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We give an elementary construction of any real closed field in terms of Nash function fields. We also give a characterization of any Archimedean field in terms of fields of Nash functions.

Introduction

In the study of Hilbert's 17th problem, orderings of a real field k are of importance (see [Alonso 1986; Alonso et al. 1984; Artin 1927; Artin and Schreier 1927a; 1927b; Bochnak and Efroymson 1980; Bröker 1982; Dubois 1981; Guangxing 2005; Marshall 2003; Prestel and Delzell 2001; Schwartz 1980]). By the Artin–Schreier theorem [Artin 1927; Artin and Schreier 1927a; 1927b], the study of such orderings amounts to considering real closures of k. The aim of this article is to construct a universal model of an arbitrary real closed field. To this end, we construct, in terms of Nash functions, all real closures of the rational function fields $k = \mathbb{Q}(\Lambda_T)$, where $\Lambda_T = (\Lambda_t : t \in T)$ and $T \neq \varnothing$ is a system of any number of variables. This suffices to achieve our purpose, because any real closed field R is order-preserving isomorphic to a real closure of some field $\mathbb{Q}(\Lambda_T)$ (Corollary 5.5). If $T = \varnothing$, then $\mathbb{Q}(\Lambda_T) = \mathbb{Q}$, and the above is obvious. We assume the Kuratowski–Zorn lemma, so the set T can be well-ordered, provided $T \neq \varnothing$.

L. Bröker [1982] proved in his ultrafilter theorem that there exists a one-to-one correspondence between the family of ultrafilters and the family of orderings in $\mathbb{Q}(\Lambda_T)$, or equivalently with the real closures of $\mathbb{Q}(\Lambda_T)$. We prove that there exists a one-to-one correspondence between the family of orderings in $\mathbb{Q}(\Lambda_T)$ and the family of *plain filters* (Theorem 5.2, Proposition 2.4, and Corollary 2.5). By a plain filter we mean a filter Ω of subsets of $\mathbb{R}^T$ with these properties:

(1) Any $U \in \Omega$ is a nonempty open connected semialgebraic set.

(2) For any algebraic set $V \subsetneq \mathbb{R}^T$, where $V = P^{-1}(0)$ and $P \in \mathbb{Q}[\Lambda_T]$, some connected component of $\mathbb{R}^T \setminus V$ belongs to Ω.

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(3) For any $U_1, U_2 \in \Omega$, there exists $U_3 \in \Omega$ such that $U_3 \subset U_1 \cap U_2$.

The correspondence between orderings and plain filters is as follows: For any ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$, there exists a unique plain filter Ω such that $f \succ 0$ if and only if $f > 0$ on some $U \in \Omega$, where $>$ is the usual ordering on $\mathbb{R}$. Conversely, any plain filter Ω determines a unique ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ in this way.

The main result of this article is Theorem 5.2, where we give a construction of any real closure of $\mathbb{Q}(\Lambda_T)$ in terms of Nash functions. The main idea and motivation for the above considerations was a geometric construction of the algebraic closure of $\mathbb{C}(\Lambda_1, \dots, \Lambda_m)$ [Spodzieja 1996]. More precisely, for any plain filter Ω of open connected semialgebraic sets and any $U \in \Omega$, the ring $\mathcal{N}(U)$ of $\mathbb{Q}$-Nash functions (see Section 1) on U is a domain. In $\bigcup_{U\in\Omega} \mathcal{N}(U)$, we introduce an equivalence relation $\sim$: $(f_1 : U_1 \to \mathbb{R}) \sim (f_2 : U_2 \to \mathbb{R})$ if and only if $f_1|_{U_3} = f_2|_{U_3}$ for some $U_3 \in \Omega$. The set $\mathcal{N}_\Omega$ of equivalence classes of $\sim$ with the usual operations of addition and multiplication is a field, which is a real closure of $\mathbb{Q}(\Lambda_T)$ (see Theorem 5.2, and compare [Spodzieja 1996, Theorem 2.4 and Corollary 2.5]). One can view $\mathcal{N}_\Omega$ as the inverse limit of the étale topology $\bigcup_{U\in\Omega} \mathcal{N}(U)$ of $\mathbb{R}^T$ [Grothendieck 1967].

In Section 3, we prove that an ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ is Archimedean if and only if the set $\bigcap_{U\in\Omega} U$ is nonempty for the plain filter Ω determining $\succ$; and if that is the case, this set has exactly one point (Theorem 3.1). In Section 4, we give some examples of non-Archimedean orderings corresponding to the one in [Spodzieja 1996].

1. Preliminaries

Let $\mathbb{K}$ be the field $\mathbb{Q}$ of rational, $\mathbb{R}$ of real, or $\mathbb{C}$ of complex numbers. Let T be a nonempty set. We denote by $\Lambda_T = (\Lambda_t : t \in T)$ a system of independent variables Λ_t, by $\mathbb{K}[\Lambda_T]$ the ring of polynomials in Λ_T over $\mathbb{K}$, and by $\mathbb{K}(\Lambda_T)$ the quotient field of $\mathbb{K}[\Lambda_T]$. Note that for any $P \in \mathbb{K}(\Lambda_T)$, we have $P \in \mathbb{K}(\Lambda_{t_1}, \dots, \Lambda_{t_m})$ for some finite number of indices $t_1, \dots, t_m \in T$.

We denote by $\mathbb{K}^T$ the set of all functions $T \to \mathbb{K}$ equipped with the unique topology for which all projections $\mathbb{K}^T \ni x \mapsto x(t) \in \mathbb{K}$, $t \in T$ are continuous.

Let $\mathbb{L}$ be a subfield of $\mathbb{K}$. A subset of $\mathbb{K}^T$ is called $\mathbb{L}$-*algebraic*, or simply *algebraic* if $\mathbb{L} = \mathbb{K}$, when it is defined by a finite system of equations $P = 0$, where $P \in \mathbb{L}[\Lambda_T]$. Any $\mathbb{L}$-algebraic set in $\mathbb{K}^T$ is of the form $\{x \in \mathbb{K}^T : (x(t_1), \dots, x(t_m)) \in V\}$, where $m \in \mathbb{N}$, $t_1, \dots, t_m \in T$, and $V \subset \mathbb{K}^m$ is an $\mathbb{L}$-algebraic subset of $\mathbb{K}^m$.

