Pacific Journal of Mathematics

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Volume 264 No. 2 August 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 264, No. 2, 2013
dx.doi.org/10.2140/pjm.2013.264.333

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We obtain upper bounds for the eigenvalues of the Schrödinger operator $L = \Delta_g + q$ depending on integral quantities of the potential q and a conformal invariant called the *min-conformal volume*. When the Schrödinger operator L is positive, integral quantities of q appearing in upper bounds can be replaced by the mean value of the potential q. The upper bounds we obtain are compatible with the asymptotic behavior of the eigenvalues. We also obtain upper bounds for the eigenvalues of the weighted Laplacian or the Bakry–Émery Laplacian $\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g$ using two approaches: first, we use the fact that Δ_ϕ is unitarily equivalent to a Schrödinger operator and we get an upper bound in terms of the L^2-norm of $\nabla_g \phi$ and the min-conformal volume; second, we use its variational characterization and we obtain upper bounds in terms of the L^∞-norm of $\nabla_g \phi$ and a new conformal invariant. The second approach leads to a Buser type upper bound and also gives upper bounds that do not depend on ϕ when the Bakry–Émery Ricci curvature is nonnegative.

1. Introduction and statement of results

We study upper bound estimates for the eigenvalues of Schrödinger operators and weighted Laplace operators or Bakry–Émery Laplace operators.

The author has benefited from the support of the *boursier du gouvernement Français* during her stay in Tours.

MSC2010: 35P15, 47A75, 58J50.

Keywords: Schrödinger operator, Bakry–Émery Laplace operator, eigenvalue, upper bound, conformal invariant.

The Schrödinger operator. Let (M, g) be a compact Riemannian manifold of dimension m and $q \in C^0(M)$. The eigenvalues of the Schrödinger operator $L := \Delta_g + q$ acting on functions constitute a nondecreasing, semibounded sequence of real numbers going to infinity:

$$\lambda_1(\Delta_g + q) \le \lambda_2(\Delta_g + q) \le \cdots \le \lambda_k(\Delta_g + q) \le \cdots \nearrow \infty.$$

The well-known Weyl law, which describes the asymptotic behavior of the eigenvalues of the Laplacian [Bérard 1986], can be easily extended to the eigenvalues of Schrödinger operators on compact Riemannian manifolds:

$$\lim_{k\to\infty} \lambda_k(\Delta_g + q)\left(\frac{\mu_g(M)}{k}\right)^{2/m} = \alpha_m, \tag{1}$$

where $\alpha_m = 4\pi^2\omega_m^{-2/m}$ and ω_m is the volume of the unit ball in $\mathbb{R}^m$. This says that the normalized eigenvalues $\lambda_k(\Delta_g + q)(\mu_g(M)/k)^{2/m}$ asymptotically tend to a constant depending only on the dimension. However, upper bounds of normalized eigenvalues in general cannot be independent of geometric invariants and the potential q; see [Colbois and Dodziuk 1994] or the introduction of [Hassannezhad 2011]. We shall obtain upper bounds for normalized eigenvalues depending on some geometric invariants and integral quantities of the potential q. These upper bounds are compatible with the asymptotic behavior in (1); that is, they tend asymptotically to a constant depending only on the dimension as k goes to infinity.

Numerous articles have studied how the eigenvalues of L can be controlled in terms of geometric invariants of the manifold and quantities depending on the potential. From the variational characterization of eigenvalues, it is easy to see that

$$\lambda_1(\Delta_g + q) \le \frac{1}{\mu_g(M)} \int_M q \, d\mu_g.$$

For the second eigenvalue $\lambda_2(\Delta_g + q)$, El Soufi and Ilias [1992, Theorem 2.2] obtained an upper bound in terms of the mean value of the potential q and a conformal invariant:

$$\lambda_2(\Delta_g + q) \le m\left(\frac{V_c([g])}{\mu_g(M)}\right)^{2/m} + \frac{\int_M q \, d\mu_g}{\mu_g(M)}, \tag{2}$$

where $V_c([g])$ is the conformal volume defined by Li and Yau [1982] which only depends on the conformal class of g, denoted by $[g]$.

For a compact orientable Riemannian surface (Σ_γ, g) of genus γ, as a consequence of inequality (2), they obtained the following inequality, where $\lfloor\ \rfloor$ denotes the floor function:

$$\lambda_2(\Delta_g + q) \le \frac{8\pi}{\mu_g(\Sigma_\gamma)}\left\lfloor \frac{\gamma+3}{2} \right\rfloor + \frac{\int_{\Sigma_\gamma} q \, d\mu_g}{\mu_g(\Sigma_\gamma)}. \tag{3}$$

For higher eigenvalues of Schrödinger operators, Grigor'yan, Netrusov and Yau [Grigor'yan et al. 2004] proved a general and abstract result that can be stated in the case of Schrödinger operators as follows. Given positive constants N and C_0, assume that a compact Riemannian manifold (M, g) has the $(2, N)$-covering property (that is, each ball of radius r can be covered by N balls of radius $r/2$) and $\mu_g(B(x, r)) \leq C_0 r^2$ for every $x \in M$ and every $r > 0$. Then, for every $q \in C^0(M)$, we have (see [Grigor'yan et al. 2004, Theorem 1.2 (1.14)])

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \delta^{-1} \int_M q^+ \, d\mu_g - \delta \int_M q^- \, d\mu_g}{\mu_g(M)}, \tag{4}$$

where $\delta \in (0, 1)$ is a constant which depends only on N, $C > 0$ is a constant which depends on N and C_0, and $q^{\pm} = \max\{|\pm q|, 0\}$.

Moreover, if L is a positive operator, then we have (see [Grigor'yan et al. 2004, Theorem 5.15])

$$\lambda_k(\Delta_g + q) \leq \frac{Ck + \int_M q \, d\mu_g}{\epsilon \mu_g(M)}, \tag{5}$$

where $\epsilon \in (0, 1)$ depends only on N and C depends on N and C_0.

The above inequalities in dimension two have a special feature as follows. Let Σ_γ be a compact orientable Riemannian surface of genus γ. Then, for every Riemannian metric g on Σ_γ and every $q \in C^0(\Sigma_\gamma)$, we have (see [Grigor'yan et al. 2004, Theorem 5.4])

$$\lambda_k(\Delta_g + q) \leq \frac{Q(\gamma + 1)k + \delta^{-1} \int_{\Sigma_\gamma} q^+ d\mu_g - \delta \int_{\Sigma_\gamma} q^- d\mu_g}{\mu_g(\Sigma_\gamma)},$$

where $\delta \in (0, 1)$ and $Q > 0$ are absolute constants.

Inequalities (4) and (5) are not compatible with the asymptotic behavior regarding the power of k, except in dimension two. Yet, for surfaces, the limit of the above upper bound for normalized eigenvalues depends on the genus γ as k goes to infinity. Therefore, it is not compatible with (1).

We obtain upper bounds which generalize and improve the above inequalities without imposing any condition on the metric and which are compatible with the asymptotic behavior. Before stating our theorem, we need to recall the definition of the *min-conformal volume*. For a compact Riemannian manifold (M, g), its *min-conformal volume* is defined as follows (see [Hassannezhad 2011]):

$$V([g]) = \inf\{\mu_{g_0}(M) : g_0 \in [g],\ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Theorem 1.1. *There exist positive constants $\alpha_m \in (0, 1)$, B_m, and C_m depending only on m such that, for every compact m-dimensional Riemannian manifold (M, g), every potential $q \in C^0(M)$, and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g+q) \le \frac{\alpha_m^{-1}\int_M q^+\,d\mu_g - \alpha_m \int_M q^-\,d\mu_g}{\mu_g(M)} + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}. \tag{6}$$

In particular, when the potential q is nonnegative, one has

$$\lambda_k(\Delta_g+q) \le A_m \frac{\int_M q\,d\mu_g}{\mu_g(M)} + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}, \tag{7}$$

where $A_m = \alpha_m^{-1}$.

