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**A GEOMETRIC MODEL OF AN ARBITRARY  
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## A GEOMETRIC MODEL OF AN ARBITRARY REAL CLOSED FIELD

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**We give an elementary construction of any real closed field in terms of Nash function fields. We also give a characterization of any Archimedean field in terms of fields of Nash functions.**

### Introduction

In the study of Hilbert's 17th problem, orderings of a real field  $k$  are of importance (see [Alonso 1986; Alonso et al. 1984; Artin 1927; Artin and Schreier 1927a; 1927b; Bochnak and Efroymsen 1980; Bröker 1982; Dubois 1981; Guangxing 2005; Marshall 2003; Prestel and Delzell 2001; Schwartz 1980]). By the Artin–Schreier theorem [Artin 1927; Artin and Schreier 1927a; 1927b], the study of such orderings amounts to considering real closures of  $k$ . The aim of this article is to construct a universal model of an arbitrary real closed field. To this end, we construct, in terms of Nash functions, all real closures of the rational function fields  $k = \mathbb{Q}(\Lambda_T)$ , where  $\Lambda_T = (\Lambda_t : t \in T)$  and  $T \neq \emptyset$  is a system of any number of variables. This suffices to achieve our purpose, because any real closed field  $R$  is order-preserving isomorphic to a real closure of some field  $\mathbb{Q}(\Lambda_T)$  (Corollary 5.5). If  $T = \emptyset$ , then  $\mathbb{Q}(\Lambda_T) = \mathbb{Q}$ , and the above is obvious. We assume the Kuratowski–Zorn lemma, so the set  $T$  can be well-ordered, provided  $T \neq \emptyset$ .

L. Bröker [1982] proved in his ultrafilter theorem that there exists a one-to-one correspondence between the family of ultrafilters and the family of orderings in  $\mathbb{Q}(\Lambda_T)$ , or equivalently with the real closures of  $\mathbb{Q}(\Lambda_T)$ . We prove that there exists a one-to-one correspondence between the family of orderings in  $\mathbb{Q}(\Lambda_T)$  and the family of *plain filters* (Theorem 5.2, Proposition 2.4, and Corollary 2.5). By a plain filter we mean a filter  $\Omega$  of subsets of  $\mathbb{R}^T$  with these properties:

- (1) Any  $U \in \Omega$  is a nonempty open connected semialgebraic set.
- (2) For any algebraic set  $V \subsetneq \mathbb{R}^T$ , where  $V = P^{-1}(0)$  and  $P \in \mathbb{Q}[\Lambda_T]$ , some connected component of  $\mathbb{R}^T \setminus V$  belongs to  $\Omega$ .

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(3) For any  $U_1, U_2 \in \Omega$ , there exists  $U_3 \in \Omega$  such that  $U_3 \subset U_1 \cap U_2$ .

The correspondence between orderings and plain filters is as follows: For any ordering  $\succ$  of  $\mathbb{Q}(\Lambda_T)$ , there exists a unique plain filter  $\Omega$  such that  $f \succ 0$  if and only if  $f > 0$  on some  $U \in \Omega$ , where  $>$  is the usual ordering on  $\mathbb{R}$ . Conversely, any plain filter  $\Omega$  determines a unique ordering  $\succ$  of  $\mathbb{Q}(\Lambda_T)$  in this way.

The main result of this article is [Theorem 5.2](#), where we give a construction of any real closure of  $\mathbb{Q}(\Lambda_T)$  in terms of Nash functions. The main idea and motivation for the above considerations was a geometric construction of the algebraic closure of  $\mathbb{C}(\Lambda_1, \dots, \Lambda_m)$  [[Spodzieja 1996](#)]. More precisely, for any plain filter  $\Omega$  of open connected semialgebraic sets and any  $U \in \Omega$ , the ring  $\mathcal{N}(U)$  of  $\mathbb{Q}$ -Nash functions (see [Section 1](#)) on  $U$  is a domain. In  $\bigcup_{U \in \Omega} \mathcal{N}(U)$ , we introduce an equivalence relation  $\sim : (f_1 : U_1 \rightarrow \mathbb{R}) \sim (f_2 : U_2 \rightarrow \mathbb{R})$  if and only if  $f_1|_{U_3} = f_2|_{U_3}$  for some  $U_3 \in \Omega$ . The set  $\mathcal{N}_\Omega$  of equivalence classes of  $\sim$  with the usual operations of addition and multiplication is a field, which is a real closure of  $\mathbb{Q}(\Lambda_T)$  (see [Theorem 5.2](#), and compare [[Spodzieja 1996](#), Theorem 2.4 and Corollary 2.5]). One can view  $\mathcal{N}_\Omega$  as the inverse limit of the étale topology  $\bigcup_{U \in \Omega} \mathcal{N}(U)$  of  $\mathbb{R}^T$  [[Grothendieck 1967](#)].

In [Section 3](#), we prove that an ordering  $\succ$  of  $\mathbb{Q}(\Lambda_T)$  is Archimedean if and only if the set  $\bigcap_{U \in \Omega} U$  is nonempty for the plain filter  $\Omega$  determining  $\succ$ ; and if that is the case, this set has exactly one point ([Theorem 3.1](#)). In [Section 4](#), we give some examples of non-Archimedean orderings corresponding to the one in [[Spodzieja 1996](#)].

### 1. Preliminaries

Let  $\mathbb{K}$  be the field  $\mathbb{Q}$  of rational,  $\mathbb{R}$  of real, or  $\mathbb{C}$  of complex numbers. Let  $T$  be a nonempty set. We denote by  $\Lambda_T = (\Lambda_t : t \in T)$  a system of independent variables  $\Lambda_t$ , by  $\mathbb{K}[\Lambda_T]$  the ring of polynomials in  $\Lambda_T$  over  $\mathbb{K}$ , and by  $\mathbb{K}(\Lambda_T)$  the quotient field of  $\mathbb{K}[\Lambda_T]$ . Note that for any  $P \in \mathbb{K}(\Lambda_T)$ , we have  $P \in \mathbb{K}(\Lambda_{t_1}, \dots, \Lambda_{t_m})$  for some finite number of indices  $t_1, \dots, t_m \in T$ .

We denote by  $\mathbb{K}^T$  the set of all functions  $T \rightarrow \mathbb{K}$  equipped with the unique topology for which all projections  $\mathbb{K}^T \ni x \mapsto x(t) \in \mathbb{K}, t \in T$  are continuous.

