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LINEAR RESTRICTION ESTIMATES FOR THE WAVE EQUATION WITH AN INVERSE SQUARE POTENTIAL

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We study modified linear restriction estimates associated with the wave equation with an inverse square potential. In particular, we show that the classical linear restriction estimates hold in their almost sharp range when the initial data is radial.

1. Introduction and statement of main result

In this paper, we study a modified restriction estimate associated with the wave equation perturbed by an inverse square potential. More precisely, we consider the following wave equation with a singular potential:

$$\begin{cases} \partial_t^2 u - \Delta u + \dfrac{a}{|x|^2} u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ a\in\mathbb{R}, \\ u(t,x)|_{t=0} = 0, & \partial_t u(t,x)|_{t=0} = f(x). \end{cases} \tag{1-1}$$

The scale-covariant elliptic operator $P_a := -\Delta + a/|x|^2$ appearing in (1-1) plays a key role in many problems of physics and geometry. The heat and Schrödinger flows for the elliptic operator P_a have been studied in the theory of combustion [Vazquez and Zuazua 2000] and in quantum mechanics [Kalf et al. 1975]. The wave equation (1-1) arises in the study of the wave propagation on conic manifolds [Cheeger and Taylor 1982]. There has been a lot of interest in developing Strichartz estimates both for the Schrödinger and wave equations with the inverse square potential; we refer the reader to [Burq et al. 2003; 2004; Planchon et al. 2003b; 2003a; Miao et al. 2013b]. However, as far as we know, there are few results about the restriction estimates associated with the operator P_a arising in the study of eigenfunction estimates of P_a. Here, we address some restriction issues in special settings associated with the operator P_a.

In the case $a = 0$ — the linear wave equation with no potential — we can solve

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the equation by the Fourier transform formula

$$u(t,x)=\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}f=\frac{1}{2i}\int_{\mathbb{R}^n}e^{2\pi i x\cdot\xi}\bigl(e^{2\pi i t|\xi|}-e^{-2\pi i t|\xi|}\bigr)\hat{f}(\xi)\frac{d\xi}{|\xi|},\tag{1-2}$$

where the Fourier transform is defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^n}e^{-2\pi i x\cdot\xi}f(x)\,dx.$$

It is well known that the spacetime norm estimate of $u(t,x)$ is connected with the linear adjoint cone restriction estimate

$$\|(F\,d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)}\le C_{p,q,n,S}\|F\|_{L^p(S,d\sigma)},\tag{1-3}$$

where F is a Schwartz function and the inverse spacetime Fourier transform of the measure $F\,d\sigma$ is defined by

$$(F\,d\sigma)^\vee(t,x)=\int_S F(\tau,\xi)e^{2\pi i(x\cdot\xi+t\tau)}\,d\sigma(\xi)=\int_{\mathbb{R}^n}F(|\xi|,\xi)e^{2\pi i(x\cdot\xi+t|\xi|)}\frac{d\xi}{|\xi|}.$$

Here, the set S is a nonempty smooth compact subset of the cone

$$\{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n:\tau=|\xi|\}\quad\text{with}\quad n\ge 2.$$

The canonical measure $d\sigma$ is the pull back of the measure $d\xi/|\xi|$ under the projection map $(\tau,\xi)\mapsto\xi$. By the decay of $(d\sigma)^\vee$ and the Knapp counterexample, the two necessary conditions for (1-2) are

$$q>\frac{2n}{n-1}\quad\text{and}\quad\frac{n+1}{q}\le\frac{n-1}{p'}\tag{1-4}$$

(see [Stein 1979; Tao 2003a]). The corresponding linear adjoint restriction conjecture for cones asserts that:

Conjecture 1.1. *The inequality* (1-3) *holds with constants depending on n, p, q, and S if and only if the inequalities* (1-4) *are satisfied.*

Even though there is a large amount of literature focused on this problem, it remains open for $n\ge 4$. For progress on this conjecture, we refer the readers to [Taberner 1985; Strichartz 1977; Tao 2001; 2003a; 2003b; Tao et al. 1998; Wolff 2001]. Shao [2009a] provided two simple and novel arguments to prove that Conjecture 1.1 holds true for the spatial rotation invariant functions which are supported on the cone. Motivated by [Shao 2009a], Miao et al. [2012] utilized expansions in spherical harmonics and analyzed the asymptotic behavior of the Bessel function to generalize Shao's result for cone cases by establishing, assuming (1-4), that

$$\|(F\,d\sigma)^\vee\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}dr}L^2_\theta(\mathbb{S}^{n-1}))}\le C_{p,q,n,S}\|F\|_{L^p(S,d\sigma)}.$$

In the case $a \neq 0$, the spacetime Fourier transform is no longer so useful; one can instead establish an approximate parametrix for the fundamental solution and try to obtain good control over it. In our case we resort to expansions in spherical harmonics and Hankel transforms, for technical reasons involving the singular potential; compare [Burq et al. 2003; Planchon et al. 2003b; Miao et al. 2013b]. Although the harmonic expansion expression leads to some loss of angular regularity in the restriction estimates, it allows us to show the restriction estimates when q is close to $2n/(n-1)$. A key ingredient in this process is to explore the oscillatory properties of the Bessel function and $e^{it|\xi|}$ to overcome the difficulties arising from the low decay of the Bessel function $J_\nu(r)$ when $\nu \sim r$. Finally, by using the properties of the hypergeometric function shown in [Planchon et al. 2003b], we prove an inequality involving the Hankel transform to obtain the desired result.

Main Theorem. *Assume $n \geq 2$ and $a > -\frac{1}{4}(n-2)^2$, and let u be the solution of* (1-1). *Suppose that $p > 1$ and*

$$q = \frac{p'(n+1)}{n-1} > \frac{2n}{n-1}. \tag{1-5}$$

Then there exists a constant C, depending only on p, q, n, and a, satisfying the following conditions:

(i) *If f is a radial Schwartz function*, *then*

$$\|u(t,x)\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \leq C \left\| |\xi|^{-\frac{1}{p}} \hat{f} \right\|_{L^p(\mathbb{R}^n)}. \tag{1-6}$$

(ii) *If f is a Schwartz function* (*may not be radial*) *and $p \geq 2$*, *then*

$$\|u(t,x)\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}\,dr}L^2_\theta(\mathbb{S}^{n-1}))} \leq C \left\| |\xi|^{-\frac{1}{p}} \hat{f} \right\|_{L^p(\mathbb{R}^n)}. \tag{1-7}$$

Remarks. (i) This extends the classical restriction estimate associated with the Laplace operator to a restriction estimate associated with $-\Delta + a/|x|^2$. We obtain more estimates than the Strichartz estimates of [Burq et al. 2003; Planchon et al. 2003b], which focus on $p = 2$. The theorem can also be viewed as an extension of the result in [Chen et al. 2012] about the operator $-\Delta + a/r^2$ acting on $L^2\big((0,\infty); r^{n-1}\,dr\big)$.

(ii) The theorem means that we almost show that the classical linear restriction estimates hold for radial functions in the conjecture range.

(iii) When $a = 0$, we recover the cone restriction result in [Shao 2009b]. When supp $\hat{f}$ is compact, we can extend the result to $q \geq p'(n+1)/(n-1)$, which is the same range as in the cone restriction conjecture.