If $\mathbb{L}$ is a subfield of $\mathbb{R}$, then we assume that $\mathbb{L}$ is an ordered field with order induced from $\mathbb{R}$.

Let $\mathbb{L}$ be a subfield of $\mathbb{R}$. A subset of $\mathbb{R}^T$ is called $\mathbb{L}$-*semialgebraic* when it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geq 0$, where $P \in \mathbb{L}[\Lambda_T]$. Analogously to the above, any $\mathbb{L}$-semialgebraic set in $\mathbb{R}^T$ is of the form

$\{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in X\}$, where $m \in \mathbb{N}$, $t_1, \dots, t_m \in T$, and $X \subset \mathbb{R}^m$ is an $\mathbb{L}$-semialgebraic subset of $\mathbb{R}^m$. A set is called *open basic* $\mathbb{L}$*-semialgebraic* if it has the form $\{x \in \mathbb{R}^T : g_i(x) > 0,\ i = 1, \dots, n\}$, for some $n \in \mathbb{N}$ and $g_i \in \mathbb{L}[\Lambda_T]$, $i = 1, \dots, n$.

We now list some basic properties of algebraic and semialgebraic sets in infinite-dimensional real vector spaces, which follow easily from their analogues in finite-dimensional spaces [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymson 1980; Efroymson 1974; 1976; 1981; Mostowski 1976; Prestel and Delzell 2001; Tancredi and Tognoli 2006; Tworzewski 1990].

Proposition 1.1. *Let* $\mathbb{L}$ *be a subfield of* $\mathbb{R}$ (*or* $\mathbb{K}$ *in* (a)).

(a) *The family of* $\mathbb{L}$*-algebraic sets in* $\mathbb{K}^T$ *is closed with respect to union and intersection of a finite number of sets.*

(b) *The family of* $\mathbb{L}$*-semialgebraic sets in* $\mathbb{R}^T$ *is closed with respect to complement, union, and intersection of a finite number of sets.*

(c) (Tarski–Seidenberg) *Let* $\pi_{t_1,\dots,t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \dots, x(t_m)) \in \mathbb{R}^m$, *where* $t_1, \dots, t_m \in T$. *If* $X \subset \mathbb{R}^T$, $Y \subset \mathbb{R}^m$ *are* $\mathbb{L}$*-semialgebraic sets, then* $\pi_{t_1,\dots,t_m}(X)$ *and* $\pi^{-1}_{t_1,\dots,t_m}(Y)$ *are* $\mathbb{L}$*-semialgebraic sets, too.*

(d) *For any* $\mathbb{L}$*-semialgebraic set* $X \subset \mathbb{R}^T$, *the interior* $\operatorname{Int} X$, *closure* $\overline{X}$, *and the boundary* ∂X *are* $\mathbb{L}$*-semialgebraic sets.*

Let $\mathbb{L}$ be a subfield of $\mathbb{R}$. A function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^T$ is an open $\mathbb{L}$-semialgebraic set, is called an $\mathbb{L}$*-Nash function* if f is analytic and there exists a nonzero polynomial $P \in \mathbb{L}[\Lambda_T, Z]$ such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$. In fact, f depends on a finite number of variables, so the analyticity of f is clear. The ring of $\mathbb{L}$-Nash functions in U is denoted by $\mathcal{N}^{\mathbb{L}}(U)$.

The next result follows via R. Thom's lemma (see for instance [Bochnak et al. 1987, Proposition 2.5.4 and the arguments of Theorems 2.3.6 and 2.4.4]) from the fact that any $\mathbb{L}$-semialgebraic set in a finite-dimensional vector space over $\mathbb{R}$ is the disjoint union of a finite number of $\mathbb{L}$-semialgebraic sets which are homeomorphic to Cartesian products of intervals.

Proposition 1.2. *Let* $\mathbb{L}$ *be a subfield of* $\mathbb{R}$. *Any connected component of an* $\mathbb{L}$*-semialgebraic subset of* $\mathbb{R}^T$ *is* $\mathbb{L}$*-semialgebraic.*

A function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}^T$ is an open set, is called a $\mathbb{C}$*-Nash function* if f is holomorphic and there exists a nonzero polynomial $P \in \mathbb{C}[\Lambda_T, Z]$ such that $P(\lambda, f(\lambda)) = 0$ for $\lambda \in U$. The ring of $\mathbb{C}$-Nash functions in U is denoted by $\mathcal{N}^{\mathbb{C}}(U)$.

For the basic properties of Nash functions and semialgebraic sets in finite-dimensional vector spaces, see, for instance, [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymson 1980; Efroymson 1974; 1976; 1981; Mostowski

1976; Nash 1952; Tancredi and Tognoli 2006; Tworzewski 1990]. From these properties, we immediately obtain:

Proposition 1.3. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $\mathbb{L}$ be a subfield of $\mathbb{K}$, and let $U \subset \mathbb{K}^T$ be an open connected set. Then $\mathcal{N}^{\mathbb{K}}(U)$ is a domain, provided U is semialgebraic when $\mathbb{K} = \mathbb{R}$. In particular $\mathcal{N}^{\mathbb{Q}}(U)$ is a domain.*

2. Orderings in $\mathbb{Q}(\Lambda_T)$

Let T be a nonempty set. A family Ω of subsets of $\mathbb{R}^T$ will be called a *c-filter* (connected sets filter) if it satisfies these conditions:

(i) Any $U \in \Omega$ is a nonempty open connected $\mathbb{Q}$-semialgebraic set.

(ii) For any $\mathbb{Q}$-algebraic set $V \subsetneq \mathbb{R}^T$, there exists $U \in \Omega$ such that $V \cap U = \varnothing$.

(iii) For any $U_1, U_2 \in \Omega$, there exists $U_3 \in \Omega$ such that $U_3 \subset U_1 \cap U_2$.

