM$, the quadratic form $(g - d\mathfrak{r}_{\gamma_0}^2 - dz_{\gamma_0}^2)(p)$ applied to $V$ can be expressed in the chart $x$, up to $O(r^3)$ terms, as the sum of two quadratic polynomials in $V$, namely $\sum_{a,b=2}^{n-1} \big(\frac{1}{2}\delta_{ab} - \frac{1}{3} r^2 R_{a1b1}(0, x^n)\big) V^a V^b$ and

$$\sum_{a=2}^{n-1} \big(\tfrac{1}{2} V_a^2 - \tfrac{4}{3} r^2 R_{a1n1}(0, x^n) V^a V^n\big) - r^2 R_{n1n1}(0, x^n)(V^n)^2.$$

By taking $r > 0$ small enough, and using $R_{n1n1}(0, x^n) \leqslant \bar{\kappa}_0 < 0$ for the second polynomial, we can make each polynomial nonnegative. □

**Proposition 2.4** ($\gamma_0$ is minimizing). *Take $r_5 > 0$ as in Lemma 2.3. The length of each piecewise $C^1$ path $t \in [0, 1] \to c(t) \in M$ ranging in $T_{r_5}(\gamma_0)$ with $z(c(0)) = 0$ and $z(c(1)) = \ell_0$ must be at least equal to $\ell_0$. Furthermore, if equality holds and $\mathfrak{r} \circ c(t) = 0$ for some $t \in [0, 1]$ then $c$, reparametrized by an arc-length parameter suitably shifted to avoid jumps*[7] *on each subinterval of $[0, 1]$ in the interior of which $c$ is $C^1$ and $dc/dt \neq 0$, coincides with $\gamma_0$.*

*Proof.* Let $c$ be a path as stated and $x$ a Fermi chart along $\gamma_0$. From Lemma 2.3, the length of $c$ satisfies

$$\ell \geqslant \int_0^1 \sqrt{\left(\frac{d}{dt}(\rho \circ c)\right)^2 + \left(\frac{dc^n}{dt}\right)^2}\, dt.$$

Therefore, if $\int_0^1 \big|(d/dt)(\rho \circ c)\big|\, dt \neq 0$, we have $\ell > \int_0^1 |dc^n/dt|\, dt \geqslant \ell_0$ as asserted. Moreover, if $\ell = \ell_0$, we see that $(d/dt)(\rho \circ c)$ must vanish, hence also $(\rho \circ c)$ anytime it does at some $t \in [0, 1]$. In that case, the images of $c$ and $\gamma_0$ coincide, so $|dc/dt| = \|dc/dt\| = |dc^n/dt|$ and $\int_0^1 |dc^n/dt|\, dt = \ell_0 = c^n(1) - c^n(0) = \int_0^1 (dc^n/dt)\, dt$. The latter equality implies that $dc^n/dt \geqslant 0$, so the path $c$, reparametrized by arc length as stated, must indeed coincide with $\gamma_0$. □

[7] By taking the initial value of the parameter on a subinterval equal to (zero, of course, on the first subinterval and elsewhere to) the final value of the parameter on the preceding subinterval.

**Proposition 2.5** (long geodesics have a negative Jacobi operator). *Given $\delta > 0$, we can find $r_6 \in (0, \min(R_1, r_1)]$ such that, for each $r \in (0, r_6)$ and each unit-speed geodesic $\sigma \in [0, \ell] \to \gamma(\sigma) \in T_r(\gamma_0)$ with length $\ell \geqslant \delta$, the Jacobi operator $\mathcal{J}_\gamma$ is negative, or else $\max_{s\in[0,\ell]} \kappa_\gamma^{n-1}(s) < 0$.*

*Proof.* Let $k = k(r)$ be the affine function of $r$ defined in Corollary 1.3 and $r^+$ be the positive root of the quadratic equation $rk(r) + \bar{\kappa}_0 = 0$; the proposition holds with $r_6 = \min(R_1, r_1, r^+)$ by Corollary 1.3. □