(iv) Equation (1-6) gives a Strichartz-type estimate

$$\|u(t,x)\|_{L^{2(n+1)/(n-1)}_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \leq C \left\| |\nabla|^{-\frac{1}{2}} f \right\|_{L^2(\mathbb{R}^n)}$$

for the radial solution. The method used here generalizes the result for the radial initial data to a linear finite combination of products of the Hankel transform of radial functions and spherical harmonics. We hope to remove the whole radial assumption in (1-6) in the future, at least for $q \geq 2(n+3)/(n+1)$.

(v) If $\hat{f} \subset \{\xi : N \leq |\xi| \leq 2N\}$ and f is radial, the method here can be employed to obtain the Strichartz estimate

$$\|u(t,x)\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \leq CN^{\frac{n-2}{2}-\frac{n+1}{q}}\|f\|_{L^2(\mathbb{R}^n)} \quad \text{for } q > \frac{2n}{n-1}.$$

We remark that the pair (q,q) is allowed to be out of the admissible requirement in [Planchon et al. 2003b], it is however consistent with the admissible range due to [Miao et al. 2013b].

(vi) We rely heavily on the harmonic expansion formula to give the expression of the solution due to the potential, which causes the restriction $p \geq 2$. It is possible that the resolvent expression can be used to remove this restriction.

Now we introduce some notation. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some constant C, which may vary from line to line and depend on various parameters. We write $A \sim B$ to mean that $A \lesssim B \lesssim A$.

If the constant C depends on parameters other than p, q, n, and S, we denote this fact explicitly using subscripts. For instance, C_ϵ should be understood as a positive constant depending on ϵ in addition to (possibly) p, q, n, and S.

Pairs of conjugate indices are written as p and p', where

$$\frac{1}{p}+\frac{1}{p'} = 1 \quad \text{and} \quad 1 \leq p \leq \infty.$$

This paper is organized as follows: In Section 2, we present some simple facts about the Hankel transforms and the Bessel functions and also recall the van der Corput lemma. Section 3 is devoted to the proof of the Main Theorem via expansions in spherical harmonics and an analysis of the asymptotic behavior of the Bessel function. Finally, in an Appendix, we show an inequality used in Section 3, involving the Hankel transforms.

2. Preliminaries

Before turning to Hankel transforms and Bessel functions, we recall the expansion formula in spherical harmonics. For details, refer to [Stein and Weiss 1971]. For convenience, we write

$$\xi = \rho\omega \quad \text{and} \quad x = r\theta \quad \text{with } \omega, \theta \in \mathbb{S}^{n-1}.$$

We denote by $\mathcal{H}^k$ the space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$, whose dimension is given by

$$d(0)=1, \qquad d(k)=\frac{2k+n-2}{k}C^{k-1}_{n+k-3}\simeq \langle k\rangle^{n-2} \quad \text{for } k>0.$$

Note that if $n=2$ this dimension is 2 for all k.

Any $g\in L^2(\mathbb{R}^n)$ can be expanded in spherical harmonics as

$$g(x)=\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} a_{k,l}(r)Y_{k,l}(\theta), \tag{2-1}$$

where $\{Y_{k,1},\dots,Y_{k,d(k)}\}$ is an orthogonal basis of $\mathcal{H}^k$. We have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1})=\bigoplus_{k=0}^{\infty}\mathcal{H}^k,$$

and by orthogonality,

$$\|g(x)\|_{L^2_\theta}=\|a_{k,l}(r)\|_{l^2_{k,l}}. \tag{2-2}$$

The Hankel transform formula (see Theorem 3.10 in [Stein and Weiss 1971], for instance) relates the Fourier transform of g to spherical harmonics. It reads

$$\hat g(\rho\omega)=\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} 2\pi i^k Y_{k,l}(\omega)\rho^{-\frac{n-2}{2}}\int_0^\infty J_{k+\frac{n-2}{2}}(2\pi r\rho)a_{k,l}(r)r^{\frac{n}{2}}\,dr. \tag{2-3}$$

Here the Bessel function $J_k(r)$ of order k is defined by

$$J_k(r)=\frac{\left(\frac{r}{2}\right)^k}{\Gamma\left(k+\frac12\right)\Gamma\left(\frac12\right)}\int_{-1}^{1} e^{isr}\left(1-s^2\right)^{\frac{2k-1}{2}}\,ds, \quad \text{with } k>-\tfrac12 \text{ and } r>0.$$

A simple computation gives the rough estimate

$$|J_k(r)|\le \frac{Cr^k}{2^k\Gamma\left(k+\frac12\right)\Gamma\left(\frac12\right)}\left(1+\frac{1}{k+\frac12}\right), \tag{2-4}$$

where C is an absolute constant. These estimates will be mainly used when $r\lesssim 1$. Another well known asymptotic expansion about the Bessel function is

$$J_k(r)=r^{-\frac12}\sqrt{\frac{2}{\pi}}\cos\left(r-\frac{k\pi}{2}-\frac{\pi}{4}\right)+O_k(r^{-3/2}) \quad \text{as } r\to\infty, \tag{2-5}$$

but with a constant depending on k (see [Stein and Weiss 1971]). As pointed out in [Stein 1993], if one seeks a uniform bound for large r and k, then the best one can do is $|J_k(r)|\le Cr^{-\frac13}$. To investigate the asymptotic behavior in k and r, we recall

Schläfli's integral representation [Watson 1944] of the Bessel function. For $r \in \mathbb{R}^+$ and $k > -\frac{1}{2}$,

$$J_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ir\sin\theta - ik\theta}\,d\theta - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-(r\sinh s + ks)}\,ds \\ =: \tilde{J}_k(r) - E_k(r). \tag{2-6}$$

We remark that $E_k(r) = 0$ when $k \in \mathbb{Z}^+$. A simple computation gives, for $r > 0$,

$$|E_k(r)| = \left|\frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-(r\sinh s + ks)}\,ds\right| \le C(r+k)^{-1}. \tag{2-7}$$

Next, we recall some properties of the Bessel function $J_k(r)$ from [Stein 1993; Stempak 2000]; see [Miao et al. 2013a] for a detailed proof.

Lemma 2.1 (asymptotics of the Bessel function). *Assume $k \gg 1$. Let $J_k(r)$ be the Bessel function of order k defined as above. Then there exist a large constant C and a small constant c independent of k and r such that*

$$|J_k(r)| \le Ce^{-c(k+r)} \qquad \text{when } r \le k/2, \tag{2-8}$$

$$|J_k(r)| \le Ck^{-\frac{1}{3}}\big(k^{-\frac{1}{3}}|r-k|+1\big)^{-\frac{1}{4}} \qquad \text{when } k/2 \le r \le 2k, \tag{2-9}$$

$$J_k(r) = r^{-\frac{1}{2}}\sum_{\pm} a_\pm(r)e^{\pm ir} + E(r) \qquad \text{when } r \ge 2k, \tag{2-10}$$

where $|a_\pm(r)| \le C$ and $|E(r)| \le Cr^{-1}$.

We define

$$\mu(k) = \frac{n-2}{2} + k, \quad \nu(k) = \sqrt{\mu^2(k) + a} \quad \text{with } a > -\frac{(n-2)^2}{4}. \tag{2-11}$$

For the sake of simplicity, we sometimes write ν instead of $\nu(k)$. Let g be a Schwartz function defined on $\mathbb{R}^n$. We define the Hankel transform of order ν:

$$(\mathcal{H}_\nu g)(\rho\omega) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) g(r\omega) r^{n-1}\,dr. \tag{2-12}$$

If the function g is radial, we can drop the dependence on ω from both sides.

