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WITH AN INVERSE SQUARE POTENTIAL**

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# LINEAR RESTRICTION ESTIMATES FOR THE WAVE EQUATION WITH AN INVERSE SQUARE POTENTIAL

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**We study modified linear restriction estimates associated with the wave equation with an inverse square potential. In particular, we show that the classical linear restriction estimates hold in their almost sharp range when the initial data is radial.**

## 1. Introduction and statement of main result

In this paper, we study a modified restriction estimate associated with the wave equation perturbed by an inverse square potential. More precisely, we consider the following wave equation with a singular potential:

$$(1-1) \quad \begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad a \in \mathbb{R}, \\ u(t, x)|_{t=0} = 0, \quad \partial_t u(t, x)|_{t=0} = f(x). \end{cases}$$

The scale-covariant elliptic operator  $P_a := -\Delta + a/|x|^2$  appearing in (1-1) plays a key role in many problems of physics and geometry. The heat and Schrödinger flows for the elliptic operator  $P_a$  have been studied in the theory of combustion [Vazquez and Zuazua 2000] and in quantum mechanics [Kalf et al. 1975]. The wave equation (1-1) arises in the study of the wave propagation on conic manifolds [Cheeger and Taylor 1982]. There has been a lot of interest in developing Strichartz estimates both for the Schrödinger and wave equations with the inverse square potential; we refer the reader to [Burq et al. 2003; 2004; Planchon et al. 2003b; 2003a; Miao et al. 2013b]. However, as far as we know, there are few results about the restriction estimates associated with the operator  $P_a$  arising in the study of eigenfunction estimates of  $P_a$ . Here, we address some restriction issues in special settings associated with the operator  $P_a$ .

In the case  $a = 0$  — the linear wave equation with no potential — we can solve

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the equation by the Fourier transform formula

$$(1-2) \quad u(t, x) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f = \frac{1}{2i} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} (e^{2\pi i t|\xi|} - e^{-2\pi i t|\xi|}) \hat{f}(\xi) \frac{d\xi}{|\xi|},$$

where the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.$$

It is well known that the spacetime norm estimate of  $u(t, x)$  is connected with the linear adjoint cone restriction estimate

$$(1-3) \quad \|(F d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C_{p,q,n,S} \|F\|_{L^p(S, d\sigma)},$$

where  $F$  is a Schwartz function and the inverse spacetime Fourier transform of the measure  $F d\sigma$  is defined by

$$(F d\sigma)^\vee(t, x) = \int_S F(\tau, \xi) e^{2\pi i(x \cdot \xi + t\tau)} d\sigma(\xi) = \int_{\mathbb{R}^n} F(|\xi|, \xi) e^{2\pi i(x \cdot \xi + t|\xi|)} \frac{d\xi}{|\xi|}.$$

Here, the set  $S$  is a nonempty smooth compact subset of the cone

$$\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|\} \quad \text{with } n \geq 2.$$

The canonical measure  $d\sigma$  is the pull back of the measure  $d\xi/|\xi|$  under the projection map  $(\tau, \xi) \mapsto \xi$ . By the decay of  $(d\sigma)^\vee$  and the Knapp counterexample, the two necessary conditions for (1-2) are

$$(1-4) \quad q > \frac{2n}{n-1} \quad \text{and} \quad \frac{n+1}{q} \leq \frac{n-1}{p'}$$

(see [Stein 1979; Tao 2003a]). The corresponding linear adjoint restriction conjecture for cones asserts that:

**Conjecture 1.1.** *The inequality (1-3) holds with constants depending on  $n, p, q,$  and  $S$  if and only if the inequalities (1-4) are satisfied.*