Let  $\mathbb{L}$  be a subfield of  $\mathbb{K}$ . A subset of  $\mathbb{K}^T$  is called  $\mathbb{L}$ -algebraic, or simply algebraic if  $\mathbb{L} = \mathbb{K}$ , when it is defined by a finite system of equations  $P = 0$ , where  $P \in \mathbb{L}[\Lambda_T]$ . Any  $\mathbb{L}$ -algebraic set in  $\mathbb{K}^T$  is of the form  $\{x \in \mathbb{K}^T : (x(t_1), \dots, x(t_m)) \in V\}$ , where  $m \in \mathbb{N}, t_1, \dots, t_m \in T$ , and  $V \subset \mathbb{K}^m$  is an  $\mathbb{L}$ -algebraic subset of  $\mathbb{K}^m$ .

If  $\mathbb{L}$  is a subfield of  $\mathbb{R}$ , then we assume that  $\mathbb{L}$  is an ordered field with order induced from  $\mathbb{R}$ .

Let  $\mathbb{L}$  be a subfield of  $\mathbb{R}$ . A subset of  $\mathbb{R}^T$  is called  $\mathbb{L}$ -semialgebraic when it is defined by a finite alternative of finite systems of inequalities  $P > 0$  or  $P \geq 0$ , where  $P \in \mathbb{L}[\Lambda_T]$ . Analogously to the above, any  $\mathbb{L}$ -semialgebraic set in  $\mathbb{R}^T$  is of the form

$\{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in X\}$ , where  $m \in \mathbb{N}$ ,  $t_1, \dots, t_m \in T$ , and  $X \subset \mathbb{R}^m$  is an  $\mathbb{L}$ -semialgebraic subset of  $\mathbb{R}^m$ . A set is called *open basic  $\mathbb{L}$ -semialgebraic* if it has the form  $\{x \in \mathbb{R}^T : g_i(x) > 0, i = 1, \dots, n\}$ , for some  $n \in \mathbb{N}$  and  $g_i \in \mathbb{L}[\Lambda_T]$ ,  $i = 1, \dots, n$ .

We now list some basic properties of algebraic and semialgebraic sets in infinite-dimensional real vector spaces, which follow easily from their analogues in finite-dimensional spaces [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymsen 1980; Efroymsen 1974; 1976; 1981; Mostowski 1976; Prestel and Delzell 2001; Tancredi and Tognoli 2006; Tworzewski 1990].

**Proposition 1.1.** *Let  $\mathbb{L}$  be a subfield of  $\mathbb{R}$  (or  $\mathbb{K}$  in (a)).*

- (a) *The family of  $\mathbb{L}$ -algebraic sets in  $\mathbb{K}^T$  is closed with respect to union and intersection of a finite number of sets.*
- (b) *The family of  $\mathbb{L}$ -semialgebraic sets in  $\mathbb{R}^T$  is closed with respect to complement, union, and intersection of a finite number of sets.*
- (c) (Tarski–Seidenberg) *Let  $\pi_{t_1, \dots, t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \dots, x(t_m)) \in \mathbb{R}^m$ , where  $t_1, \dots, t_m \in T$ . If  $X \subset \mathbb{R}^T$ ,  $Y \subset \mathbb{R}^m$  are  $\mathbb{L}$ -semialgebraic sets, then  $\pi_{t_1, \dots, t_m}(X)$  and  $\pi_{t_1, \dots, t_m}^{-1}(Y)$  are  $\mathbb{L}$ -semialgebraic sets, too.*
- (d) *For any  $\mathbb{L}$ -semialgebraic set  $X \subset \mathbb{R}^T$ , the interior  $\text{Int } X$ , closure  $\bar{X}$ , and the boundary  $\partial X$  are  $\mathbb{L}$ -semialgebraic sets.*

Let  $\mathbb{L}$  be a subfield of  $\mathbb{R}$ . A function  $f : U \rightarrow \mathbb{R}$ , where  $U \subset \mathbb{R}^T$  is an open  $\mathbb{L}$ -semialgebraic set, is called an  *$\mathbb{L}$ -Nash function* if  $f$  is analytic and there exists a nonzero polynomial  $P \in \mathbb{L}[\Lambda_T, Z]$  such that  $P(\lambda, f(\lambda)) = 0$  for  $\lambda \in U$ . In fact,  $f$  depends on a finite number of variables, so the analyticity of  $f$  is clear. The ring of  $\mathbb{L}$ -Nash functions in  $U$  is denoted by  $\mathcal{N}^{\mathbb{L}}(U)$ .

The next result follows via R. Thom’s lemma (see for instance [Bochnak et al. 1987, Proposition 2.5.4 and the arguments of Theorems 2.3.6 and 2.4.4]) from the fact that any  $\mathbb{L}$ -semialgebraic set in a finite-dimensional vector space over  $\mathbb{R}$  is the disjoint union of a finite number of  $\mathbb{L}$ -semialgebraic sets which are homeomorphic to Cartesian products of intervals.

**Proposition 1.2.** *Let  $\mathbb{L}$  be a subfield of  $\mathbb{R}$ . Any connected component of an  $\mathbb{L}$ -semialgebraic subset of  $\mathbb{R}^T$  is  $\mathbb{L}$ -semialgebraic.*

A function  $f : U \rightarrow \mathbb{C}$ , where  $U \subset \mathbb{C}^T$  is an open set, is called a  *$\mathbb{C}$ -Nash function* if  $f$  is holomorphic and there exists a nonzero polynomial  $P \in \mathbb{C}[\Lambda_T, Z]$  such that  $P(\lambda, f(\lambda)) = 0$  for  $\lambda \in U$ . The ring of  $\mathbb{C}$ -Nash functions in  $U$  is denoted by  $\mathcal{N}^{\mathbb{C}}(U)$ .

For the basic properties of Nash functions and semialgebraic sets in finite-dimensional vector spaces, see, for instance, [Benedetti and Risler 1990; Bochnak et al. 1987; Bochnak and Efroymsen 1980; Efroymsen 1974; 1976; 1981; Mostowski

1976; Nash 1952; Tancredi and Tognoli 2006; Tworzewski 1990]. From these properties, we immediately obtain:

**Proposition 1.3.** *Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ , let  $\mathbb{L}$  be a subfield of  $\mathbb{K}$ , and let  $U \subset \mathbb{K}^T$  be an open connected set. Then  $\mathcal{N}^{\mathbb{K}}(U)$  is a domain, provided  $U$  is semialgebraic when  $\mathbb{K} = \mathbb{R}$ . In particular  $\mathcal{N}^{\mathbb{Q}}(U)$  is a domain.*

## 2. Orderings in $\mathbb{Q}(\Lambda_T)$

Let  $T$  be a nonempty set. A family  $\Omega$  of subsets of  $\mathbb{R}^T$  will be called a *c-filter* (connected sets filter) if it satisfies these conditions:

- (i) Any  $U \in \Omega$  is a nonempty open connected  $\mathbb{Q}$ -semialgebraic set.
- (ii) For any  $\mathbb{Q}$ -algebraic set  $V \subsetneq \mathbb{R}^T$ , there exists  $U \in \Omega$  such that  $V \cap U = \emptyset$ .
- (iii) For any  $U_1, U_2 \in \Omega$ , there exists  $U_3 \in \Omega$  such that  $U_3 \subset U_1 \cap U_2$ .