We remark that if g has the expansion (2-1), it follows from (2-3) that

$$\hat{g}(\xi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d(k)} 2\pi i^k Y_{k,l}(\omega)(\mathcal{H}_{\mu(k)} a_{k,l})(\rho). \tag{2-13}$$

The following properties of the Hankel transform are proved in [Burq et al. 2003; Planchon et al. 2003b]:

Lemma 2.2. *Let $\mathcal{H}_\nu$ be as above and let*

$$A_{\nu(k)} := -\partial_r^2 - \frac{n-1}{r}\partial_r + \Big(\nu^2(k) - \Big(\frac{n-2}{2}\Big)^2\Big) r^{-2}.$$

(i) $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$.

(ii) $\mathcal{H}_\nu$ *is self-adjoint*: $\mathcal{H}_\nu = \mathcal{H}_\nu^*$.

(iii) $\mathcal{H}_\nu$ *is an* L^2 *isometry*: $\|\mathcal{H}_\nu \phi\|_{L^2_\xi} = \|\phi\|_{L^2_x}$.

(iv) $\mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi)$, *for* $\phi \in L^2$.

We conclude this section by recalling van der Corput's lemma [Stein 1993]:

Lemma 2.3. *Let ϕ be a smooth real-valued function defined on an interval $[a, b]$, and assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a, b]$. Assume moreover that either $k \geq 2$, or $k = 1$ and $\phi'(x)$ is monotonic. Then*

$$\left| \int_a^b e^{i\lambda\phi(x)}\, dx \right| \leq c_k \lambda^{-\frac{1}{k}}, \tag{2-14}$$

with c_k independent of ϕ and λ.

3. Proof of the main theorem

In this section, we will use the asymptotic properties of the Bessel function and the stationary phase argument to establish two estimates for the Hankel transform. A key ingredient is to effectively exploit the oscillatory property of the Bessel function and $e^{it|\xi|}$ to obtain more decay.

The Hankel transform and the solution. Let us consider (1-1) in polar coordinates. Write $v(t, r, \theta) = u(t, r\theta)$ and $g(r, \theta) = f(r\theta) = f(x)$. Then $v(t, r, \theta)$ satisfies

$$\begin{cases} \partial_{tt} v - \partial_{rr} v - \dfrac{n-1}{r}\partial_r v - \dfrac{1}{r^2}\Delta_\theta v + \dfrac{a}{r^2} v = 0, \\ v(0, r, \theta) = 0, \quad \partial_t v(0, r, \theta) = g(r, \theta). \end{cases} \tag{3-1}$$

We use the spherical harmonic expansion to write

$$g(r, \theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(r) Y_{k,l}(\theta). \tag{3-2}$$

Using separation of variables, we can write v as a superposition

$$v(t, r, \theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} v_{k,l}(t, r) Y_{k,l}(\theta), \tag{3-3}$$

where $v_{k,l}$ satisfies

$$\begin{cases} \partial_{tt} v_{k,l} - \partial_{rr} v_{k,l} - \dfrac{n-1}{r} \partial_r v_{k,l} + \dfrac{k(k+n-2)+a}{r^2} v_{k,l} = 0, \\ v_{k,l}(0,r) = 0, \quad \partial_t v_{k,l}(0,r) = a_{k,l}(r) \end{cases} \tag{3-4}$$

for each $k, l \in \mathbb{N}$, $1 \le l \le d(k)$. Define

$$A_{\nu(k)} := -\partial_r^2 - \frac{n-1}{r}\partial_r + \frac{\nu^2(k) - \left(\frac{n-2}{2}\right)^2}{r^2}. \tag{3-5}$$

Then we are reduced to considering the system

$$\begin{cases} \partial_{tt} v_{k,l} + A_{\nu(k)} v_{k,l} = 0, \\ v_{k,l}(0,r) = 0, \quad \partial_t v_{k,l}(0,r) = a_{k,l}(r). \end{cases} \tag{3-6}$$

Applying the Hankel transform to (3-6), we have, by Lemma 2.2,

$$\begin{cases} \partial_{tt} \tilde{v}_{k,l} + \rho^2 \tilde{v}_{k,l} = 0, \\ \tilde{v}_{k,l}(0,\xi) = 0, \quad \partial_t \tilde{v}_{k,l}(0,\xi) = b_{k,l}(\rho), \end{cases} \tag{3-7}$$

where

$$\tilde{v}_{k,l}(t,\rho) = (\mathcal{H}_\nu v_{k,l})(t,\rho), \quad b_{k,l}(\rho) = (\mathcal{H}_\nu a_{k,l})(\rho). \tag{3-8}$$

Solving this ODE and using the Hankel transform, we obtain

$$\begin{aligned} v_{k,l}(t,r) &= \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) \tilde{v}_{k,l}(t,\rho) \rho^{n-2}\, d\rho \\ &= \frac{1}{2i} \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) \big(e^{it\rho} - e^{-it\rho}\big) b_{k,l}(\rho) \rho^{n-2}\, d\rho. \end{aligned}$$

Therefore, we get

$$\begin{aligned} u(x,t) &= v(t,r,\theta) \\ &= \sum_{k=0}^\infty \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) \sin(t\rho) b_{k,l}(\rho) \rho^{n-2}\, d\rho \\ &= \sum_{k=0}^\infty \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \mathcal{H}_{\nu(k)} \big[\rho^{-1} \sin(t\rho) b_{k,l}(\rho)\big](r). \end{aligned} \tag{3-9}$$

Estimates of Hankel transforms. We now turn to some key estimates needed for proving the main theorem.

Proposition 3.1. *Let $R \gg 1$ and let φ be a smooth function supported in the interval $I := \left[\frac{1}{2}, 1\right]$ and taking values in* $[0, 1]$. *Then*

$$\left\| \left(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left| \int_0^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho)\, d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_t(\mathbb{R}; L^2_r([R,2R]))} \leq C \left\| \left(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_\rho(I)}, \tag{3-10}$$

where C is a constant independent of R.

Proof. Using the Plancherel theorem in t, we have

$$\text{LHS of (3-10)} \lesssim \left\| \left(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \| J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) \|^2_{L^2_\rho} \right)^{\frac{1}{2}} \right\|_{L^2_r([R,2R])}. \tag{3-11}$$

With this, it is easy to verify (3-10) if we can prove that

$$\int_R^{2R} |J_k(r)|^2\, dr \leq C, \tag{3-12}$$

where $R \gg 1$ and C is independent of k and R. To prove (3-12), we write

$$\int_R^{2R} |J_k(r)|^2\, dr = \int_{I_1} |J_k(r)|^2\, dr + \int_{I_2} |J_k(r)|^2\, dr + \int_{I_3} |J_k(r)|^2\, dr, \tag{3-13}$$

where

$$\begin{aligned} I_1 &= [R, 2R] \cap [0, k/2], \\ I_2 &= [R, 2R] \cap [k/2, 2k], \\ I_3 &= [R, 2R] \cap [2k, \infty]. \end{aligned}$$

By using (2-8) and (2-10) in Lemma 2.1, we have

$$\int_{I_1} |J_k(r)|^2\, dr \leq C \int_{I_1} e^{-cr} r\, dr \leq Ce^{-cR}, \quad \int_{I_3} |J_k(r)|^2\, dr \leq C. \tag{3-14}$$

For the remaining interval, we write

$$\int_{I_2} |J_k(r)|^2\, dr \leq \int_{\frac{k}{2}}^{2k} |J_k(r)|^2\, dr \leq C \int_{\frac{k}{2}}^{2k} k^{-\frac{2}{3}} \left(1 + k^{-\frac{1}{3}} |r - k|\right)^{-\frac{1}{2}}\, dr \leq C,$$

where the last inequality follows from the fact that the integral is uniformly bounded (by $2 + \sqrt{2}$) for all $k > 0$. Together with (3-14), this yields (3-12). □

Proposition 3.2. *Suppose $R \gg 1$. Let φ be a smooth function supported in the interval $I := \left[\frac{1}{2}, 1\right]$ and taking values in* $[0, 1]$.