**Proposition 2.6** (each geodesic is minimizing). *One can find a small real $r_7 > 0$ such that, for each unit-speed geodesic $s \in [0, \ell] \to \gamma(s) \in M$ and each piecewise $C^1$ path $t \in [0, 1] \to c(t) \in M$, both ranging in $T_{r_7}(\gamma_0)$ with $c(0) = \gamma(0)$, $c(1) = \gamma(\ell)$, the length of c must be at least equal to $\ell$. Moreover, equality holds if and only if c, reparametrized by a suitable arc length parameter on each subinterval of* $[0, 1]$ *in the interior of which c is $C^1$ and $dc/dt \neq 0$, coincides with $\gamma$.*

*Proof.* Let $\gamma$ be a geodesic of length $\ell$ as stated. The proposition is obvious if $\ell < \mathrm{i}$. If $\ell \geqslant \mathrm{i}$, which we suppose in the proof, we may use Propositions 1.2 and 1.4 read with $\delta = \mathrm{i}$; the radii $r_1$ and $r_2$ are understood accordingly and we take $r_7 \leqslant r_2$. In this situation, we know that $\gamma$ is an embedding and there exists a Fermi chart $x_\gamma$ along $\gamma$ whose domain $T_\eta(\gamma)$ contains $T(\gamma_0, \gamma)$ but the $(\gamma_0, \gamma)$-exceptional points.

Our next task is the main one; namely, we must specify how the radius $\eta$ of that tubular domain can be controlled by $r_7$. By inspecting the proof of Proposition 1.4, we see (sticking to its notations, except for $\gamma^*$ now written $\gamma$, so $m^* = \gamma(0)$, $u^* = (d\gamma/ds)(0)$) that such a control amounts to a similar one on

$$\left\|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\right\|^2 = \sum_{i=1}^{n} \left(\mathcal{V}_0^{*i}(x^*, \Pi_0 u_0^*, x)\right)^2,$$

where $x^*$, $\Pi_0 u_0^*$, $x$ satisfy the bounds (9) now read with $\epsilon = r_7$ and where $\mathcal{V}_0^{*n}$ has to be defined by

$$\mathcal{V}_0^{*n} = -\frac{1}{u_0^{*n}} \sum_{\alpha=1}^{n-1} u_0^{*\alpha} \mathcal{V}_0^{*\alpha} \quad \text{with } u_0^{*n} = \pm\sqrt{1 - \sum_{\alpha=1}^{n-1} (u_0^{*\alpha})^2}.$$

Furthermore, as $r_7 \downarrow 0$, we know that $\sum_{\alpha=1}^{n-1} (\mathcal{V}_0^{*\alpha})^2$ tends to zero. All we require is thus a uniform positive lower bound on $|u_0^{*n}|$. Such a bound will follow from (6) and Lemma 1.5. Indeed, the former combined with Proposition 1.1 implies here that

$$\left|\frac{d\gamma^n}{ds}\right| \geqslant 1 - \left(c_1 \Theta_0^2 + \frac{1}{\theta_0^2}\left(\frac{4}{\mathrm{i}} + 2c_1\Theta_0^2(\ell_0 + \mathrm{i})\right)^2\right) r_7^2,$$

which in turn yields $|u_0^{*n}| \geqslant |d\gamma^n/ds| - c_1 \Theta_0^2 r_7^2$. Thus we get

$$|u_0^{*n}| \geqslant 1 - \left(2c_1\Theta_0^2 + \frac{1}{\theta_0^2}\left(\frac{4}{\mathfrak{i}} + 2c_1\Theta_0^2(\ell_0+\mathfrak{i})\right)^2\right) r_7^2.$$