**Proposition 2.1.** *Let  $\Omega$  be a c-filter of subsets of  $\mathbb{R}^T$ . The set  $\partial\Omega := \bigcap_{U \in \Omega} \bar{U}$  has at most one point. Moreover, whenever  $T$  is a finite set,  $\partial\Omega \neq \emptyset$  if and only if there exists a bounded set  $U \in \Omega$ .*

*Proof.* If  $x_1, x_2 \in \partial\Omega$  with  $x_1 \neq x_2$ , then for some polynomial  $f \in \mathbb{Q}[\Lambda_T]$ , we have  $f(x_1) < 0 < f(x_2)$ . Hence, for some  $W \in \Omega$  such that  $W \cap f^{-1}(0) = \emptyset$ , we have both  $f(x) < 0$  and  $f(x) > 0$  for some  $x \in W$ . This contradiction gives the first part of the assertion.

Now let  $T = \{t_1, \dots, t_m\}$ . Suppose that  $\partial\Omega \neq \emptyset$  and each  $W \in \Omega$  is an unbounded set. Take  $x_0 \in \partial\Omega$ , and let  $f = (\Lambda_T) = \Lambda_{t_1}^2 + \dots + \Lambda_{t_m}^2 - r$ , where  $r \in \mathbb{Q}$  and  $r > x_0^2(t_1) + \dots + x_0^2(t_m)$ . Then  $f^{-1}(0) \cap W = \emptyset$  for some  $W \in \Omega$ . Since  $W$  is a connected unbounded set,  $x_0$  is not an accumulation point of  $W$ . This contradicts the choice of  $x_0$ . Now assume that some  $W \in \Omega$  is bounded. Then it is easy to see that there exists a sequence of nonempty compact sets  $C_1 \supset C_2 \supset \dots$  with diameters decreasing to 0 and such that  $U \cap C_n \neq \emptyset$  for all  $U \in \Omega$  and  $n \in \mathbb{N}$ . Then there exists  $x \in \bigcap_{n \in \mathbb{N}} C_n$  belonging to  $\partial\Omega$ . □

Let us fix a c-filter  $\Omega$  and define a relation  $\succ_{\Omega}$  in  $\mathbb{Q}(\Lambda_T)$  by

$$f \succ_{\Omega} 0 \iff \text{there exists } U \in \Omega \text{ such that } f(x) > 0 \text{ for all } x \in U,$$

$$f \succ_{\Omega} g \iff f - g \succ_{\Omega} 0.$$

Let  $\Omega$  be a family of subsets of  $\mathbb{R}^T$ . If an ordering  $\succ$  of  $\mathbb{Q}(\Lambda_T)$  satisfies  $f \succ 0$  if and only if  $f > 0$  on some  $U \in \Omega$ , we say that  $\Omega$  *determines the ordering*  $\succ$ .

**Lemma 2.2.** *The relation  $\succ_{\Omega}$  is an ordering in  $\mathbb{Q}(\Lambda_T)$ , or in other words, a total ordering satisfying*

$$f \succ_{\Omega} g \implies f + h \succ_{\Omega} g + h \quad \text{and} \quad f \succ_{\Omega} 0, g \succ_{\Omega} 0 \implies fg \succ_{\Omega} 0.$$

*Proof.* The relation  $\succ_{\Omega}$  is well-defined. Indeed, if  $f \in \mathbb{Q}(\Lambda_T)$  and  $f \neq 0$ , then the union of the sets of zeros and poles of  $f$  is contained in some  $\mathbb{Q}$ -algebraic set  $V \subsetneq \mathbb{R}^m$ . Hence, by (i) and (ii), for some  $U \in \Omega$ , the values  $f(x)$  have a fixed sign for all  $x \in U$ . Moreover, if for some  $U_1, U_2 \in \Omega$  we have  $f(x) > 0$  for  $x \in U_1$  and  $f(x) \leq 0$  for  $x \in U_2$ , then  $0 < f(x) \leq 0$  for  $x \in U_1 \cap U_2$ , and  $U_1 \cap U_2 \neq \emptyset$  by (iii). This is impossible. It is easy to see that the remaining conditions are also satisfied.  $\square$

**Proposition 2.3.** *Let  $\Omega_1, \Omega_2$  be c-filters. If the orderings  $\succ_{\Omega_1}$  and  $\succ_{\Omega_2}$  are equal, then  $\Omega = \{U \cup W : U \in \Omega_1, W \in \Omega_2\}$  is a c-filter determining the ordering  $\succ_{\Omega_1}$ .*

*Proof.* Since  $\Omega_1$  and  $\Omega_2$  are c-filters, it suffices to prove that  $U \cap W \neq \emptyset$  for all  $U \in \Omega_1$  and  $W \in \Omega_2$ . Suppose  $U \cap W = \emptyset$  for some  $U \in \Omega_1$  and  $W \in \Omega_2$ . Let  $U = U_1 \cup \dots \cup U_k \cup V$  be a decomposition of  $U$  into disjoint basic open semialgebraic sets  $U_1, \dots, U_k$  and a semialgebraic set  $V$  included in an algebraic set. By (i) and (ii), there exists  $U' \in \Omega_1$  such that  $U' \subset U_i$  for some  $i \in \{1, \dots, k\}$ . Since  $U_i = \{x \in \mathbb{R}^T : f_j(x) > 0, j = 1, \dots, n\}$  for some  $f_1, \dots, f_n \in \mathbb{Q}[\Lambda_T]$ , by the assumption we have  $f_1, \dots, f_n \succ_{\Omega_1} 0$ , and so there exists  $W_1 \in \Omega_2$  such that  $f_j(x) > 0$  for all  $x \in W_1$  and  $j = 1, \dots, n$ . By (iii), there exists  $W_2 \in \Omega_2$  such that  $W_2 \subset W \cap W_1$  and  $f_j(x) > 0$  for all  $j = 1, \dots, n$  and  $x \in W_2$ . Thus  $W_2 \subset U$ , which contradicts the assumption.  $\square$

Now let  $\succ$  be an ordering in  $\mathbb{Q}(\Lambda_T)$ , and let

$$\mathcal{U}_{\succ} = \left\{ \bigcap_{i=1}^n f_i^{-1}((0, +\infty)) \subset \mathbb{R}^T : f_i \in \mathbb{Q}(\Lambda_T), f_i \succ 0 \text{ for } i = 1, \dots, n, n \in \mathbb{N} \right\},$$

where we regard  $f \in \mathbb{Q}(\Lambda_T)$  as a function  $f : \mathbb{R}^T \rightarrow \mathbb{R}$ . By the definition of  $\mathcal{U}_{\succ}$  and the Tarski transfer principle (see [Tarski 1948; Seidenberg 1954]), we find that  $\emptyset \notin \mathcal{U}_{\succ}$ . Moreover, the relation  $\succ$  is defined by

$$f \succ 0 \iff \text{there exists } U \in \mathcal{U}_{\succ} \text{ such that } f(x) > 0 \text{ for all } x \in U.$$

The sets of the family  $\mathcal{U}_{\succ}$  may be disconnected, so  $\mathcal{U}_{\succ}$  is not a c-filter. We will prove that the ordering  $\succ$  is defined by some c-filter.

