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DIRICHLET EQUATIONS

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We consider a $(p, 2)$-equation with a Carathéodory reaction $f(z, x)$ which is resonant at $\pm\infty$ and has constant sign, z-dependent zeros. Using variational methods, together with truncation and comparison techniques and Morse theory, we establish the existence of five nontrivial smooth solutions (four of constant sign and the fifth nodal). If the reaction $f(z, x)$ is C^1 in $x \in \mathbb{R}$, then we produce a second nodal solution for a total of six nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the nonlinear Dirichlet problem

$$(1) \qquad -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad 2 < p.$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u(z) = \operatorname{div}(\|Du(z)\|^{p-2} Du(z)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Problem (1) is important in quantum physics in connection with Derrick's model [Derrick 1964] for the existence of solitons, which was investigated in more detail by Benci, D'Avenia, Fortunato, and Pisani [Benci et al. 2000]. Recently, such equations attracted the interest of people working on nonlinear partial differential equations and some existence and multiplicity results were proved in [Cingolani and Degiovanni 2005; Cingolani and Vannella 2003; Sun 2012]. All consider nonresonant equations. In contrast, in this work we deal with the resonant case. More precisely, we assume that, asymptotically at $\pm\infty$, we have resonance with respect to the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. In problem (1) the reaction $f(z, x)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable, and, for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous) and has positive and negative zeros which in general depend on $z \in \Omega$. Problems driven by the p-Laplacian, and with a

reaction that has zeros, were studied by Bartsch, Liu, and Weth [Bartsch et al. 2005] (they assume that the zeros are constant) and by Iturriaga, Massa, Sánchez, and Ubilla [Iturriaga et al. 2010] (they have variable zeros). In both works the reaction $f(z, \cdot)$ is $(p-1)$-superlinear.

Here, we prove the existence of at least five nontrivial smooth solutions and provide sign information for all of them (two are positive, two are negative and the fifth is nodal). Moreover, by strengthening the regularity of $f(z, \cdot)$ (namely, assuming that $f(z, \cdot) \in C^1(\mathbb{R})$), we produce a second nodal solution for a total of six nontrivial smooth solutions, all with precise sign information.

Our approach is variational based on the critical point theory, coupled with suitable truncation and comparison techniques and with Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools that we will use in this work.

2. Mathematical background

Let X be a Banach space. By X^* we denote the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* if the following is true:

C-condition. Every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$(2) \qquad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

This compactness-type condition is in general weaker than the usual Palais–Smale condition ("PS-condition" for short). However, it suffices to have a deformation theorem and from it derive the minimax theory of certain critical values of φ (see, for example, [Gasiński and Papageorgiou 2006]). In particular, we can state the following theorem, known in the literature as the mountain pass theorem [ibid., p. 648].

Theorem 1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$, $\rho > 0$, $\|x_0 - x_1\| > \rho$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x) : \|x - x_0\| = \rho] = \eta_\rho$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ.

In the analysis of problem (1), in addition to the Sobolev spaces

$$W_0^{1,p}(\Omega), \quad H_0^1(\Omega),$$

we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \Omega\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}$$

(here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$).

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in $x \in \mathbb{R}$; i.e.,

$$|f_0(z, x)| \leq \hat{a}(z) + \hat{c}|x|^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $\hat{a} \in L^\infty(\Omega)_+$, $\hat{c} > 0$, and

$$1 < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(z, x) = \int_0^x f_0(z, s)\, ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next theorem is a particular case of a more general result of [Gasiński and Papageorgiou 2012].

Theorem 2. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}) \text{ with } \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0; i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Remark. We should mention that the first such result was proved by Brézis and Nirenberg [1993] and was later extended by García Azorero, Peral Alonso, and Manfredi [García Azorero et al. 2000].

Let $h, \hat{h} \in L^{\infty}(\Omega)$. We write $h \prec \hat{h}$ if, for every compact $K \subseteq \Omega$, we can find $\varepsilon > 0$ such that

$$h(z) + \varepsilon \le \hat{h}(z) \quad \text{for a.a. } z \in K.$$

Clearly, if $h, \hat{h} \in C(\Omega)$ and $h(z) < \hat{h}(z)$ for all $z \in \Omega$, then $h \prec \hat{h}$. A straightforward modification of the proof of Proposition 2.6 of [Arcoya and Ruiz 2006] in order to accommodate the extra linear term $-\Delta u$ gives the following strong comparison principle.

Proposition 3. *If $\xi \ge 0$, $h, \hat{h} \in L^{\infty}(\Omega)$, $h \prec \hat{h}$, $u, v \in C_0^1(\overline{\Omega})$ are solutions of*

$$-\Delta_p u(z) - \Delta u(z) + \xi |u(z)|^{p-2} u(z) = h(z),$$

$$-\Delta_p v(z) - \Delta v(z) + \xi |v(z)|^{p-2} v(z) = \hat{h}(z) \quad \text{in } \Omega,$$

and $v \in \operatorname{int} C_+$, then $v - u \in \operatorname{int} C_+$.

Proof. We follow [Arcoya and Ruiz 2006] (see Proposition 2.6).

By nonlinear regularity, $u, v \in C^{1,\beta}(\overline{\Omega})$ $(0 < \beta < 1)$.

We have

$$A_p(u) + A(u) + \xi |u|^{p-2} u = h \le \hat{h} = A_p(v) + A(v) + \xi v^{p-1} \quad \text{in } W^{-1,p'}(\Omega).$$

Acting with $(u - v)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\left\langle A_p(u) - A_p(v), (u-v)^+ \right\rangle + \left\langle A(u) - A(v), (u-v)^+ \right\rangle$$
$$+ \int_{\Omega} \xi (|u|^{p-2} u - v^{p-1})(u-v)^+ \, dz \le 0,$$

which implies that $\|D(u-v)^+\|_2^2 \le 0$, since A_p is monotone; hence $u \le v$.

First we show that $u(z) \le v(z)$ for all $z \in \Omega$. For this purpose, we introduce

$$D_0 = \{z \in \Omega : u(z) = v(z)\} \quad \text{and} \quad D_1 = \{z \in \Omega : Du(z) = Dv(z) = 0\}.$$

We show that $D_0 \subseteq D_1$. So, let $z_0 \in D_0$. Since $u \le v$, the function $z \mapsto (u-v)(z)$ attains its maximum at $z_0 \in D_0$ and so we have $Du(z_0) = Dv(z_0)$. If $Du(z_0) \ne 0$, then we can find $\overline{B}_\rho(z_0) \subseteq \Omega$ such that

$$\|Du(z)\| > 0, \quad \|Dv(z)\| > 0, \quad (Du(z), Dv(z))_{\mathbb{R}^N} > 0 \quad \text{for all } z \in \overline{B}_\rho(z_0).$$

We set $w = v - u \in C_+ \setminus \{0\}$. Then w satisfies the linear elliptic equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i} \left(\eta_{ij}(z) \frac{\partial w}{\partial z_j} \right) = -\xi(v^{p-1} - |u|^{p-2} u) + \hat{h} - h.$$

In this equation the coefficients $\eta_{ij}(\,\cdot\,)$ are given by

$$\eta_{ij}(z) = \delta_{ij}(\|Du(z)\|^{p-2} + 1) + (p-2)\|Du(z)\|^{p-4}\frac{\partial u}{\partial z_i}(z)\frac{\partial u}{\partial z_j}(z)$$

for all $z \in \bar{B}_\rho(z_0)$ (see [Arcoya and Ruiz 2006, p. 854]). Hence $\eta_{ij} \in C^\beta(\bar{B}_\rho(z_0))$ with $\beta \in (0, 1)$ and the η_{ij} form a uniformly elliptic operator by taking $\rho \in (0, 1)$ even smaller if necessary. Then the strong maximum principle (see [Gilbarg and Trudinger 2001; Vázquez 1984]) implies that

$$u(z) < v(z) \quad \text{for all } z \in \bar{B}_\rho(z_0),$$

which contradicts the fact that $z_0 \in D_0$. So, we infer that $D_0 \subseteq D_1$.

Since by hypothesis $v \in \text{int } C_+$, we see that D_1 is compact and so D_0 is compact. So, we can find $\Omega_1 \subseteq \Omega$ open and smooth such that

$$D_0 \subseteq \Omega_1 \subseteq \bar{\Omega}_1 \subseteq \Omega.$$

We can find $\varepsilon > 0$ such that

$$u(z) + \varepsilon < v(z) \quad \text{for all } z \in \partial\Omega_1,$$
$$h(z) + \varepsilon < \hat{h}(z) \quad \text{for a.a. } z \in \Omega_1.$$

Let $\delta \in (0, \min\{\varepsilon, 1\})$ be such that

$$\xi \left| |s|^{p-2}s - |s'|^{p-2}s' \right| < \varepsilon \quad \text{for all } s, s' \in [-\|u\|_\infty, \|v\|_\infty] \text{ with } |s - s'| < 2\delta.$$

Then we have

$$-\Delta_p(u+\delta) - \Delta(u+\delta) + \xi|u+\delta|^{p-2}(u+\delta) = -\Delta_p(u) - \Delta(u) + \xi|u+\delta|^{p-2}(u+\delta)$$
$$= \xi[|u+\delta|^{p-2}(u+\delta) - |u|^{p-2}u] + h$$
$$\leq h + \varepsilon \leq \hat{h} = -\Delta_p v - \Delta v + \xi v^{p-1},$$

which implies $u + \delta \leq v$ in Ω_1, by the weak maximum principle.

Since $D_0 \subseteq \Omega_1$, we infer that the boundary point theorem is valid for uniformly elliptic operators with Hölder continuous coefficients (see [Finn and Gilbarg 1957, Lemma 7, p. 31; Gilbarg and Trudinger 2001, p. 46]). So, for every $z_0 \in \partial\Omega$, we have

$$\frac{\partial w}{\partial n}(z_0) < 0,$$

and therefore $v - u \in \text{int } C_+$. □

We now recall some basic facts concerning the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. We consider the nonlinear eigenvalue problem

(3)
$$\begin{cases} -\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ if the problem (3) has a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$; that solution is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. The smallest eigenvalue $\hat{\lambda}_1(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ has the following properties (see [Anane 1987; Anane and Tsouli 1996; García Azorero and Peral Alonso 1987]):

- $\hat{\lambda}_1(p)$ is positive and isolated.