Defining $\mathfrak{r}_1 > 0$ by, say,

$$\mathfrak{r}_1^2\left(2c_1\Theta_0^2 + \frac{1}{\theta_0^2}\left(\frac{4}{\mathfrak{i}} + 2c_1\Theta_0^2(\ell_0+\mathfrak{i})\right)^2\right) = \frac{1}{2},$$

and taking $r_7 \leqslant \mathfrak{r}_1$, we obtain $|u_0^{*n}| \geqslant \frac{1}{2}$. Now, it is clear that $\|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\|$ tends to zero as $r_7 \downarrow 0$. Here, among the arguments of $\mathcal{V}_0^*$, we are given the first one, since $x^* = x(\gamma(0))$; similarly for the second one, since $\Pi_0 u_0^*$ is defined out of $(d\gamma/ds)(0)$; the sole variable is the third one, since $x = x(m)$ with $m \in T(\gamma_0, \gamma) \cap T_\eta(\gamma)$. Moreover, using the aforementioned Fermi chart $x_\gamma$, the identity

$$\rho(x_\gamma) = \|\mathcal{V}_0^*(x^*, \Pi_0 u_0^*, x)\|$$

holds. So $\rho(x_\gamma) \downarrow 0$ as $r_7 \downarrow 0$, which shows that the implicit function theorem used in the proof of Proposition 1.4 allows us to let $\eta$ go to zero as $r_7 \downarrow 0$.

Besides, Proposition 2.5 read with $\delta = \mathfrak{i}$ implies that, if we take $r_7 < r_6$, the Jacobi operator of $\gamma$ is negative.

We conclude that there exists $r_7 > 0$ small enough such that, if $\gamma$ ranges in $T_{r_7}(\gamma_0)$, the radius $\eta$ of the tube about $\gamma$ provided by Proposition 1.4 may be taken small enough such that Lemma 2.3 and Proposition 2.4 hold for the geodesic $\gamma$ in $T_\eta(\gamma)$.

Now, we are in position to complete the proof of Proposition 2.6. Let $c$ be a path as stated. By the definition of $T(\gamma_0, \gamma)$, the smallness of $r_7$ (hence of $\eta$) and the property of $T_\eta(\gamma)$ proved in Proposition 1.4, there exists a closed interval contained in $[0, 1]$ such that the restriction $\bar{c}$ of $c$ to this interval fulfills the assumption of Proposition 2.4 (read in $T_\eta(\gamma)$ instead of $T_{r_5}(\gamma_0)$). So we get the inequalities $L =$ length of $c \geqslant$ length of $\bar{c} \geqslant \ell =$ length of $\gamma$, which proves the first part of the proposition. Moreover, if $L = \ell$, the images of the paths $c$ and $\bar{c}$ must coincide, so $\bar{c}$ shares with $\gamma$ the same endpoints and the last part of Proposition 2.6 follows from that of Proposition 2.4. □

**Corollary 2.7** (each geodesic is uniquely determined by its endpoints). *Take $r_7 > 0$ as in Proposition 2.6. For each $(p, q) \in T_{r_7}(\gamma_0) \times T_{r_7}(\gamma_0)$, there exists at most one unit-speed geodesic of $\gamma : [0, \ell] \to M$ entirely lying in $T_{r_7}(\gamma_0)$ with $\gamma(0) = p$, $\gamma(\ell) = q$.*

*Proof.* We argue by contradiction. If two distinct unit-speed geodesics of $M$ entirely lying in $T_{r_7}(\gamma_0)$ had the same endpoints, Proposition 2.6 would imply that the length of each geodesic is at least equal to the length of the other; so the geodesics would have equal length. Still by Proposition 2.6, the geodesics would thus coincide, which is absurd. □

## 3. Proof of Theorem 0.2

*Reduction of the proof.* We only have to prove the existence of a radius $r > 0$ such that each pair of points of the tube $T_r(\gamma_0)$, located as stated in Theorem 0.2, can be joined by a geodesic with interior lying in $(T_r(\gamma_0))^\circ$. Indeed, suppose we have done so. Then, each such geodesic must be unique (by Corollary 2.7) and minimizing among piecewise $C^1$ paths sharing the same endpoints and lying in $T_r(\gamma_0)$ (by Proposition 2.6), so the proof is complete.