**Proposition 2.4.** *There exists a unique c-filter  $\Omega$  with the following properties:*

- (a) *For any  $f \in \mathbb{Q}(\Lambda_T)$ , we have  $f \succ_{\Omega} 0$  if and only if  $f \succ 0$ .*
- (b) *For any  $U \in \Omega$ , there exists a  $\mathbb{Q}$ -algebraic set  $V \subsetneq \mathbb{R}^T$  such that  $U$  is a connected component of  $\mathbb{R}^T \setminus V$ .*
- (c) *For any  $\mathbb{Q}$ -algebraic set  $V \subsetneq \mathbb{R}^T$ , some connected component of  $\mathbb{R}^T \setminus V$  belongs to  $\Omega$ .*

*Proof.* Let  $\mathcal{F}$  be the family of all connected components of sets  $U \in \mathcal{U}_{\succ}$ .

**Claim 1.** Every  $U \in \mathcal{U}_>$  has a connected component  $U_0$  such that  $U_0 \cap W \neq \emptyset$  for any  $W \in \mathcal{U}_>$ .

Let  $U \in \mathcal{U}_>$  and let  $U = U_1 \cup \dots \cup U_n$  be the decomposition into connected components. Assume to the contrary that there exist  $W_1, \dots, W_n \in \mathcal{U}_>$  such that  $U_i \cap W_i = \emptyset$  for  $i = 1, \dots, n$ . Then  $U \cap W_1 \cap \dots \cap W_n = \emptyset$ , which is impossible. This gives Claim 1.

**Claim 2.** Each  $U \in \mathcal{U}_>$  has exactly one connected component  $S_U$  that intersects every  $W \in \mathcal{U}_>$ .

Let  $U \in \mathcal{U}_>$ , and let  $U_1, \dots, U_p$  be the connected components of  $U$ . Then

$$(1) \quad U = \bigcap_{l=1}^s \{x \in \mathbb{R}^T : g_l(x) > 0\}$$

for some nonzero polynomials  $g_l \in \mathbb{Q}[\Lambda_T]$ , with  $g_l > 0$  for  $l = 1, \dots, s$ , and

$$U_i = [f_i^{-1}(0) \cap U_i] \cup \bigcup_{j=1}^n \bigcap_{k=1}^m \{x \in \mathbb{R}^T : f_{i,j,k}(x) > 0\}, \quad i = 1, \dots, p,$$

for some nonzero polynomials  $f_i, f_{i,j,k} \in \mathbb{Q}[\Lambda_T]$ . Denote by  $\epsilon_{i,j,k}$  the sign of  $f_{i,j,k}$  in the ordering  $>$ . Then  $\epsilon_{i,j,k} \neq 0$  and  $\epsilon_{i,j,k} f_{i,j,k} > 0$  for any  $i, j, k$ . Observe that for some  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, n\}$ , we have  $f_{i,j,k} > 0$  for  $k = 1, \dots, m$ . Indeed, in the opposite case,

$$\emptyset = \bigcap_{l=1}^s \bigcap_{i=1}^p \bigcap_{j=1}^n \bigcap_{k=1}^m \{x \in \mathbb{R}^T : g_l(x) > 0, \epsilon_{i,j,k} f_{i,j,k}(x) > 0\} \in \mathcal{U}_> ,$$

which is impossible. So, for some  $i_0 \in \{1, \dots, p\}$  and  $j_0 \in \{1, \dots, n\}$ ,

$$U' = \bigcap_{k=1}^m \{x \in \mathbb{R}^T : f_{i_0,j_0,k}(x) > 0\} \in \mathcal{U}_> ,$$

and  $U' \cap U_j = \emptyset$  for  $j \neq j_0$ . Hence, by Claim 1,  $S_U = U_{j_0}$  is the unique connected component of  $U$  satisfying Claim 2.

**Claim 3.** The family  $\Omega = \{S_U : U \in \mathcal{U}_>\}$  is a *c-filter*.

Since for every  $\mathbb{Q}$ -algebraic set  $V \subset \mathbb{R}^T$  there exists  $U \in \mathcal{U}_>$  such that  $U \cap V = \emptyset$ , we have  $S_U \cap V = \emptyset$ . Hence, it suffices to prove that for any  $S_{U_1}, S_{U_2} \in \Omega$ , there exists  $S_{U_3} \in \Omega$  contained in  $S_{U_1} \cap S_{U_2}$ . Indeed, by the argument of Claim 2, there exist  $W_1, W_2 \in \mathcal{U}_>$  such that  $W_1 \subset S_{U_1}$  and  $W_2 \subset S_{U_2}$ . Hence,  $S_{W_1 \cap W_2} \subset W_1 \cap W_2 \subset S_{U_1} \cap S_{U_2}$  and  $S_{W_1 \cap W_2} \in \Omega$ .

**Claim 4.** The *c-filter*  $\Omega$  defined in Claim 3 satisfies the assertion of [Proposition 2.4](#).

Part (a) is obvious.

Let  $U \in \mathcal{U}_{>}$  be of the form (1),  $f = g_1 \dots g_s$ , and  $V = f^{-1}(0)$ . Then, by the definition of  $S_U$ , we see that  $S_U$  is a connected component of  $\mathbb{R}^T \setminus V$ . This gives (b).

Let  $V = f^{-1}(0)$  be a  $\mathbb{Q}$ -algebraic subset of  $\mathbb{R}^T$ . Then  $U = \{x \in \mathbb{R}^T : f^2(x) > 0\} = \mathbb{R}^T \setminus V \in \mathcal{U}_{>}$  and  $S_U \in \Omega$  is a connected component of  $\mathbb{R}^T \setminus V$ . This gives (c) and completes the proof.  $\square$

We call the c-filter  $\Omega$  defined in Proposition 2.4 the *plain filter* for the ordering  $>$  and denote it by  $\Omega_{>}$ .

From Proposition 2.4, we immediately obtain:

**Corollary 2.5.** *The mapping  $> \mapsto \Omega_{>}$  is a one-to-one correspondence between the set of orderings of  $\mathbb{Q}(\Lambda_T)$  and the set of plain filters.*

**Remark 2.6.** From the ultrafilter theorem [Bröker 1982], we see that for any ultrafilter  $\mathcal{F}$  of subsets of  $\mathbb{R}^T$ , there exists a plain filter  $\Omega \subset \mathcal{F}$ .