Proposition 2.1. *Let Ω be a c-filter of subsets of $\mathbb{R}^T$. The set $\partial\Omega := \bigcap_{U \in \Omega} \overline{U}$ has at most one point. Moreover, whenever T is a finite set, $\partial\Omega \neq \varnothing$ if and only if there exists a bounded set $U \in \Omega$.*

Proof. If $x_1, x_2 \in \partial\Omega$ with $x_1 \neq x_2$, then for some polynomial $f \in \mathbb{Q}[\Lambda_T]$, we have $f(x_1) < 0 < f(x_2)$. Hence, for some $W \in \Omega$ such that $W \cap f^{-1}(0) = \varnothing$, we have both $f(x) < 0$ and $f(x) > 0$ for some $x \in W$. This contradiction gives the first part of the assertion.

Now let $T = \{t_1, \dots, t_m\}$. Suppose that $\partial\Omega \neq \varnothing$ and each $W \in \Omega$ is an unbounded set. Take $x_0 \in \partial\Omega$, and let $f = (\Lambda_T) = \Lambda_{t_1}^2 + \cdots + \Lambda_{t_m}^2 - r$, where $r \in \mathbb{Q}$ and $r > x_0^2(t_1) + \cdots + x_0^2(t_m)$. Then $f^{-1}(0) \cap W = \varnothing$ for some $W \in \Omega$. Since W is a connected unbounded set, x_0 is not an accumulation point of W. This contradicts the choice of x_0. Now assume that some $W \in \Omega$ is bounded. Then it is easy to see that there exists a sequence of nonempty compact sets $C_1 \supset C_2 \supset \cdots$ with diameters decreasing to 0 and such that $U \cap C_n \neq \varnothing$ for all $U \in \Omega$ and $n \in \mathbb{N}$. Then there exists $x \in \bigcap_{n \in \mathbb{N}} C_n$ belonging to $\partial\Omega$. □

Let us fix a c-filter Ω and define a relation $\succ_\Omega$ in $\mathbb{Q}(\Lambda_T)$ by

$$f \succ_\Omega 0 \iff \text{there exists } U \in \Omega \text{ such that } f(x) > 0 \text{ for all } x \in U,$$
$$f \succ_\Omega g \iff f - g \succ_\Omega 0.$$

Let Ω be a family of subsets of $\mathbb{R}^T$. If an ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ satisfies $f \succ 0$ if and only if $f > 0$ on some $U \in \Omega$, we say that Ω *determines the ordering* $\succ$.

Lemma 2.2. *The relation $\succ_\Omega$ is an ordering in $\mathbb{Q}(\Lambda_T)$, or in other words, a total ordering satisfying*

$$f \succ_\Omega g \Rightarrow f + h \succ_\Omega g + h \quad \textit{and} \quad f \succ_\Omega 0,\ g \succ_\Omega 0 \Rightarrow fg \succ_\Omega 0.$$

Proof. The relation $\succ_\Omega$ is well-defined. Indeed, if $f \in \mathbb{Q}(\Lambda_T)$ and $f \neq 0$, then the union of the sets of zeros and poles of f is contained in some $\mathbb{Q}$-algebraic set $V \subsetneq \mathbb{R}^m$. Hence, by (i) and (ii), for some $U \in \Omega$, the values $f(x)$ have a fixed sign for all $x \in U$. Moreover, if for some $U_1, U_2 \in \Omega$ we have $f(x) > 0$ for $x \in U_1$ and $f(x) \leq 0$ for $x \in U_2$, then $0 < f(x) \leq 0$ for $x \in U_1 \cap U_2$, and $U_1 \cap U_2 \neq \varnothing$ by (iii). This is impossible. It is easy to see that the remaining conditions are also satisfied. □

Proposition 2.3. *Let Ω_1, Ω_2 be c-filters. If the orderings $\succ_{\Omega_1}$ and $\succ_{\Omega_2}$ are equal, then $\Omega = \{U \cup W : U \in \Omega_1,\ W \in \Omega_2\}$ is a c-filter determining the ordering $\succ_{\Omega_1}$.*

Proof. Since Ω_1 and Ω_2 are c-filters, it suffices to prove that $U \cap W \neq \varnothing$ for all $U \in \Omega_1$ and $W \in \Omega_2$. Suppose $U \cap W = \varnothing$ for some $U \in \Omega_1$ and $W \in \Omega_2$. Let $U = U_1 \cup \cdots \cup U_k \cup V$ be a decomposition of U into disjoint basic open semialgebraic sets $U_1, \ldots, U_k$ and a semialgebraic set V included in an algebraic set. By (i) and (ii), there exists $U' \in \Omega_1$ such that $U' \subset U_i$ for some $i \in \{1, \ldots, k\}$. Since $U_i = \{x \in \mathbb{R}^T : f_j(x) > 0,\ j = 1, \ldots, n\}$ for some $f_1, \ldots, f_n \in \mathbb{Q}[\Lambda_T]$, by the assumption we have $f_1, \ldots, f_n \succ_{\Omega_1} 0$, and so there exists $W_1 \in \Omega_2$ such that $f_j(x) > 0$ for all $x \in W_1$ and $j = 1, \ldots, n$. By (iii), there exists $W_2 \in \Omega_2$ such that $W_2 \subset W \cap W_1$ and $f_j(x) > 0$ for all $j = 1, \ldots, n$ and $x \in W_2$. Thus $W_2 \subset U$, which contradicts the assumption. □

Now let $\succ$ be an ordering in $\mathbb{Q}(\Lambda_T)$, and let

$$\mathcal{U}_\succ = \left\{ \bigcap_{i=1}^{n} f_i^{-1}((0, +\infty)) \subset \mathbb{R}^T : f_i \in \mathbb{Q}(\Lambda_T),\ f_i \succ 0 \text{ for } i = 1, \ldots, n,\ n \in \mathbb{N} \right\},$$

where we regard $f \in \mathbb{Q}(\Lambda_T)$ as a function $f : \mathbb{R}^T \to \mathbb{R}$. By the definition of $\mathcal{U}_\succ$ and the Tarski transfer principle (see [Tarski 1948; Seidenberg 1954]), we find that $\varnothing \notin \mathcal{U}_\succ$. Moreover, the relation $\succ$ is defined by

$$f \succ 0 \iff \text{there exists } U \in \mathcal{U}_\succ \text{ such that } f(x) > 0 \text{ for all } x \in U.$$

The sets of the family $\mathcal{U}_\succ$ may be disconnected, so $\mathcal{U}_\succ$ is not a c-filter. We will prove that the ordering $\succ$ is defined by some c-filter.