*Strategy.* Fixing $p \in T_r(\gamma_0)$, let us consider the subsets

$$\mathfrak{X}_p^+ = \{m \in T_r(\gamma_0), m \neq p, z(m) \geqslant z(p) \text{ and, if } z(p) = 0 \text{ or } \ell_0, z(m) \neq z(p)\},$$
$$\mathfrak{X}_p^- = \{m \in T_r(\gamma_0), m \neq p, z(m) \leqslant z(p) \text{ and, if } z(p) = 0 \text{ or } \ell_0, z(m) \neq z(p)\}.$$

Assuming $z(p) < \ell_0$, we will prove Theorem 0.2 for $q \in \mathfrak{X}_p^+$. Assuming $z(p) > 0$, we would prove it similarly for $q \in \mathfrak{X}_p^-$. Let us proceed to the proof itself. We distinguish two cases.

Case 1: $z(q) - z(p) < \mathfrak{c}/2$. For $\lambda \in [0, 1]$, set $p_\lambda^\perp = [\gamma_0, p](\lambda\mathfrak{r}(p))$ and $q_\lambda^\perp = [\gamma_0, q](\lambda\mathfrak{r}(q))$. Take $r < \mathfrak{c}/2$. Then, for each $\lambda \in [0, 1]$, the points $p_\lambda^\perp$ and $q_\lambda^\perp$ lie in the Riemannian ball $\{m \in M, d(m_0^\perp, m) < \varrho\}$ with center $m_0^\perp = \gamma_0\big(\frac{1}{2}(z(p) + z(q))\big)$ and radius $\varrho = \mathfrak{c}/2 + r < \mathfrak{c}$. Hence there exists a unique minimizing geodesic $c_\lambda : [0, 1] \to M$ going from $p_\lambda^\perp$ to $q_\lambda^\perp$ and such that, for each $t \in [0, 1]$, the map $\lambda \in [0, 1] \to c_\lambda(t) \in M$ is smooth. We must prove that $c_1((0, 1)) \subset (T_r(\gamma_0))^\circ$. To do so, let us argue by connectedness on the set

$$\Lambda = \{\lambda \in [0, 1], c_\lambda((0, 1)) \subset (T_r(\gamma_0))^\circ\}.$$

By construction, $\Lambda$ is nonempty ($0 \in \Lambda$) and relatively open in $[0, 1]$, so we only have to prove that $\Lambda$ is closed. Letting $(\lambda_i)_{i\in\mathbb{N}}$ be a sequence of $\Lambda$ and $\lambda_\infty = \lim_{i\to\infty} \lambda_i \in [0, 1]$, it amounts to prove that $c_{\lambda_\infty}((0, 1)) \subset (T_r(\gamma_0))^\circ$. By continuity, the geodesic $c_{\lambda_\infty}$ ranges in $T_r(\gamma_0)$. If $c_{\lambda_\infty}(\theta) \in C_r(\gamma_0)$ for some $\theta \in (0, 1)$, Proposition 2.2 implies that $c_{\lambda_\infty}$ is constant, so $p_{\lambda_\infty}^\perp = q_{\lambda_\infty}^\perp$. But the latter yields $p = q$, contradicting the assumption $q \in \mathfrak{X}_p^+$.

We are left with ruling out the following property:

$$z(c_{\lambda_\infty}(\theta)) = 0 \text{ or } \ell_0 \quad \text{for some } \theta \in (0, 1). \tag{15}$$

To do so, given $\delta > 0$, we distinguish two subcases as stated in Theorem 0.2.

*Subcase 1*: $n = 2$. If (15) held, the vector $(dc_{\lambda_\infty}/dt)(\theta)$ would necessarily belong to $\ker dz \setminus \{0\}$. But then, the geodesic $t \mapsto c_{\lambda_\infty}(t)$ would stay for all $t \in [0, 1]$ in the end of the tube given by the equation $z = z(c_{\lambda_\infty}(\theta))$ because, when $n = 2$, the latter is *totally geodesic*. We reach a contradiction, since we have assumed that $z(p) < \ell_0$ and, if $z(p) = 0$, $z(q) \neq 0$.