We also obtain upper bounds for eigenvalues of positive Schrödinger operators. Note that the positivity of the Schrödinger operator $L = \Delta_g + q$ implies that $\int_M q$ is nonnegative, and q here may not be nonnegative. The following upper bound generalizes inequalities (5) and (7).

Theorem 1.2. *There exist constants $A_m > 1$, B_m, and C_m depending only on m such that if $L = \Delta_g + q$ with $q \in C^0(M)$ is a positive operator, then, for every compact m-dimensional Riemannian manifold (M^m, g) and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g+q) \le A_m \frac{\int_M q\,d\mu_g}{\mu_g(M)} + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}.$$

Given the Schrödinger operator $L = \Delta_g + q$, for every $\varepsilon > 0$, the Schrödinger operator $\widetilde{L} = \Delta_g + q - \lambda_1(L) + \varepsilon$ is positive and $\lambda_k(\widetilde{L}) = \lambda_k(L) - \lambda_1(L) + \varepsilon$. When ε goes to zero, Theorem 1.1 leads to the following.

Corollary 1.3. *Under the assumptions of Theorem 1.1, we get*

$$\lambda_k(\Delta_g+q) \le A_m \frac{\int_M q\,d\mu_g}{\mu_g(M)} + (1-A_m)\lambda_1(\Delta_g+q) + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}.$$

In the two-dimensional case, for a compact orientable Riemannian surface (Σ_γ, g) of genus γ, thanks to the uniformization and Gauss–Bonnet theorems, one has $V([g]) \le 4\pi\gamma$. Therefore, in compact orientable Riemannian surfaces, one can replace the min-conformal volume by the topological invariant $4\pi\gamma$ in the above inequalities.

Corollary 1.4. *There exist absolute constants $a \in (0, 1)$, A, and B such that, for every compact orientable Riemannian surface (Σ_γ, g) of genus γ, every potential $q \in C^0(M)$, and every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_g+q)\mu_g(\Sigma_\gamma) \le \int_{\Sigma_\gamma} (aq^+ - a^{-1}q^-)\,d\mu_g + A\gamma + Bk. \tag{8}$$

And if L is a positive operator,

$$\lambda_k(\Delta_g + q)\mu_g(\Sigma_\gamma) \le a \int_{\Sigma_\gamma} q\, d\mu_g + A\gamma + Bk.$$

An interesting application of Theorem 1.1 is the case of weighted Laplace operators or Bakry–Émery Laplace operators.

Bakry–Émery Laplacian. Let (M, g) be a Riemannian manifold and $\phi \in C^2(M)$. The corresponding weighted Laplace operator Δ_ϕ is defined by

$$\Delta_\phi = \Delta_g + \nabla_g \phi \cdot \nabla_g.$$

This operator is associated with the quadratic functional $\int_M |\nabla_g f|^2 e^{-\phi}\, d\mu_g$, that is,

$$\int_M \Delta_\phi f h e^{-\phi}\, d\mu_g = \int_M \langle \nabla_g f, \nabla_g h\rangle e^{-\phi}\, d\mu_g.$$

This operator is an elliptic operator on $C_c^\infty(M) \subseteq L^2(e^{-\phi}\, d\mu_g)$ and can be extended to a selfadjoint operator with the weighted measure $e^{-\phi}\, d\mu_g$. In this sense, it arises as a generalization of the Laplacian. The weighted Laplace operator Δ_ϕ is also known as the diffusion operator or the Bakry–Émery Laplace operator which is used to study the diffusion process; see, for instance, the pioneering work of Bakry and Émery [1985] or [Lott 2007; Lott and Villani 2009]. The triple (M, g, ϕ) is called a Bakry–Émery manifold, where $\phi \in C^2(M)$ and (M, g) is a Riemannian manifold with the weighted measure $e^{-\phi} d\mu_g$; see [Lu and Rowlett 2012; Rowlett 2010]. The interplay between the geometry of M and the behavior of ϕ is mostly taken into account by means of a new notion of curvature called the Bakry–Émery Ricci tensor[1], which is defined by

$$\mathrm{Ricci}_\phi = \mathrm{Ricci}_g + \mathrm{Hess}\, \phi.$$

Our aim is to find upper bounds for the eigenvalues of Δ_ϕ denoted by $\lambda_k(\Delta_\phi)$ in terms of the geometry of M and of properties of ϕ.

Upper bounds for the first eigenvalue $\lambda_1(\Delta_\phi)$ of complete noncompact Riemannian manifolds have been recently considered in several works; see [Munteanu and Wang 2012; Setti 1998; Su and Zhang 2012; Wu 2010; 2012]. These upper bounds depend on the L^∞-norm of $\nabla_g \phi$ and a lower bound of the Bakry–Émery Ricci tensor.

Let (M, g, ϕ) be a complete noncompact Bakry–Émery manifold of dimension m with $\mathrm{Ricci}_\phi \ge -\kappa^2(m-1)$ and $|\nabla_g \phi| \le \sigma$ for some constants $\kappa \ge 0$ and $\sigma > 0$. Then we have, by [Su and Zhang 2012, Proposition 2.1] (see also [Munteanu and

[1] The Bakry–Émery Ricci tensor Ricci_ϕ is also referred to as the ∞-Bakry–Émery Ricci tensor. We denote Ricci_ϕ and $\mathrm{Hess}\, \phi$ by $\mathrm{Ricci}_\phi(M, g)$ and $\mathrm{Hess}_g\, \phi$ wherever any confusion might occur.

Wang 2012; Wu 2010; 2012])

$$\lambda_1(\Delta_\phi) \leq \tfrac{1}{4}((m-1)\kappa+\sigma)^2. \tag{9}$$

In particular, if $\mathrm{Ricci}_\phi \geq 0$, we have

$$\lambda_1(\Delta_\phi) \leq \tfrac{1}{4}\sigma^2. \tag{10}$$

We consider compact Bakry–Émery manifolds and we present two approaches to obtain upper bounds for the eigenvalues of the Bakry–Émery Laplace operator in terms of the geometry of M and the properties of ϕ.

First approach. One can see that Δ_ϕ is unitarily equivalent to the Schrödinger operator $L=\Delta_g+\frac{1}{2}\Delta_g\phi+\frac{1}{4}|\nabla_g\phi|^2$; see, for example, [Setti 1998, p. 28]. Therefore, as a consequence of Theorem 1.2, we obtain an upper bound for $\lambda_k(\Delta_\phi)$ in terms of the min-conformal volume and the L^2-norm of $\nabla_g\phi$.

Theorem 1.5. *There exist constants A_m, B_m, and C_m depending on $m\in\mathbb{N}^*$, such that, for every m-dimensional compact Bakry–Émery manifold (M,g,ϕ), we have*

$$\lambda_k(\Delta_\phi) \leq A_m\frac{1}{\mu_g(M)}\|\nabla_g\phi\|^2_{L^2(M)} + B_m\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + C_m\left(\frac{k}{\mu_g(M)}\right)^{2/m}.$$

It is worth noticing that in full generality it is not possible to obtain upper bounds which do not depend on ϕ; see, for instance, [Su and Zhang 2012, Section 2]. However, we will see that for compact manifolds with nonnegative Bakry–Émery Ricci curvature we can find upper bounds which do not depend on ϕ (see Corollary 1.8).