Proposition 2.4. *There exists a unique c-filter Ω with the following properties*:

(a) *For any $f \in \mathbb{Q}(\Lambda_T)$, we have $f \succ_\Omega 0$ if and only if $f \succ 0$.*

(b) *For any $U \in \Omega$, there exists a $\mathbb{Q}$-algebraic set $V \subsetneq \mathbb{R}^T$ such that U is a connected component of $\mathbb{R}^T \setminus V$.*

(c) *For any $\mathbb{Q}$-algebraic set $V \subsetneq \mathbb{R}^T$, some connected component of $\mathbb{R}^T \setminus V$ belongs to Ω.*

Proof. Let $\mathcal{F}$ be the family of all connected components of sets $U \in \mathcal{U}_\succ$.

Claim 1. *Every $U \in \mathcal{U}_\succ$ has a connected component U_0 such that $U_0 \cap W \neq \varnothing$ for any $W \in \mathcal{U}_\succ$.*

Let $U \in \mathcal{U}_\succ$ and let $U = U_1 \cup \cdots \cup U_n$ be the decomposition into connected components. Assume to the contrary that there exist $W_1, \ldots, W_n \in \mathcal{U}_\succ$ such that $U_i \cap W_i = \varnothing$ for $i = 1, \ldots, n$. Then $U \cap W_1 \cap \cdots \cap W_n = \varnothing$, which is impossible. This gives Claim 1.

Claim 2. *Each $U \in \mathcal{U}_\succ$ has exactly one connected component S_U that intersects every $W \in \mathcal{U}_\succ$.*

Let $U \in \mathcal{U}_\succ$, and let $U_1, \ldots, U_p$ be the connected components of U. Then

$$U = \bigcap_{l=1}^{s} \{x \in \mathbb{R}^T : g_l(x) > 0\} \tag{1}$$

for some nonzero polynomials $g_l \in \mathbb{Q}[\Lambda_T]$, with $g_l \succ 0$ for $l = 1, \ldots, s$, and

$$U_i = [f_i^{-1}(0) \cap U_i] \cup \bigcup_{j=1}^{n} \bigcap_{k=1}^{m} \{x \in \mathbb{R}^T : f_{i,j,k}(x) > 0\}, \quad i = 1, \ldots, p,$$

for some nonzero polynomials f_i, $f_{i,j,k} \in \mathbb{Q}[\Lambda_T]$. Denote by $\epsilon_{i,j,k}$ the sign of $f_{i,j,k}$ in the ordering $\succ$. Then $\epsilon_{i,j,k} \neq 0$ and $\epsilon_{i,j,k} f_{i,j,k} \succ 0$ for any i, j, k. Observe that for some $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$, we have $f_{i,j,k} \succ 0$ for $k = 1, \ldots, m$. Indeed, in the opposite case,

$$\varnothing = \bigcap_{l=1}^{s} \bigcap_{i=1}^{p} \bigcap_{j=1}^{n} \bigcap_{k=1}^{m} \{x \in \mathbb{R}^T : g_l(x) > 0,\ \epsilon_{i,j,k} f_{i,j,k}(x) > 0\} \in \mathcal{U}_\succ,$$

which is impossible. So, for some $i_0 \in \{1, \ldots, p\}$ and $j_0 \in \{1, \ldots, n\}$,

$$U' = \bigcap_{k=1}^{m} \{x \in \mathbb{R}^T : f_{i_0,j_0,k}(x) > 0\} \in \mathcal{U}_\succ,$$

and $U' \cap U_j = \varnothing$ for $j \neq j_0$. Hence, by Claim 1, $S_U = U_{j_0}$ is the unique connected component of U satisfying Claim 2.

Claim 3. *The family $\Omega = \{S_U : U \in \mathcal{U}_\succ\}$ is a c-filter.*

Since for every $\mathbb{Q}$-algebraic set $V \subset \mathbb{R}^T$ there exists $U \in \mathcal{U}_\succ$ such that $U \cap V = \varnothing$, we have $S_U \cap V = \varnothing$. Hence, it suffices to prove that for any $S_{U_1}, S_{U_2} \in \Omega$, there exists $S_{U_3} \in \Omega$ contained in $S_{U_1} \cap S_{U_2}$. Indeed, by the argument of Claim 2, there exist $W_1, W_2 \in \mathcal{U}_\succ$ such that $W_1 \subset S_{U_1}$ and $W_2 \subset S_{U_2}$. Hence, $S_{W_1 \cap W_2} \subset W_1 \cap W_2 \subset S_{U_1} \cap S_{U_2}$ and $S_{W_1 \cap W_2} \in \Omega$.

Claim 4. *The c-filter Ω defined in Claim 3 satisfies the assertion of Proposition 2.4.*

Part (a) is obvious.

Let $U \in \mathcal{U}_\succ$ be of the form (1), $f = g_1 \dots g_s$, and $V = f^{-1}(0)$. Then, by the definition of S_U, we see that S_U is a connected component of $\mathbb{R}^T \setminus V$. This gives (b).

Let $V = f^{-1}(0)$ be a $\mathbb{Q}$-algebraic subset of $\mathbb{R}^T$. Then $U = \{x \in \mathbb{R}^T : f^2(x) > 0\} = \mathbb{R}^T \setminus V \in \mathcal{U}_\succ$ and $S_U \in \Omega$ is a connected component of $\mathbb{R}^T \setminus V$. This gives (c) and completes the proof. □

We call the c-filter Ω defined in Proposition 2.4 the *plain filter* for the ordering $\succ$ and denote it by $\Omega_\succ$.

From Proposition 2.4, we immediately obtain:

Corollary 2.5. *The mapping $\succ \mapsto \Omega_\succ$ is a one-to-one correspondence between the set of orderings of $\mathbb{Q}(\Lambda_T)$ and the set of plain filters.*

Remark 2.6. From the ultrafilter theorem [Bröker 1982], we see that for any ultrafilter $\mathcal{F}$ of subsets of $\mathbb{R}^T$, there exists a plain filter $\Omega \subset \mathcal{F}$.

Remark 2.7. It is easy to observe that the statements of this section hold with $\mathbb{Q}$ replaced by $\mathbb{R}$.