*Subcase 2*: *$n > 2$ and either $|z(p) - z(q)| \geqslant \varsigma$ or $\varsigma \leqslant z(p) \leqslant z(q) \leqslant \ell_0 - \varsigma$.* If $|z(p) - z(q)| \geqslant \varsigma$, the length $\ell_{\lambda_\infty}$ of the geodesic $c_{\lambda_\infty}$ must be bounded below by $\varsigma$ due to Lemma 2.3. It follows that $dc^n_{\lambda_\infty}/dt > 0$ if $r > 0$ is taken small enough, due to Proposition 1.2 read with $\delta = \varsigma$. So, in that case, the property (15) cannot hold. If instead $\varsigma \leqslant z(p) \leqslant z(q) \leqslant \ell_0 - \varsigma$, with $|z(p) - z(q)| < \varsigma$, the latter inequality yields $\ell_{\lambda_\infty} \leqslant \varsigma + 2r$, while the former pinching combined with Lemma 2.3 yields $\ell_{\lambda_\infty} \geqslant 2\varsigma$ if (15) holds. In that case, we get the lower bound $r \geqslant \varsigma$ which is absurd, provided $r < \varsigma$. In either case, we conclude that (15) cannot occur for $r > 0$ small enough.

Having proved that $\lambda_\infty \in \Lambda$, we conclude that $\Lambda$ is closed and hence equal to $[0, 1]$. In particular, $1 \in \Lambda$ so Case 1 is settled.

<u>Case 2</u>: $z(q) - z(p) \geqslant \mathfrak{c}/2$. Here, reading the constant $r_1$ from Proposition 1.2 with $\delta = \mathfrak{c}/2$, we take $r > 0$ small as done in Proposition 2.5. Furthermore, we consider the subset of the interval $[z(p), \ell_0]$ defined by

$$\mathfrak{Z}_p^+ = \big\{\mathfrak{z} \in [z(p), \ell_0], \forall m \in \mathfrak{X}_p^+, z(m) = \mathfrak{z} \Longrightarrow T_r(\gamma_0) \text{ is strongly convex for } (p, m)\big\}.$$

By construction, if $\mathfrak{z} \in \mathfrak{Z}_p^+$, the whole interval $[z(p), \mathfrak{z}]$ must lie in $\mathfrak{Z}_p^+$ and, by Case 1, we know that $\mathfrak{Z}_p^+$ contains the interval $\big[z(p), z(p) + \mathfrak{c}/2\big)$. In the next two lemmas, we prove that $\mathfrak{Z}_p^+$ is both closed and relatively open in $[z(p), \ell_0]$. Granted it is, by connectedness, it must coincide with $[z(p), \ell_0]$; hence Theorem 0.2 is established when $z(p) < \ell_0$ and $q \in \mathfrak{X}_p^+$. The proof when $z(p) > 0$ and $q \in \mathfrak{X}_p^-$ is similar. $\square$

**Lemma 3.1.** *The subset $\mathfrak{Z}_p^+$ is closed.*

**Lemma 3.2.** *The subset $\mathfrak{Z}_p^+$ is relatively open in $[z(p), \ell_0]$.*

*Proof of Lemma 3.1.* Let $(z_i)_{i \in \mathbb{N}}$ be a sequence of $\mathfrak{Z}_p^+$; set $\mathfrak{z} = \lim_{i \to \infty} z_i \in [z(p), \ell_0]$. We must prove that $\mathfrak{z} \in \mathfrak{Z}_p^+$, so we may assume with no loss of generality that $\mathfrak{z} \geqslant z(p) + \mathfrak{c}/2$. Fix $m \in \mathfrak{X}_p^+$ satisfying $z(m) = \mathfrak{z}$ and let $(m_i)_{i \in \mathbb{N}}$ be a sequence of $\mathfrak{X}_p^+$ such that, for all $i \in \mathbb{N}$, $z(m_i) = z_i$ and $\lim_{i \to \infty} m_i = m$. For each $i \in \mathbb{N}$, set $t \in [0, 1] \to c_i(t) \in M$ for the unique minimizing geodesic such that $c_i(0) = p$, $c_i(1) = m_i$ and $c_i((0, 1)) \subset (T_r(\gamma_0))^\circ$. By Proposition 1.1, the sequence $\big((dc_i/dt)(0)\big)_{i \in \mathbb{N}}$ is bounded in $T_pM$; it thus converges toward a vector $V \in T_pM$. By continuity of the map $\exp_p : T_pM \to M$, the geodesic $t \in [0, 1] \to \exp_p(tV) \in M$ (let us denote it by $c$) satisfies $c(0) = p$, $c(1) = m$ and $c([0, 1]) \subset T_r(\gamma_0)$. For each $t \in (0, 1)$, Proposition 1.2 implies that $z(c(t)) \in (z(p), z(m))$ while, taking $r \leqslant r_4$, we know that $\mathfrak{r}(c(t)) < r$ by Proposition 2.2, so the inclusion $c((0, 1)) \subset (T_r(\gamma_0))^\circ$ must hold. Finally, by Proposition 2.6 and Corollary 2.7, the geodesic $c$ must be minimizing and unique in $T_r(\gamma_0)$. In other words, we have proved that $T_r(\gamma_0)$ is strongly convex for $(p, m)$. Since the point $m$ is arbitrary, we conclude that $\mathfrak{z} \in \mathfrak{Z}_p^+$ as desired. $\square$