Even though there is a large amount of literature focused on this problem, it remains open for  $n \geq 4$ . For progress on this conjecture, we refer the readers to [Taberner 1985; Strichartz 1977; Tao 2001; 2003a; 2003b; Tao et al. 1998; Wolff 2001]. Shao [2009a] provided two simple and novel arguments to prove that Conjecture 1.1 holds true for the spatial rotation invariant functions which are supported on the cone. Motivated by [Shao 2009a], Miao et al. [2012] utilized expansions in spherical harmonics and analyzed the asymptotic behavior of the Bessel function to generalize Shao’s result for cone cases by establishing, assuming (1-4), that

$$\|(F d\sigma)^\vee\|_{L^q_t(\mathbb{R}; L^q_{r^{n-1}} L^2_\theta(\mathbb{S}^{n-1}))} \leq C_{p,q,n,S} \|F\|_{L^p(S, d\sigma)}.$$

In the case  $a \neq 0$ , the spacetime Fourier transform is no longer so useful; one can instead establish an approximate parametrix for the fundamental solution and try to obtain good control over it. In our case we resort to expansions in spherical harmonics and Hankel transforms, for technical reasons involving the singular potential; compare [Burq et al. 2003; Planchon et al. 2003b; Miao et al. 2013b]. Although the harmonic expansion expression leads to some loss of angular regularity in the restriction estimates, it allows us to show the restriction estimates when  $q$  is close to  $2n/(n - 1)$ . A key ingredient in this process is to explore the oscillatory properties of the Bessel function and  $e^{it|\xi|}$  to overcome the difficulties arising from the low decay of the Bessel function  $J_\nu(r)$  when  $\nu \sim r$ . Finally, by using the properties of the hypergeometric function shown in [Planchon et al. 2003b], we prove an inequality involving the Hankel transform to obtain the desired result.

**Main Theorem.** *Assume  $n \geq 2$  and  $a > -\frac{1}{4}(n - 2)^2$ , and let  $u$  be the solution of (1-1). Suppose that  $p > 1$  and*

$$(1-5) \quad q = \frac{p'(n + 1)}{n - 1} > \frac{2n}{n - 1}.$$

*Then there exists a constant  $C$ , depending only on  $p, q, n$ , and  $a$ , satisfying the following conditions:*

(i) *If  $f$  is a radial Schwartz function, then*

$$(1-6) \quad \|u(t, x)\|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C \|\ |\xi|^{-\frac{1}{p}} \hat{f} \|_{L^p(\mathbb{R}^n)}.$$

(ii) *If  $f$  is a Schwartz function (may not be radial) and  $p \geq 2$ , then*

$$(1-7) \quad \|u(t, x)\|_{L^q_t(\mathbb{R}; L^q_{r^{n-1} dr} L^2_\theta(\mathbb{S}^{n-1}))} \leq C \|\ |\xi|^{-\frac{1}{p}} \hat{f} \|_{L^p(\mathbb{R}^n)}.$$

**Remarks.** (i) This extends the classical restriction estimate associated with the Laplace operator to a restriction estimate associated with  $-\Delta + a/|x|^2$ . We obtain more estimates than the Strichartz estimates of [Burq et al. 2003; Planchon et al. 2003b], which focus on  $p = 2$ . The theorem can also be viewed as an extension of the result in [Chen et al. 2012] about the operator  $-\Delta + a/r^2$  acting on  $L^2((0, \infty); r^{n-1} dr)$ .

(ii) The theorem means that we almost show that the classical linear restriction estimates hold for radial functions in the conjecture range.

(iii) When  $a = 0$ , we recover the cone restriction result in [Shao 2009b]. When  $\text{supp } \hat{f}$  is compact, we can extend the result to  $q \geq p'(n + 1)/(n - 1)$ , which is the same range as in the cone restriction conjecture.

(iv) Equation (1-6) gives a Strichartz-type estimate

$$\|u(t, x)\|_{L^{2(n+1)/(n-1)}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C \|\ |\nabla|^{-\frac{1}{2}} f \|_{L^2(\mathbb{R}^n)}$$

for the radial solution. The method used here generalizes the result for the radial initial data to a linear finite combination of products of the Hankel transform of radial functions and spherical harmonics. We hope to remove the whole radial assumption in (1-6) in the future, at least for  $q \geq 2(n + 3)/(n + 1)$ .