3. Archimedean orderings in $\mathbb{Q}(\Lambda_T)$

Let $\succ$ be an ordering of $\mathbb{Q}(\Lambda_T)$. Then one can assume that T is linearly ordered by

$$t_1 \succ t_2 \iff \Lambda_{t_1} \succ \Lambda_{t_2}.$$

If $f \succ g$, then we also write $g \prec f$.

Theorem 3.1. *The following conditions are equivalent*:

(a) *The field $(\mathbb{Q}(\Lambda_T), \succ)$ is Archimedean.*

(b) *There exists $x_\succ \in \partial\Omega_\succ$ such that the set of coordinates of $x_\succ$ is algebraically independent over $\mathbb{Q}$.*

(c) *There exists $x_\succ \in \partial\Omega_\succ$ such that $f \succ 0$ if and only if $f(x_\succ) > 0$.*

(d) *There exists $x_\succ \in \partial\Omega_\succ$ such that $x_\succ \in U$ for any $U \in \Omega_\succ$.*

Proof. Assume (a). Then for any $t_1, \dots, t_n \in T$ with $t_1 \prec \dots \prec t_n$, and for the projection $\pi_{t_1,\dots,t_n} : \mathbb{R}^T \mapsto (x(t_1), \dots, x(t_n)) \in \mathbb{R}^n$, the family

$$\Omega_{t_1,\dots,t_n} = \{\pi_{t_1,\dots,t_n}(U) : U \in \Omega\} \tag{2}$$

determines an Archimedean order in $\mathbb{Q}(\Lambda_{t_1}, \dots, \Lambda_{t_n})$. Thus for some $W \in \Omega_{t_1,\dots,t_n}$, the function $f = \Lambda_{t_1}^2 + \dots + \Lambda_{t_n}^2$ is bounded on W. So the set W is bounded. Hence, by Proposition 2.1, there exists $(x_1, \dots, x_n) \in \partial\Omega_{t_1,\dots,t_n}$. Since the projections $\pi_{t_1,\dots,t_n}$ are open, it is easy to observe that, for $t_{k_1}, \dots, t_{k_j} \in \{t_1, \dots, t_n\}$ with $t_{k_1} \prec \dots \prec t_{k_j}$, we have $(x_{k_1}, \dots, x_{k_j}) \in \partial\Omega_{t_{k_1},\dots,t_{k_j}}$. Consequently, there exists $x \in \mathbb{R}^T$ such that for any $t_1, \dots, t_n \in T$ with $t_1 \prec \cdots \prec t_n$, we have $\pi_{t_1,\dots,t_n}(x) \in \partial\Omega_{t_1,\dots,t_n}$. Summing up, $x \in \partial\Omega$. The set of coordinates of x is algebraically independent over $\mathbb{Q}$: otherwise, $f(x) = 0$ for some nonzero polynomial $f \in \mathbb{Q}[\Lambda_T]$, and so f is infinitesimal. This contradicts (a) and gives (b).

Assume (b). Then any nonzero $f \in \mathbb{Q}(\Lambda_T)$ with $f \succ 0$ is defined at $x_\succ$. Moreover, $f(x_\succ) \neq 0$, so $f(x_\succ) > 0$. Conversely, assume that $f(x_\succ) > 0$. Then obviously for some connected component U of $f^{-1}(0, +\infty)$, we have $U \in \Omega_\succ$ and $f(x) > 0$ for $x \in U$. Summing up, we obtain (c).

The implication (c) $\Rightarrow$ (d) is trivial.

Now assume (d). Then we immediately obtain (b), and hence, no $f \in \mathbb{Q}(\Lambda_T)$ is infinitesimal, and the field $(\mathbb{Q}(\Lambda_T), \succ)$ is Archimedean. This gives (a) and completes the proof. □

Remark 3.2. The assertion of Theorem 3.1 also holds for every c-filter determining $\succ$ in place of the plain filter $\Omega_\succ$.

Theorem 3.1 implies:

Corollary 3.3. *Let T be a finite set. Then the set of Archimedean orderings of $\mathbb{Q}(\Lambda_T)$ is a dense subset of the space of orderings in $\mathbb{Q}(\Lambda_T)$ in the path topology* (see, for instance, [Marshall 2008]) *of the real spectrum* $\operatorname{Sper}(\mathbb{Q}[\Lambda_T])$.

4. Examples of non-Archimedean orderings

Let m be a fixed positive integer and Λ a system of m variables $\Lambda_1, \dots, \Lambda_m$.

Take any $P \in \mathbb{R}[\Lambda]$. Let $\Gamma_P \subset \mathbb{R}^m$ be a set defined by

$$\Gamma_P = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m : P(\lambda_1, \dots, \lambda_{m-1}, \lambda_m + \gamma) = 0 \text{ for some } \gamma \in [0, \infty)\}.$$

We define a polynomial $\omega(P) \in \mathbb{R}[\Lambda_1, \dots, \Lambda_{m-1}]$ (or a number $\omega(P) \in \mathbb{R}$ if $m = 1$) by $\omega(P) = 0$ for $P = 0$, and $\omega(P) = P_0$ for $P \neq 0$, where

$$P = P_0 \Lambda_m^d + P_1 \Lambda_m^{d-1} + \cdots + P_d$$

and $P_i \in \mathbb{R}[\Lambda_1, \dots, \Lambda_{m-1}]$ (or $P_i \in \mathbb{R}$ if $m = 1$) for $i = 0, \dots, d$ and $P_0 \neq 0$.

Let us define sets $W_P \subset \mathbb{R}^m$, for $P \in \mathbb{R}[\Lambda]$. The definition will be inductive with respect to the number of variables $\Lambda_1, \dots, \Lambda_m$. For $P \in \mathbb{R}[\Lambda]$, we put

$$W_P = \begin{cases} \mathbb{R} \setminus \Gamma_P \subset \mathbb{R} & \text{if } m = 1, \\ (\mathbb{R}^m \setminus \Gamma_P) \cap (W_{\omega(P)} \times \mathbb{R}) \subset \mathbb{R}^m & \text{if } m > 1. \end{cases} \tag{3}$$

By the Tarski–Seidenberg theorem — see Proposition 1.1(c) — the sets W_P are semialgebraic for all $P \in \mathbb{R}[\Lambda]$.

Analogously to Theorem 1.1 of [Spodzieja 1996], we prove the following proposition, which gives an example of c-filter.