*Proof of Lemma 3.2.* Pick $\mathfrak{z} \in \mathfrak{Z}_p^+$ and $m \in \mathfrak{X}_p^+$ with $z(m) = \mathfrak{z}$. We may take $\mathfrak{z} \in [z(p) + \mathfrak{c}/2, \ell_0)$ without loss, due to Lemma 3.1. Let $t \in [0, 1] \to c_m(t) \in M$ be the geodesic such that $c_m(0) = p$, $c_m(1) = m$ and $c_m((0, 1)) \subset (T_r(\gamma_0))^\circ$. By Proposition 2.5, the Jacobi operator of $c_m$ is negative. Therefore the tangent map $d(\exp_p)\big((dc_m/dt)(0)\big) : T_pM \to T_mM$ is invertible [Aubin 1998, pp. 17–18; do Carmo 1992, pp. 117, 149; Milnor 1963, pp. 98, 100]. The inverse function theorem [Lang 2002] yields a real $\epsilon_m > 0$ such that each point $m'$ lying in the Riemannian ball $B(m, \epsilon_m)$ can be joined to the point $p$ by a unique geodesic $t \in [0, 1] \to c_{m'}(t) = \exp_p(tV') \in M$ with $V' \in T_pM$ close to $V_m = (dc_m/dt)(0)$. Possibly shrinking $\epsilon_m > 0$, we take it such that $z(p) + \mathfrak{c}/4 \leqslant z < \ell_0$ on $\overline{B(m, \epsilon_m)}$. Since the level set $T_r(\gamma_0) \cap \{z = \mathfrak{z}\}$ is compact, it can be covered by the union of finitely many balls $B_i = B(m_i, \epsilon_i)$, $i \in \{1, \dots, N\}$, each constructed like the ball $B(m, \epsilon_m)$. There exists $\theta > 0$ such that the level set $T_r(\gamma_0) \cap \{z = \mathfrak{z} + \theta\}$ remains covered by $\bigcup_{i=1}^N B_i$.

**Claim.** *The subset $\mathfrak{Z}_p^+$ contains $\mathfrak{z} + \theta$.*

The claim, provisionally taken for granted, implies that $[z(p), \mathfrak{z} + \theta] \subset \mathfrak{Z}_p^+$, so Lemma 3.2, indeed, holds. □