(v) If  $\hat{f} \subset \{\xi : N \leq |\xi| \leq 2N\}$  and  $f$  is radial, the method here can be employed to obtain the Strichartz estimate

$$\|u(t, x)\|_{L^q_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq CN^{\frac{n-2}{2} - \frac{n+1}{q}} \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for } q > \frac{2n}{n-1}.$$

We remark that the pair  $(q, q)$  is allowed to be out of the admissible requirement in [Planchon et al. 2003b], it is however consistent with the admissible range due to [Miao et al. 2013b].

(vi) We rely heavily on the harmonic expansion formula to give the expression of the solution due to the potential, which causes the restriction  $p \geq 2$ . It is possible that the resolvent expression can be used to remove this restriction.

Now we introduce some notation. We use  $A \lesssim B$  to denote the statement that  $A \leq CB$  for some constant  $C$ , which may vary from line to line and depend on various parameters. We write  $A \sim B$  to mean that  $A \lesssim B \lesssim A$ .

If the constant  $C$  depends on parameters other than  $p, q, n$ , and  $S$ , we denote this fact explicitly using subscripts. For instance,  $C_\epsilon$  should be understood as a positive constant depending on  $\epsilon$  in addition to (possibly)  $p, q, n$ , and  $S$ .

Pairs of conjugate indices are written as  $p$  and  $p'$ , where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad 1 \leq p \leq \infty.$$

This paper is organized as follows: In Section 2, we present some simple facts about the Hankel transforms and the Bessel functions and also recall the van der Corput lemma. Section 3 is devoted to the proof of the Main Theorem via expansions in spherical harmonics and an analysis of the asymptotic behavior of the Bessel function. Finally, in an Appendix, we show an inequality used in Section 3, involving the Hankel transforms.

## 2. Preliminaries

Before turning to Hankel transforms and Bessel functions, we recall the expansion formula in spherical harmonics. For details, refer to [Stein and Weiss 1971]. For convenience, we write

$$\xi = \rho\omega \quad \text{and} \quad x = r\theta \quad \text{with } \omega, \theta \in \mathbb{S}^{n-1}.$$

We denote by  $\mathcal{H}^k$  the space of spherical harmonics of degree  $k$  on  $\mathbb{S}^{n-1}$ , whose

dimension is given by

$$d(0) = 1, \quad d(k) = \frac{2k + n - 2}{k} C_{n+k-3}^{k-1} \simeq \langle k \rangle^{n-2} \quad \text{for } k > 0.$$

Note that if  $n = 2$  this dimension is 2 for all  $k$ .

Any  $g \in L^2(\mathbb{R}^n)$  can be expanded in spherical harmonics as

$$(2-1) \quad g(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(r) Y_{k,l}(\theta),$$

where  $\{Y_{k,1}, \dots, Y_{k,d(k)}\}$  is an orthogonal basis of  $\mathcal{H}^k$ . We have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k,$$

and by orthogonality,

$$(2-2) \quad \|g(x)\|_{L^2_\theta} = \|a_{k,l}(r)\|_{l^2_{k,l}}.$$

The Hankel transform formula (see Theorem 3.10 in [Stein and Weiss 1971], for instance) relates the Fourier transform of  $g$  to spherical harmonics. It reads

$$(2-3) \quad \hat{g}(\rho\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} 2\pi i^k Y_{k,l}(\omega) \rho^{-\frac{n-2}{2}} \int_0^\infty J_{k+\frac{n-2}{2}}(2\pi r\rho) a_{k,l}(r) r^{\frac{n}{2}} dr.$$

Here the Bessel function  $J_k(r)$  of order  $k$  is defined by

$$J_k(r) = \frac{\left(\frac{r}{2}\right)^k}{\Gamma(k + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{isr} (1-s^2)^{\frac{2k-1}{2}} ds, \quad \text{with } k > -\frac{1}{2} \text{ and } r > 0.$$