In the two-dimensional case, as a result of Corollary 1.4, we obtain the following.

Corollary 1.6. *There exist absolute constants $a\in(0,1)$, A, and B such that, for every compact orientable Riemannian surface (Σ_γ,g) of genus γ and every $k\in\mathbb{N}^*$,*

$$\lambda_k(\Delta_\phi)\mu_g(\Sigma_\gamma) \leq a\|\nabla_g\phi\|^2_{L^2(\Sigma_\gamma)} + A\gamma + Bk.$$

Second approach. This approach is based on using the technique introduced in [Hassannezhad 2011], which was successfully applied for the Laplace operator Δ_g on Riemannian manifolds [Hassannezhad 2011, Theorem 1.1]. We obtain upper bounds for eigenvalues of Δ_ϕ in terms of a conformal invariant. We also obtain a Buser type upper bound for $\lambda_k(\Delta_\phi)$ (see Corollary 1.9).

Definition 1.1. Let (M,g,ϕ) be a compact Bakry–Émery manifold. We define the ϕ-min-conformal volume as

$$V_\phi([g]) = \inf\{\mu_\phi(M,g_0) : g_0\in[g],\ \mathrm{Ricci}_\phi(M,g_0)\geq -(m-1)\}, \tag{11}$$

where $\mu_\phi(M,g_0)$ is the weighted measure[2] of M with respect to the metric g_0.

[2]For a Bakry–Émery manifold (M,g,ϕ), when μ_ϕ is the weighted measure with respect to the

Note that up to dilations[3] there is always a Riemannian metric $g_0 \in [g]$ such that $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$. We are now ready to state our theorem.

Theorem 1.7. *There exist positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that, for every compact Bakry–Émery manifold (M, g, ϕ) with $|\nabla_g \phi| \leq \sigma$, for some $\sigma \geq 0$ and for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A(m)\max\{\sigma^2, 1\}\left(\frac{V_\phi([g])}{\mu_\phi(M)}\right)^{2/m} + B(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}. \tag{12}$$

If a metric g is conformally equivalent to a metric g_0 with $\mathrm{Ricci}_\phi(M, g_0) \geq 0$, $V_\phi([g]) = 0$. Thus an immediate consequence of Theorem 1.7 is the following.

Corollary 1.8. *There exists a positive constant $A(m)$ depending only on $m \in \mathbb{N}^*$ such that, for every compact Bakry–Émery manifold (M, g, ϕ) with $V_\phi([g]) = 0$ and for every $k \in \mathbb{N}^*$,*

$$\lambda_k(\Delta_\phi) \leq A(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}. \tag{13}$$

The above upper bound is similar to the upper bound for the eigenvalues of the Laplacian in Riemannian manifolds (M, g) when $V([g]) = 0$; see [Korevaar 1993].

If $\mathrm{Ricci}_\phi(M) \geq -\kappa^2(m-1)$ for some $\kappa \geq 0$, then, for $g_0 = \kappa^2 g$, one has $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$ and $V_\phi([g]) \leq \mu_\phi(M, g_0) = \kappa^m \mu_\phi(M, g)$. Replacing in inequality (12), we get a Buser type upper bound for the eigenvalues of the Bakry–Émery Laplacian.

Corollary 1.9 (Buser type upper bound). *There are positive constants $A(m)$ and $B(m)$ depending only on $m \in \mathbb{N}^*$ such that, for every compact Bakry–Émery manifold (M, g, ϕ) with $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$, and for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Delta_\phi) \leq A(m)\max\{\sigma^2, 1\}\kappa^2 + B(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

A weaker version of Corollary 1.9 can be proved directly by the classic idea used by Buser [1979] and Li and Yau [1980]. We refer the reader to the appendix, where we give a simple and direct proof.

Remark 1.1. All of the results mentioned above for compact manifolds are also valid when one considers bounded subdomains of complete manifolds with the Neumann boundary condition.

metric g, we simply denote the weighted measure of a measurable subset A of M by $\mu_\phi(A)$ instead of $\mu_\phi(A, g)$.

[3]Note that $\mathrm{Hess}\,\phi$ and Ricci_g do not change under dilations. If $\mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g$, for all $\alpha > 0$, $\mathrm{Ricci}_\phi(M, g_0) := \mathrm{Ricci}_\phi(M, \alpha g) = \mathrm{Ricci}_\phi(M, g) \geq -\kappa^2(m-1)g = -(\kappa^2/\alpha)(m-1)g_0$.

2. Preliminaries and technical tools

Basic definitions. A *capacitor* is a pair of Borel sets (F, G) in a topological space satisfying $F \subsetneq G$.

We say that a metric space (X, d) satisfies the $(\kappa, N; \rho)$-*covering property* if each ball of radius $0 < r \leq \rho$ can be covered by N balls of radius r/κ. We sometimes call this the local covering property when $\rho < \infty$.

For any $x \in X$ and $0 \leq r \leq R$, we define the annulus $A(x, r, R)$ as

$$A(x, r, R) := B(x, R) \setminus B(x, r) = \{y \in X : r \leq d(x, y) < R\}.$$

Note that $A(x, 0, R) = B(x, R)$. If $F = A(x, r, R)$ and $\lambda \geq 1$, we define ${}^{\lambda}F := A(x, \lambda^{-1}r, \lambda R)$. For $F \subseteq X$ and $r > 0$, we denote by F^r the r-neighborhood of F:

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

Here we state the key method that we use in order to obtain our results. This method was introduced in [Hassannezhad 2011] and was inspired by two elaborate constructions given in [Colbois and Maerten 2008; Grigor'yan et al. 2004]. It leads to the construction of a "nice" family of capacitors, crucial to estimating the eigenvalues of Schrödinger operators and Bakry–Émery operators via capacities.

Capacity on Riemannian manifolds. For each capacitor (F, G) in a Riemannian manifold (M, g) of dimension m, we define the capacity and the m-capacity by

$$\text{(14)}\quad \operatorname{cap}_g(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 \, d\mu_g \quad \text{and} \quad \operatorname{cap}^{(m)}_{[g]}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m \, d\mu_g,$$

respectively, where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that $\operatorname{supp} \varphi \subset G$, $0 \leq \phi \leq 1$, and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, $\operatorname{cap}_g(F, G) = \operatorname{cap}^{(m)}_{[g]}(F, G) = +\infty$.

Proposition 2.1 ([Hassannezhad 2012, Theorem 1.2.1]; see also [Hassannezhad 2011]). *Let (X, d, μ) be a metric measure space with a nonatomic Borel measure μ satisfying the $(2, N; \rho)$-covering property. Then, for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties*:

(i) $\mu(F_i) \geq \nu := \mu(X)/(8c^2 n)$, *where c is a constant depending only on N.*

(ii) *The G_i are mutually disjoint.*

(iii) *The family $\mathcal{A}$ is such that either*

  (a) *all the F_i are annuli with outer radii smaller than ρ and $G_i = {}^2F_i$, or*

  (b) *all the F_i are domains in X and $G_i = F_i^{r_0}$ with $r_0 = \frac{1}{1600}\rho$.*

As a consequence of this proposition, we have:

Lemma 2.2. *Let (M^m, g, μ) be a compact Riemannian manifold with a nonatomic Borel measure μ. Then there exist positive constants $c(m) \in (0, 1)$ and $\alpha(m)$ depending only on the dimension such that, for every $k \in \mathbb{N}^*$, there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of mutually disjoint capacitors with the following properties*:

(I) $\mu(F_i) > c(m)\dfrac{\mu(M)}{k}$.