*Proof of the claim.* Pick $m' \in \mathfrak{X}_p^+$ with $z(m') = \mathfrak{z} + \theta$. There exists $i \in \{1, \dots, N\}$ such that $m' \in B_i$. So $m' = \exp_p(V') \in M$ for a unique vector $V' \in T_pM$ close to $V_i = (dc_{m_i}/dt)(0)$. Moreover, there exists a unique geodesic path $\lambda \in [0, 1] \to \mathfrak{m}(\lambda) \in M$ ranging in $B_i$ such that $\mathfrak{m}(0) = m_i$, $\mathfrak{m}(1) = m'$. Let $\lambda \in [0, 1] \to V_\lambda \in T_pM$ be the corresponding path, derived (like $V'$) from the inverse function theorem as done above, such that $\exp_p(V_\lambda) \equiv \mathfrak{m}(\lambda)$. Set $t \in [0, 1] \to \gamma_\lambda(t) \in M$ for the geodesic path given by $\gamma_\lambda(t) = \exp_p(tV_\lambda)$. From the pinching $z(p) + \mathfrak{c}/4 \leqslant z(\mathfrak{m}(\lambda)) < \ell_0$ combined with Proposition 2.2, we know that $\mathfrak{m}((0, 1)) \subset (T_r(\gamma_0))^\circ$. Let us argue by connectedness on the subset of the interval $[0, 1]$ given by

$$\mathrm{L} = \big\{\lambda \in [0, 1], \gamma_\lambda((0, 1)) \subset (T_r(\gamma_0))^\circ\big\},$$

which is nonempty ($0 \in \mathrm{L}$). The closedness of L can readily be established, arguing as we did for that of $\mathfrak{Z}_p^+$. Let us focus on proving that L is relatively open in $[0, 1]$. If $\lambda \in \mathrm{L}$, the continuity of $\exp_p$ implies the existence of $\mu > 0$ such that $\gamma_{\lambda'}([0, 1]) \subset T_{2r}(\gamma_0)$ for each $\lambda' \in (\lambda - \mu, \lambda + \mu) \cap [0, 1]$. By Lemma 2.3, taking $2r \leqslant r_5$, we know that the length of the geodesic $\gamma_{\lambda'}$ is at least equal to $\mathfrak{c}/4$. By Proposition 1.2 read in $T_{2r}(\gamma_0)$ with $\delta = \mathfrak{c}/4$, we can take $r > 0$ small enough such that $d\gamma_{\lambda'}/dt > 0$; hence $z(\gamma_{\lambda'}((0, 1])) \subset (z(p), \ell_0)$. Furthermore, taking $2r \leqslant r_4$ and applying Proposition 2.2, we get $\mathfrak{r}(\gamma_{\lambda'}(t)) < r$ for $t \in (0, 1)$. It follows that $\lambda' \in \mathrm{L}$; in other words, L is relatively open in $[0, 1]$. By connectedness, we get $\mathrm{L} = [0, 1]$. In particular, $1 \in \mathrm{L}$, from which we readily infer that $m' \in \mathfrak{X}_p^+$. Since $m'$ is arbitrary, we conclude $\mathfrak{z} + \theta \in \mathfrak{Z}_p^+$, as claimed. □

## 4. Proof of Corollary 0.3

The assumption made in Theorem 0.2 on the geodesic $\gamma_0$ is an *open* condition. Given a small real $\varsigma > 0$, we can thus find $r > 0$ such that Theorem 0.2 holds for the geodesic $s \in [-r, \ell_0 + r] \to \gamma_r(s) \in M$ defined as the extension of the geodesic $\gamma_0$ to the interval $[-r, \ell_0 + r]$. There still exists a Fermi map about the extended geodesic $\gamma_r$; let us stick to our preceding notations for this map. It is important to note the inclusion

$$N_r(\gamma_0) \subset T_r(\gamma_r) \tag{16}$$

which follows from those of $\overline{B(\gamma_0(0), r)}$ and $\overline{B(\gamma_0(\ell_0), r)}$ in $T_r(\gamma_r)$ combined with the identity $N_r(\gamma_0) \equiv T_r(\gamma_0) \cup \overline{B(\gamma_0(0), r)} \cup \overline{B(\gamma_0(\ell_0), r)}$. Given a pair of points $(p, q)$ in $N_r(\gamma_0)$, say, with $z(p) \leqslant z(q)$, we must prove that $N_r(\gamma_0)$ is strongly convex for $(p, q)$. To do so, it suffices to construct a geodesic path from $p$ to $q$ ranging in $(N_r(\gamma_0))^\circ$. Indeed, by (16) combined with Proposition 2.6 and Corollary 2.7 applied in $T_r(\gamma_r)$, such a geodesic path will necessarily be minimizing and unique in $N_r(\gamma_0)$. From Theorem 0.2 applied in $T_r(\gamma_0) \subset N_r(\gamma_0)$, we only have to treat the following two cases.