- $\hat{\lambda}_1(p)$ is simple (its eigenspace is one-dimensional).

- $$\hat{\lambda}_1(p) = \inf\left[\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \not\equiv 0 \right].$$

In this variational characterization of $\hat{\lambda}_1(p)$, the infimum is realized on the corresponding one-dimensional eigenspace. Moreover, it is clear from the third property above that the elements of the one-dimensional eigenspace do not change sign. In the sequel, by $\hat{u}_{1,p} \in W_0^{1,p}(\Omega)$, we denote the L^p-normalized (i.e., $\|\hat{u}_{1,p}\|_p = 1$) positive eigenfunction corresponding to the eigenvalue $\hat{\lambda}_1(p) > 0$. The nonlinear regularity theory (see, for example, [Gasiński and Papageorgiou 2006, pp. 737–738]), implies that $\hat{u}_{1,p} \in C_+ \setminus \{0\}$. Then the nonlinear maximum principle of [Vázquez 1984] says that $\hat{u}_{1,p} \in \mathrm{int}\, C_+$. Since the spectrum $\sigma(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ is closed and $\hat{\lambda}_1(p) > 0$ is isolated, the second eigenvalue $\hat{\lambda}_2(p) = \inf[\lambda \in \sigma(p) : \lambda > \hat{\lambda}_1(p)]$ is also well-defined.

If $N = 1$ (ordinary differential equation), then $\sigma(p) = \{\hat{\lambda}_k(p)\}_{k \geq 1} \subseteq (0, +\infty)$, where each $\hat{\lambda}_k(p)$ is a simple eigenvalue, $\hat{\lambda}_k(p) \to +\infty$ as $k \to +\infty$ and the corresponding eigenfunctions $\{\hat{u}_{k,p}\}_{k \geq 1}$ have exactly $k - 1$ zeros (see, for example, [Gasiński and Papageorgiou 2006, p. 761]).

If $N \geq 2$ (partial differential equation), then the Ljusternik–Schnirelmann minimax scheme via the Krasnoselskii genus gives us a whole strictly increasing sequence of eigenvalues $\{\hat{\lambda}_k(p)\}_{k \geq 1}$ such that $\hat{\lambda}_k(p) \to +\infty$ as $k \to +\infty$. It is not known if this is the complete list of eigenvalues. If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues of $(-\Delta, H_0^1(\Omega))$.

Next we recall some basic definitions and facts from Morse theory and from [Cingolani and Vannella 2003; 2007], which we will need in order to produce a second nodal solution.

So, as before, let X be a Banach space and (Y_1, Y_2) a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k-th-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). For $k < 0$, $H_k(Y_1, Y_2) = 0$.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the sets

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \quad K_\varphi = \{x \in X : \varphi'(x) = 0\}, \quad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The critical groups of $\varphi \in C^1(X)$ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$) are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}\big) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Now suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The second deformation theorem (see, for example, [Gasiński and Papageorgiou 2006, p. 628]), implies that this definition is independent of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite and define

$$M(t, x) = \sum_{k \geq 0} \text{rank}\, C_k(\varphi, x) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geq 0} \text{rank}\, C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$(4) \qquad \sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t),$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with integer coefficients β_k.

Let H be a Hilbert space, x a point in H, and U a neighborhood of x. Let $\varphi \in C^2(U)$. If $x \in K_\varphi$, then the Morse index $\mu = \mu(x)$ of x is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite. The nullity $\nu(x)$ of $x \in K_\varphi$ is the dimension of $\ker \varphi''(x)$. We say that $x \in K_\varphi$ is nondegenerate if $\varphi''(x)$ is invertible (i.e., $\nu(x) = 0$). If $\varphi \in C^2(U)$ and $x \in K_\varphi$ is nondegenerate with Morse index μ, then

$$C_k(\varphi, x) = \delta_{k,\mu} \mathbb{Z} \quad \text{for all } k \geq 0,$$

where $\delta_{k,\mu}$ is the Kronecker symbol.

As mentioned in the introduction, to produce a second nodal solution, we will use some facts from [Cingolani and Vannella 2003; 2007]. Suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that, for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

$$|f_x'(z, x)| \leq \tilde{\alpha}(z) + \tilde{c}|x|^{r-2} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $\tilde{\alpha} \in L^\infty(\Omega)_+$, $\tilde{c} > 0$ and $p \leq r < p^*$. We set $F(z, x) = \int_0^x f(z, s)\, ds$ and consider the C^2-functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

For all $u, v, y \in W_0^{1,p}(\Omega)$, we have (see [Cingolani and Vannella 2003])

$$\langle \varphi''(u)v, y \rangle = \int_\Omega (1 + \|Du\|^{p-2})(Dv, Dy)_{\mathbb{R}^N}\, dz$$
$$+ (p-2)\int_\Omega \|Du\|^{p-4}(Du, Dv)_{\mathbb{R}^N}(Du, Dy)_{\mathbb{R}^N}\, dz - \int_\Omega f_x'(z, u)vy\, dz.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair consisting of the spaces

$$W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \quad \text{and} \quad W_0^{1,p}(\Omega), \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

Suppose that $u_0 \in K_\varphi$. Nonlinear regularity theory (see [Ladyzhenskaya and Ural'tseva 1968; Lieberman 1991]) implies that $u_0 \in C_0^1(\overline{\Omega})$. It follows that

$$b(\cdot) = \|Du_0(\cdot)\|^{(p-4)/2} Du_0(\cdot) \in L^\infty(\Omega, \mathbb{R}^N).$$

Let H_b be the completion of $C_c^\infty(\Omega)$ under the inner product

$$(v, y)_b = \int_\Omega \left[(1 + \|b\|^2)(Dv, Dy)_{\mathbb{R}^N} + (p-2)(b, Dv)_{\mathbb{R}^N}(b, Dy)_{\mathbb{R}^N}\right] dz.$$

Denote by $\|\cdot\|_b$ the corresponding norm. Clearly $\|\cdot\|_b$ is equivalent to the usual Sobolev norm of $H_0^1(\Omega)$, so H_b and $H_0^1(\Omega)$ are isomorphic. Since $p > 2$, $W_0^{1,p}(\Omega)$ is embedded continuously into H_b. Let $L_b \in \mathcal{L}(H_b, H_b^*)$ be defined by

$$\langle L_b(v), y \rangle_b = (v, y)_b - \int_\Omega f_x'(z, u_0)vy\, dz \quad \text{for all } v, y \in H_b.$$

Then L_b is a Fredholm operator of index zero and it is the extension of $\varphi''(u_0)$ on H_b. We consider the orthogonal direct sum decomposition

$$H_b = H^- \oplus H^0 \oplus H^+,$$

where H^-, H^0, H^+ are the negative, null and positive spaces according to the spectral decomposition of L_b in $L^2(\Omega)$. Then H^- and H^0 are finite-dimensional and, since $u_0 \in C_0^1(\overline{\Omega})$, standard regularity theory implies that

$$H^- \oplus H^0 \subseteq W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

We set $V = H^- \oplus H^0$ and $W = W_0^{1,p}(\Omega) \cap H^+$. Then $W_0^{1,p}(\Omega) = V \oplus W$ and, by [Cingolani and Vannella 2003, p. 279], there exists $c > 0$ such that

$$\langle \varphi''(u_0)v, v \rangle \geq c\|v\|_b^2 \quad \text{for all } v \in W.$$

In what follows, for every $r \in (1, +\infty)$, we denote by

$$A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega), \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

the nonlinear map defined by

$$(5) \qquad \langle A_r(u), y \rangle = \int_\Omega \|Du\|^{r-2}(Du, Dy)_{\mathbb{R}^N} \, dz \quad \text{for all } u, y \in W_0^{1,r}(\Omega).$$

If $r = 2$, then we set $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. The next result can be found in [Gasiński and Papageorgiou 2006, pp. 745–746].

Proposition 4. *If $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is defined by (5), then A_r is continuous, monotone (hence maximal monotone) and of type $(S)_+$; that is, if u_n converges weakly to u in $W_0^{1,p}(\Omega)$ and $\limsup_{n \to +\infty} \langle A_r(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W_0^{1,r}(\Omega)$.*

Throughout this paper by $\| \cdot \|$ we denote the norm of $W_0^{1,p}(\Omega)$. By virtue of Poincaré's inequality, $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$. By $\| \cdot \|$ we will also denote the norm of $\mathbb{R}^N$. No confusion is possible, since it will always be clear from the context which norm we mean.

For $x \in \mathbb{R}$, we define

$$x^\pm = \max\{\pm x, 0\}.$$

Then for $u \in W_0^{1,p}(\Omega)$ we set

$$u^\pm(\cdot) = u(\cdot)^\pm.$$

We know that $u^\pm \in W_0^{1,p}(\Omega)$ and

$$|u| = u^+ + u^- \quad \text{and} \quad u = u^+ - u^-.$$

By $|\cdot|_{\mathbb{R}^N}$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Finally, if $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, if $(z, x) \to g(z, x)$ is a Carathéodory function), then we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3. Constant sign solutions

In this section, we produce four nontrivial smooth solutions of constant sign, two positive and two negative. The hypotheses on the reaction $f(z, x)$ are the following:

Hypotheses H. (i) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

 (ii) $f(z, 0) = 0$ a.e. in Ω.

(iii) $|f(z, x)| \leq \alpha(z) + c|x|^{p-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$.

(iv) If $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\lim_{x \to \pm\infty} \frac{pF(z, x)}{|x|^p} = \hat{\lambda}_1(p) \quad \text{uniformly for a.a. } z \in \Omega,$$

and, for some $\tau > 2$,

$$\limsup_{x \to \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\tau} \le \hat{\beta} < 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

(v) There exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that

$$w_-(z) \le c_- < 0 < c_+ \le w_+(z) \quad \text{for all } z \in \bar{\Omega},$$

$$\operatorname*{ess\,sup}_{\Omega} f(\,\cdot\,, w_+(\,\cdot\,)) \le 0 \le \operatorname*{ess\,inf}_{\Omega} f(\,\cdot\,, w_-(\,\cdot\,)),$$

and $A_p(w_-) + A(w_-) \le 0 \le A_p(w_+) + A(w_+)$ in $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$.