(II) $\operatorname{cap}_g(F_i, G_i) \le \dfrac{\mu_g(M)}{k}\left[\dfrac{1}{r_0^2}\left(\dfrac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\dfrac{k}{\mu_g(M)}\right)^{2/m}\right]$, *with* $r_0 = \frac{1}{1600}$.

Proof of Lemma 2.2. Take the metric measure space (M, d_{g_0}, μ), where $g_0 \in [g]$ with $\mathrm{Ricci}_{g_0} \ge -(m-1)$ and d_{g_0} is the distance associated to the Riemannian metric g_0. It is easy to verify that (M, d_{g_0}, μ) has the $(2, N; 1)$-covering property, where N is a constant depending only on the dimension [Hassannezhad 2011]. Therefore, Proposition 2.1 implies that for every $k \in \mathbb{N}^*$ there is a family of $3k$ mutually disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{3k}$ satisfying the following properties (see [Grigor'yan et al. 2004, Proposition 3.1] for more justification):

- $\mu(F_i) > c(m)\mu(M)/k$, where $c(m) \in (0, 1)$ is a positive constant depending only on the dimension.
- Either
  (a) all the F_i are annuli with outer radii smaller than 1 and $\operatorname{cap}^{(m)}_{[g]}(F_i, {}^2F_i) \le Q_m$, where the constant Q_m depends only on the dimension, and $G_i = {}^2F_i$;
  or
  (b) all the F_i are domains in M and $G_i = F_i^{r_0}$, where $r_0 = \frac{1}{1600}$.

Hence, the family of $\{(F_i, G_i)\}_{i=1}^{3k}$ has property (I). We now show that at least k of the capacitors satisfy property (II). We first find an upper bound for the m-capacity $\operatorname{cap}^{(m)}_{[g]}(F_i, G_i)$. If all the F_i are annuli, we already have an estimate by property (a). If the F_i are domains, one can define a family of functions $\varphi_i \in \mathcal{T}(F_i, G_i)$, $1 \le i \le 3k$, such that $|\nabla_{g_0}\varphi_i| \le 1/r_0$. Then

$$\operatorname{cap}^{(m)}_{[g]}(F_i, G_i) \le \int_M |\nabla_{g_0}\varphi_i|^m \, d\mu_{g_0} \le \frac{1}{r_0^m}\mu_{g_0}(G_i).$$

Since $G_1, \dots, G_{3k}$ are mutually disjoint, there exist at least $2k$ of them so that $\mu_{g_0}(G_i) \le \mu_{g_0}(M)/k$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \le \mu_g(M)/k$. Therefore, up to reordering, we assume that the first k of them (that is, $G_1, \dots, G_k$) satisfy the inequalities

$$\mu_g(G_i) \le \mu_g(M)/k \quad \text{and} \quad \mu_{g_0}(G_i) \le \mu_{g_0}(M)/k.$$

Hence, in general, there exist k capacitors (F_i, G_i), $1 \le i \le k$, with

$$\operatorname{cap}_{[g]}^{(m)}(F_i, G_i) \le Q_m + \frac{1}{r_0^m} \frac{\mu_{g_0}(M)}{k}.$$

The left side of this inequality is a conformal invariant. Now, taking the infimum over $g_0 \in [g]$ with $\operatorname{Ricci}_{g_0} \ge -(m-1)$, we get

$$\operatorname{cap}_{[g]}^{(m)}(F_i, G_i) \le Q_m + \frac{1}{r_0^m} \frac{V([g])}{k}.$$

Now, for every $\varepsilon > 0$, we consider plateau functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$, with

$$\int_M |\nabla_g f_i|^m \, d\mu_g \le \operatorname{cap}_{[g]}^{(m)}(F_i, G_i) + \varepsilon.$$

Therefore,

$$\begin{aligned}
(15)\quad \operatorname{cap}_g(F_i, G_i) &\le \int_M |\nabla_g f_i|^2 d\mu_g \le \left(\int_M |\nabla_g f_i|^m d\mu_g\right)^{2/m} \left(\int_M 1_{\operatorname{supp} f_i} \, d\mu_g\right)^{1-2/m} \\
&\le \left(\operatorname{cap}_{[g]}^{(m)}(F_i, G_i) + \varepsilon\right)^{2/m} \mu_g(G_i)^{1-2/m} \\
&\le \left(Q_m + \frac{1}{r_0^m} \frac{V([g])}{k} + \varepsilon\right)^{2/m} \mu_g(G_i)^{1-2/m} \\
&\le \left[Q_m^{2/m} + \frac{1}{r_0^2}\left(\frac{V([g])}{k}\right)^{2/m} + \varepsilon^{2/m}\right]\left(\frac{\mu_g(M)}{k}\right)^{1-2/m}.
\end{aligned}$$

where the last inequality is due to the well-known fact that

$$(a+b)^s \le a^s + b^s$$

when a, b are nonnegative real numbers and $0 < s \le 1$. Letting ε tend to zero, we obtain property (II). This completes the proof. □

Capacity on Bakry–Émery manifolds. In an analogous way, we define the capacity in a Bakry–Émery manifold (M, g, ϕ). For each capacitor (F, G) in a Bakry–Émery manifold (M, g, ϕ) of dimension m, the capacity and the m-capacity are defined as

$$(16)\quad \operatorname{cap}_\phi(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_\phi \quad \text{and} \quad \operatorname{cap}_\phi^{(m)}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m \, d\mu_\phi,$$

respectively, where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all functions $\varphi \in C_0^\infty(M)$ such that $\operatorname{supp} \varphi \subset G$, $0 \le \phi \le 1$ and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, $\operatorname{cap}_\phi(F, G) = \operatorname{cap}_\phi^{(m)}(F, G) = +\infty$.

We prove a similar lemma below (Lemma 2.2). We first show that every compact Bakry–Émery manifold satisfies the assumptions of Proposition 2.1. Thanks to a volume comparison theorem for Bakry–Émery manifolds, which we quote next, we can show that such have the local covering property (see Lemma 2.4).

Theorem 2.3 (volume comparison theorem [Wei and Wylie 2009]). *Let (M, g, ϕ) be a compact Bakry–Émery manifold with $Ricci_\phi \geq \alpha(m-1)$. If $\partial_r \phi \geq -\sigma$ with respect to geodesic polar coordinates centered at x, then, for every $0 < r \leq R$, we have (assume $R \leq \pi/2\sqrt{\alpha}$ if $\alpha > 0$)*

$$\frac{\mu_\phi(B(x,R))}{\mu_\phi(B(x,r))} \leq e^{\sigma R} \frac{v(m,R,\alpha)}{v(m,r,\alpha)}, \tag{17}$$

and, in particular, letting r tend to zero yields

$$\mu_\phi(B(x,R)) \leq e^{\sigma R} v(m,R,\alpha), \tag{18}$$

where $v(m,r,\alpha)$ is the volume of a ball of radius r in the simply connected space form of constant sectional curvature α.