Proposition 4.1. *The family* $\mathcal{W} = \{W_P : P \in \mathbb{R}[\Lambda]\}$ *satisfies these conditions*:

R_0. $W_P \subset \{\lambda \in \mathbb{R}^m : P(\lambda) \neq 0\}$.
R_1. $W_P \cap W_Q = W_{PQ}$.
R_2. *For* $P \neq 0$, W_P *is an unbounded subset of* $\mathbb{R}^m$.
R_3. *For* $P \neq 0$, W_P *is an open, connected and simply connected set.*

Moreover, one can demand that

R_4. $W_P = \mathbb{R}^m$ *for* $P = \text{const}$, $P \neq 0$.

In particular, $\mathcal{W}$ *contains the c-filter*

$$\Omega = \{W_P : P \in \mathbb{Q}[\Lambda]\}.$$

Lemma 4.2. *Let* $1 \leq i_1 < \cdots < i_m \leq n$, *and let* $P \in \mathbb{R}[\Lambda_{i_1}, \ldots, \Lambda_{i_m}]$. *Let* $Q \in \mathbb{R}[\Lambda_1, \ldots, \Lambda_n]$ *be a polynomial of the form*

$$Q(x_1, \ldots, x_n) = P(x_{i_1}, \ldots, x_{i_m}), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4}$$

Then $W_P \subset \mathbb{R}^m$, $W_Q \subset \mathbb{R}^n$, *and*

$$W_Q \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_{i_1}, \ldots, x_{i_m}) \in W_P\}.$$

Proof. For $P = 0$ or $n = m$, the assertion is trivial. Assume that $P \neq 0$ and $n > m$. Consider the case $n = m + 1$. Then there exists $1 \leq j \leq n$ such that

$$(\Lambda_{i_1}, \ldots, \Lambda_{i_m}) = (\Lambda_1, \ldots, \Lambda_{n-j}, \Lambda_{n-j+2}, \ldots, \Lambda_n),$$

under the obvious convention for $j = 1$ and $j = n$. Denote the i-th iteration of ω by ω^i, where $\omega^0(P) = P$. Then, for $(x_1, \ldots, x_{n-i}) \in \mathbb{R}^{n-i}$,

$$\omega^i(Q)(x_1, \ldots, x_{n-i}) = \begin{cases} \omega^i(P)(x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) & \text{if } 0 \leq i \leq j-2, \\ \omega^i(P)(x_1, \ldots, x_{n-j}) & \text{if } i = j-1, \\ \omega^{i-1}(P)(x_1, \ldots, x_{n-i}) & \text{if } j \leq i \leq n. \end{cases}$$

Hence,

$$\Gamma_{\omega^i(Q)} = \{(x_1, \ldots, x_{n-i}) \in \mathbb{R}^{n-i} : (x_1, \ldots, x_{n-j}, x_{n-j+2}, \ldots, x_{n-i}) \in \Gamma_{\omega^i(P)}\}$$

for $0 \leq i \leq j-2$, and

$$\Gamma_{\omega^{j-1}(Q)} = \{(x_1, \ldots, x_{n-j+1}) \in \mathbb{R}^{n-j+1} : (x_1, \ldots, x_{n-j}) \in \Gamma_{\omega^{j-1}(P)}\}$$

and $\Gamma_{\omega^i(Q)} = \Gamma_{\omega^{i-1}(P)}$ for $j \leq i \leq n$. In particular, $W_{\omega^i(Q)} = W_{\omega^{i-1}(P)}$ for $j \leq i \leq n$.

Summing up, by (3),

$$
\begin{aligned}
W_Q &= \bigcap_{i=0}^{n} [(\mathbb{R}^{n-i} \setminus \Gamma_{\omega^i(Q)}) \times \mathbb{R}^i] \\
&= \bigcap_{i=0}^{j-2} \{(x_1,\dots,x_n) \in \mathbb{R}^n : (x_1,\dots,x_{n-j},x_{n-j+2},\dots,x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i(P)}\} \\
&\qquad \cap \{(x_1,\dots,x_n) \in \mathbb{R}^n : (x_1,\dots,x_{n-j}) \in \mathbb{R}^{n-j} \setminus \Gamma_{\omega^{j-1}(P)}\} \cap [W_{\omega^j(Q)} \times \mathbb{R}^j] \\
&\subset \bigcap_{i=0}^{j-2} \{(x_1,\dots,x_n) \in \mathbb{R}^n : (x_1,\dots,x_{n-j},x_{n-j+2},\dots,x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i(P)}\} \\
&\qquad \cap [W_{\omega^{j-1}(P)} \times \mathbb{R}^j] \\
&= \{(x_1,\dots,x_n) \in \mathbb{R}^n : (x_{i_1},\dots,x_{i_m}) \in W_P\}.
\end{aligned}
$$

This gives the assertion for $n = m+1$. Hence, by an easy induction with respect to $n-m$, we obtain the assertion. □

Let T be a linearly ordered set and let $\succ$ be the ordering of T.

For any $t_1,\dots,t_m \in T$, $t_1 \prec \cdots \prec t_m$, we define a projection map

$$\pi_{t_1,\dots,t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1),\dots,x(t_m)) \in \mathbb{R}^m.$$

Define a family Ω of semialgebraic subsets U of $\mathbb{R}^T$ by

$$U = (\pi_{t_1,\dots,t_m})^{-1}(W_P), \tag{5}$$

where $m \in \mathbb{N}$, $t_1,\dots,t_m \in T$, $t_1 \prec \cdots \prec t_m$, and $P \in \mathbb{Q}[\Lambda_{t_1},\dots,\Lambda_{t_m}] \setminus \{0\}$.

Proposition 4.3. *The family* Ω *is a c-filter.*

Proof. By Proposition 4.1 (condition R_2), any $U \in \Omega$ is a nonempty set.

Let $V \subsetneq \mathbb{R}^T$ be a $\mathbb{Q}$-algebraic set, and let $P \in \mathbb{Q}[\Lambda_T] \setminus \{0\}$ be such that $V = \{x \in \mathbb{R}^T : P(x) = 0\}$. Then $P \in \mathbb{Q}[\Lambda_{t_1},\dots,\Lambda_{t_m}] \setminus \{0\}$ for some $t_1,\dots,t_m \in T$, $t_1 \prec \cdots \prec t_m$, and $U = (\pi_{t_1,\dots,t_m})^{-1}(W_P)$. Applying Proposition 4.1 (condition R_0), we obtain that U satisfies (i).