(i) *If K is finite, there exists a constant C_K independent of R such that*

$$\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|\int_0^\infty e^{-it\rho}J_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\,d\rho\right|^2\right)^{\frac{1}{2}}\right\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))} \le C_K R^{-\frac{1}{2}}\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}|b_{k,l}(\rho)|^2\right)^{\frac{1}{2}}\right\|_{L_\rho^1(I)}. \tag{3-15}$$

(ii) *If K is infinite, there exists a constant C independent of R such that*

$$\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|\int_0^\infty e^{-it\rho}J_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\,d\rho\right|^2\right)^{\frac{1}{2}}\right\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))} \le C R^{-\frac{1}{2}}\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}|b_{k,l}(\rho)|^2\right)^{\frac{1}{2}}\right\|_{L_\rho^2(I)}. \tag{3-16}$$

Proof. We first prove (3-15). Recalling (2-5) we can write $|J_{\nu(k)}(r)| \le C_K r^{-\frac{1}{2}}$ when $r \gg 1$. By the Minkowski inequality and the Hausdorff–Young inequality in t, there exists a constant C_K independent of R such that

$$\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|\int_0^\infty e^{-it\rho}J_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\,d\rho\right|^2\right)^{\frac{1}{2}}\right\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))} \le C_K R^{-\frac{1}{2}}\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|b_{k,l}(\rho)\right|^2\right)^{\frac{1}{2}}\right\|_{L_\rho^1(I)}.$$

This proves (3-15). When K is infinite, we need to show a precise estimate uniform in K. We utilize the Schläfli's integral representation of the Bessel function (2-6) to write $J_{\nu(k)}(r\rho) = E_{\nu(k)}(r\rho) + \tilde{J}_{\nu(k)}(r\rho)$. By (2-7), the Minkowski inequality, and the Hausdorff–Young inequality in t, there exists a constant C independent of K and R such that

$$\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|\int_0^\infty e^{-it\rho}E_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\,d\rho\right|^2\right)^{\frac{1}{2}}\right\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))} \le C R^{-1}\left\|\left(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\left|b_{k,l}(\rho)\right|^2\right)^{\frac{1}{2}}\right\|_{L_\rho^1(I)}.$$

Thus, it remains to prove (3-16) with $J_{\nu(k)}$ replaced by $\tilde{J}_{\nu(k)}$. We consider $0<\delta\ll 1$ to be fixed later, and write $[-\pi,\pi] = I_1 \cup I_2 \cup I_3$, with

$$\begin{aligned} I_1 &= \{\theta : |\theta| \le \delta\}, \\ I_2 &= [-\pi, -\pi/2-\delta] \cup [\pi/2+\delta, \pi], \\ I_3 &= [-\pi,\pi] \setminus (I_1 \cup I_2). \end{aligned}$$

We define

$$\Phi_{r,k}(\theta)=\sin\theta-\frac{k\theta}{r}, \tag{3-17}$$

and let $\chi_\delta(\theta)$ be a smooth function satisfying

$$\chi_\delta(\theta)=\begin{cases}1, & \theta\in[-\delta,\delta],\\ 0, & \theta\notin[-2\delta,2\delta].\end{cases} \tag{3-18}$$

Then write

$$\tilde{J}_k(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ir\Phi_{r,k}(\theta)}\,d\theta=\tilde{J}_k^1(r)+\tilde{J}_k^2(r)+\tilde{J}_k^3(r), \tag{3-19}$$

with

$$\begin{aligned}\tilde{J}_k^1(r)&:=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ir\Phi_{r,k}(\theta)}\chi_\delta(\theta)\,d\theta,\\ \tilde{J}_k^2(r)&:=\frac{1}{2\pi}\int_{I_2}e^{ir\Phi_{r,k}(\theta)}\,d\theta,\\ \tilde{J}_k^3(r)&:=\frac{1}{2\pi}\int_{I_3}e^{ir\Phi_{r,k}(\theta)}(1-\chi_\delta(\theta))\,d\theta.\end{aligned}$$

When $\theta\in I_2$, the function $\Phi'_{r,k}(\theta)=\cos\theta-k/r$ is monotonic in the intervals $[-\pi,-\pi/2-\delta]$ and $[\pi/2+\delta,\pi]$, and satisfies

$$|\Phi'_{r,k}(\theta)|\geq\frac{k}{r}+|\cos\theta|\geq\sin\delta.$$

Then van der Corput's lemma (Lemma 2.3) gives, uniformly in k,

$$\left|\frac{1}{2\pi}\int_{I_2}e^{ir\Phi_{r,k}(\theta)}\,d\theta\right|\leq c_\delta r^{-1}. \tag{3-20}$$

When $\theta\in I_3$, we have $|\Phi''_{r,k}(\theta)|\geq\sin\delta$, and Lemma 2.3 again yields that

$$\left|\frac{1}{2\pi}\int_{I_3}e^{ir\Phi_{r,k}(\theta)}(1-\chi_\delta(\theta))\,d\theta\right|\leq c_\delta r^{-\frac{1}{2}} \tag{3-21}$$

uniformly in k.

Using arguments similar to those above, it follows from (3-20) and (3-21) that

$$\begin{aligned}\bigg\|\bigg(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\bigg|\int_0^\infty e^{-it\rho}\big(\tilde{J}^2_{\nu(k)}(r\rho)+\tilde{J}^3_{\nu(k)}(r\rho)\big)b_{k,l}(\rho)\varphi(\rho)\,d\rho\bigg|^2\bigg)^{\frac{1}{2}}\bigg\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))}\\ \lesssim R^{-\frac{1}{2}}\bigg\|\bigg(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\big|b_{k,l}(\rho)\big|^2\bigg)^{\frac{1}{2}}\bigg\|_{L^1_\rho(I)}.\end{aligned}$$

To establish (3-16) with $J_{\nu(k)}$ replaced by $\tilde{J}^1_{\nu(k)}$, we need to use the oscillation of

$e^{it\rho}$ effectively. To this end, we write the Fourier series as $b_{k,l}(\rho)=\sum_j b^j_{k,l}e^{i\frac{\pi}{2}\rho j}$, where

$$b^j_{k,l}=\frac{1}{4}\int_0^4 e^{-i\frac{\pi}{2}\rho j}b_{k,l}(\rho)\,d\rho. \tag{3-22}$$