<u>Case 1</u>: $z(q) - z(p) \geqslant \varsigma$ and either $z(p) < 0$ or $z(q) > \ell_0$. By Theorem 0.2, the tube $T_r(\gamma_r)$ is strongly convex for $(p, q)$. Let $t \in [0, 1] \to \gamma(t) \in M$ denote the geodesic from $\gamma(0) = p$ to $\gamma(1) = q$ such that $\gamma((0, 1)) \subset (T_r(\gamma_r))^\circ$. We must prove that $\gamma((0, 1)) \subset (N_r(\gamma_0))^\circ$. By Proposition 1.2, we know that $d(z \circ \gamma)/dt > 0$ while, by Proposition 2.2, we have $\mathfrak{r} \circ \gamma < r$ on $(0, 1)$. We may assume with no loss of generality the existence of $T \in (0, 1)$ such that either $z(\gamma(T)) = 0$ or $z(\gamma(T)) = \ell_0$. If the former occurs, the restriction of $\gamma$ to the subinterval $[0, T]$ is minimizing in $T_r(\gamma_r) \cap \{-r \leqslant z \leqslant 0\}$ among piecewise $C^1$ paths joining $p$ to $\gamma(T)$. Besides, the ball $\overline{B(\gamma_0(0), r)}$ being strongly convex, there exists a unique minimizing geodesic $\tau \in [0, 1] \to c(\tau) \in M$ such that $c(0) = p$, $c(1) = \gamma(T)$, $c((0, 1]) \subset (B(\gamma_0(0), r))^\circ$. By uniqueness and due to (16), these geodesics must coincide: $c(\tau) \equiv \gamma(\tau T)$. In particular, we do have $\gamma((0, T]) \subset (B(\gamma_0(0), r))^\circ$. Similarly, if the latter occurs, the restriction of $\gamma$ to the subinterval $[T, 1]$ is minimizing in $T_r(\gamma_r) \cap \{\ell_0 \leqslant z \leqslant \ell_0 + r\}$ among piecewise $C^1$ paths joining $\gamma(T)$ to $q$. The ball $\overline{B(\gamma_0(\ell_0), r)}$ being strongly convex, there exists a unique minimizing geodesic $\tau \in [0, 1] \to c(\tau) \in M$ such that $c(0) = \gamma(T)$, $c(1) = q$ and $c([0, 1)) \subset (B(\gamma_0(\ell_0), r))^\circ$. Again, these geodesics must coincide: $c(\tau) \equiv \gamma(\tau + (1 - \tau)T)$. In particular, we do have $\gamma([T, 1)) \subset (B(\gamma_0(\ell_0), r))^\circ$. Case 1 is settled.

<u>Case 2</u>: $z(q) - z(p) < \varsigma$ and either $z(p) < \varsigma$ or $z(q) > \ell_0 - \varsigma$. Here, we may assume that the points $p$ and $q$ lie in the closure of a strongly convex ball $B$ and argue as in Case 1 of the proof of Theorem 0.2, with $T_r(\gamma_0)$ now replaced by $N_r(\gamma_0)$. Doing so, the present proof is reduced to ruling out the analogue of (15),

namely the property

$$c_{\lambda_\infty}(\theta)\in\left[\partial B(\gamma_0(0),r)\cap\{z<0\}\right]\cup\left[\partial B(\gamma_0(\ell_0),r)\cap\{z>\ell_0\}\right]\quad\text{for some }\theta\in(0,1).$$

This can be done by observing that the geodesic $t\in[0,1]\to c_{\lambda_\infty}(t)\in M$ is *minimizing* from $p^{\perp}_{\lambda_\infty}$ to $q^{\perp}_{\lambda_\infty}$ and by relying on the inclusion (16) combined with the strong convexity of the balls $\overline{B(\gamma_0(0),r)}$ and $\overline{B(\gamma_0(\ell_0),r)}$; we leave it as an exercise.

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