Let $U_1, U_2 \in \Omega$. Let $t_1,\dots,t_m,u_1,\dots,u_n \in T$ satisfy $t_1 \prec \cdots \prec t_m$ and $u_1 \prec \cdots \prec u_n$, and assume moreover that for some $P \in \mathbb{Q}[\Lambda_{t_1},\dots,\Lambda_{t_m}]$ and $Q \in \mathbb{Q}[\Lambda_{u_1},\dots,\Lambda_{u_n}]$ we have $U_1 = (\pi_{t_1,\dots,t_m})^{-1}(W_P)$ and $U_2 = (\pi_{u_1,\dots,u_n})^{-1}(W_Q)$. Let $v_1,\dots,v_s \in T$, $v_1 \prec \cdots \prec v_s$, be such that $\{t_1,\dots,t_m\} \cup \{u_1,\dots,u_n\} \subset \{v_1,\dots,v_s\}$, and let $\overline{P}, \overline{Q} \in \mathbb{R}[\Lambda_{v_1},\dots,\Lambda_{v_s}]$ be polynomials of the form (4) determined by P and Q, respectively. Then, by Proposition 4.1 (condition R_1) and Lemma 4.2,

$$(\pi_{v_1,\dots,v_s})^{-1}(W_{\overline{PQ}}) = (\pi_{v_1,\dots,v_s})^{-1}(W_{\overline{P}}) \cap (\pi_{v_1,\dots,v_s})^{-1}(W_{\overline{Q}}) \subset U_1 \cap U_2.$$

This gives (ii).

Take any $U \in \Omega$. There exist $t_1, \dots, t_m \in T$ and $P \in \mathbb{R}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$ such that $t_1 \prec \dots \prec t_m$ and $U = (\pi_{t_1,\dots,t_m})^{-1}(W_P)$. By Proposition 4.1 (condition R_3), U satisfies (iii). This completes the proof. □

From the definition of the family Ω, we immediately obtain:

Corollary 4.4. *For any $t_1, t_2 \in T$, we have $t_1 \succ t_2$ if and only if $\Lambda_{t_1} \succ_\Omega \Lambda_{t_2}$.*

Let $Q \in \mathbb{Q}[\Lambda_T] \setminus \{0\}$ and let Ω_Q be a family of semialgebraic subsets U of $\mathbb{R}^T$ defined by

$$U = (\pi_{t_1,\dots,t_m})^{-1}(W_P \cap W_Q), \tag{6}$$

where $m \in \mathbb{N}$, $t_1, \dots, t_m \in T$, $t_1 \prec \dots \prec t_m$, and $PQ \in \mathbb{Q}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$. By Proposition 4.3, we have:

Corollary 4.5. *Ω_Q is a c-filter.*

Let $X \subset \mathbb{R}^T$ be an open semialgebraic set and let $\mathring{x} \in X$ be a point with rational coordinates. There exist $t_1, \dots, t_k \in T$, $t_1 \prec \dots \prec t_k$, and an open semialgebraic set $Y \subset \mathbb{R}^k$ such that $X = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_k)) \in Y\}$. Hence, there exists $r > 0$ such that

$$B := \{x \in \mathbb{R}^T : \max_{i=1,\dots,k} |x(t_i) - \mathring{x}(t_i)| < r\} \subset X.$$

Let

$$P_0 = \Lambda_{t_1} \dots \Lambda_{t_k}(\Lambda_{t_1}^2 + \dots + \Lambda_{t_k}^2 - 1/r^2),$$

let $U_0 = (\pi_{t_1,\dots,t_k})^{-1}(W_{P_0})$, and let $F : U_0 \to \mathbb{R}^T$ be a mapping defined by

$$F(x)(t) = \begin{cases} \mathring{x}(t) + 1/x(t) & \text{for } x \in U_0,\ t \in \{t_1, \dots, t_k\}, \\ x(t) & \text{for } x \in U_0,\ t \in T \setminus \{t_1, \dots, t_k\}. \end{cases}$$

Proposition 4.6. *$\{F(U) : U \in \Omega_{P_0}\}$ is a c-filter subset of X. In particular, for any open semialgebraic set $Y \subset \mathbb{R}^T$, there exists c-filter subsets of Y.*

Proof. By Lemma 4.2, any set $U \in \Omega_{P_0}$ is a subset of U_0. Moreover, F is an open semialgebraic mapping, so $F(U)$ is semialgebraic for $U \in \Omega_{P_0}$. Hence, $\{F(U) : U \in \Omega_{P_0}\}$ satisfies conditions (i)–(iii). □

From Proposition 4.6 and Theorem 3.1, we have that:

Corollary 4.7. *The set of c-filters defined in Proposition 4.6 is a dense subset of the space of orderings in $\mathbb{Q}(\Lambda_T)$ in the path topology of the real spectrum* Sper($\mathbb{Q}[\Lambda_T]$). *Moreover, any ordering determined by such a c-filter is not Archimedean.*

Remark 4.8. It is easy to see that the results of this section hold if we replace $\mathbb{Q}$ by $\mathbb{R}$.

5. Fields of Nash functions

Let T be a nonempty set. We denote by $\mathcal{N}(X)$ the domain of $\mathbb{Q}$-Nash functions on an open connected semialgebraic set $X \subset \mathbb{R}^T$.