(vi) For every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z, x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

(vii) There exist integer $m \ge 2$ and functions $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_m(2) \le \eta(z) \le \hat{\eta}(z) \le \hat{\lambda}_{m+1}(2) \quad \text{a.e. in } \Omega, \quad \hat{\lambda}_m(2) \ne \eta, \quad \hat{\lambda}_{m+1}(2) \ne \hat{\eta}$$

and

$$\eta(z) \le \liminf_{x \to 0} \frac{f(z, x)}{x} \le \limsup_{x \to 0} \frac{f(z, x)}{x} \le \hat{\eta}(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Remarks. Hypothesis H(iv) implies that, asymptotically at $\pm\infty$, we have resonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ from the right. Hence the energy functional of the problem, as we will see, is indefinite. Hypothesis H(v) is satisfied if we can find $c_- < 0 < c_+$ such that $f(z, c_+) = f(z, c_-) = 0$ a.e. in Ω.

Example. The following function satisfies the hypotheses H (for simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} \eta(x - |x|^{r-2}x) & \text{if } |x| \le 1, \\ \hat{\lambda}_1(p)(|x|^{p-2}x - |x|^{\tau-2}x) & \text{if } |x| > 1, \end{cases}$$

with $\eta \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$, $m \ge 2$ and $r > 2$, $1 < \tau < p$.

We introduce the following truncations of $f(z, \cdot)$:

$$(6) \qquad \hat{f}_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z, x) & \text{if } 0 \le x \le w_+(z), \\ f(z, w_+(z)) & \text{if } w_+(z) < x \end{cases}$$

and

$$\hat{f}_-(z, x) = \begin{cases} f(z, w_-(z)) & \text{if } x < w_-(z), \\ f(z, x) & \text{if } w_-(z) \le x \le 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

Both are Carathéodory functions. Let $\hat{F}_\pm(z, x) = \int_0^x \hat{f}_\pm(z, s)\, ds$ and consider the C^1-functionals $\hat{\varphi}_\pm : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}_\pm(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\varphi \in C^1(W_0^{1,p}(\Omega))$.

First, we produce two nontrivial constant sign smooth solutions of (1).

Proposition 5. *If hypotheses* H(iii), (v), (vi), (vii) *hold, then problem* (1) *has at least the two nontrivial constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+,$$

and both are local minimizers of φ.

Proof. First we produce the positive solution.

From (6) we see that $\hat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we can check easily that $\hat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$(7) \qquad \hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)] = m_+.$$

By virtue of hypothesis H(vii), we can find $\vartheta > \hat{\lambda}_1(2)$ and $0 < \delta < \min\{c_+, -c_-\}$ such that

$$(8) \qquad F(z, x) \ge \tfrac{1}{2}\vartheta x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta.$$

Let $t \in (0, 1)$ be small such that $t\hat{u}_{1,2}(z) \in [0, \delta]$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_{1,2} \in \operatorname{int} C_+$). Then

$$\hat{\varphi}_+(t\hat{u}_{1,2}) = \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}\hat{\lambda}_1(2) - \int_\Omega \hat{F}_+(z, t\hat{u}_{1,2})\, dz$$

$$\le \frac{t^p}{p}\|D\hat{u}_{1,2}\|_p^p + \frac{t^2}{2}[\hat{\lambda}_1(2) - \vartheta]$$

(see (8) and recall that $\|\hat{u}_{1,2}\|_2 = 1$).

Since $\vartheta > \hat{\lambda}_1(2)$ and $p > 2$, by choosing $t \in (0, 1)$ even smaller if necessary, we have $\hat{\varphi}_+(t\hat{u}_{1,2}) < 0$, which implies $\hat{\varphi}_+(u_0) = \hat{m}_+ < 0 = \hat{\varphi}_+(0)$ (see (7)); hence $u_0 \neq 0$.

From (7) we have $\hat{\varphi}'_+(u_0) = 0$, which implies

$$(9) \qquad\qquad A_p(u_0) + A(u_0) = N_{\hat{f}_+}(u_0).$$

On (9) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain $u_0 \geq 0$, $u_0 \neq 0$ (see (6)). Also, we act with $(u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\langle A_p(u_0), (u_0 - w_+)^+ \rangle + \langle A(u_0), (u_0 - w_+)^+ \rangle$$
$$= \int_\Omega \hat{f}_+(z, u_0)(u_0 - w_+)^+ \, dz = \int_\Omega f(z, w_+)(u_0 - w_+)^+ \, dz \quad \text{(see (6))}$$
$$\leq \langle A_p(w_+) + A(w_+), (u_0 - w_+)^+ \rangle,$$

by hypothesis H(v). Therefore

$$\int_{\{u_0 > w_+\}} \left(\|Du_0\|^{p-2} Du_0 - \|Dw_+\|^{p-2} Dw_+, \, Du_0 - Dw_+ \right)_{\mathbb{R}^N} dz$$
$$+ \|D(u_0 - w_+)^+\|_2^2 \leq 0.$$

It follows that $u_0 \leq w_+$.

So, we have proved that

$$u_0 \in [0, w_+] = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq w_+(z) \text{ a.e. in } \Omega\}.$$

Then (9) becomes $A_p(u_0) + A(u_0) = N_f(u_0)$ (see (6)), and hence

$$(10) \qquad -\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \quad \text{a.e. in } \Omega, \quad u_0|_{\partial\Omega} = 0.$$

From (10) and [Ladyzhenskaya and Ural'tseva 1968, Theorem 7.1, p. 286], we have $u_0 \in L^\infty(\Omega)$. We can apply the regularity result of [Lieberman 1991, p. 320] and have $u_0 \in C_+ \setminus \{0\}$. Note that

$$A_p(u_0) + A(u_0) - N_f(u_0) = 0 \leq A_p(w_+) + A(w_+) - N_f(w_+) \quad \text{in } W^{-1,p'}(\Omega),$$

by H(v), and, for a.a. $z \in \Omega$ and all $x, y \in [-\rho, \rho]$ with $x > y$, we have, by H(vi),

$$f(z, x) - f(z, y) \geq -\xi_\rho(x - y),$$

Let $a(\xi) = \|\xi\|^{p-2}\xi + \xi$ for all $\xi \in \mathbb{R}^N$. Then $a \in C^1(\mathbb{R}^N)$,

$$\nabla a(\xi) = \|\xi\|^{p-2}\left(I + (p-2)\frac{\xi \otimes \xi}{\|\xi\|^2} \right) + I$$

and

$$\text{div } a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have $(\nabla a(\xi)y, y)_{\mathbb{R}^N} \geq \|y\|^2$ for all ξ, $y \in \mathbb{R}^N$ and so we can apply [Pucci and Serrin 2007, Theorem 2.5.3, p. 37] and infer, via H(v), that

$$u_0(z) < w_+(z) \quad \text{for all } z \in \bar{\Omega}.$$

Let $\rho = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$. By virtue of H(vi) and (10), we have

$$-\Delta_p u_0(z) - \Delta u_0(z) + \hat{\xi}_\rho u_0(z)^{p-1} = f(z, u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1} \geq 0 \quad \text{a.e. in } \Omega,$$

and hence

$$\Delta_p u_0(z) + \Delta u_0(z) \leq \hat{\xi}_\rho u_0(z)^{p-1} \quad \text{a.e. in } \Omega.$$

Invoking the boundary point theorem of Pucci and Serrin [2007, Theorem 5.5.1, p. 120] we have $u_0 \in \text{int } C_+$. Therefore

$$u_0 \in \text{int}_{C_0^1(\bar{\Omega})}[0, w_+].$$

It is clear from (6) that $\hat{\varphi}_+|_{[0,w_+]} = \varphi|_{[0,w_+]}$. Therefore u_0 is a local $C_0^1(\bar{\Omega})$-minimizer of φ and so by Theorem 2 it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ.

Similarly, working this time with $\hat{\varphi}_-$, we produce another constant sign smooth solution $v_0 \in -\text{int } C_+$ which is a local minimizer of φ. $\qquad \square$

Using $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$, we can produce two more nontrivial constant sign smooth solutions.

Proposition 6. *If hypotheses* H *hold and* K_φ *is finite, problem* (1) *has at least four nontrivial constant sign smooth solutions*

$$u_0, \hat{u} \in \text{int } C_+ \qquad \text{with } \hat{u} - u_0 \in \text{int } C_+$$
$$v_0, \hat{v} \in -\text{int } C_+ \quad \text{with } v_0 - \hat{v} \in \text{int } C_+.$$

Proof. From Proposition 5 we already have two solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$.

Next we produce the second nontrivial positive smooth solution. To this end, we introduce the following truncation of $f(z, \cdot)$:

$$(11) \qquad h_+(z, x) = \begin{cases} f(z, u_0(z)) & \text{if } x \leq u_0(z), \\ f(z, x) & \text{if } u_0(z) < x. \end{cases}$$

This is a Carathéodory function. We set $H_+(z, x) = \int_0^x h_+(z, s)\,ds$ and consider the C^1-functional $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega H_+(z, u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Claim 1. *The functional* ψ_+ *satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

(12) $|\psi_+(u_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$

and

(13) $(1 + \|u_n\|)\psi'_+(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$.

From (13) we have

(14) $\left| \langle A_p(u_n), g \rangle + \langle A(u_n), g \rangle - \int_\Omega h_+(z, u_n) g \, dz \right| \leq \dfrac{\varepsilon_n \|g\|}{1 + \|u_n\|}$

for all $g \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \downarrow 0$.

In (14) we choose $g = -u_n^- \in W_0^{1,p}(\Omega)$. Then we get

$$\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 - \int_\Omega f(z, u_0)(-u_n^-) \, dz \leq \varepsilon_n \quad \text{for all } n \geq 1,$$

by (11); this implies that $\|Du_n^-\|_p^p \leq c_1 \|u_n^-\|$ for some $c_1 > 0$ and all $n \geq 1$ (by H(iii)), and we conclude, since $p > 1$, that

(15) $\{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

We will show that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Arguing by contradiction, because of (15) and by passing to a suitable subsequence if necessary, we may assume that $\|u_n^+\| \to \infty$. We set $y_n = u_n^+/\|u_n^+\|$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

(16) $y_n \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$ as $n \to \infty$,

where $\xrightarrow{w}$ indicates weak convergence. From (14), we have

(17) $\left| \langle A_p(y_n), g \rangle + \dfrac{1}{\|u_n^+\|^{p-2}} \langle A(y_n), g \rangle - \int_\Omega \dfrac{h_+(z, u_n^+)}{\|u_n^+\|^{p-1}} g \, dz \right| \leq \varepsilon'_n \|g\|,$

with $\varepsilon'_n \to 0$ (see (15)).