Then $\sum_j |b^j_{k,l}|^2=\|b_{k,l}(\rho)\|^2_{L^2_\rho(I)}$. For simplicity, we use the scaling argument to reduce the problem by replacing t and r by $2\pi t$ and $2\pi r$ respectively, and define (3-23)

$$\psi^k_{t-j/4}(r)=\frac{1}{2\pi}\int_0^\infty e^{-2\pi i(t-\frac{j}{4})\rho}\int_{\mathbb{R}} e^{2\pi i\rho r\sin\theta-i\nu(k)\theta}\chi_\delta(\theta)\,d\theta\varphi(\rho)\,d\rho.$$

Let $m=t-j/4$. Then we write

$$\begin{aligned}\psi^k_m(r)&=\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{2\pi i\rho(r\sin\theta-m)}e^{-i\nu(k)\theta}\chi_\delta(\theta)\varphi(\rho)\,d\rho\,d\theta\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\check{\varphi}(r\sin\theta-m)e^{-i\nu(k)\theta}\chi_\delta(\theta)\,d\theta.\end{aligned} \tag{3-24}$$

For our purpose, we need to investigate the asymptotic behavior of the function $\psi^k_m(r)$. We consider two subcases:

(a) $4R\le |m|$. $R\ge 1$, hence $|m|\ge 4$. Since $\check{\varphi}$ is a Schwartz function, we have

$$|\check{\varphi}(r\sin\theta-m)|\le C_N(1+|r\sin\theta-m|)^{-N}\quad\text{for all } N>0.$$

On the other hand, we have

$$|r\sin\theta-m|\ge |m|-r|\sin\theta|\ge \tfrac{1}{100}|m|,$$

since $r\le 2R\le |m|$ and $|\theta|\le 2\delta$. Thus, (3-24) gives

$$|\psi^k_m(r)|\le C_{\delta,N}(1+|m|)^{-N}. \tag{3-25}$$

Keeping in mind that $m=t-j/4$, we have

$$\begin{aligned}&\Bigg\|\Bigg(\sum_{k=0}^\infty\sum_{l=1}^{d(k)}\Bigg|\sum_{\substack{j:\\4R\le|t-j/4|}} b^j_{k,l}\psi^k_{t-j/4}(r)\Bigg|^2\Bigg)^{\frac12}\Bigg\|_{L^\infty_t(\mathbb{R};L^\infty_r([R,2R]))}\\&\le C_{\delta,N}R^{-N}\Bigg\|\Bigg(\sum_{k=0}^\infty\sum_{l=1}^{d(k)}\Bigg|\sum_{\substack{j:\\4R\le|t-j/4|}} |b^j_{k,l}|(1+|t-\tfrac{j}{4}|)^{-N}\Bigg|^2\Bigg)^{\frac12}\Bigg\|_{L^\infty_t(\mathbb{R};L^\infty_r([R,2R]))}.\end{aligned}$$

By the Cauchy–Schwarz inequality, and choosing N large enough, the above is bounded by

$$C_{\delta,N}R^{-N}\Big\|\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\sum_{j}|b_{k,l}^{j}|^2(1+|t-\tfrac{j}{4}|)^{-N}\Big)^{\frac12}\Big\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))}$$
$$\le C_{\delta,N}R^{-N}\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\sum_{j}|b_{k,l}^{j}|^2\Big)^{\frac12}\lesssim R^{-N}\Big\|\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\big|b_{k,l}(\rho)\big|^2\Big)^{\frac12}\Big\|_{L_\rho^2(I)}.$$

(b) $|m|<4R$. Again, since $\check{\varphi}$ is a Schwartz function,

$$|\check{\varphi}(r\sin\theta-m)|\le C_N(1+|r\sin\theta-m|)^{-N}\quad\text{for all } N>0. \tag{3-26}$$

By (3-24), this gives

$$\big|\psi_m^k(r)\big|\le\frac{C_N}{2\pi}\Big(\int_{\substack{\{\theta:|\theta|<2\delta,\\ |r\sin\theta-m|\ge1\}}}d\theta+\int_{\substack{\{\theta:|\theta|<2\delta,\\ |r\sin\theta-m|\ge1\}}}(1+|r\sin\theta-m|)^{-N}\,d\theta\Big).$$

Let $y=r\sin\theta-m$; then

$$\big|\psi_m^k(r)\big|\le\frac{C_N}{2\pi r}\Big(\int_{\{y:|y|\le1\}}dy+\int_{\{y:|y|\ge1\}}(1+|y|)^{-N}\,dy\Big)\lesssim\frac{1}{r}. \tag{3-27}$$

For fixed t, R, we define the set $A=\{j\in\mathbb{Z}:|t-j/4|\le4R\}$. It is easy to see the cardinality of A is $O(R)$. Thus, it follows from (3-27) and the Cauchy–Schwarz inequality that

$$\Big\|\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\Big|\sum_{j\in A}b_{k,l}^{j}\psi_{t-j/4}^{k}(r)\Big|^2\Big)^{\frac12}\Big\|_{L_t^\infty(\mathbb{R};L_r^\infty([R,2R]))}$$
$$\le C_{\delta,N}R^{-\frac12}\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\sum_{j}|b_{k,l}^{j}|^2\Big)^{\frac12}\lesssim R^{-\frac12}\Big\|\Big(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\big|b_{k,l}(\rho)\big|^2\Big)^{\frac12}\Big\|_{L_\rho^2(I)}\ \square$$

Proposition 3.3. *Let φ be a smooth function supported on $I=\big[\frac12,1\big]$ and taking values in* $[0,1]$, *and let R be a positive real number. Assume* (1-5) *is satisfied, and consider the quantity*

$$Q=\Big\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\Big|\int_0^\infty e^{-it\rho}J_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\,d\rho\Big|^2\Big)^{\frac12}\Big\|_{L_t^q(\mathbb{R};L^q_{r^{n-1}dr}([R,2R]))}.$$

(*Recall the definition of $\nu=\nu(k)$ in* (2-11).)

(i) *When K is finite, there exists a constant C_K independent of R such that*

$$Q\le C_K\min\big\{R^{\frac{n}{q}},R^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\big\}\Big\|\Big(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}|b_{k,l}(\rho)|^2\Big)^{\frac12}\Big\|_{L_\rho^p(I)}. \tag{3-28}$$

(ii) *When K is infinite, there exists a constant C independent of R such that*

$$Q \le C \min\left\{R^{\frac{n}{q}}, R^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\right\} \left\| \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_\rho(I)}. \tag{3-29}$$

Proof. We first consider the case $R \lesssim 1$. The Minkowski inequality and the Hausdorff–Young inequality in t show that

$$Q \lesssim \left\| r^{-\frac{n-2}{2}} \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \| J_{\nu(k)}(r\rho) b_{k,l}(\rho)\varphi(\rho)\|^2_{L^{q'}_\rho} \right)^{\frac{1}{2}} \right\|_{L^q_{r^{n-1}\,dr}([R,2R])}.$$

Hence, by (2-4), there exists a constant C independent of K such that

$$Q \le C \left(\int_R^{2R} r^{-\frac{(n-2)q}{2}} \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \left| \frac{(8\pi r)^\nu}{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \right|^2 \|\varphi(\rho) b_{k,l}(\rho)\|^2_{L^{q'}_\rho} \right)^{\frac{q}{2}} r^{n-1}\,dr \right)^{\frac{1}{q}}$$

and

$$Q \le C R^{\frac{n}{q}} \left\| \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \left|\varphi(\rho) b_{k,l}(\rho)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^{q'}_\rho(I)} r.$$