**Remark 2.7.** It is easy to observe that the statements of this section hold with  $\mathbb{Q}$  replaced by  $\mathbb{R}$ .

### 3. Archimedean orderings in $\mathbb{Q}(\Lambda_T)$

Let  $>$  be an ordering of  $\mathbb{Q}(\Lambda_T)$ . Then one can assume that  $T$  is linearly ordered by

$$t_1 > t_2 \iff \Lambda_{t_1} > \Lambda_{t_2}.$$

If  $f > g$ , then we also write  $g < f$ .

**Theorem 3.1.** *The following conditions are equivalent:*

- (a) *The field  $(\mathbb{Q}(\Lambda_T), >)$  is Archimedean.*
- (b) *There exists  $x_{>} \in \partial\Omega_{>}$  such that the set of coordinates of  $x_{>}$  is algebraically independent over  $\mathbb{Q}$ .*
- (c) *There exists  $x_{>} \in \partial\Omega_{>}$  such that  $f > 0$  if and only if  $f(x_{>}) > 0$ .*
- (d) *There exists  $x_{>} \in \partial\Omega_{>}$  such that  $x_{>} \in U$  for any  $U \in \Omega_{>}$ .*

*Proof.* Assume (a). Then for any  $t_1, \dots, t_n \in T$  with  $t_1 < \dots < t_n$ , and for the projection  $\pi_{t_1, \dots, t_n} : \mathbb{R}^T \mapsto (x(t_1), \dots, x(t_n)) \in \mathbb{R}^n$ , the family

$$(2) \quad \Omega_{t_1, \dots, t_n} = \{\pi_{t_1, \dots, t_n}(U) : U \in \Omega\}$$

determines an Archimedean order in  $\mathbb{Q}(\Lambda_{t_1}, \dots, \Lambda_{t_n})$ . Thus for some  $W \in \Omega_{t_1, \dots, t_n}$ , the function  $f = \Lambda_{t_1}^2 + \dots + \Lambda_{t_n}^2$  is bounded on  $W$ . So the set  $W$  is bounded. Hence, by Proposition 2.1, there exists  $(x_1, \dots, x_n) \in \partial\Omega_{t_1, \dots, t_n}$ . Since the projections  $\pi_{t_1, \dots, t_n}$  are open, it is easy to observe that, for  $t_{k_1}, \dots, t_{k_j} \in \{t_1, \dots, t_n\}$  with  $t_{k_1} < \dots < t_{k_j}$ , we have  $(x_{k_1}, \dots, x_{k_j}) \in \partial\Omega_{t_{k_1}, \dots, t_{k_j}}$ . Consequently, there



exists  $x \in \mathbb{R}^T$  such that for any  $t_1, \dots, t_n \in T$  with  $t_1 < \dots < t_n$ , we have  $\pi_{t_1, \dots, t_n}(x) \in \partial\Omega_{t_1, \dots, t_n}$ . Summing up,  $x \in \partial\Omega$ . The set of coordinates of  $x$  is algebraically independent over  $\mathbb{Q}$ : otherwise,  $f(x) = 0$  for some nonzero polynomial  $f \in \mathbb{Q}[\Lambda_T]$ , and so  $f$  is infinitesimal. This contradicts (a) and gives (b).

Assume (b). Then any nonzero  $f \in \mathbb{Q}(\Lambda_T)$  with  $f > 0$  is defined at  $x_{>}$ . Moreover,  $f(x_{>}) \neq 0$ , so  $f(x_{>}) > 0$ . Conversely, assume that  $f(x_{>}) > 0$ . Then obviously for some connected component  $U$  of  $f^{-1}(0, +\infty)$ , we have  $U \in \Omega_{>}$  and  $f(x) > 0$  for  $x \in U$ . Summing up, we obtain (c).

The implication (c)  $\Rightarrow$  (d) is trivial.

Now assume (d). Then we immediately obtain (b), and hence, no  $f \in \mathbb{Q}(\Lambda_T)$  is infinitesimal, and the field  $(\mathbb{Q}(\Lambda_T), >)$  is Archimedean. This gives (a) and completes the proof.  $\square$

**Remark 3.2.** The assertion of [Theorem 3.1](#) also holds for every c-filter determining  $>$  in place of the plain filter  $\Omega_{>}$ .

[Theorem 3.1](#) implies:

**Corollary 3.3.** *Let  $T$  be a finite set. Then the set of Archimedean orderings of  $\mathbb{Q}(\Lambda_T)$  is a dense subset of the space of orderings in  $\mathbb{Q}(\Lambda_T)$  in the path topology (see, for instance, [\[Marshall 2008\]](#)) of the real spectrum  $\text{Sper}(\mathbb{Q}[\Lambda_T])$ .*

### 4. Examples of non-Archimedean orderings

Let  $m$  be a fixed positive integer and  $\Lambda$  a system of  $m$  variables  $\Lambda_1, \dots, \Lambda_m$ .

Take any  $P \in \mathbb{R}[\Lambda]$ . Let  $\Gamma_P \subset \mathbb{R}^m$  be a set defined by

$$\Gamma_P = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m : P(\lambda_1, \dots, \lambda_{m-1}, \lambda_m + \gamma) = 0 \text{ for some } \gamma \in [0, \infty)\}.$$

We define a polynomial  $\omega(P) \in \mathbb{R}[\Lambda_1, \dots, \Lambda_{m-1}]$  (or a number  $\omega(P) \in \mathbb{R}$  if  $m = 1$ ) by  $\omega(P) = 0$  for  $P = 0$ , and  $\omega(P) = P_0$  for  $P \neq 0$ , where

$$P = P_0\Lambda_m^d + P_1\Lambda_m^{d-1} + \dots + P_d$$

and  $P_i \in \mathbb{R}[\Lambda_1, \dots, \Lambda_{m-1}]$  (or  $P_i \in \mathbb{R}$  if  $m = 1$ ) for  $i = 0, \dots, d$  and  $P_0 \neq 0$ .

Let us define sets  $W_P \subset \mathbb{R}^m$ , for  $P \in \mathbb{R}[\Lambda]$ . The definition will be inductive with respect to the number of variables  $\Lambda_1, \dots, \Lambda_m$ . For  $P \in \mathbb{R}[\Lambda]$ , we put

$$(3) \quad W_P = \begin{cases} \mathbb{R} \setminus \Gamma_P \subset \mathbb{R} & \text{if } m = 1, \\ (\mathbb{R}^m \setminus \Gamma_P) \cap (W_{\omega(P)} \times \mathbb{R}) \subset \mathbb{R}^m & \text{if } m > 1. \end{cases}$$

By the Tarski–Seidenberg theorem — see [Proposition 1.1\(c\)](#) — the sets  $W_P$  are semialgebraic for all  $P \in \mathbb{R}[\Lambda]$ .