Lemma 2.4. *Let (M, g, ϕ) be a compact Bakry–Émery manifold with $Ricci_\phi \geq -\kappa^2(m-1)$ and $|\nabla_g \phi| \leq \sigma$ for some $\kappa \geq 0$ and $\sigma \geq 0$. There exist constants $N(m) \in \mathbb{N}^*$ and $\xi = \xi(\sigma,\kappa) > 0$ such that (M, g, ϕ) satisfies the $(2, N; \xi)$-covering property. Moreover, there exists a positive constant $C(m)$ such that, for every $0 \leq r < R \leq \xi$ and $x \in M$, the annulus $A = A(x,r,R)$ satisfies $\operatorname{cap}_\phi^{(m)}(A, {}^2A)) \leq C(m)$.*

Proof. Take $\xi = \min\{1/\sigma, 1/\kappa\}$. (Take $\xi = \infty$ if $\sigma = \kappa = 0$.) We first show that (M, μ_ϕ) has the doubling property for $r < 4\xi$, that is,

$$\mu_\phi(B(x,r)) \leq c\mu_\phi(B(x,r/2)), \quad 0 < r < 4\xi,$$

for some positive constant c. From this, it is easy to deduce that (M, μ_ϕ) has the $(2, N; \xi)$-covering property, for example with $N = c^4$. To prove the doubling property, according to inequality (17) we have

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq e^{\sigma r} \frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} = e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)}.$$

Take $\tilde{r} := \kappa r$. Then, for $0 < r < 4\xi = 4\min\{1/\sigma, 1/\kappa\}$, we have

$$e^{\sigma r} \frac{v(m,\kappa r,-1)}{v(m,\kappa r/2,-1)} \leq e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)} \leq c(m),$$

where

$$c(m) := \sup_{\tilde{r} \in (0,4)} e^4 \frac{v(m,\tilde{r},-1)}{v(m,\tilde{r}/2,-1)}.$$

Thus

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \leq c(m) \quad \text{for every } 0 < r < \xi.$$

Therefore, (M, g, ϕ) has the $(2, N; \xi)$-covering property for $N = c^4(m)$.

To estimate the capacity of an annulus, we now follow the same argument as in [Hassannezhad 2011, p. 3430]. Let $A = A(x, r, R)$ and let $f \in \mathcal{T}(A, {}^2A)$ be given by

$$f(y) = \begin{cases} 1 & \text{if } y \in A(x, r, R), \\ 2d_{g_0}(y, B(x, r/2))/r & \text{if } y \in A(x, r/2, r) \text{ and } r \neq 0, \\ 1 - d_{g_0}(y, B(x, R))/R & \text{if } y \in A(x, R, 2R), \\ 0 & \text{if } y \in M \setminus A(x, r/2, 2R). \end{cases} \tag{19}$$

We have

$$|\nabla_{g_0} f| \leq \begin{cases} 2/r & \text{on } B(x, r) \setminus B(x, r/2), \\ 1/R & \text{on } B(x, 2R) \setminus B(x, R). \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{cap}_\phi^{(m)}(A, {}^2A) &\leq \int_M |\nabla_g f|^m d\mu_\phi \leq \left(\frac{2}{r}\right)^m \mu_\phi(A(x,r/2,r)) + \left(\frac{1}{R}\right)^m \mu_\phi(A(x,R,2R)) \\ &\leq \left(\frac{2}{r}\right)^m \mu_\phi(B(x, r)) + \left(\frac{1}{R}\right)^m \mu_\phi(B(x, 2R)). \end{aligned}$$

Having inequality (18), we get

$$\begin{aligned} \mathrm{cap}_\phi^{(m)}(A, {}^2A) &\leq \left(\frac{2}{r}\right)^m e^{\sigma r} v(m, r, -\kappa^2) + \left(\frac{1}{R}\right)^m e^{2\sigma R} v(m, 2R, -\kappa^2) \\ &= \left(\frac{2}{\kappa r}\right)^m e^{\sigma r} v(m, \kappa r, -1) + \left(\frac{1}{\kappa R}\right)^m e^{2\sigma R} v(m, 2\kappa R, -1). \end{aligned}$$

Take $\tilde{r} := \kappa r$ and $\widetilde{R} := \kappa R$. Then, for every $0 < r < R \leq 2\xi = 2 \min\{1/\sigma, 1/\kappa\}$, we get

$$\mathrm{cap}_\phi^{(m)}(A, {}^2A) \leq \left(\frac{2}{\tilde{r}}\right)^m e^2 v(m, \tilde{r}, -1) + \left(\frac{1}{\widetilde{R}}\right)^m e^4 v(m, 2\widetilde{R}, -1). \tag{20}$$

Setting $C(m)$ to the supremum of the expression on the right side over $\tilde{r}, \widetilde{R} \in (0, 2)$ completes the proof. □

Lemma 2.5. *Let (M^m, g, ϕ) be a compact Bakry–Émery manifold with $|\nabla_g \phi| \leq \sigma$ for some $\sigma \geq 0$. There exist positive constants $c(m) \in (0, 1)$ and $\alpha(m)$ depending only on the dimension such that, for every $k \in \mathbb{N}^*$, there exists a family $\{(F_i, G_i)\}_{i=1}^k$ of capacitors with the following properties*:

(I) $\mu_\phi(F_i) > c(m) \dfrac{\mu_\phi(M)}{k}$,

(II) $\mathrm{cap}_\phi(F_i, G_i) \leq \dfrac{\mu_\phi(M)}{k} \left[\dfrac{1}{r_0^2} \left(\dfrac{V_\phi([g])}{\mu_\phi(M)}\right)^{2/m} + \alpha(m) \left(\dfrac{k}{\mu_\phi(M)}\right)^{2/m} \right]$, *where* $1/r_0 = 1600 \max\{\sigma, 1\}$.

Proof. We consider the Bakry–Émery manifold (M, g, ϕ) as the metric measure space (M, d_{g_0}, μ_ϕ) where $g_0 \in [g]$ with $\mathrm{Ricci}_\phi(M, g_0) \geq -(m-1)$ and μ_ϕ is the weighted measure with respect to the metric g. According to Lemma 2.4, this space has the $(2, N, \xi)$-covering property with $\xi = \min\{1/\sigma, 1\}$. Having Proposition 2.1 and Lemma 2.4, and following steps analogous to those in Lemma 2.2 we see, for every $k \in \mathbb{N}^*$, there exists a family of k mutually disjoint capacitors $\{F_i, G_i\}$ satisfying the following properties:

- $\mu_\phi(F_i) \geq c(m)\mu_\phi(M)/k$, where $c(m) \in (0, 1)$ is a positive constant depending only on the dimension, and $\mu_\phi(G_i) \leq \mu_\phi(M)/k$. Either
  (a) all the F_i are annuli with outer radii smaller than ξ, $G_i = {}^2F_i$, and
$$\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq C(m),$$
  where $C(m)$ is the constant defined in (20);

  or

  (b) all the F_i are domains in M, $G_i = F_i^{r_0}$ is the r_0-neighborhood of F_i, and $\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq r_0^{-2} V_\phi([g])/k$, with $r_0 = \xi/1600$.