Hypothesis H(iii) and (11) imply that

(18) $\left\{ \dfrac{N_{h_+}(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ is bounded.

From (18) and using hypothesis H(iv), as in the proof of Proposition 30 of [Aizicovici et al. 2008], we have

(19) $\dfrac{N_{h_+}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \beta = \hat{\lambda}_1(p) y^{p-1}$ in $L^{p'}(\Omega)$.

Also, if in (17) we choose $g = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (16) and (19), we obtain

$$\lim_{n \to \infty} [\langle A_p(y_n), y_n - y \rangle + \langle A(y_n), y_n - y \rangle] = 0,$$

from which we get successively

$$\limsup_{n \to \infty} [\langle A_p(y_n), y_n - y \rangle + \langle A(y), y_n - y \rangle] \leq 0 \quad \text{(since } A \text{ is monotone)},$$

$$\limsup_{n \to \infty} \langle A_p(y_n), y_n - y \rangle \leq 0 \quad \text{(see (16))},$$

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{(see Proposition 4)}.$$

The upshot is that

$$\|y\| = 1, \quad y \geq 0. \tag{20}$$

Passing to the limit as $n \to \infty$ in (17) and using (19) and (20), we see that

$$\langle A_p(y), g \rangle = \hat{\lambda}_1(p) \int_\Omega y^{p-1} g \, dz \quad \text{for all } g \in W_0^{1,p}(\Omega),$$

since $p > 2$ and $\|u_n^+\| \to \infty$. This yields $A_p(y)\hat{\lambda}_1(p)y^{p-1}$ and so

$$-\Delta_p y(z) = \hat{\lambda}_1(p) y(z)^{p-1} \quad \text{a.e. in } \Omega, \quad y|_{\partial\Omega} = 0,$$

implying, in view of (20), that

$$y = \lambda \hat{u}_{1,p} \quad \text{for some } \lambda > 0. \tag{21}$$

Therefore $y(z) > 0$ for all $z \in \Omega$ and this implies that $u_n^+(z) \to +\infty$ for all $z \in \Omega$. Then, by virtue of hypothesis H(iv), we have

$$\limsup_{n \to \infty} \frac{f(z, u_n^+(z))u_n^+(z) - pF(z, u_n^+(z))}{|u_n^+(z)|^\tau} \leq \hat{\beta} < 0 \quad \text{for a.a. } z \in \Omega,$$

or again, in view of (11),

$$\limsup_{n \to \infty} \frac{h_+(z, u_n^+(z))u_n^+(z) - pH_+(z, u_n^+(z))}{|u_n^+(z)|^\tau} \leq \hat{\beta} < 0 \quad \text{for a.a. } z \in \Omega. \tag{22}$$

Hypothesis H(iv) and Fatou's lemma, together with (21) and (22), imply that

$$\limsup_{n \to \infty} \frac{1}{\|u_n^+\|^\tau} \int_\Omega \left[h_+(z, u_n^+)u_n^+(z) - pH_+(z, u_n^+) \right] dz < 0 \tag{23}$$

On the other hand, from (12) and (15), we have

$$-M_2 \leq \|Du_n^+\|_p^p + \frac{p}{2}\|Du_n^+\|_2^2 - \int_\Omega pH_+(z, u_n^+) \, dz \tag{24}$$

for some $M_2 > 0$ and all $n \geq 1$.

Also, if we choose $g = u_n^+ \in W_0^{1,p}(\Omega)$ in (14), then

$$(25) \qquad -\varepsilon_n \leq -\|Du_n^+\|_p^p - \|Du_n^+\|_2^2 + \int_\Omega h_+(z, u_n^+) u_n^+ \, dz \quad \text{for all } n \geq 1.$$

Adding (24) and (25), we obtain

$$-M_3 \leq \int_\Omega \left[h_+(z, u_n^+) u_n^+ - p H_+(z, u_n^+) \right] dz + \left(\frac{p}{2} - 1 \right) \|Du_n^+\|_2^2$$

for some $M_3 > 0$ and all $n \geq 1$, whence (since $p > 2$)

$$-\frac{M_3}{\|u_n^+\|^\tau} \leq \frac{1}{\|u_n^+\|^\tau} \int_\Omega \left[h_+(z, u_n^+) u_n^+ - p H_+(z, u_n^+) \right] dz + c_2 \left(\frac{p}{2} - 1 \right) \frac{1}{\|u_n^+\|^{\tau-2}}$$

for some $c_2 > 0$ and all $n \geq 1$, and finally, since $\tau > 2$ and $p > 2$,

$$(26) \qquad 0 \leq \liminf_{n \to \infty} \frac{1}{\|u_n^+\|^\tau} \int_\Omega \left[h_+(z, u_n^+) u_n^+ - p H_+(z, u_n^+) \right] dz.$$

Comparing (23) and (26), we reach a contradiction.

This proves that $\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded; hence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded, by (15). So, we may assume that

$$(27) \qquad u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega).$$

If in (14) we choose $g = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (27), then, as before, exploiting the monotonicity of A, we have

$$\limsup_{n \to \infty} \langle A_p(u_n), u_n - u \rangle \leq 0,$$

implying that $u_n \to u$ in $W_0^{1,p}(\Omega)$, by Proposition 4. Hence ψ_+ satisfies the C-condition, and this proves Claim 1. ☐

Claim 2. *The function u_0 is a local minimizer of ψ_+.*

Proof. We may assume that $K_\varphi \cap [0, w_+] = \{0, u_0\}$. Otherwise, let y be a nontrivial element of $K_\varphi \cap [0, w_+]$ distinct from u_0; as a nontrivial solution of (1), y can be taken such that $u_0 \leq y$, because (1) has a biggest solution in $[0, w_+]$ (this is shown like Proposition 4.4 in [Filippakis et al. 2009]). Therefore, we are done if such a y exists.

We introduce the following truncation of $h_+(z, \cdot)$:

$$(28) \qquad \hat{h}_+(z, x) = \begin{cases} h_+(z, x) & \text{if } x \leq w_+(z), \\ h_+(z, w_+(z)) & \text{if } w_+(z) < x. \end{cases}$$

This is a Carathéodory function. We set $\hat{H}_+(z, x) = \int_0^x \hat{h}_+(z, s) \, ds$ and consider

the C^1-functional $\hat{\psi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{H}_+(z, u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (28) it is clear that $\hat{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. Hence, we can find $\bar{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_+(\bar{u}_0) = \inf[\hat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)],$$

which is to say $\hat{\psi}_+'(\bar{u}_0) = 0$; therefore

(29) $$A_p(\bar{u}_0) + A(\bar{u}_0) = N_{\hat{h}_+}(\bar{u}_0).$$

On (29) first we act with $(u_0 - \bar{u}_0)^+ \in W_0^{1,p}(\Omega)$ and then with $(\bar{u}_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Using (11), (28) and hypothesis H(v), this leads to

$$\bar{u}_0 \in [u_0, w_+] = \left\{u \in W_0^{1,p}(\Omega) : u_0(z) \le u(z) \le w_+(z) \text{ a.e. in } \Omega\right\}.$$

Then (29) becomes $A_p(\bar{u}_0) + A(\bar{u}_0) = N_f(\bar{u}_0)$ by (11) and (28); thus $\bar{u}_0 \in K_\varphi \cap [0, w_+]$, which is to say $\bar{u}_0 = u_0$.

From Proposition 5 and its proof, we have

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad u_0(z) < w_+(z) \quad \text{for all } z \in \bar{\Omega}.$$

From (28) we infer that $\psi_+|_{[0, w_+]} = \hat{\psi}_+|_{[0, w_+]}$, so u_0 is a local $C_0^1(\bar{\Omega})$-minimizer of ψ_+. Applying Theorem 2, we see that u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of ψ_+, as we wished to show. $\qquad\square$

If $u \in K_{\psi_+}$, then

$$A_p(u) + A(u) = N_{h_+}(u).$$

Acting with $(u_0 - u)^+ \in W_0^{1,p}(\Omega)$ and using (11), we show that $u_0 \le u$. Therefore

(30) $$K_{\psi_+} \subseteq [u_0) = \left\{u \in W_0^{1,p}(\Omega) : u_0(z) \le u(z) \text{ for a.a. } z \in \Omega\right\}.$$

By virtue of Claim 2, $u_0 \in K_{\psi_+}$. Note that from (11) and (30) it follows that $K_{\psi_+} \subseteq K_\varphi$ and recall that by hypothesis K_φ is finite. So, as in [Aizicovici et al. 2008, proof of Proposition 29], we can find $\rho \in (0, 1)$ small such that

(31) $$\psi_+(u_0) < \inf[\psi_+(u) : \|u - u_0\| = \rho] = \eta_\rho^+.$$

Claim 3. $$\psi_+(t\hat{u}_{1,p}) \to -\infty \quad \text{as } t \to +\infty.$$

Proof. By virtue of hypothesis H(iv), we can find $\hat{\beta}_1 \in (\hat{\beta}, 0)$ and $M_4 > \|u_0\|_\infty$ such that $f(z, x)x - pF(z, x) \le \hat{\beta}_1 x^\tau$ for a.a. $z \in \Omega$, all $x \ge M_4$. Thus

(32) $$h_+(z, x)x - pH_+(z, x) \le \hat{\beta}_1 x^\tau + c_3$$

for a.a. $z \in \Omega$, all $x \geq M_4$, some $c_3 > 0$.

(Note that $F(z, x) = H_+(z, x) - H_+(z, u_0(z)) + f(z, u_0(z))u_0(z)$ for a.a. $z \in \Omega$, all $x \geq M_4$.)