Analogously to [Theorem 1.1](#) of [\[Spodzieja 1996\]](#), we prove the following proposition, which gives an example of c-filter.

**Proposition 4.1.** *The family  $\mathcal{W} = \{W_P : P \in \mathbb{R}[\Lambda]\}$  satisfies these conditions:*

- $R_0.$   $W_P \subset \{\lambda \in \mathbb{R}^m : P(\lambda) \neq 0\}.$
- $R_1.$   $W_P \cap W_Q = W_{PQ}.$
- $R_2.$  *For  $P \neq 0$ ,  $W_P$  is an unbounded subset of  $\mathbb{R}^m$ .*
- $R_3.$  *For  $P \neq 0$ ,  $W_P$  is an open, connected and simply connected set.*

Moreover, one can demand that

- $R_4.$   $W_P = \mathbb{R}^m$  for  $P = \text{const}$ ,  $P \neq 0$ .

In particular,  $\mathcal{W}$  contains the  $c$ -filter

$$\Omega = \{W_P : P \in \mathbb{Q}[\Lambda]\}.$$

**Lemma 4.2.** *Let  $1 \leq i_1 < \dots < i_m \leq n$ , and let  $P \in \mathbb{R}[\Lambda_{i_1}, \dots, \Lambda_{i_m}]$ . Let  $Q \in \mathbb{R}[\Lambda_1, \dots, \Lambda_n]$  be a polynomial of the form*

$$(4) \quad Q(x_1, \dots, x_n) = P(x_{i_1}, \dots, x_{i_m}), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Then  $W_P \subset \mathbb{R}^m$ ,  $W_Q \subset \mathbb{R}^n$ , and

$$W_Q \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_{i_1}, \dots, x_{i_m}) \in W_P\}.$$

*Proof.* For  $P = 0$  or  $n = m$ , the assertion is trivial. Assume that  $P \neq 0$  and  $n > m$ . Consider the case  $n = m + 1$ . Then there exists  $1 \leq j \leq n$  such that

$$(\Lambda_{i_1}, \dots, \Lambda_{i_m}) = (\Lambda_1, \dots, \Lambda_{n-j}, \Lambda_{n-j+2}, \dots, \Lambda_n),$$

under the obvious convention for  $j = 1$  and  $j = n$ . Denote the  $i$ -th iteration of  $\omega$  by  $\omega^i$ , where  $\omega^0(P) = P$ . Then, for  $(x_1, \dots, x_{n-i}) \in \mathbb{R}^{n-i}$ ,

$$\omega^i(Q)(x_1, \dots, x_{n-i}) = \begin{cases} \omega^i(P)(x_1, \dots, x_{n-j}, x_{n-j+2}, \dots, x_{n-i}) & \text{if } 0 \leq i \leq j - 2, \\ \omega^i(P)(x_1, \dots, x_{n-j}) & \text{if } i = j - 1, \\ \omega^{i-1}(P)(x_1, \dots, x_{n-i}) & \text{if } j \leq i \leq n. \end{cases}$$

Hence,

$$\Gamma_{\omega^i(Q)} = \{(x_1, \dots, x_{n-i}) \in \mathbb{R}^{n-i} : (x_1, \dots, x_{n-j}, x_{n-j+2}, \dots, x_{n-i}) \in \Gamma_{\omega^i(P)}\}$$

for  $0 \leq i \leq j - 2$ , and

$$\Gamma_{\omega^{j-1}(Q)} = \{(x_1, \dots, x_{n-j+1}) \in \mathbb{R}^{n-j+1} : (x_1, \dots, x_{n-j}) \in \Gamma_{\omega^{j-1}(P)}\}$$

and  $\Gamma_{\omega^i(Q)} = \Gamma_{\omega^{i-1}(P)}$  for  $j \leq i \leq n$ . In particular,  $W_{\omega^i(Q)} = W_{\omega^{i-1}(P)}$  for  $j \leq i \leq n$ .

Summing up, by (3),

$$\begin{aligned}
 W_Q &= \bigcap_{i=0}^n [(\mathbb{R}^{n-i} \setminus \Gamma_{\omega^i(Q)}) \times \mathbb{R}^i] \\
 &= \bigcap_{i=0}^{j-2} \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, \dots, x_{n-j}, x_{n-j+2}, \dots, x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i(P)}\} \\
 &\quad \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, \dots, x_{n-j}) \in \mathbb{R}^{n-j} \setminus \Gamma_{\omega^{j-1}(P)}\} \cap [W_{\omega^j(Q)} \times \mathbb{R}^j] \\
 &\subset \bigcap_{i=0}^{j-2} \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, \dots, x_{n-j}, x_{n-j+2}, \dots, x_{n-i}) \in \mathbb{R}^{n-i-1} \setminus \Gamma_{\omega^i(P)}\} \\
 &\quad \cap [W_{\omega^{j-1}(P)} \times \mathbb{R}^j] \\
 &= \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_{i_1}, \dots, x_{i_m}) \in W_P\}.
 \end{aligned}$$

This gives the assertion for  $n = m + 1$ . Hence, by an easy induction with respect to  $n - m$ , we obtain the assertion. □

Let  $T$  be a linearly ordered set and let  $\succ$  be the ordering of  $T$ .

For any  $t_1, \dots, t_m \in T, t_1 < \dots < t_m$ , we define a projection map

$$\pi_{t_1, \dots, t_m} : \mathbb{R}^T \ni x \mapsto (x(t_1), \dots, x(t_m)) \in \mathbb{R}^m.$$

Define a family  $\Omega$  of semialgebraic subsets  $U$  of  $\mathbb{R}^T$  by

$$(5) \quad U = (\pi_{t_1, \dots, t_m})^{-1}(W_P),$$

where  $m \in \mathbb{N}, t_1, \dots, t_m \in T, t_1 < \dots < t_m$ , and  $P \in \mathbb{Q}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$ .

**Proposition 4.3.** *The family  $\Omega$  is a c-filter.*

*Proof.* By Proposition 4.1 (condition  $R_2$ ), any  $U \in \Omega$  is a nonempty set.

Let  $V \subsetneq \mathbb{R}^T$  be a  $\mathbb{Q}$ -algebraic set, and let  $P \in \mathbb{Q}[\Lambda_T] \setminus \{0\}$  be such that  $V = \{x \in \mathbb{R}^T : P(x) = 0\}$ . Then  $P \in \mathbb{Q}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$  for some  $t_1, \dots, t_m \in T, t_1 < \dots < t_m$ , and  $U = (\pi_{t_1, \dots, t_m})^{-1}(W_P)$ . Applying Proposition 4.1 (condition  $R_0$ ), we obtain that  $U$  satisfies (i).