Secondly, we consider the case $R \gg 1$. By Prepositions 3.1 and 3.2, we use interpolation to obtain

$$\begin{aligned} &\left\| r^{-\frac{n-2}{2}} \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \left| \int_0^\infty J_\nu(r\rho) e^{it\rho} b_{k,l}(\rho)\varphi(\rho)\,d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}\,dr}([R,2R]))} \\ &\qquad \le C_K R^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)} \left\| \left(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \left|\varphi(\rho) b_{k,l}(\rho)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^{q'}_\rho(I)}. \end{aligned}$$

When K is infinite,

$$\begin{aligned} &\left\| r^{-\frac{n-2}{2}} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left| \int_0^\infty J_\nu(r\rho) e^{it\rho} b_{k,l}(\rho)\varphi(\rho)\,d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}\,dr}([R,2R]))} \\ &\qquad \le C R^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)} \left\| \left(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left|\varphi(\rho) b_{k,l}(\rho)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^2_\rho(I)}. \end{aligned}$$

In view of (1-5) and since supp $\varphi \subset \left[\frac{1}{2}, 1\right]$, this shows (3-28) and (3-29). □

Conclusion of the proof of the Main Theorem. We know that if f is radial, so is u in (3-9). To prove the Main Theorem, we need to estimate the following by (3-9):

$$\left\| \sum_{k=0}^{K} \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \mathcal{H}_\nu\left(\rho^{-1} \sin(t\rho) b_{k,l}(\rho)\right)(r) \right\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}\,dr} L^2_\theta(\mathbb{S}^{n-1}))} \tag{3-30}$$

in the cases of $K = 0$ and $K = \infty$, which correspond to the radial case and the general case respectively. To this end, we use orthogonality and apply a dyadic decomposition to (3-30) to obtain the estimate

$$
\begin{aligned}
&(3\text{-}30)\\
&\le C\Bigg(\sum_R\Bigg(\sum_M\Bigg\|\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{d(k)} J_\nu(r\rho)e^{it\rho}\chi\Big(\frac{\rho}{M}\Big)\\
&\qquad\qquad\times\bigg|\int_0^\infty (r\rho)^{-\frac{n-2}{2}} b_{k,l}(\rho)\rho^{n-2}\,d\rho\bigg|^2\Bigg)^{\frac12}\Bigg\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}dr}([R,2R]))}\Bigg)^q\Bigg)^{\frac1q},\\
&=:\Sigma
\end{aligned}
$$

where R and M are dyadic numbers and χ is a smooth function supported on $\left[\frac12, 1\right]$ and taking values in $[0,1]$. By a scaling argument, we have

$$
\begin{aligned}
\Sigma &\le C\Bigg(\sum_R\Bigg(\sum_M M^{(n-1)-\frac{n+1}{q}}\Bigg\|\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\\
&\bigg|\int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho)e^{it\rho}\chi(\rho)b_{k,l}(M\rho)\rho^{n-2}d\rho\bigg|^2\Bigg)^{\frac12}\Bigg\|_{L^q_t(\mathbb{R};L^q_{r^{n-1}dr}([RM,2RM]))}\Bigg)^q\Bigg)^{\frac1q}.
\end{aligned}
$$

Applying Proposition 3.3 with $\varphi(\rho) = \chi(\rho)\rho^{\frac n2-1}$ to the above, one can see that when K is finite,

$$
\begin{aligned}
\Sigma \le C_K\Bigg(\sum_R\Bigg(\sum_M &\min\Big\{(RM)^{\frac nq}, (RM)^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\Big\}\\
&\times M^{(n-1)-\frac{n+1}{q}}\Bigg\|\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{d(k)}\big|\chi(\rho)\rho^{\frac n2-1}b_{k,l}(M\rho)\big|^2\Bigg)^{\frac12}\Bigg\|_{L^p_\rho}\Bigg)^q\Bigg)^{\frac1q},
\end{aligned}
$$

and when K is infinite,

$$
\begin{aligned}
\Sigma \le C\Bigg(\sum_R\Bigg(\sum_M &\min\Big\{(RM)^{\frac nq}, (RM)^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\Big\}\\
&\times M^{(n-1)-\frac{n+1}{q}}\Bigg\|\Bigg(\sum_{k=0}^{\infty}\sum_{l=1}^{d(k)}\big|\chi(\rho)\rho^{\frac n2-1}b_{k,l}(M\rho)\big|^2\Bigg)^{\frac12}\Bigg\|_{L^2_\rho}\Bigg)^q\Bigg)^{\frac1q}.
\end{aligned}
$$

Since $q > 2n/(n-1)$, one has

$$
\sup_R\sum_M \min\Big\{(RM)^{\frac nq}, (RM)^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\Big\} < \infty,
$$

$$
\sup_M\sum_R \min\Big\{(RM)^{\frac nq}, (RM)^{-\frac{n-1}{2}\left(1-\frac{2n}{q(n-1)}\right)}\Big\} < \infty.
$$

Then by Schur's test lemma and the embedding $l^p \hookrightarrow l^q$ with $q > \dfrac{2n}{n-1} > p$, we have, in the case when K is finite,

$$\begin{aligned}
\Sigma &\le C_K \bigg(\sum_M M^{\left((n-1)-\frac{n+1}{q}\right)p} \bigg\| \chi(\rho) \Big(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} |b_{k,l}(M\rho)|^2 \Big)^{\frac12} \bigg\|_{L^p_\rho}^p \bigg)^{\frac1p} \\
&\le C_K \bigg(\sum_M M^{\left(\frac{n-1}{p'}-\frac{n+1}{q}\right)p} \bigg\| \chi\Big(\frac{\rho}{M}\Big) \Big(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \big|b_{k,l}(\rho)\big|^2 \Big)^{\frac12} \rho^{\frac{n-2}{p}} \bigg\|_{L^p_\rho}^p \bigg)^{\frac1p} \\
&\le C_K \bigg\| \Big(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \big|b_{k,l}(\rho)\big|^2 \Big)^{\frac12} \bigg\|_{L^p_{\rho^{n-2}\,d\rho}(\mathbb{R}^+)},
\end{aligned}$$

and in the case when K is infinite and $p \ge 2$,

$$\begin{aligned}
\Sigma &\le C \bigg(\sum_M M^{\left(\frac{n-1}{p'}-\frac{n+1}{q}\right)q} \bigg\| \chi\Big(\frac{\rho}{M}\Big) \Big(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \big|b_{k,l}(\rho)\big|^2 \Big)^{\frac12} \rho^{\frac{n-2}{2}} \bigg\|_{L^p_\rho}^q \bigg)^{\frac1q} \\
&\le C \bigg\| \Big(\sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \big|b_{k,l}(\rho)\big|^2 \Big)^{\frac12} \bigg\|_{L^p_{\rho^{n-2}\,d\rho}(\mathbb{R}^+)}.
\end{aligned}$$

By Lemma 2.2, we have $b_{k,l}(\rho) = \mathcal{H}_{\nu(k)}\mathcal{H}_{\mu(k)}[\mathcal{H}_{\mu(k)}a_{k,l}](\rho)$. To proceed we make use of the following fact, whose proof we defer to the Appendix:

Claim. *For the measure space* $(\mathbb{R}^+,\ dw(\rho))$, *where* $dw(\rho) = \rho^{n-2}\,d\rho$, *and for* $1 < p < \infty$, *denote by* $L^p(w)$ *the corresponding Lebesgue space equipped with the norm*

$$\|f\|_p = \Big(\int_0^\infty |f|^p \, dw \Big)^{\frac1p}.$$

Let $\mathcal{H}_{\nu(k)}, \mathcal{H}_{\mu(k)}$ *be the Hankel transforms defined above and suppose that*

$$\frac{n-2}{2} - \nu(0) < \frac{n-1}{p} < \frac{n-2}{2} + \mu(0) + 2. \tag{3-31}$$

Then there exists a constant C *such that, for any* $\{f_k\}_{k=0}^\infty \in L^p(w; l^2)$, *we have*

$$\bigg\| \Big(\sum_k |\mathcal{H}_{\nu(k)}\mathcal{H}_{\mu(k)} f_k|^2 \Big)^{\frac12} \bigg\|_{L^p(w)} \le C \bigg\| \Big(\sum_k |f_k|^2 \Big)^{\frac12} \bigg\|_{L^p(w)}. \tag{3-32}$$

Condition (3-31) is satisfied because $1 < p < \dfrac{2n}{n-1}$ by the Main Theorem's assumptions. Thus, applying the Claim we get

$$\Sigma \le C \bigg\| \Big(\sum_{k=0}^{K} \sum_{l=1}^{d(k)} \big|[\mathcal{H}_{\mu(k)}a_{k,l}](\rho)\big|^2 \Big)^{\frac12} \bigg\|_{L^p_{\rho^{n-2}\,d\rho}(\mathbb{R}^+)}.$$

By (2-2) and (2-13), under the conditions of the Main Theorem, we further have

$$\Sigma \le C \left\| |\xi|^{-\frac{1}{p}} \hat{f}(\xi) \right\|_{L^p_{\rho^{n-1}\,d\rho}(\mathbb{R}^+;L^2_\omega(\mathbb{S}^{n-1}))} \le C \left\| |\xi|^{-\frac{1}{p}} \hat{f}(\xi) \right\|_{L^p_\xi(\mathbb{R}^n)}.$$

This completes the proof. □

Appendix: Proof of (3-32)

Let $T_k = \mathcal{H}_{\mu(k)}\mathcal{H}_{\nu(k)}$, and set $\lambda = \dfrac{n-2}{2}$. We first show that

$$\|T_k f_k\|_{L^p(w)} \le C\|f_k\|_{L^p(w)}, \tag{A-1}$$

by following the argument used to prove Theorem 3.1 in [Planchon et al. 2003b]. By that argument, we can write

$$(T_k f_k)(\rho) = \int_0^\infty k^0_{\nu,\mu(k)}(\rho,s) f_k(s) s^{n-1}\,ds,$$

where the kernel is given by

$$k^0_{\alpha,\beta}(\rho,s) = \begin{cases} A_{\alpha,\beta}\dfrac{s^{\beta-\lambda}}{\rho^{\lambda+\beta+2}} F\Big(\dfrac{\alpha+\beta}{2}+1, \dfrac{\beta-\alpha}{2}+1; \beta+1; \Big(\dfrac{s}{\rho}\Big)^2\Big) & \text{for } s<\rho,\\ A_{\beta,\alpha}\dfrac{s^{\alpha-\lambda}}{\rho^{\lambda+\alpha+2}} F\Big(\dfrac{\beta+\alpha}{2}+1, \dfrac{\alpha-\beta}{2}+1; \alpha+1; \Big(\dfrac{\rho}{s}\Big)^2\Big) & \text{for } s>\rho,\end{cases}$$

where $F(a,b;c;d)$ is the hypergeometric function and

$$A_{\alpha,\beta} = \frac{2\Gamma(\frac{\alpha+\beta}{2}+1)}{\Gamma(\frac{\beta-\alpha}{2})\Gamma(\beta+1)}.$$

When s is near ρ, the kernel $k^0_{\nu,\mu}(\rho,s)$ behaves like $c(\rho-s)^{-1} + O(-\log|\rho-s|)$. Define

$$(\widetilde{T}_k[s^{\frac{n-1}{p}} f_k(s)])(\rho) := \int_0^\infty \widetilde{k}^0_{\nu,\mu}(\rho,s)\big[s^{\frac{n-1}{p}} f_k(s)\big]\frac{ds}{s},$$

where

$$\widetilde{k}^0_{\nu,\mu}(\rho,s) = \rho^{\frac{n-1}{p}} k^0_{\nu,\mu}(\rho,s) s^{n-\frac{n-1}{p}}.$$

Then

$$\big(\widetilde{T}_k[s^{\frac{n-1}{p}} f_k(s)]\big)(\rho) = \int_0^\infty \rho^{\frac{n-1}{p}} k^0_{\nu,\mu}(\rho,s) s^{n-\frac{n-1}{p}} [s^{\frac{n-1}{p}} f_k(s)]\frac{ds}{s} = \rho^{\frac{n-1}{p}}(T_k f_k)(\rho).$$

Note that

$$\big\|(T_k f_k)(\rho)\big\|_{L^p(w)} = \big\|\rho^{\frac{n-1}{p}}(T_k f_k)(\rho)\big\|_{L^p_{\rho^{-1}\,d\rho}}.$$

To prove (A-1), it suffices to show

$$\|\widetilde{T}_k f_k\|_{L^p(\rho^{-1}\,d\rho)} \le C\|f_k\|_{L^p(\rho^{-1}\,d\rho)}. \tag{A-2}$$

Again by the argument in [Planchon et al. 2003b], one has

$$|k^0_{\nu,\mu}(\rho,s)| = \begin{cases} O(\rho^{-\lambda-\mu-2+\epsilon}s^{-\lambda+\mu-\epsilon}) & \text{for } s<\rho,\\ O(\rho^{\nu-\lambda-\epsilon}s^{-\lambda-\nu-2+\epsilon}) & \text{for } s>\rho.\end{cases}$$

Then

$$|\widetilde{k}^0_{\nu,\mu}(\rho,s)| = \begin{cases} O\big((s/\rho)^{\lambda+\mu+2-\frac{n-1}{p}-\epsilon}\big) & \text{for } s<\rho,\\ O\big((s/\rho)^{\lambda-\nu-\frac{n-1}{p}+\epsilon}\big) & \text{for } s>\rho.\end{cases}$$

Since $\lambda-\nu<(n-1)/p<\lambda+\mu+2$, the kernel $\widetilde{k}^0_{\nu,\mu}$ is bounded in $L^1(d\rho/\rho)$. Using logarithmic coordinates, we express the operator T_k as a convolution operator with the kernel $\widetilde{k}^0_{\nu,\mu}$. When $s\sim\rho$, we recall that $\widetilde{k}^0_{\nu,\mu}$ is a Calderón–Zygmund kernel behaving like

$$c(\rho-s)^{-1}+O(-\log|\rho-s|).$$

Applying Young's inequality to the region away from $\rho\sim s$ and Calderón–Zygmund theory to the region $\rho\sim s$, we obtain (A-2), and so (A-1).

By a similar argument, we show the adjoint operator T_k^* is also bounded in $L^{p'}(w)$, provided $\lambda-\mu<(n-1)/p'<\lambda+\nu+2$, which is true for $1<p<\infty$.