A simple computation gives the rough estimate

$$(2-4) \quad |J_k(r)| \leq \frac{Cr^k}{2^k \Gamma(k + \frac{1}{2})\Gamma(\frac{1}{2})} \left(1 + \frac{1}{k + \frac{1}{2}}\right),$$

where  $C$  is an absolute constant. These estimates will be mainly used when  $r \lesssim 1$ . Another well known asymptotic expansion about the Bessel function is

$$(2-5) \quad J_k(r) = r^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{k\pi}{2} - \frac{\pi}{4}\right) + O_k(r^{-3/2}) \quad \text{as } r \rightarrow \infty,$$

but with a constant depending on  $k$  (see [Stein and Weiss 1971]). As pointed out in [Stein 1993], if one seeks a uniform bound for large  $r$  and  $k$ , then the best one can do is  $|J_k(r)| \leq Cr^{-\frac{1}{3}}$ . To investigate the asymptotic behavior in  $k$  and  $r$ , we recall

Schl\"afli's integral representation [Watson 1944] of the Bessel function. For  $r \in \mathbb{R}^+$  and  $k > -\frac{1}{2}$ ,

$$(2-6) \quad J_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir \sin \theta - ik\theta} d\theta - \frac{\sin(k\pi)}{\pi} \int_0^{\infty} e^{-(r \sinh s + ks)} ds$$

$$=: \tilde{J}_k(r) - E_k(r).$$

We remark that  $E_k(r) = 0$  when  $k \in \mathbb{Z}^+$ . A simple computation gives, for  $r > 0$ ,

$$(2-7) \quad |E_k(r)| = \left| \frac{\sin(k\pi)}{\pi} \int_0^{\infty} e^{-(r \sinh s + ks)} ds \right| \leq C(r+k)^{-1}.$$

Next, we recall some properties of the Bessel function  $J_k(r)$  from [Stein 1993; Stempak 2000]; see [Miao et al. 2013a] for a detailed proof.

**Lemma 2.1** (asymptotics of the Bessel function). *Assume  $k \gg 1$ . Let  $J_k(r)$  be the Bessel function of order  $k$  defined as above. Then there exist a large constant  $C$  and a small constant  $c$  independent of  $k$  and  $r$  such that*

$$(2-8) \quad |J_k(r)| \leq C e^{-c(k+r)} \quad \text{when } r \leq k/2,$$

$$(2-9) \quad |J_k(r)| \leq C k^{-\frac{1}{3}} (k^{-\frac{1}{3}} |r - k| + 1)^{-\frac{1}{4}} \quad \text{when } k/2 \leq r \leq 2k,$$

$$(2-10) \quad J_k(r) = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(r) e^{\pm ir} + E(r) \quad \text{when } r \geq 2k,$$

where  $|a_{\pm}(r)| \leq C$  and  $|E(r)| \leq C r^{-1}$ .

We define

$$(2-11) \quad \mu(k) = \frac{n-2}{2} + k, \quad \nu(k) = \sqrt{\mu^2(k) + a} \quad \text{with } a > -\frac{(n-2)^2}{4}.$$

For the sake of simplicity, we sometimes write  $\nu$  instead of  $\nu(k)$ . Let  $g$  be a Schwartz function defined on  $\mathbb{R}^n$ . We define the Hankel transform of order  $\nu$ :

$$(2-12) \quad (\mathcal{H}_{\nu} g)(\rho\omega) = \int_0^{\infty} (r\rho)^{-\frac{n-2}{2}} J_{\nu}(r\rho) g(r\omega) r^{n-1} dr.$$

If the function  $g$  is radial, we can drop the dependence on  $\omega$  from both sides.