Let $\succ$ be an ordering in $\mathbb{Q}(\Lambda_T)$ and let $\Omega_\succ$ be the plain filter of subsets of $\mathbb{R}^T$ determining $\succ$. Let us introduce in $\bigcup_{U\in\Omega_\succ} \mathcal{N}(U)$ a relation $\sim_\succ$ by

$$(f_1 : U_1 \to \mathbb{R}) \sim_\succ (f_2 : U_2 \to \mathbb{R}) \iff \exists_{U\in\Omega_\succ} (U \subset U_1 \cap U_2 \text{ and } f_1|_U = f_2|_U).$$

From Proposition 2.4, we immediately see that $\sim_\succ$ is an equivalence relation. The equivalence class of $\sim_\succ$ determined by $f : U \to \mathbb{R}$ is denoted by $[f]_\succ$, and the set of all such classes by $\mathcal{N}_\succ$. The set $\mathcal{N}_\succ$ is linearly ordered by

$$[f]_\succ \succ 0 \iff \exists_{U\in\Omega_\succ} (f \in \mathcal{N}(U) \text{ and } f(x) > 0 \text{ for } x \in U).$$

Proposition 5.1. *The set $\mathcal{N}_\succ$, together with the usual operations*

$$[f_1]_\succ + [f_2]_\succ = [f_1|_U + f_2|_U]_\succ\,, \quad [f_1]_\succ \cdot [f_2]_\succ = [f_1|_U f_2|_U]_\succ\,,$$

where $f_1 \in \mathcal{N}(U_1)$, $f_2 \in \mathcal{N}(U_2)$, and $U \in \Omega_\succ$, $U \subset U_1 \cap U_2$, is a real field.

Proof. Since the ring $\mathcal{N}(U)$ is a domain for any $U \in \Omega_\succ$, so is $\mathcal{N}_\succ$. We prove that any nonzero $f \in \mathcal{N}_\succ$ has an inverse in $\mathcal{N}_\succ$. Indeed, there exists $U \in \Omega_\succ$ such that $f \in \mathcal{N}(U)$. Since $f \neq 0$, the set $f^{-1}(0)$ is contained in some proper $\mathbb{Q}$-algebraic subset of $\mathbb{R}^T$. Then, by the definition of c-filter, one can assume that $f(\lambda) \neq 0$ for any $\lambda \in U$. Thus $1/f \in \mathcal{N}(U)$, so f has an inverse in $\mathcal{N}_\succ$. Summing up, $\mathcal{N}_\succ$ is a field. Since $-1 \in \mathcal{N}(U)$ is not a sum of squares in $\mathcal{N}(U)$, it follows that $-1 \in \mathcal{N}_\succ$ is not a sum of squares in $\mathcal{N}_\succ$. □

Theorem 5.2. *The field $\mathcal{N}_\succ$ is a real closure of the field* $(\mathbb{Q}(\Lambda_T), \succ)$.

Proof. Take any irreducible polynomial $P \in \mathcal{N}_\succ[Z]$ of odd degree d with respect to Z. Then there exists $U \in \Omega_\succ$ such that $P \in \mathcal{N}(U)[Z]$. Let $t_1, \dots, t_m \in T$, and let $\tilde{U} \subset \mathbb{R}^m$ be an open connected semialgebraic set such that $U = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in \tilde{U}\}$. By using the Hermite method (for $\tilde{U}$) we deduce that there exists a decomposition $U = U_1 \cup \dots \cup U_k \cup V$ of U into disjoint open basic $\mathbb{Q}$-semialgebraic sets $U_1, \dots, U_k$ and a semialgebraic set V included in an algebraic set such that $P(x, Z)$ has the same number of zeroes for all $x \in U_i$ and each of these zeroes is single. By (i) and (ii) in the definition of a c-filter, there exists $U' \in \Omega_\succ$ such that $U' \subset U_i$ for some $i \in \{1, \dots, k\}$. Then there exists $k \in \mathbb{N}$, $k > 0$ such that $P(x, Z)$ has exactly k zeroes for $x \in U'$, and so there exist functions $\xi_1, \dots, \xi_k : U' \to \mathbb{R}$ with $\xi_1(x) < \dots < \xi_k(x)$ such that $P(x, \xi_i(x)) = 0$ for $x \in U'$, $i = 1, \dots, k$. As $\xi_i(x)$ are single zeroes of $P(x, Z)$, by the Implicit Function Theorem, ξ_i is a Nash function for $i = 1, \dots, k$. As $\mathcal{N}_\succ$ is a real field

(Proposition 5.1), $\mathcal{N}_{\succ}$ is a real closed field. Since $\mathcal{N}_{\succ}$ is an algebraic extension of $\mathbb{Q}(\Lambda_T)$, by the Artin–Schreier Theorem, it is a real closure of $(\mathbb{Q}(\Lambda_T), \succ)$. □

Remark 5.3. The above results of this section also hold for an arbitrary c-filter determining $\succ$ in place of the plain filter $\Omega_{\succ}$. The results also hold if we put $\mathbb{R}$ in place of $\mathbb{Q}$.

From Theorems 3.1 and 5.2, we recover the familiar result that any Archimedean field can be embedded in $\mathbb{R}$.

Corollary 5.4. *Let $\Omega_{\succ}$ be a plain filter of subsets of $\mathbb{R}^T$ determining an Archimedean ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$, and let $x_{\succ} \in \bigcap_{U \in \Omega_{\succ}} U$. Then the mapping*

$$\mathcal{N}_{\succ} \ni f \mapsto f(x_{\succ}) \in \mathbb{R}$$

is an order-preserving monomorphism.

From Theorem 5.2, we immediately obtain:

Corollary 5.5. *Let R be a real closed field with ordering $\succ$, and let T be the transcendence basis of R over $\mathbb{Q}$ whose existence is guaranteed by the Kuratowski–Zorn lemma. Assume that $T \neq \varnothing$ and let $\Lambda_T = (\Lambda_t : t \in T)$ be a system of independent variables. Then the field R is order-preserving isomorphic to a real closure of the rational functions field $\mathbb{Q}(\Lambda_T)$, i.e., to some field $\mathcal{N}_{\succ}$.*

Remark 5.6. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Then $\mathbb{K} = R[i]$, where $i^2 = -1$, for some real closed field R. Let $T \subset R$ be the transcendence basis of $\mathbb{K}$ over $\mathbb{Q}$. Assume that $T \neq \varnothing$. Then $\mathbb{K}$ is isomorphic to an algebraic closure of $\mathbb{Q}(\Lambda_T)$. By Theorem 1.1 of [Spodzieja 1996], one can introduce a filter $\Omega_{\mathbb{C}}$ of open, connected, and simply connected semialgebraic subsets U of $\mathbb{C}^T$ satisfying conditions (i), (ii), and (iii). Then, analogously to [Spodzieja 1996], one can introduce a geometric construction of the algebraic closure of $\mathbb{Q}(\Lambda_T)$ in terms of complex Nash functions.

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