Now we are ready to show (3-32). We consider two cases:

If $1<p\le 2$, since the adjoint operator $T_k^*=\mathscr{H}_{\nu(k)}\mathscr{H}_{\mu(k)}$ is bounded in $L^{p'}(w)$, we get (3-32) by duality.

If $2\le p<\infty$, we set $q:=p/2$ (forgetting the earlier value of q). Then $q\ge 1$, and we have

$$\begin{aligned}\Big\|\Big(\sum_k |T_kf_k|^2\Big)^{\frac12}\Big\|^2_{L^p(w)} &= \Big\|\sum_k |T_kf_k|^2\Big\|_{L^q(w)}\\ &= \sup_{\substack{g\ge 0\\ g\in L^{q'}(w)}}\Big|\int_0^\infty \sum_k |T_kf_k|^2 g(\rho)\rho^{n-2}\,d\rho\Big|\\ &= \sum_k \sup_{\substack{g\ge 0\\ g\in L^{q'}(w)}}\int_0^\infty |T_kf_k|^2 g(\rho)\rho^{n-2}\,d\rho\\ &= \sum_k \|T_kf_k\|^2_{L^p(w)}.\end{aligned}$$

By (A-1), we see that

$$\begin{aligned}\Big\|\Big(\sum_k |T_kf_k|^2\Big)^{\frac12}\Big\|^2_{L^p(w)} &\le C\sum_k \|f_k\|^2_{L^p(w)}\\ &= C\sum_k \sup_{\substack{g\ge 0\\ g\in L^{q'}(w)}}\Big|\int_0^\infty |f_k|^2 g(\rho)\rho^{n-2}\,d\rho\Big|\end{aligned}$$

$$\leq C \sup_{\substack{g\geq 0\\ g\in L^{q'}(w)}} \int_0^\infty \sum_k |f_k|^2 g(\rho)\rho^{n-2}\,d\rho$$

$$\leq C \Big\| \sum_k |f_k|^2 \Big\|^2_{L^q(w)} \leq C \Big\| \Big(\sum_k |f_k|^2\Big)^{\frac{1}{2}} \Big\|^2_{L^p(w)}.$$

References

[Burq et al. 2003] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential", *J. Funct. Anal.* **203**:2 (2003), 519–549. MR 2004m:35025 Zbl 1030.35024

[Burq et al. 2004] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay", *Indiana Univ. Math. J.* **53**:6 (2004), 1665–1680. MR 2005k:35241 Zbl 1084.35014

[Cheeger and Taylor 1982] J. Cheeger and M. Taylor, "On the diffraction of waves by conical singularities, I", *Comm. Pure Appl. Math.* **35**:3 (1982), 275–331. MR 84h:35091a Zbl 0526.58049

[Chen et al. 2012] P. Chen, E. M. Ouhabaz, A. Sikora, and L. Yan, "Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner–Riesz means", preprint, 2012. arXiv 1202.4052v1

[Kalf et al. 1975] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, "On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials", pp. 182–226 in *Spectral theory and differential equations* (Dundee, 1974), edited by W. N. Everitt, Lecture Notes in Mathematics **448**, Springer, Berlin, 1975. MR 53 #1051 Zbl 0311.47021

[Miao et al. 2012] C. Miao, J. Zhang, and J. Zheng, "A note on the cone restriction conjecture", *Proc. Amer. Math. Soc.* **140**:6 (2012), 2091–2102. MR 2888196 Zbl 1250.42052

[Miao et al. 2013a] C. Miao, J. Zhang, and J. Zheng, "Linear adjoint restriction estimates for paraboloid", preprint, 2013.

[Miao et al. 2013b] C. Miao, J. Zhang, and J. Zheng, "Strichartz estimates for wave equation with inverse square potential", preprint, 2013. To appear in *Commun. Contemp. Math.*

[Planchon et al. 2003a] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Dispersive estimate for the wave equation with the inverse-square potential", *Discrete Contin. Dyn. Syst.* **9**:6 (2003), 1387–1400. MR 2005h:35212 Zbl 1047.35081

[Planchon et al. 2003b] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "L^p estimates for the wave equation with the inverse-square potential", *Discrete Contin. Dyn. Syst.* **9**:2 (2003), 427–442. MR 2003j:35195 Zbl 1031.35092

[Shao 2009a] S. Shao, "A note on the cone restriction conjecture in the cylindrically symmetric case", *Proc. Amer. Math. Soc.* **137**:1 (2009), 135–143. MR 2010f:35222 Zbl 1198.42009

[Shao 2009b] S. Shao, "Sharp linear and bilinear restriction estimates for paraboloids in the cylindrically symmetric case", *Rev. Mat. Iberoam.* **25**:3 (2009), 1127–1168. MR 2011e:35362 MR 1183:42010

[Stein 1979] E. M. Stein, "Some problems in harmonic analysis", pp. 3–20 in *Harmonic analysis in Euclidean spaces, Part 1* (Williamstown, MA, 1978), edited by G. Weiss and S. Wainger, Proc. Sympos. Pure Math. **35**, Amer. Math. Soc., Providence, RI, 1979. MR 80m:42027 Zbl 0445.42006

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, Princeton, NJ, 1993. MR 95c:42002 Zbl 0821.42001

[Stein and Weiss 1971] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, NJ, 1971. MR 46 #4102 Zbl 0232.42007

[Stempak 2000] K. Stempak, "A weighted uniform L^p-estimate of Bessel functions: a note on a paper of K. Guo: 'A uniform L^p estimate of Bessel functions and distributions supported on S^{n-1}' ", *Proc. Amer. Math. Soc.* **128**:10 (2000), 2943–2945. MR 2000m:33004 Zbl 1031.35092

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 58 #23577 Zbl 0372.35001

[Taberner 1985] B. B. Taberner, "On the restriction of the Fourier transform to a conical surface", *Trans. Amer. Math. Soc.* **292**:1 (1985), 321–333. MR 86k:42023 Zbl 0578.42006

[Tao 2001] T. Tao, "Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates", *Math. Z.* **238**:2 (2001), 215–268. MR 2003a:42010 Zbl 0992.42004

[Tao 2003a] T. Tao, "Recent progress on the restriction conjecture", preprint, 2003. arXiv 0311181

[Tao 2003b] T. Tao, "A sharp bilinear restriction estimate for paraboloids", *Geom. Funct. Anal.* **13**:6 (2003), 1359–1384. MR 2004m:47111 Zbl 1068.42011

[Tao et al. 1998] T. Tao, A. Vargas, and L. Vega, "A bilinear approach to the restriction and Kakeya conjectures", *J. Amer. Math. Soc.* **11**:4 (1998), 967–1000. MR 99f:42026 Zbl 0924.42008

[Vazquez and Zuazua 2000] J. L. Vazquez and E. Zuazua, "The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential", *J. Funct. Anal.* **173**:1 (2000), 103–153. MR 2001j:35122 Zbl 0953.35053

[Watson 1944] G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge University Press, Cambridge, 1944. MR 6,64a Zbl 0063.08184

[Wolff 2001] T. Wolff, "A sharp bilinear cone restriction estimate", *Ann. of Math.* (2) **153**:3 (2001), 661–698. MR 2002j:42019 Zbl 1125.42302

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