Hence, $\mathrm{cap}_\phi^{(m)}(F_i, G_i) \leq C(m) + r_0^{-2} V_\phi([g])/k$. Now, for every $\varepsilon > 0$, we consider a family of functions $\{f_i\}_{i=1}^k$, $f_i \in \mathcal{T}(F_i, G_i)$ such that
$$\int_M |\nabla_g f_i|^m e^{-\phi}\, d\mu_g \leq \mathrm{cap}_\phi^{(m)}(F_i, G_i) + \varepsilon.$$
We repeat the same argument as before.
$$\begin{aligned}
\mathrm{cap}_\phi(F_i, G_i) &\leq \int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \\
&\leq \left(\int_M |\nabla_g f_i|^m e^{-\phi} d\mu_g\right)^{2/m} \left(\int_M 1_{\mathrm{supp}\, f_i} e^{-\phi} d\mu_g\right)^{1-2/m} \\
&\leq \left[C(m)^{2/m} + \frac{1}{r_0^2}\left(\frac{V_\phi([g])}{k}\right)^{2/m} + \varepsilon^{2/m}\right]\left(\frac{\mu_\phi(M)}{k}\right)^{1-2/m}.
\end{aligned}$$
Having $1/r_0 = 1600/\xi = 1600 \max\{\sigma, 1\}$ and letting ε tend to zero, we obtain property (II). This completes the proof. □

3. Eigenvalues of Schrödinger operators

In this section, we prove Theorems 1.1 and 1.2. The idea of the proof is to construct a suitable family of test functions to be used in the variational characterization of the eigenvalues. Due to the min-max Theorem, we have the following variational characterization for the eigenvalues of the Schrödinger operator $L = \Delta_g + q$:

$$\lambda_k(\Delta_g + q) = \min_{V_k} \max_{0 \neq f \in V_k} \frac{\int_M |\nabla_g f|^2 \, d\mu_g + \int_M f^2 q \, d\mu_g}{\int_M f^2 \, d\mu_g},$$

where V_k is a k-dimensional linear subspace of $H^1(M)$ and μ_g is the Riemannian measure corresponding to the metric g.

According to this variational formula, for every family $\{f_i\}_{1=1}^k$ of disjointly supported test functions, one has

$$\lambda_k(\Delta_g + q) \leq \max_{i \in \{1,\dots,k\}} \frac{\int_M |\nabla_g f_i|^2 \, d\mu_g + \int_M f_i^2 q \, d\mu_g}{\int_M f_i^2 \, d\mu_g}. \tag{21}$$

The potential $q \in C^0(M)$ is a signed function (notice that we can assume $q \in L^1(M)$ as well). We define a signed measure σ associated to the potential q by

$$\sigma(A) = \int_A q \, d\mu_g \quad \text{for every measurable subset } A \text{ of } X.$$

For any signed measure ν we write $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are the positive and negative parts of ν, respectively. For any signed measure ν and $0 \leq \delta \leq 1$ we define a new signed measure ν_δ as $\nu_\delta := \delta \nu^+ - \nu^-$.

Let μ and ν be two signed measures on M. Then, according to [Grigor'yan et al. 2004, Lemma 4.3], we have

$$(\mu + \nu)_\delta \geq \mu_\delta + \nu_\delta. \tag{22}$$

Proof of Theorem 1.1. For a real number $\lambda \in \mathbb{R}$ define $\mu_\lambda := (\lambda \mu_g - \sigma)^+$ as a nonatomic Borel measure on M. We apply Lemma 2.2 to (M, g, μ_λ). Thus, for every $k \in \mathbb{N}^*$ and every $\lambda \in \mathbb{R}$, there exists a family $\{(F_i, G_i)\}_{i=1}^{2k}$ of $2k$ capacitors satisfying properties (I) and (II) of Lemma 2.2.

From now on, we take $\lambda := \lambda_k = \lambda_k(L)$. Property (I) yields

$$(\lambda_k \mu_g - \sigma)^+(F_i) \geq c(m) \frac{(\lambda_k \mu_g - \sigma)^+(M)}{2k}.$$

The measure $(\lambda_k \mu_g - \sigma)^-$ is also nonatomic. Since G_i are mutually disjoint, up to reordering, the first k of them satisfy

$$(\lambda_k \mu_g - \sigma)^-(G_i) \leq \frac{(\lambda_k \mu_g - \sigma)^-(M)}{k}, \quad i \in \{1, \dots, k\}.$$

Therefore

$$\begin{aligned}(\lambda_k \mu_g - \sigma)^-(G_i) &- (\lambda_k \mu_g - \sigma)^+(F_i) \\ &\leq \frac{(\lambda_k \mu_g - \sigma)^-(M)}{k} - c(m) \frac{(\lambda_k \mu_g - \sigma)^+(M)}{2k}.\end{aligned} \tag{23}$$

For every $\epsilon > 0$ and every $1 \le i \le k$, we choose $f_i \in \mathcal{T}(F_i, G_i)$ such that

$$\int_M |\nabla_g f_i|^2 \, d\mu_g \le \operatorname{cap}_g(F_i, G_i) + \epsilon. \tag{24}$$

Inequality (21) implies that there exists $i \in \{1, \ldots, k\}$ so that

$$\lambda_k \int_M f_i^2 \, d\mu_g \le \int_M |\nabla_g f_i|^2 \, d\mu_g + \int_M f_i^2 q \, d\mu_g.$$

Hence, having Lemma 2.2 and inequality (23), we get

$$\begin{aligned}
0 &\le \int_M |\nabla_g f_i|^2 \, d\mu_g - \int_M f_i^2 (\lambda_k - q) \, d\mu_g \\
&\le \operatorname{cap}_g(F_i, G_i) + \epsilon - \int_M f_i^2 (\lambda_k - q) \, d\mu_g \\
&\le \frac{\mu_g(M)}{2k} \left[\frac{1}{r_0^2} \left(\frac{V([g])}{\mu_g(M)} \right)^{2/m} + \alpha(m) \left(\frac{2k}{\mu_g(M)} \right)^{2/m} \right] + \epsilon \\
&\quad + \int_M f_i^2 (\lambda_k - q)^- \, d\mu_g - \int_M f_i^2 (\lambda_k - q)^+ \, d\mu_g \\
&\le \frac{\mu_g(M)}{2k} \left[\frac{1}{r_0^2} \left(\frac{V([g])}{\mu_g(M)} \right)^{2/m} + \alpha(m) \left(\frac{2k}{\mu_g(M)} \right)^{2/m} \right] + \epsilon \\
&\quad + \frac{(\lambda_k \mu_g - \sigma)^-(M)}{k} - c(m) \frac{(\lambda_k \mu_g - \sigma)^+(M)}{2k}.
\end{aligned} \tag{25}$$

We now estimate the last two terms of this inequality considering two alternatives.

Case 1. If $\lambda_k = \lambda_k(L)$ is positive, then, applying (22) for the measure $\lambda_k \mu_g$ and signed measure $-\sigma$ with $\delta = c(m)/2$, we get

$$\begin{aligned}
&\frac{c(m)}{2} (\lambda_k \mu_g - \sigma)^+(M) - (\lambda_k \mu_g - \sigma)^-(M) \\
&\qquad\qquad \ge \frac{c(m)}{2} \sigma^-(M) - \sigma^+(M) + \frac{c(m)}{2} \lambda_k \mu_g(M).
\end{aligned} \tag{26}$$

Substituting (26) in (25) and letting ϵ tend to zero gives

$$\begin{aligned}
\lambda_k \le \;& \frac{(2/c(m))\sigma^+(M) - \sigma^-(M)}{\mu_g(M)} \\
&+ \frac{1}{c(m) r_0^2} \left(\frac{V([g])}{\mu_g(M)} \right)^{2/m} + \frac{\alpha(m)}{c(m)} \left(\frac{2k}{\mu_g(M)} \right)^{2/m}.
\end{aligned} \tag{27}$$