Let  $U_1, U_2 \in \Omega$ . Let  $t_1, \dots, t_m, u_1, \dots, u_n \in T$  satisfy  $t_1 < \dots < t_m$  and  $u_1 < \dots < u_n$ , and assume moreover that for some  $P \in \mathbb{Q}[\Lambda_{t_1}, \dots, \Lambda_{t_m}]$  and  $Q \in \mathbb{Q}[\Lambda_{u_1}, \dots, \Lambda_{u_n}]$  we have  $U_1 = (\pi_{t_1, \dots, t_m})^{-1}(W_P)$  and  $U_2 = (\pi_{u_1, \dots, u_n})^{-1}(W_Q)$ . Let  $v_1, \dots, v_s \in T, v_1 < \dots < v_s$ , be such that  $\{t_1, \dots, t_m\} \cup \{u_1, \dots, u_n\} \subset \{v_1, \dots, v_s\}$ , and let  $\bar{P}, \bar{Q} \in \mathbb{R}[\Lambda_{v_1}, \dots, \Lambda_{v_s}]$  be polynomials of the form (4) determined by  $P$  and  $Q$ , respectively. Then, by Proposition 4.1 (condition  $R_1$ ) and Lemma 4.2,

$$(\pi_{v_1, \dots, v_s})^{-1}(W_{\bar{P}\bar{Q}}) = (\pi_{v_1, \dots, v_s})^{-1}(W_{\bar{P}}) \cap (\pi_{v_1, \dots, v_s})^{-1}(W_{\bar{Q}}) \subset U_1 \cap U_2.$$

This gives (ii).

Take any  $U \in \Omega$ . There exist  $t_1, \dots, t_m \in T$  and  $P \in \mathbb{R}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$  such that  $t_1 < \dots < t_m$  and  $U = (\pi_{t_1, \dots, t_m})^{-1}(W_P)$ . By [Proposition 4.1](#) (condition  $R_3$ ),  $U$  satisfies [\(iii\)](#). This completes the proof.  $\square$

From the definition of the family  $\Omega$ , we immediately obtain:

**Corollary 4.4.** *For any  $t_1, t_2 \in T$ , we have  $t_1 \succ t_2$  if and only if  $\Lambda_{t_1} \succ_{\Omega} \Lambda_{t_2}$ .*

Let  $Q \in \mathbb{Q}[\Lambda_T] \setminus \{0\}$  and let  $\Omega_Q$  be a family of semialgebraic subsets  $U$  of  $\mathbb{R}^T$  defined by

$$(6) \quad U = (\pi_{t_1, \dots, t_m})^{-1}(W_P \cap W_Q),$$

where  $m \in \mathbb{N}$ ,  $t_1, \dots, t_m \in T$ ,  $t_1 < \dots < t_m$ , and  $PQ \in \mathbb{Q}[\Lambda_{t_1}, \dots, \Lambda_{t_m}] \setminus \{0\}$ . By [Proposition 4.3](#), we have:

**Corollary 4.5.**  *$\Omega_Q$  is a  $c$ -filter.*

Let  $X \subset \mathbb{R}^T$  be an open semialgebraic set and let  $\hat{x} \in X$  be a point with rational coordinates. There exist  $t_1, \dots, t_k \in T$ ,  $t_1 < \dots < t_k$ , and an open semialgebraic set  $Y \subset \mathbb{R}^k$  such that  $X = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_k)) \in Y\}$ . Hence, there exists  $r > 0$  such that

$$B := \{x \in \mathbb{R}^T : \max_{i=1, \dots, k} |x(t_i) - \hat{x}(t_i)| < r\} \subset X.$$

Let

$$P_0 = \Lambda_{t_1} \dots \Lambda_{t_k} (\Lambda_{t_1}^2 + \dots + \Lambda_{t_k}^2 - 1/r^2),$$

let  $U_0 = (\pi_{t_1, \dots, t_k})^{-1}(W_{P_0})$ , and let  $F : U_0 \rightarrow \mathbb{R}^T$  be a mapping defined by

$$F(x)(t) = \begin{cases} \hat{x}(t) + 1/x(t) & \text{for } x \in U_0, t \in \{t_1, \dots, t_k\}, \\ x(t) & \text{for } x \in U_0, t \in T \setminus \{t_1, \dots, t_k\}. \end{cases}$$

**Proposition 4.6.**  *$\{F(U) : U \in \Omega_{P_0}\}$  is a  $c$ -filter subset of  $X$ . In particular, for any open semialgebraic set  $Y \subset \mathbb{R}^T$ , there exists  $c$ -filter subsets of  $Y$ .*

*Proof.* By [Lemma 4.2](#), any set  $U \in \Omega_{P_0}$  is a subset of  $U_0$ . Moreover,  $F$  is an open semialgebraic mapping, so  $F(U)$  is semialgebraic for  $U \in \Omega_{P_0}$ . Hence,  $\{F(U) : U \in \Omega_{P_0}\}$  satisfies conditions [\(i\)](#)–[\(iii\)](#).  $\square$

From [Proposition 4.6](#) and [Theorem 3.1](#), we have that:

**Corollary 4.7.** *The set of  $c$ -filters defined in [Proposition 4.6](#) is a dense subset of the space of orderings in  $\mathbb{Q}[\Lambda_T]$  in the path topology of the real spectrum  $\text{Sper}(\mathbb{Q}[\Lambda_T])$ . Moreover, any ordering determined by such a  $c$ -filter is not Archimedean.*

**Remark 4.8.** It is easy to see that the results of this section hold if we replace  $\mathbb{Q}$  by  $\mathbb{R}$ .

### 5. Fields of Nash functions

Let  $T$  be a nonempty set. We denote by  $\mathcal{N}(X)$  the domain of  $\mathbb{Q}$ -Nash functions on an open connected semialgebraic set  $X \subset \mathbb{R}^T$ .