Without loss of generality, we may assume that $\tau < p$ (see H(iv)). We have

$$\frac{d}{dx}\left(\frac{H_+(z, x)}{x^p}\right) = \frac{h_+(z, x)x^p - px^{p-1}H_+(z, x)}{x^{2p}}$$

$$= \frac{h_+(z, x)x - pH_+(z, x)}{x^{p+1}} \leq \frac{\hat{\beta}_1 x^\tau + c_3}{x^{p+1}} \quad \text{(see (32))}$$

$$= \hat{\beta}_1 x^{\tau - p - 1} + \frac{c_3}{x^{p+1}}.$$

It follows that

$$\frac{H_+(z, x)}{x^p} - \frac{H_+(z, y)}{y^p} \leq -\frac{\hat{\beta}_1}{p - \tau}\left(\frac{1}{x^{p-\tau}} - \frac{1}{y^{p-\tau}}\right) - \frac{c_3}{p}\left(\frac{1}{x^p} - \frac{1}{y^p}\right)$$

for a.a. $z \in \Omega$, all $x \geq y \geq M_4$.

Letting $x \to +\infty$, using hypothesis H(iv) and recalling that $\tau < p$, we obtain

$$\frac{\hat{\lambda}_1(p)}{p} - \frac{H_+(z, y)}{y^p} \leq \frac{\hat{\beta}_1}{p - \tau}\frac{1}{y^{p-\tau}} + \frac{c_3}{p}\frac{1}{y^p} \quad \text{for a.a. } z \in \Omega, \text{ all } y \geq M_4,$$

or, upon multiplication by y^p and with $c_4 = c_3/p$,

$$(33) \qquad \frac{\hat{\lambda}_1(p)}{p}y^p - H_+(z, y) \leq \frac{\hat{\beta}_1}{p - \tau}y^\tau + c_4 \quad \text{for a.a. } z \in \Omega, \text{ all } y \geq 0.$$

Then, for $t > 0$, we have

$$(34) \qquad \psi_+(t\hat{u}_{1,p}) = \frac{t^p}{p}\hat{\lambda}_1(p)\|\hat{u}_{1,p}\|_p^p + \frac{t^2}{2}\|D\hat{u}_{1,p}\|_2^2 - \int_\Omega H_+(z, t\hat{u}_{1,p})\,dz$$

$$\leq \frac{\hat{\beta}_1}{p - \tau}t^\tau\|\hat{u}_{1,p}\|_\tau^\tau + \frac{t^2}{2}\|D\hat{u}_{1,p}\|_2^2 + c_4|\Omega|_N \quad \text{(see (33))}.$$

Since $\tau > 2$ (see H(iv)) and $\hat{\beta}_1 < 0$, it follows from (34) that $\psi_+(t\hat{u}_{1,p}) \to -\infty$ as $t \to +\infty$. This proves Claim 3. $\qquad \square$

Claims 1, 3 and (31) permit the use of Theorem 1, the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$(35) \qquad\qquad \psi_+(u_0) < \eta_\rho^+ \leq \psi_+(\hat{u})$$

(see (31)) and

$$(36) \qquad\qquad \psi_+'(\hat{u}) = 0.$$

From (35) we see that $\hat{u} \neq u_0$, while from (36) we have $\hat{u} \in [u_0)$ (see (30)).

Therefore $\hat{u}$ is the second nontrivial positive solution of (1) (see (11)). Moreover, nonlinear regularity theory (see [Lieberman 1991]) implies that $\hat{u} \in \text{int } C_+$, $u_0 \leq \hat{u}$, $u_0 \neq \hat{u}$. From the tangency principle of [Pucci and Serrin 2007, p. 35], we have

$$u_0(z) < \hat{u}(z) \quad \text{for all } z \in \Omega.$$

Let $\rho = \|\hat{u}\|_\infty$ and let $\hat{\xi} > \hat{\xi}_\rho$ ($\hat{\xi}_\rho > 0$ as postulated by hypothesis H(vi)). We set

$$h(z) = f(z, u_0(z)) + \hat{\xi} u_0(z)^{p-1} \quad \text{and} \quad \hat{h}(z) = f(z, \hat{u}(z)) + \hat{\xi} \hat{u}(z)^{p-1}.$$

Clearly, $h, \hat{h} \in L^\infty(\Omega)_+$, $h \prec \hat{h}$ (see H(vi) and recall $u_0(z) < \hat{u}(z)$ for all $z \in \Omega$). Moreover, $\hat{u} \in \text{int } C_+$ and so we can use Proposition 3 and infer that $\hat{u} - u_0 \in \text{int } C_+$.

Similarly, consider the truncation

$$h_-(z, x) = \begin{cases} f(z, x) & \text{if } x < v_0(z), \\ f(z, v_0(z)) & \text{if } v_0(z) \leq x. \end{cases}$$

Arguing as before, we produce a second nontrivial negative solution $\hat{v} \in -\text{int } C_+$ such that $v_0 - \hat{v} \in \text{int } C_+$. $\qquad\square$

4. Nodal solutions

In this section we produce nodal solutions for problem (1). Under the current hypotheses H, we will produce a nodal solution, and subsequently, by strengthening the regularity on $f(z, \cdot)$ (see hypotheses $\widehat{H}$ below), we will generate a second nodal solution. In this section, Morse theory is a basic tool.

Our strategy is the following. First we will show that problem (1) has extremal constant sign solutions; i.e., there is a smallest nontrivial positive solution u_+ of (1) and a biggest nontrivial negative solution v_- of (1). By truncating $f(z, \cdot)$ at $\{v_-(z), u_+(z)\}$ and using variational methods and Morse theoretic techniques, we show that problem (1) has nontrivial solutions in the order interval $[v_-, u_+]$ distinct from v_- and u_+. The extremality of v_- and u_+ implies that such solutions are necessarily nodal. The nonhomogeneity of the differential operator $u \to -\Delta_p u - \Delta u$ creates difficulties, which we have to overcome. To this end, note that hypotheses H(iii), (vii) imply that we can find $c_5 > \hat{\lambda}_1(2)$ and $c_6 > 0$ such that

$$f(z, x)x \geq c_5 x^2 - c_6 |x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

This growth estimate leads to the following Dirichlet problem

$$(37) \qquad -\Delta_p u(z) - \Delta u(z) = c_5 u(z) - c_6 |u(z)|^{p-2} u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Proposition 7. *Problem* (37) *has a unique nontrivial positive solution* $u_* \in \text{int } C_+$ *and, since* (37) *is odd,* $v_* = -u_* \in -\text{int } C_+$ *is the unique nontrivial negative solution of* (37).

Proof. We consider the C^1-functional $\gamma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{c_5}{2}\|u^+\|_2^2 + \frac{c_5}{p}\|u^+\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $p > 2$, it is clear that γ_+ is coercive. Also, γ_+ is sequentially weakly lower semicontinuous. Therefore, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$(38) \qquad\qquad \gamma_+(u_*) = \inf\,[\gamma_+(u) : u \in W_0^{1,p}(\Omega)] = m_*^+.$$

Since $c_5 > \hat{\lambda}_1(2)$ and $p > 2$, for $t \in (0, 1)$ small, we have $\gamma_+(t\hat{u}_{1,2}) < 0$, which implies $\gamma_+(u_*) = m_*^+ < 0 = \gamma_+(0)$ by (38); hence $u_* \neq 0$.

From (38) we have

$$(39) \qquad\qquad\qquad\qquad \gamma_+'(u_*) = 0$$

and therefore

$$A_p(u_*) + A(u_*) = c_5 u_*^+ - c_6 (u_*^+)^{p-1}.$$

On (39) we act with $-u_*^- \in W_0^{1,p}(\Omega)$ and infer that $u_* \geq 0$, $u_* \neq 0$. Hence $A_p(u_*) + A(u_*) = c_5 u_* - c_6 u_*^{p-1}$, and so

$$-\Delta_p u_*(z) - \Delta u_*(z) = c_5 u_*(z) - c_6 u_*(z)^{p-1} \quad \text{a.e. in } \Omega, \quad u_*|_{\partial\Omega} = 0.$$

Nonlinear regularity theory (see [Ladyzhenskaya and Ural'tseva 1968; Lieberman 1991]) implies that $u_* \in C_+ \setminus \{0\}$. Moreover, from the strong maximum principle of [Pucci and Serrin 2007, p. 34], we have $u_*(z) > 0$ for all $z \in \Omega$. Then

$$\Delta_p u_*(z) + \Delta u_*(z) \leq c_6 u_*(z)^{p-1} \quad \text{a.e. in } \Omega,$$

which in view of [Pucci and Serrin 2007, p. 120] leads to

$$u_* \in \text{int}\, C_+.$$

This establishes the existence of a nontrivial positive smooth solution of (37).

Next we show the uniqueness of $u_* \in \text{int}\, C_+$. To this end, we consider the integral functional $\beta_+ : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$(40) \qquad \beta_+(u) = \begin{cases} \frac{1}{p}\|Du^{1/2}\|_p^p + \frac{1}{2}\|Du^{1/2}\|_2^2 & \text{if } u \geq 0,\ u^{1/2} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $G_0(t) = t^p/p + t^2/2$ for all $t \geq 0$. Clearly G_0 is strictly convex and strictly increasing. We set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. From (40) we have

$$(41) \qquad \beta_+(u) = \begin{cases} \int_\Omega G(Du^{1/2})\,dz & \text{if } u \geq 0,\ u^{1/2} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let u_1, $u_2 \in \mathrm{dom}\, \beta_+$ and set $y_1 = u_1^{1/2}$, $y_2 = u_2^{1/2}$. Then y_1, $y_2 \in W_0^{1,p}(\Omega)$. We define

$$y_3 = (tu_1 + (1-t)u_2)^{1/2} \in W_0^{1,p}(\Omega) \quad \text{with } t \in [0,1].$$

Then Lemma 4 of [Benguria et al. 1981] (see also [Díaz and Saá 1987, Lemma 1]) implies that

$$\|Dy_3(z)\| \leq \left(t\|Dy_1(z)\|^2 + (1-t)\|Dy_2(z)\|^2\right)^{1/2} \text{ a.e. in } \Omega,$$

or again, since G_0 is increasing,

$$(42) \quad G_0(\|Dy_3(z)\|) \leq G_0\left(\left(t\|Dy_1(z)\|^2 + (1-t)\|Dy_2(z)\|^2\right)^{1/2}\right) \quad \text{a.e. in } \Omega.$$

The right-hand side is bounded above by $tG_0(\|Dy_1(z)\|) + (1-t)G_0(\|Dy_2(z)\|)$, since $t \to G_0(t^{1/2})$ is convex. So from (42) we obtain successively

$$G(Dy_3(z)) \leq tG(Dy_1(z)) + (1-t)G(Dy_2(z)) \qquad \text{a.e. in } \Omega,$$

$$G\left(D(tu_1 + (1-t)u_2)^{1/2}(z)\right) \leq tG(Du_1^{1/2}(z)) + (1-t)G(Du_2^{1/2}(z)) \quad \text{a.e. in } \Omega,$$

and finally, using (41), the convexity of β_+.