Case 2. If $\lambda_k = \lambda_k(L)$ is nonpositive, applying (22) for the signed measures $\lambda_k \mu_g$ and $-\sigma$ with $\delta = c(m)/2$ implies

$$\frac{c(m)}{2} (\lambda_k \mu_g - \sigma)^+(M) - (\lambda_k \mu_g - \sigma)^-(M) \ge \frac{c(m)}{2} \sigma^-(M) - \sigma^+(M) + \lambda_k \mu_g(M).$$

Substituting this in (25) and letting ϵ go to zero gives

$$\text{(28)}\quad \lambda_k \le \frac{\sigma^+(M)-(c(m)/2)\sigma^-(M)}{\mu_g(M)} + \frac{1}{2r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \frac{\alpha(m)}{2}\left(\frac{2k}{\mu_g(M)}\right)^{2/m}.$$

Therefore, $\lambda_k(L)$ is smaller than the sum of the right sides of inequalities (27) and (28). We finally obtain inequality (6) with, for example, $\alpha_m = c(m)/4$. □

Proof of Theorem 1.2. We partly follow the spirit of' the proof of [Grigor'yan et al. 2004, Theorem 5.15]. Take the metric measure space (M, g, μ_g). By Lemma 2.2, for every $k \in N^*$ there is a family of $2k$ disjoint capacitors $\{(F_i, G_i)\}_{i=1}^{2k}$ that satisfies properties (I) and (II) of Lemma 2.2. For every $\varepsilon > 0$, let $\{f_i\}_{i=1}^{2k}$ be a family of test functions with $2f_i \in \mathcal{T}(F_i, G_i)$ and $4\int_M |\nabla_g f_i|^2\, d\mu_g \le \operatorname{cap}_g(F_i, G_i) + \varepsilon$. We claim that this family satisfies the following property:

$$\text{(29)}\qquad \sum_{i=1}^{2k}\int_M f_i^2 q\, d\mu_g \le \sum_{i=1}^{2k}\int_M |\nabla_g f_i|^2\, d\mu_g + \int_M q\, d\mu_g.$$

If we have inequality (29),

$$\text{(30)}\qquad \begin{aligned}\sum_{i=1}^{2k}\int_M (|\nabla_g f_i|^2 + f_i^2 q)\, d\mu_g &\le 2\sum_{i=1}^{2k}\int_M |\nabla_g f_i|^2\, d\mu_g + \int_M q\, d\mu_g\\ &\le k \max_i \operatorname{cap}_g(F_i, G_i) + k\varepsilon + \int_M q\, d\mu_g.\end{aligned}$$

By the assumption, $\int_M (|\nabla_g f_i|^2 + f_i^2 q)\, d\mu_g$ is positive for each $1 \le i \le 2k$. Therefore, at least k of them (up to reordering we assume that it's the first k) satisfy the inequality

$$\text{(31)}\qquad \int_M (|\nabla_g f_i|^2 + f_i^2 q)\, d\mu_g \le \max_i \operatorname{cap}_g(F_i, G_i) + \varepsilon + \frac{\int_M q\, d\mu_g}{k}.$$

Inequality (31), together with the bounds of $\operatorname{cap}_g(F_i, G_i)$ and $\mu_g(F_i)$ given in Lemma 2.2 and properties (I) and (II), leads to

$$\begin{aligned}\lambda_k(L) &\le \max_i \frac{\int_M |\nabla_g f_i|^2 d\mu_g + \int_M f_i^2 q\, d\mu_g}{\int_M f_i^2\, d\mu_g}\\ &\le \frac{\max_i \operatorname{cap}_g(F_i, G_i) + \varepsilon + (1/k)\int_M q\, d\mu_g}{\mu_g(F_i)}\\ &\le \frac{1}{c(m) r_0^2}\left(\frac{V([g])}{\mu_g(M)}\right)^{2/m} + \alpha(m)\left(\frac{2k}{\mu_g(M)}\right)^{2/m} + \frac{2k\varepsilon}{c(m)\mu_g(M)} + \frac{2\int_M q\, d\mu_g}{c(m)\mu_g(M)}.\end{aligned}$$

Hence we get the desired inequality as ε tends to zero. It remains to prove inequality (29) which is proved in [Grigor'yan et al. 2004, Section 5]; however, for the reader's convenience we repeat the proof. We define the function h by the identity

$$\sum_{i=1}^{2k} f_i^2 + h^2 = 1. \tag{32}$$

Since $f_1, \dots, f_{2k}$ are disjointly supported and $0 \le f_i \le 1/2$, $h \ge 1/2$. We now estimate the left side of inequality (29).

$$\begin{aligned}\int_M \Big(\sum_{i=1}^{2k} f_i^2 + h^2 - h^2\Big) q \, d\mu_g &= \int_M q \, d\mu_g - \int_M h^2 q \, d\mu_g \\ &\le \int_M q \, d\mu_g + \int_M |\nabla h|^2 \, d\mu_g,\end{aligned} \tag{33}$$

where the last inequality comes from the fact that the Schrödinger operator L is positive. Identity (32) implies

$$-2h\nabla_g h = -\nabla_g h^2 = \sum_{i=1}^{2k} \nabla_g f_i^2 = 2\sum_{i=1}^{2k} f_i \nabla_g f_i.$$

Therefore,

$$|\nabla_g h|^2 \le |2h\nabla_g h|^2 = \sum_{i=1}^{2k} |\nabla_g f_i^2|^2 = 4\sum_{i=1}^{2k} |f_i \nabla_g f_i|^2 \le \sum_{i=1}^{2k} |\nabla_g f_i|^2. \tag{34}$$

Combining inequalities (33) and (34) we get inequality (29). □

4. Eigenvalues of Bakry–Émery Laplace operators

In this section we consider eigenvalues of the Bakry–Émery Laplace operator Δ_ϕ on a Bakry–Émery manifold (M, g, ϕ), where M is a compact m-dimensional Riemannian manifold and $\phi \in C^2(M)$. We denote the weighted measure on M by μ_ϕ with

$$\mu_\phi(A) = \int_A e^{-\phi} \, d\mu_g \quad \text{for every Borel subset } A \text{ of } M.$$

Proof of Theorem 1.5. As we mentioned in the introduction, one can see that $\Delta_\phi = \Delta_g + \nabla_g\phi \cdot \nabla_g$ is unitarily equivalent to the positive Schrödinger operator $L = \Delta_g + \frac{1}{2}\Delta_g\phi + \frac{1}{4}|\nabla_g\phi|^2$. Therefore, Theorem 1.2 yields

$$\begin{aligned}\lambda_k(\Delta_\phi) \le A_m \frac{1}{\mu_g(M)} \int_M \big(\tfrac{1}{2}\Delta_g\phi + \tfrac{1}{4}|\nabla_g\phi|^2\big) \, d\mu_g \\ + B_m \Big(\frac{V([g])}{\mu_g(M)}\Big)^{2/m} + C_m \Big(\frac{k}{\mu_g(M)}\Big)^{2/m}.\end{aligned}$$

Now Stokes' theorem implies that $\int_M \Delta_g \phi \, d\mu_g = 0$. This gives the result. □

For the proof of Theorem 1.7, we use the characteristic variational formula for the Bakry–Émery Laplacian; see for example [Lu and Rowlett 2012, Proposition 1; Rowlett 2010, Proposition 4].

$$\lambda_k(\Delta_\phi) = \inf_{V_k} \sup_{f \in V_k} \frac{\int_M |\nabla_g f|^2 e^{-\phi} \, d\mu_g}{\int_M f^2 e^{-\phi} \, d\mu_g}, \tag{35}$$

where V_k is a k-dimensional linear subspace of $H^1(M, \mu_\phi)$.