Let  $\succ$  be an ordering in  $\mathbb{Q}(\Lambda_T)$  and let  $\Omega_\succ$  be the plain filter of subsets of  $\mathbb{R}^T$  determining  $\succ$ . Let us introduce in  $\bigcup_{U \in \Omega_\succ} \mathcal{N}(U)$  a relation  $\sim_\succ$  by

$$(f_1 : U_1 \rightarrow \mathbb{R}) \sim_\succ (f_2 : U_2 \rightarrow \mathbb{R}) \iff \exists U \in \Omega_\succ (U \subset U_1 \cap U_2 \text{ and } f_1|_U = f_2|_U).$$

From Proposition 2.4, we immediately see that  $\sim_\succ$  is an equivalence relation. The equivalence class of  $\sim_\succ$  determined by  $f : U \rightarrow \mathbb{R}$  is denoted by  $[f]_\succ$ , and the set of all such classes by  $\mathcal{N}_\succ$ . The set  $\mathcal{N}_\succ$  is linearly ordered by

$$[f]_\succ \succ 0 \iff \exists U \in \Omega_\succ (f \in \mathcal{N}(U) \text{ and } f(x) > 0 \text{ for } x \in U).$$

**Proposition 5.1.** *The set  $\mathcal{N}_\succ$ , together with the usual operations*

$$[f_1]_\succ + [f_2]_\succ = [f_1|_U + f_2|_U]_\succ, \quad [f_1]_\succ \cdot [f_2]_\succ = [f_1|_U f_2|_U]_\succ,$$

where  $f_1 \in \mathcal{N}(U_1)$ ,  $f_2 \in \mathcal{N}(U_2)$ , and  $U \in \Omega_\succ$ ,  $U \subset U_1 \cap U_2$ , is a real field.

*Proof.* Since the ring  $\mathcal{N}(U)$  is a domain for any  $U \in \Omega_\succ$ , so is  $\mathcal{N}_\succ$ . We prove that any nonzero  $f \in \mathcal{N}_\succ$  has an inverse in  $\mathcal{N}_\succ$ . Indeed, there exists  $U \in \Omega_\succ$  such that  $f \in \mathcal{N}(U)$ . Since  $f \neq 0$ , the set  $f^{-1}(0)$  is contained in some proper  $\mathbb{Q}$ -algebraic subset of  $\mathbb{R}^T$ . Then, by the definition of c-filter, one can assume that  $f(\lambda) \neq 0$  for any  $\lambda \in U$ . Thus  $1/f \in \mathcal{N}(U)$ , so  $f$  has an inverse in  $\mathcal{N}_\succ$ . Summing up,  $\mathcal{N}_\succ$  is a field. Since  $-1 \in \mathcal{N}(U)$  is not a sum of squares in  $\mathcal{N}(U)$ , it follows that  $-1 \in \mathcal{N}_\succ$  is not a sum of squares in  $\mathcal{N}_\succ$ . □

**Theorem 5.2.** *The field  $\mathcal{N}_\succ$  is a real closure of the field  $(\mathbb{Q}(\Lambda_T), \succ)$ .*

*Proof.* Take any irreducible polynomial  $P \in \mathcal{N}_\succ[Z]$  of odd degree  $d$  with respect to  $Z$ . Then there exists  $U \in \Omega_\succ$  such that  $P \in \mathcal{N}(U)[Z]$ . Let  $t_1, \dots, t_m \in T$ , and let  $\tilde{U} \subset \mathbb{R}^m$  be an open connected semialgebraic set such that  $U = \{x \in \mathbb{R}^T : (x(t_1), \dots, x(t_m)) \in \tilde{U}\}$ . By using the Hermite method (for  $\tilde{U}$ ) we deduce that there exists a decomposition  $U = U_1 \cup \dots \cup U_k \cup V$  of  $U$  into disjoint open basic  $\mathbb{Q}$ -semialgebraic sets  $U_1, \dots, U_k$  and a semialgebraic set  $V$  included in an algebraic set such that  $P(x, Z)$  has the same number of zeroes for all  $x \in U_i$  and each of these zeroes is single. By (i) and (ii) in the definition of a c-filter, there exists  $U' \in \Omega_\succ$  such that  $U' \subset U_i$  for some  $i \in \{1, \dots, k\}$ . Then there exists  $k \in \mathbb{N}$ ,  $k > 0$  such that  $P(x, Z)$  has exactly  $k$  zeroes for  $x \in U'$ , and so there exist functions  $\xi_1, \dots, \xi_k : U' \rightarrow \mathbb{R}$  with  $\xi_1(x) < \dots < \xi_k(x)$  such that  $P(x, \xi_i(x)) = 0$  for  $x \in U'$ ,  $i = 1, \dots, k$ . As  $\xi_i(x)$  are single zeroes of  $P(x, Z)$ , by the Implicit Function Theorem,  $\xi_i$  is a Nash function for  $i = 1, \dots, k$ . As  $\mathcal{N}_\succ$  is a real field

(**Proposition 5.1**),  $\mathcal{N}_>$  is a real closed field. Since  $\mathcal{N}_>$  is an algebraic extension of  $\mathbb{Q}(\Lambda_T)$ , by the Artin–Schreier Theorem, it is a real closure of  $(\mathbb{Q}(\Lambda_T), >)$ .  $\square$

**Remark 5.3.** The above results of this section also hold for an arbitrary  $c$ -filter determining  $>$  in place of the plain filter  $\Omega_>$ . The results also hold if we put  $\mathbb{R}$  in place of  $\mathbb{Q}$ .

From Theorems 3.1 and 5.2, we recover the familiar result that any Archimedean field can be embedded in  $\mathbb{R}$ .

**Corollary 5.4.** *Let  $\Omega_>$  be a plain filter of subsets of  $\mathbb{R}^T$  determining an Archimedean ordering  $>$  of  $\mathbb{Q}(\Lambda_T)$ , and let  $x_> \in \bigcap_{U \in \Omega_>} U$ . Then the mapping*

$$\mathcal{N}_> \ni f \mapsto f(x_>) \in \mathbb{R}$$

*is an order-preserving monomorphism.*

From **Theorem 5.2**, we immediately obtain:

**Corollary 5.5.** *Let  $R$  be a real closed field with ordering  $>$ , and let  $T$  be the transcendence basis of  $R$  over  $\mathbb{Q}$  whose existence is guaranteed by the Kuratowski–Zorn lemma. Assume that  $T \neq \emptyset$  and let  $\Lambda_T = (\Lambda_t : t \in T)$  be a system of independent variables. Then the field  $R$  is order-preserving isomorphic to a real closure of the rational functions field  $\mathbb{Q}(\Lambda_T)$ , i.e., to some field  $\mathcal{N}_>$ .*

**Remark 5.6.** Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero. Then  $\mathbb{K} = R[i]$ , where  $i^2 = -1$ , for some real closed field  $R$ . Let  $T \subset R$  be the transcendence basis of  $\mathbb{K}$  over  $\mathbb{Q}$ . Assume that  $T \neq \emptyset$ . Then  $\mathbb{K}$  is isomorphic to an algebraic closure of  $\mathbb{Q}(\Lambda_T)$ . By Theorem 1.1 of [Spodzieja 1996], one can introduce a filter  $\Omega_{\mathbb{C}}$  of open, connected, and simply connected semialgebraic subsets  $U$  of  $\mathbb{C}^T$  satisfying conditions (i), (ii), and (iii). Then, analogously to [Spodzieja 1996], one can introduce a geometric construction of the algebraic closure of  $\mathbb{Q}(\Lambda_T)$  in terms of complex Nash functions.

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