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of the auxiliary problem (37). From the first part of the proof we have $u \in \mathrm{int}\, C_+$. Therefore $u^2 \in \mathrm{dom}\, \beta_+$. Also, if $h \in C_0^1(\bar{\Omega})$ and $t \in (-1,1)$ is small, then $u^2 + th \in \mathrm{dom}\, \beta_+$. So, the Gâteaux derivative of β_+ at u^2 in the direction h exists. The chain rule and the density of $C_0^1(\bar{\Omega})$ in $W_0^{1,p}(\Omega)$ imply

$$(43) \qquad \beta_+'(u^2)(h) = \int_\Omega \frac{-\Delta_p u - \Delta u}{u} h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Similarly, if $v \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (37), then $v \in \mathrm{int}\, C_+$ and we have

$$(44) \qquad \beta_+'(v^2)(h) = \int_\Omega \frac{-\Delta_p v - \Delta v}{v} h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Since β_+ is convex, its Gâteaux derivative is monotone, and so, from (43) and (44), we have

$$0 \leq \left\langle \beta_+'(u^2) - \beta_+'(v^2), u^2 - v^2 \right\rangle_{L^1}$$

$$= \int_\Omega \left(\frac{-\Delta_p u - \Delta u}{u} - \frac{-\Delta_p v - \Delta v}{v} \right)(u^2 - v^2)\, dz$$

$$= \int_\Omega \left(\frac{c_5 u - c_6 u^{p-1}}{u} - \frac{c_5 v - c_6 v^{p-1}}{v} \right)(u^2 - v^2)\, dz$$

$$= c_6 \int_\Omega (v^{p-1} - u^{p-1})(u^2 - v^2)\, dz \leq 0.$$

Therefore $u = v$, showing that $u_* \in \text{int } C_+$ is the unique nontrivial positive solution of (37).

Since (37) is odd, we conclude that $v_* = -u_* \in -\text{int } C_+$ is the unique nontrivial negative solution of (37).   □

Having this proposition, we can now establish the existence of extremal nontrivial constant sign solutions for problem (1).

Proposition 8. *If hypotheses* H *hold, then problem* (1) *has a smallest nontrivial positive solution* $u_+ \in \text{int } C_+$ *and a biggest nontrivial negative solution* $v_- \in -\text{int } C_+$.

Proof. Recall that the set of nontrivial positive solutions of (1) is downward directed (i.e., if u_1, u_2 are nontrivial positive solutions of (1), then there exists a nontrivial positive solution u of (1) such that $u \leq u_1$ and $u \leq u_2$; see [Filippakis et al. 2009, Lemma 4.2 and Proposition 4.4]). So, in order to produce the smallest nontrivial positive solution of (1), it suffices to consider the set

$$S_+ = \{u \in W_0^{1,p}(\Omega) : u \text{ is a nontrivial solution of (1)}, u \in [0, w_+]\}.$$

From Proposition 5, we know that S_+ is nonempty and $S_+ \subseteq \text{int } C_+$.

Let $\bar{u} \in S_+$ and consider the Carathéodory function

$$(45) \qquad e_+(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ c_5 x - c_6 x^{p-1} & \text{if } 0 \leq x \leq \bar{u}(z), \\ c_5 \bar{u}(z) - c_6 \bar{u}(z)^{p-1} & \text{if } \bar{u}(z) < x. \end{cases}$$

We set $E_+(z, x) = \int_0^x e_+(z, s)\, ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega E_+(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (45) that $\sigma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$(46) \qquad \sigma_+(\tilde{u}) = \inf[\sigma_+(u) : u \in W_0^{1,p}(\Omega)].$$

As before (see the proof of Proposition 7), since $c_5 > \hat{\lambda}_1(2)$ and $p > 2$, for $t \in (0, 1)$ small we have $\sigma_+(t\tilde{u}_{1,2}) < 0$, and therefore $\sigma_+(\tilde{u}) < 0 = \sigma_+(0)$; hence $\tilde{u} \neq 0$. From (46) this implies $\sigma_+'(\tilde{u}) = 0$; therefore

$$(47) \qquad A_p(\tilde{u}) + A(\tilde{u}) = N_{e_+}(\tilde{u}).$$

On (47) we act with $-\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\tilde{u} \geq 0$, $\tilde{u} \neq 0$ (see (45)). Also

on (47) we act with $(\tilde{u} - \bar{u})^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\langle A_p(\tilde{u}), (\tilde{u} - \bar{u})^+ \rangle + \langle A(\tilde{u}), (\tilde{u} - \bar{u})^+ \rangle$$
$$= \int_\Omega e_+(z, \tilde{u})(\tilde{u} - \bar{u})^+ \, dz = \int_\Omega (c_5 \bar{u} - c_6 \bar{u}^{p-1})(\tilde{u} - \bar{u})^+ \, dz \quad \text{(see (45))}$$
$$\leq \int_\Omega f(z, \bar{u})(\tilde{u} - \bar{u})^+ \, dz = \langle A_p(\bar{u}), (\tilde{u} - \bar{u})^+ \rangle + \langle A(\bar{u}), (\tilde{u} - \bar{u})^+ \rangle;$$

this implies

$$\int_{\{\tilde{u} > \bar{u}\}} \left(\|D\tilde{u}\|^{p-2} D\tilde{u} - \|D\bar{u}\|^{p-2} D\bar{u}, D\tilde{u} - D\bar{u} \right)_{\mathbb{R}^N} dz + \|D(\tilde{u} - \bar{u})^+\|_2^2 \leq 0,$$

and so $\tilde{u} \leq \bar{u}$.

So, we have proved that

$$\tilde{u} \in [0, \bar{u}] = \{u \in W_0^{1,p}(\Omega) : 0 \leq \tilde{u}(z) \leq \bar{u}(z) \text{ a.e. in } \Omega\}, \quad \tilde{u} \neq 0.$$

From (45) and (47) it follows that

$$-\Delta_p \tilde{u}(z) - \Delta \tilde{u}(z) = c_5 \tilde{u}(z) - c_6 \tilde{u}(z)^{p-1} \quad \text{a.e. in } \Omega, \quad \tilde{u}|_{\partial\Omega} = 0, \quad \tilde{u} \geq 0, \quad \tilde{u} \neq 0,$$

whence $\tilde{u} = u_*$ by Proposition 7, and therefore $\tilde{u} \leq \bar{u}$.

Since $\bar{u} \in S_+$ is arbitrary, we conclude that

$$(48) \qquad\qquad u_* \leq u \quad \text{for all } u \in S_+.$$

Now let $C \subseteq S_+$ be a chain (i.e., a totally ordered subset of S_+). Then we can find $\{u_n\}_{n \geq 1} \subseteq C$ such that $\inf C = \inf_{n \geq 1} u_n$; (see [Dunford and Schwartz 1958, p. 336]).

We have

$$(49) \qquad A_p(u_n) + A(u_n) = N_f(u_n), \quad u_n \in [u_*, w_+] \quad \text{for all } n \geq 1$$

by (48), so $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

So, we may assume that

$$(50) \qquad\qquad u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega).$$

On (49) we act with $u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (50). Then

$$\lim_{n \to \infty} \left(\langle A_p(u_n), u_n - u \rangle + \langle A(u_n), u_n - u \rangle \right) = 0,$$

and so (reasoning as in Claim 1 in the proof of Proposition 6)

$$(51) \qquad\qquad u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

So, if in (49) we pass to the limit as $n \to \infty$ and use (51), then (48) yields

$$A_p(u) + A(u) = N_f(u), \quad u_* \leq u,$$

which leads to $u \in S_+$, $u = \inf C$.

Because C is an arbitrary chain, the Kuratowski–Zorn lemma gives the existence of a minimal element $u_+ \in S_+$ of S_+. But recall that S_+ is downward directed. So, if $u \in S_+$, we can find $y \in S_+$ such that $y \leq u$, $y \leq u_+$. The minimality of u_+ implies that $u_+ = y$ and so $u_+ \leq u$. Since $u \in S_+$ is arbitrary, we conclude that u_+ is the smallest nontrivial positive solution of (1).

Similarly, let S_- be the set of nontrivial negative solutions of (1) in $[w_-, 0]$. Then S_- is upward directed (i.e., if $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$; see [Filippakis et al. 2009, Lemma 4.3]). Let $\overline{v} \in S_-$ and consider the Carathéodory function

$$e_-(z, x) = \begin{cases} c_5 \overline{v}(z) - c_6 |\overline{v}(z)|^{p-2} \overline{v}(z) & \text{if } x < \overline{v}(z), \\ c_5 x - c_6 |x|^{p-2} x & \text{if } \overline{v}(z) \leq x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

We set $E_-(z, x) = \int_0^x e_-(z, s) \, ds$ and consider the C^1-functional $\sigma_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_-(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega E_-(z, u(z)) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as above, we produce $v_- \in -\operatorname{int} C_+$, the smallest nontrivial negative solution of (1). □

To implement the strategy outlined in the beginning of this section and produce a nodal solution, we need to be able to identify the nonzero critical points of φ distinct from u_* and v_* which are in the order interval $[v_*, u_*]$. This can be done using critical groups. For this reason, we compute the critical groups of φ at the origin.

Proposition 9. *If hypotheses* H *hold, then* $C_k(\varphi, 0) = \delta_{k,d_m} \mathbb{Z}$ *for all* $k \geq 0$ *with* $d_m = \dim \bigoplus_{i=1}^m E(\hat{\lambda}_i(2))$.