Proof of Theorem 1.7. According to Lemma 2.5, for $k \in \mathbb{N}^*$ we have a family of k capacitors satisfying properties (I) and (II). For every $\varepsilon > 0$, take $f_i \in \mathcal{T}(F_i, G_i)$, $1 \le i \le k$, so that

$$\int_M |\nabla_g f_i|^2 e^{-\phi} d\mu_g \le \operatorname{cap}_\phi(F_i, G_i) + \varepsilon.$$

Hence, the characteristic variational formula (35) gives

$$\lambda_k(\Delta_\phi) \le \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi} \, d\mu_g}{\int_M f_i^2 e^{-\phi} \, d\mu_g} \le \max_i \frac{\operatorname{cap}_\phi(F_i, G_i) + \varepsilon}{\mu_\phi(F_i)}.$$

Having the properties (I) and (II), we get

$$\lambda_k(\Delta_\phi) \le A(m) \max\{\sigma^2, 1\} \left(\frac{V_\phi([g])}{\mu_\phi(M)} \right)^{2/m} + B(m) \left(\frac{k}{\mu_\phi(M)} \right)^{2/m} + \frac{k\varepsilon}{c(m)\mu_\phi(M)}.$$

Letting ε go to zero, we get the desired inequality. □

Appendix: Buser type upper bound on Bakry–Émery manifolds

Here, we present a direct and simple proof of a weaker version of Corollary 1.9. The idea behind this proof was used by Buser [1979, Satz 7], Cheng [1975], and Li and Yau [1980] in the case of the Laplace–Beltrami operator. It is based on constructing a family of balls as capacitors which will be the support of test functions. We can successfully apply this idea in the case of the Bakry–Émery Laplace operator.

Theorem A.1 (Buser type upper bound). *Let (M, g, ϕ) be a compact Bakry–Émery manifold with* $\operatorname{Ricci}_\phi(M) > -\kappa^2(m-1)$ *and $|\nabla_g \phi| \le \sigma$ for some $\kappa \ge 0$ and $\sigma \ge 0$. There are positive constants $A(m)$ and $B(m)$ such that, for every $k \in \mathbb{N}^*$,*

$$\lambda_k(\Delta_\phi) \le A(m) \max\{\sigma, \kappa\}^2 + B(m) \left(\frac{k}{\mu_\phi(M)} \right)^{2/m}.$$

To see that the above theorem is weaker than Corollary 1.9, consider the case where $\operatorname{Ricci}_\phi(M, g)$ is nonnegative. Indeed, the upper bound in Theorem A.1 still depends on σ while Corollary 1.9 provides an upper bound which depends only on the dimension.

Proof. Since $\mathrm{Ricci}_\phi(M) > -\kappa^2(m-1)$ and $|\nabla_g \phi| \le \sigma$, the comparison theorem gives us the following inequalities for every $0 < r \le \xi = \min\{1/\sigma, 1/\kappa\}$ (with $\xi = \infty$ if $\sigma = \kappa = 0$):

$$\frac{\mu_\phi(B(x,r))}{\mu_\phi(B(x,r/2))} \le e^{\sigma r} \frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} \le \sup_{r\in(0,\xi)} e^{\sigma r} \frac{v(m,r,-\kappa^2)}{v(m,r/2,-\kappa^2)} =: c_1(m)$$

and

$$\mu_\phi(B(x,r)) \le e^{\sigma r} v(m,r,-\kappa^2) \le \sup_{s\in(0,\xi)} e^{\sigma s} v(m,s,-\kappa^2) r^m =: c_2(m) r^m. \tag{36}$$

Given $k \in \mathbb{N}^*$, let $\rho(k)$ be the positive number defined by

$$\rho(k) = \sup\{r : \text{there exist } p_1,\dots,p_k \in M \text{ with } d_g(p_i,p_j) > r \text{ for all } i \ne j\}.$$

We consider two cases.

Case 1. Let $\rho(k) \ge \xi$. For every $r < \xi$, there are k points $p_1,\dots,p_k$ with $B(p_i,r/2)\,\mathrm{cap}\,B(p_j,r/2) = \varnothing$ for all $i \ne j$. For each $i \in \{1,\dots,k\}$, we consider a plateau function $f_i \in \mathcal{T}(B(p_i,r/4), B(p_i,r/2))$, $1 \le i \le k$, defined as in (19). Then, for every $1 \le i \le k$ and every $r < \xi$,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\,d\mu_g}{\int_M f_i^2 e^{-\phi}\,d\mu_g} \le \frac{16}{r^2} \frac{\mu_\phi(B(p_i,r/2))}{\mu_\phi(B(p_i,r/4))} \le c_1(m)\frac{16}{r^2}.$$

Therefore, letting r tend to ξ, one has

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\,d\mu_g}{\int_M f_i^2 e^{-\phi}\,d\mu_g} \le c_1(m)\frac{16}{\xi^2} \le A(m)\max\{\sigma,\kappa\}^2.$$

Case 2. Let $\rho(k) < \xi$. Take $r < \rho(k)$ very close to $\rho(k)$. As in Case 1, there are k points $p_1,\dots,p_k$ with $B(p_i,r/2)\,\mathrm{cap}\,B(p_j,r/2) = \varnothing$ for all $i \ne j$. Repeating the same argument, we get, for every $1 \le i \le k$,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\,d\mu_g}{\int_M f_i^2 e^{-\phi}\,d\mu_g} \le c_1(m)\frac{16}{r^2}.$$

Therefore, for every $1 \le i \le k$,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\,d\mu_g}{\int_M f_i^2 e^{-\phi}\,d\mu_g} \le c_1(m)\frac{16}{\rho(k)^2}.$$

We now estimate $\rho(k)$. Let $\rho(k) < s < \xi$ and n be the maximal number of points $q_1,\dots,q_n \in M$ so that $d(q_i,q_j) > s$ for all $i \ne j$. Of course $n \le k$ and because of the maximality of n, the balls $\{B(q_i,s)\}_{i=1}^n$ cover M. Hence, according to inequality (36),

$$\mu_\phi(M) \le \sum_{i=1}^{n} \mu_\phi(B(q_i, s)) \le nc_2(m)s^m \le kc_2(m)s^m.$$

Thus, letting s tend to $\rho(k)$, we get

$$\frac{1}{\rho(k)^2} \le c_2(m)^{2/m}\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

Therefore,

$$\frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\, d\mu_g}{\int_M f_i^2 e^{-\phi}\, d\mu_g} \le 16c_1(m)c_2(m)^{2/m}\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}.$$

In conclusion, we obtain

$$\lambda_k(\Delta_\phi) \le \max_i \frac{\int_M |\nabla_g f_i|^2 e^{-\phi}\, d\mu_g}{\int_M f_i^2 e^{-\phi}\, d\mu_g} \le A(m)\max\{\sigma, \kappa\}^2 + B(m)\left(\frac{k}{\mu_\phi(M)}\right)^{2/m}. \quad \square$$

Acknowledgements

This paper is part of the author's PhD thesis under the direction of Professors Bruno Colbois (Neuchâtel University), Ahmad El Soufi (François Rabelais University), and Alireza Ranjbar-Motlagh (Sharif University of Technology). The author acknowledges their support and encouragement and thanks Bruno Colbois and Ahmad El Soufi for suggesting the problem and helpful discussions. She is also very grateful to the referee for helpful comments.

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Received July 16, 2012. Revised December 26, 2012.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 264 No. 2 August 2013

0030-8730(201308)264:2;1-2