We remark that if  $g$  has the expansion (2-1), it follows from (2-3) that

$$(2-13) \quad \hat{g}(\xi) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} 2\pi i^k Y_{k,l}(\omega) (\mathcal{H}_{\mu(k)} a_{k,l})(\rho).$$

The following properties of the Hankel transform are proved in [Burq et al. 2003; Planchon et al. 2003b]:

**Lemma 2.2.** *Let  $\mathcal{H}_v$  be as above and let*

$$A_v(k) := -\partial_r^2 - \frac{n-1}{r} \partial_r + \left( v^2(k) - \left( \frac{n-2}{2} \right)^2 \right) r^{-2}.$$

- (i)  $\mathcal{H}_v = \mathcal{H}_v^{-1}$ .
- (ii)  $\mathcal{H}_v$  is self-adjoint:  $\mathcal{H}_v = \mathcal{H}_v^*$ .
- (iii)  $\mathcal{H}_v$  is an  $L^2$  isometry:  $\|\mathcal{H}_v \phi\|_{L_\xi^2} = \|\phi\|_{L_x^2}$ .
- (iv)  $\mathcal{H}_v(A_v \phi)(\xi) = |\xi|^2 (\mathcal{H}_v \phi)(\xi)$ , for  $\phi \in L^2$ .

We conclude this section by recalling van der Corput’s lemma [Stein 1993]:

**Lemma 2.3.** *Let  $\phi$  be a smooth real-valued function defined on an interval  $[a, b]$ , and assume  $|\phi^{(k)}(x)| \geq 1$  for all  $x \in [a, b]$ . Assume moreover that either  $k \geq 2$ , or  $k = 1$  and  $\phi'(x)$  is monotonic. Then*

$$(2-14) \quad \left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq c_k \lambda^{-\frac{1}{k}},$$

with  $c_k$  independent of  $\phi$  and  $\lambda$ .

### 3. Proof of the main theorem

In this section, we will use the asymptotic properties of the Bessel function and the stationary phase argument to establish two estimates for the Hankel transform. A key ingredient is to effectively exploit the oscillatory property of the Bessel function and  $e^{it|\xi|}$  to obtain more decay.

**The Hankel transform and the solution.** Let us consider (1-1) in polar coordinates. Write  $v(t, r, \theta) = u(t, r\theta)$  and  $g(r, \theta) = f(r\theta) = f(x)$ . Then  $v(t, r, \theta)$  satisfies

$$(3-1) \quad \begin{cases} \partial_{tt} v - \partial_{rr} v - \frac{n-1}{r} \partial_r v - \frac{1}{r^2} \Delta_\theta v + \frac{a}{r^2} v = 0, \\ v(0, r, \theta) = 0, \quad \partial_t v(0, r, \theta) = g(r, \theta). \end{cases}$$

We use the spherical harmonic expansion to write

$$(3-2) \quad g(r, \theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(r) Y_{k,l}(\theta).$$

Using separation of variables, we can write  $v$  as a superposition

$$(3-3) \quad v(t, r, \theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} v_{k,l}(t, r) Y_{k,l}(\theta),$$



where  $v_{k,l}$  satisfies

$$(3-4) \quad \begin{cases} \partial_{tt} v_{k,l} - \partial_{rr} v_{k,l} - \frac{n-1}{r} \partial_r v_{k,l} + \frac{k(k+n-2)+a}{r^2} v_{k,l} = 0, \\ v_{k,l}(0, r) = 0, \quad \partial_t v_{k,l}(0, r) = a_{k,l}(r) \end{cases}$$

for each  $k, l \in \mathbb{N}, 1 \leq l \leq d(k)$ . Define

$$(3-5) \quad A_{v(k)} := -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{v^2(k) - \left(\frac{n-2}{2}\right)^2}{r^2}.$$

Then we are reduced to considering the system

$$(3-6) \quad \begin{cases} \partial_{tt} v_{k,l} + A_{v(k)} v_{k,l} = 0, \\ v_{k,l}(0, r) = 0, \quad \partial_t v_{k,l}(0, r) = a_{k,l}(r). \end{cases}$$