Proof. Let $\mu \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$ and consider the C^2-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \frac{\mu}{2} \|u\|_2^2 \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the homotopy $h_0 : [0, 1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h_0(t, u) = t\varphi(u) + (1 - t)\varphi_0(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Clearly $h_0(0, \cdot) = \varphi_0(\cdot)$ and $h_1(0, \cdot) = \varphi(\cdot)$.

It is easy to see that, since $p > 2$, φ_0 satisfies the C-condition. Also, reasoning as in Claim 1 in the proof of Proposition 6, via hypothesis H(iv), we show that φ satisfies the C-condition.

Suppose we can find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(52) \qquad t_n \to t, \, u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad (h_0)'_u(t_n, u_n) = 0 \quad \text{for all } n \geq 1.$$

From the equality in (52), we have

$$A_p(u_n) + A(u_n) = t_n N_f(u_n) + (1 - t_n)\mu u_n,$$

and therefore

$$(53) \quad -\Delta_p u_n(z) - \Delta u_n(z) = t_n f(z, u_n(z)) + (1 - t_n)\mu u_n(z) \text{a.e. in } \Omega, \quad u_n|_{\partial\Omega} = 0.$$

Since $\mu \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$, we have

$$t f(z, x) x + (1 - t)\mu x^2 \geq c_5 x^2 - c_6 |x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \, t \in [0, 1],$$

where $c_5 > \hat{\lambda}_1(2)$ and $c_6 > 0$ are as before (see (37)). Then from (53) and the proof of Proposition 8, we have $u_* \leq u_n$ for all $n \geq 1$, which contradicts (52). Therefore (52) cannot happen and so the homotopy invariance of critical groups (see, for example, [Chang 2005]) implies that $C_k(h_0(0, \cdot), 0) = C_k(h_0(1, \cdot), 0)$ for all $k \geq 0$, whence

$$(54) \qquad\qquad C_k(\varphi_0, 0) = C_k(\varphi, 0) \quad \text{for all } k \geq 0.$$

Note that $\varphi_0''(0) = A - \mu I$ (see [Cingolani and Vannella 2003, p. 277]) and recall that $\mu \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$. Invoking Theorem 1.1 of [Cingolani and Vannella 2003], we have $C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geq 0$, with $d_m = \dim \bigoplus_{i=1}^m E(\hat{\lambda}_i(2))$. Using (54) concludes the proof. $\qquad\square$

Now we have all the necessary tools to complete our strategy and produce a nodal solution.

Proposition 10. *If hypotheses H hold, problem* (1) *has a nodal solution* $y_0 \in C_0^1(\overline{\Omega})$ *such that*

$$u_+ - y_0 \in \text{int } C_+ \quad \text{and} \quad y_0 - v_- \in \text{int } C_+.$$

Proof. Let $u_+ \in \text{int } C_+$ and $v_- \in -\text{int } C_+$ be the two extremal nontrivial constant sign solutions of (1) produced in Proposition 8. Using these two solutions, we introduce the following truncation of the reaction $f(z, \cdot)$:

$$(55) \qquad g(z, x) = \begin{cases} f(z, v_-(z)) & \text{if } x < v_-(z), \\ f(z, x) & \text{if } v_-(z) \leq x \leq u_+(z), \\ f(z, u_+(z)) & \text{if } u_+(z) < x. \end{cases}$$

This is a Carathéodory function. We set $G(z, x) = \int_0^x g(z, s)\, ds$. Also, let $g_\pm(z, x) = g(z, \pm x^\pm)$ and $G_\pm(z, x) = \int_0^x g_\pm(z, s)\, ds$. Then we introduce the C^1-functionals $\xi^*, \xi_\pm^* : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\xi^*(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G(z, u(z))\, dz,$$

$$\xi_\pm^*(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_\pm(z, u(z))\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Claim 1. $K_{\xi^*} \subseteq [v_-, u_+]$, $K_{\xi_+^*} = \{0, u_+\}$, $K_{\xi_-^*} = \{0, v_-\}$.

Proof. Let $u \in K_{\xi^*}$. Then we have

(56) $$A_p(u) + A(u) = N_g(u).$$

On (56) we act with $(u - u_+)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\langle A_p(u), (u - u_+)^+ \rangle + \langle A(u), (u - u_+)^+ \rangle$$
$$= \int_\Omega g(z, u)(u - u_+)^+\, dz = \int_\Omega f(z, u_+)(u - u_+)^+\, dz \quad \text{(see (55))}$$
$$= \langle A_p(u_+), (u - u_+)^+ \rangle + \langle A(u_+), (u - u_+)^+ \rangle,$$

so that

$$\int_{\{u > u_+\}} \left(\|Du\|^{p-2}Du - \|Du_+\|^{p-2}Du_+, Du - Du_+\right)_{\mathbb{R}^N} dz + \|D(u - u_+)^+\|_2^2 = 0$$

and therefore $u \leq u_+$.

Similarly, acting on (56) with $(v_- - u)^+ \in W_0^{1,p}(\Omega)$, we show that $v_- \leq u$. Therefore $K_{\xi^*} \subseteq [v_-, u_+]$.

In a similar fashion, we show that $K_{\xi_+^*} \subseteq [0, u_+]$. Clearly $\{0, u_+\} \subseteq K_{\xi_+^*}$. The extremality of u_+ implies that $K_{\xi_+^*} = \{0, u_+\}$. Similarly, $K_{\xi_-^*} = \{v_-, 0\}$. This proves Claim 1. □

Claim 2. *The functions $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ are both local minimizers of the functional ξ^*.*

Proof. It is clear from (55) that ξ_+^* is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\xi_+^*(\tilde{u}) = \inf\,[\xi_+^*(u) : u \in W_0^{1,p}(\Omega)].$$

As in the proof of Proposition 5, using hypothesis H(vii) and the fact that $2 < p$, we have $\xi_+^*(t\hat{u}_{1,2}) < 0$ for $t \in (0, 1)$ small, which give $\xi_+^*(\tilde{u}) < 0 = \xi_+^*(0)$; hence $\tilde{u} \neq 0$, showing that $\tilde{u} = u_+$ by Claim 1.

But $u_+ \in \text{int}\, C_+$ and $\xi^*|_{C_+} = \xi_+^*|_{C_+}$ (see (55)). Therefore u_+ is a local $C_0^1(\bar{\Omega})$-minimizer of ξ^*; hence it is a local $W_0^{1,p}(\Omega)$-minimizer of ξ^* (see Theorem 2). Similarly for $v_- \in -\text{int}\, C_+$, using this time the functional ξ_-^*. □

We assume that K_{ξ^*} is finite (otherwise, we already have an infinity of distinct nodal solutions). Also, without any loss of generality, we assume that $\xi^*(v_-) \leq \xi_+^*(u_+)$ (the analysis is similar if the opposite inequality holds). By virtue of Claim 2, as in [Aizicovici et al. 2008, proof of Proposition 29] we can find $\rho \in (0, 1)$ small such that

$$(57)\quad \xi^*(v_-) \leq \xi^*(u_+) < \inf\,[\xi^*(u) : \|u - u_+\| = \rho] = \eta_\rho^* \quad \text{and} \quad \|v_- - u_+\| > \rho.$$

Note that ξ^* is coercive (see (55)); hence it satisfies the C-condition. This fact and (57) permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$(58)\qquad\qquad y_0 \in K_{\xi^*} \quad \text{and} \quad \eta_\rho^* \leq \xi^*(y_0).$$

From (57), (58) and Claim 1, we have

$$(59)\qquad\qquad y_0 \in [v_-, u_+], \quad y_0 \notin \{v_-, u_+\}.$$

Since y_0 is a critical point of ξ^* of mountain pass type, we have

$$(60)\qquad\qquad C_1(\xi^*, y_0) \neq 0.$$

Using the homotopy invariance of critical groups, we have $C_k(\xi^*, 0) = C_k(\varphi, 0)$ for all $k \geq 0$, which gives (see Proposition 9)

$$(61)\qquad\qquad C_k(\xi^*, 0) = \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \geq 0.$$

From (60), (61) and since $d_m \geq 2$, we infer that $y_0 \neq 0$. Then the extremality of u_+ and v_- and the fact that $y_0 \in [v_-, u_+]$ imply that $y_0 \in C_0^1(\bar{\Omega})$ (see [Lieberman 1991]) is a nodal solution of (1).

Using the tangency principle of [Pucci and Serrin 2007, p. 35], we have

$$(62)\qquad\qquad v_-(z) < y_0(z) < u_+(z) \quad \text{for all } z \in \Omega.$$

Let $\rho = \max\{\|u_+\|, \|v_-\|\}$ and let $\xi_\rho > 0$ as postulated by hypothesis H(vi). Then, for $\tilde{\xi} > \xi_\rho$, we have

$$
\begin{aligned}
-\Delta_p y_0(z) &- \Delta y_0(z) + \tilde{\xi}|y_0(z)|^{p-2} y_0(z) \\
&= f(z, y_0(z)) + \tilde{\xi}|y_0(z)|^{p-2} y_0(z) \leq f(z, u_+(z)) + \tilde{\xi} u_+(z)^{p-1} \\
&= -\Delta_p u_+(z) - \Delta u_+(z) + \tilde{\xi} u_+(z)^{p-1} \quad \text{a.e. in } \Omega.
\end{aligned}
$$

Setting $h(z) = f(z, y_0(z)) + \tilde{\xi}|y_0(z)|^{p-2}y_0(z)$ and $\hat{h} = f(z, u_+(z)) + \tilde{\xi}u_+(z)^{p-1}$, we see that $h, \hat{h} \in L^\infty(\Omega)$ and $h \prec \hat{h}$. Since $u_+ \in \text{int } C_+$, we can apply Proposition 3 and infer that $u_+ - y_0 \in \text{int } C_+$. Similarly we show that $y_0 - v_- \in \text{int } C_+$. $\square$

So, we can state the following multiplicity theorem concerning problem (1). We stress that the result is proved without assuming any differentiability on the function $x \to f(z, x)$ (see hypotheses H). In addition our multiplicity theorem provides precise sign information for all the solutions produced.