Applying the Hankel transform to (3-6), we have, by Lemma 2.2,

$$(3-7) \quad \begin{cases} \partial_{tt} \tilde{v}_{k,l} + \rho^2 \tilde{v}_{k,l} = 0, \\ \tilde{v}_{k,l}(0, \xi) = 0, \quad \partial_t \tilde{v}_{k,l}(0, \xi) = b_{k,l}(\rho), \end{cases}$$

where

$$(3-8) \quad \tilde{v}_{k,l}(t, \rho) = (\mathcal{H}_v v_{k,l})(t, \rho), \quad b_{k,l}(\rho) = (\mathcal{H}_v a_{k,l})(\rho).$$

Solving this ODE and using the Hankel transform, we obtain

$$\begin{aligned} v_{k,l}(t, r) &= \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{v(k)}(r\rho) \tilde{v}_{k,l}(t, \rho) \rho^{n-2} d\rho \\ &= \frac{1}{2i} \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{v(k)}(r\rho) (e^{it\rho} - e^{-it\rho}) b_{k,l}(\rho) \rho^{n-2} d\rho. \end{aligned}$$

Therefore, we get

$$(3-9) \quad \begin{aligned} u(x, t) &= v(t, r, \theta) \\ &= \sum_{k=0}^\infty \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_{v(k)}(r\rho) \sin(t\rho) b_{k,l}(\rho) \rho^{n-2} d\rho \\ &= \sum_{k=0}^\infty \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \mathcal{H}_{v(k)}[\rho^{-1} \sin(t\rho) b_{k,l}(\rho)](r). \end{aligned}$$

**Estimates of Hankel transforms.** We now turn to some key estimates needed for proving the main theorem.

**Proposition 3.1.** *Let  $R \gg 1$  and let  $\varphi$  be a smooth function supported in the interval  $I := [\frac{1}{2}, 1]$  and taking values in  $[0, 1]$ . Then*

$$(3-10) \quad \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left| \int_0^{\infty} e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^2(\mathbb{R}; L_r^2([R, 2R])} \\ \leq C \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_{\rho}^2(I)},$$

where  $C$  is a constant independent of  $R$ .

*Proof.* Using the Plancherel theorem in  $t$ , we have

$$(3-11) \quad \text{LHS of (3-10)} \lesssim \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \|J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho)\|_{L_{\rho}^2}^2 \right)^{\frac{1}{2}} \right\|_{L_r^2([R, 2R])}.$$

With this, it is easy to verify (3-10) if we can prove that

$$(3-12) \quad \int_R^{2R} |J_k(r)|^2 dr \leq C,$$

where  $R \gg 1$  and  $C$  is independent of  $k$  and  $R$ . To prove (3-12), we write

$$(3-13) \quad \int_R^{2R} |J_k(r)|^2 dr = \int_{I_1} |J_k(r)|^2 dr + \int_{I_2} |J_k(r)|^2 dr + \int_{I_3} |J_k(r)|^2 dr,$$

where

$$I_1 = [R, 2R] \cap [0, k/2], \\ I_2 = [R, 2R] \cap [k/2, 2k], \\ I_3 = [R, 2R] \cap [2k, \infty].$$

By using (2-8) and (2-10) in Lemma 2.1, we have

$$(3-14) \quad \int_{I_1} |J_k(r)|^2 dr \leq C \int_{I_1} e^{-cr} r dr \leq C e^{-cR}, \quad \int_{I_3} |J_k(r)|^2 dr \leq C.$$

For the remaining interval, we write

$$\int_{I_2} |J_k(r)|^2 dr \leq \int_{\frac{k}{2}}^{2k} |J_k(r)|^2 dr \leq C \int_{\frac{k}{2}}^{2k} k^{-\frac{2}{3}} (1 + k^{-\frac{1}{3}} |r - k|)^{-\frac{1}{2}} dr \leq C,$$

where the last inequality follows from the fact that the integral is uniformly bounded (by  $2 + \sqrt{2}$ ) for all  $k > 0$ . Together with (3-14), this yields (3-12).  $\square$