Theorem 11. *If hypotheses* H *hold, the problem* (1) *has at least five nontrivial smooth solutions*:

$$u_0, \hat{u} \in \text{int } C_+ \qquad with \ u_0 - \hat{u} \in \text{int } C_+,$$

$$v_0, \hat{v} \in -\text{int } C_+ \quad with \ v_0 - \hat{v} \in \text{int } C_+,$$

and

$$y_0 \in C_0^1(\overline{\Omega}) \quad nodal \ s.t. \quad y_0 - v_0 \in \text{int } C_+, \quad u_0 - y_0 \in \text{int } C_+.$$

Next, by strengthening the regularity condition on $f(z, \cdot)$, we will be able to produce a second nodal solution.

The new hypotheses on the reaction $f(z, x)$ are the following:

Hypotheses $\widehat{\text{H}}$.   (i) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function.

(ii) For a.a. $z \in \Omega$, we have $f(z, 0) = 0$ and $f(z, \cdot) \in C^1(\mathbb{R})$.

(iii) $|f'_x(z, x)| \leq \alpha(z) + c|x|^{r-2}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$ and $p \leq r < p^*$.

(iv) If $F(z, x) = \displaystyle\int_0^x f(z, s) \, ds$, then

$$\lim_{x \to \pm\infty} \frac{pF(z, x)}{|x|^p} = \hat{\lambda}_1(p) \quad \text{uniformly for a.a. } z \in \Omega$$

and, for some $\tau > 2$,

$$\lim_{x \to \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\tau} \leq \hat{\beta} < 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

(v) There exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$w_-(z) \leq c_- < 0 < c_+ \leq w_+(z) \quad \text{for all } z \in \overline{\Omega},$$

$$\text{ess sup}_\Omega f(\cdot, w_+(\cdot)) \leq 0 \leq \text{ess inf}_\Omega f(\cdot, w_-(\cdot))$$

and

$$A_p(w_-) + A(w_-) \leq 0 \leq A_p(w_+) + A(w_+) \quad \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$$

(where $1/p + 1/p' = 1$).

(vi) For every $\rho > 0$, there exists $\xi_\rho > 0$ such that, for a.a. $z \in \Omega$, the function $x \to f(z, x) + \xi_\rho |x|^{p-2} x$ is nondecreasing on $[-\rho, \rho]$.

(vii) There exists integer $m \geq 2$ such that

$$\hat{\lambda}_m(2) \leq f_x'(z, 0) \leq \hat{\lambda}_{m+1}(2) \quad \text{a.e. in } \Omega, \quad \hat{\lambda}_m(2) \neq f_x'(z, \cdot), \quad \hat{\lambda}_{m+1}(2) \neq f_x'(z, \cdot),$$

$$f_x'(z, 0) = \lim_{x \to 0} \frac{f(z, x)}{x} \quad \text{uniformly for a.a. } z \in \Omega.$$

In what follows, we use the notation and the functionals introduced in the proof of Proposition 10.

Proposition 12. *If hypotheses $\widehat{\mathrm{H}}$ hold, then problem* (1) *has a second nodal solution* $\hat{y} \in C_0^1(\overline{\Omega})$ *such that*

$$u_+ - \hat{y} \in \operatorname{int} C_+ \quad \text{and} \quad \hat{y} - v_- \in \operatorname{int} C_+.$$

Proof. We assume that $K_{\xi_+^*}$ is finite (otherwise we already have an infinity of nodal solutions). From the proof of Proposition 10, we have

$$\{0, u_+, v_-, y_0\} \subseteq K_{\xi_+^*} \subseteq [v_-, u_+].$$

We know that $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ are local minimizers of the functional ξ_* (see Claim 2 in the proof of Proposition 10). So, we have

$$(63) \qquad C_k(\xi^*, u_+) = C_k(\xi^*, v_-) = \delta_{k,0} \mathbb{Z} \quad \text{for all } k \geq 0.$$

Also, from (61) we have

$$(64) \qquad C_k(\xi^*, 0) = \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \geq 0.$$

Moreover, since ξ^* is coercive (see (55)), we have

$$(65) \qquad C_k(\xi^*, \infty) = \delta_{k,0} \mathbb{Z} \quad \text{for all } k \geq 0.$$

Claim 1. $C_k(\varphi, y_0) = C_k(\xi^*, y_0)$ *for all $k \geq 0$.*

Proof. We consider the homotopy $\tilde{h} : [0, 1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{h}(t, u) = (1 - t)\xi^*(u) + t\varphi(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

We have $\tilde{h}(0, \cdot) = \xi^*(\cdot)$ and $\tilde{h}(1, \cdot) = \varphi(\cdot)$ and both functionals satisfy the C-condition. Let $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$(66) \qquad t_n \to t, u_n \to y_0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad (\tilde{h})_u'(t_n, u_n) = 0 \quad \text{for all } n \geq 1.$$

From the equation in (66), we have

$$A_p(u_n) + A(u_n) = (1 - t_n)N_g(u_n) + t_n N_f(u_n) \quad \text{for all } n \geq 1,$$

by (55). Hence

(67)
$$\begin{cases} -\Delta_p u_n(z) - \Delta u_n(z) = (1 - t_n)g(z, u_n(z)) + t_n f(z, u_n(z)) & \text{a.e. in } \Omega, \\ \quad\qquad\qquad u_n|_{\partial\Omega} = 0 & \text{for all } n \geq 1. \end{cases}$$

From (67) and [Lieberman 1991], we know that we can find $\gamma \in (0, 1)$ and $M > 0$ such that

(68)
$$u_n \in C_0^{1,\gamma}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\gamma}(\overline{\Omega})} \leq M \quad \text{for all } n \geq 1.$$

From (68) and the compact embedding of $C_0^{1,\gamma}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, by passing to a suitable subsequence if necessary, we may assume by (66) that

(69)
$$u_n \to y_0 \quad \text{in } C^1(\overline{\Omega}).$$

Since $y_0 \in \text{int}_{C_0^{1,\gamma}(\overline{\Omega})}[v_-, u_+]$ (see Theorem 11), from (69) it follows that

$$u_n \in [v_-, u_+], \quad u_n \neq v_-, \quad u_n \neq u_+ \quad \text{for all } n \geq n_0;$$

this, by (55), gives $\{u_n\}_{n \geq n_0} \subseteq K_{\xi^*}$, which contradicts our hypothesis that K_{ξ^*} is finite. So, (66) cannot happen, and, from the homotopy invariance of critical groups, we have

$$C_k(\tilde{h}(0, \cdot), y_0) = C_k(\tilde{h}(1, \cdot), y_0) \quad \text{for all } k \geq 0,$$

which yields the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

From (60) and Claim 1, we have

(70)
$$C_1(\varphi, y_0) \neq 0.$$

Claim 2. $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \geq 0$.

Proof. From [Cingolani and Vannella 2007, Lemma 2.2], we know that we can find $\rho > 0$ and a C^2-function $\vartheta : V \cap \overline{B}_\rho \to \mathbb{R}$ (recall $V = H^- \oplus H^0$ (see Section 2), while $\overline{B}_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho\}$) such that

$$\langle \vartheta''(0)v, u \rangle = \langle \varphi''(y_0)v, u \rangle \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

In addition $\vartheta''(0)$ is a Fredholm operator and $\ker \vartheta''(0) = H^0$. From [Cingolani and Vannella 2003, p. 286], we have

(71)
$$C_k(\varphi, y_0) = C_k(\vartheta, 0) \quad \text{for all } k \geq 0.$$

Then (70), (71) imply that

(72)
$$C_1(\vartheta, 0) \neq 0,$$

and so

(73)
$$d_- = \dim H^- \leq 1.$$

Let $d_0 = \dim H^0$. We consider two cases:

<u>Case I:</u> $d_0 = 0$. In this case, $u = 0$ is a nondegenerate critical point of ϑ with Morse index d_-. Hence $C_k(\vartheta, 0) = \delta_{k,d_-}\mathbb{Z}$ for all $k \geq 0$ (see Section 2). In view of (72) we then have $d_- = 1$, so $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see (71)).

<u>Case II:</u> $d_0 > 0$. In this case, $u = 0$ is a degenerate critical point of ϑ. From (73) we see that $d_- = 1$ or $d_- = 0$.

If $d_- = 1$, then, from [Cingolani and Vannella 2003, p. 286], we have $C_k(\vartheta, 0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$, so $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see (71)).

If $d_- = 0$, then, from (72) and [Cingolani and Degiovanni 2009], we have $C_k(\vartheta, 0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$, so $C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$ (see (71)).

This proves Claim 2. $\qquad\square$

Claims 1 and 2 imply that

$$(74) \qquad\qquad C_k(\xi^*, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0.$$

Suppose that $K_{\xi_+^*} = \{0, u_+, v_-, y_0\}$. Then, from (63), (64), (65), (74) and the Morse relation (see (4)) with $t = -1$, we have $(-1)^{d_m} = 0$, a contradiction. Therefore, we can find $\hat{y} \in K_{\xi_+^*}$, $\hat{y} \notin \{0, u_+, v_-, y_0\}$. We have $\hat{y} \in [v_-, u_+]$ (see Claim 1 in the proof of Proposition 10) and so $\hat{y}$ is nodal. Moreover, $\hat{y} \in C_0^1(\overline{\Omega})$ (nonlinear regularity) and, as we did for y_0 (see the proof of Proposition 10), we show that $\hat{y} \in \text{int}_{C_0^1(\overline{\Omega})}[v_-, u_+]$. $\qquad\square$

Now we can state the second multiplicity theorem for problem (1).

Theorem 13. *If hypotheses $\widehat{H}$ hold, then problem* (1) *has at least six nontrivial smooth solutions*

$$u_0, \hat{u} \in \text{int}\, C_+ \qquad \text{with } \hat{u} - u_0 \in \text{int}\, C_+,$$

$$v_0, \hat{v} \in -\text{int}\, C_+ \quad \text{with } v_0 - \hat{v} \in \text{int}\, C_+$$

and $y_0, \hat{y} \in C_0^1(\overline{\Omega})$ nodal with $u_0 - y_0, u_0 - \hat{y} \in \text{int}\, C_+$ and $y_0 - v_0, \hat{y} - v_0 \in \text{int}\, C_+$.

Acknowledgement

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