**Proposition 3.2.** *Suppose  $R \gg 1$ . Let  $\varphi$  be a smooth function supported in the interval  $I := [\frac{1}{2}, 1]$  and taking values in  $[0, 1]$ .*

(i) *If  $K$  is finite, there exists a constant  $C_K$  independent of  $R$  such that*

$$(3-15) \quad \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ \leq C_K R^{-\frac{1}{2}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^1(I)}.$$

(ii) If  $K$  is infinite, there exists a constant  $C$  independent of  $R$  such that

$$(3-16) \quad \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ \leq CR^{-\frac{1}{2}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2(I)}.$$

*Proof.* We first prove (3-15). Recalling (2-5) we can write  $|J_{\nu(k)}(r)| \leq C_K r^{-\frac{1}{2}}$  when  $r \gg 1$ . By the Minkowski inequality and the Hausdorff–Young inequality in  $t$ , there exists a constant  $C_K$  independent of  $R$  such that

$$\left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ \leq C_K R^{-\frac{1}{2}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^1(I)}.$$

This proves (3-15). When  $K$  is infinite, we need to show a precise estimate uniform in  $K$ . We utilize the Schlöfli’s integral representation of the Bessel function (2-6) to write  $J_{\nu(k)}(r\rho) = E_{\nu(k)}(r\rho) + \tilde{J}_{\nu(k)}(r\rho)$ . By (2-7), the Minkowski inequality, and the Hausdorff–Young inequality in  $t$ , there exists a constant  $C$  independent of  $K$  and  $R$  such that

$$\left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty e^{-it\rho} E_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ \leq CR^{-1} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^1(I)}.$$

Thus, it remains to prove (3-16) with  $J_{\nu(k)}$  replaced by  $\tilde{J}_{\nu(k)}$ . We consider  $0 < \delta \ll 1$  to be fixed later, and write  $[-\pi, \pi] = I_1 \cup I_2 \cup I_3$ , with

$$I_1 = \{\theta : |\theta| \leq \delta\}, \\ I_2 = [-\pi, -\pi/2 - \delta] \cup [\pi/2 + \delta, \pi], \\ I_3 = [-\pi, \pi] \setminus (I_1 \cup I_2).$$

We define

$$(3-17) \quad \Phi_{r,k}(\theta) = \sin \theta - \frac{k\theta}{r},$$

and let  $\chi_\delta(\theta)$  be a smooth function satisfying

$$(3-18) \quad \chi_\delta(\theta) = \begin{cases} 1, & \theta \in [-\delta, \delta], \\ 0, & \theta \notin [-2\delta, 2\delta]. \end{cases}$$

Then write

$$(3-19) \quad \tilde{J}_k(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\Phi_{r,k}(\theta)} d\theta = \tilde{J}_k^1(r) + \tilde{J}_k^2(r) + \tilde{J}_k^3(r),$$

with

$$\begin{aligned} \tilde{J}_k^1(r) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ir\Phi_{r,k}(\theta)} \chi_\delta(\theta) d\theta, \\ \tilde{J}_k^2(r) &:= \frac{1}{2\pi} \int_{I_2} e^{ir\Phi_{r,k}(\theta)} d\theta, \\ \tilde{J}_k^3(r) &:= \frac{1}{2\pi} \int_{I_3} e^{ir\Phi_{r,k}(\theta)} (1 - \chi_\delta(\theta)) d\theta. \end{aligned}$$

When  $\theta \in I_2$ , the function  $\Phi'_{r,k}(\theta) = \cos \theta - k/r$  is monotonic in the intervals  $[-\pi, -\pi/2 - \delta]$  and  $[\pi/2 + \delta, \pi]$ , and satisfies

$$|\Phi'_{r,k}(\theta)| \geq \frac{k}{r} + |\cos \theta| \geq \sin \delta.$$

Then van der Corput's lemma ([Lemma 2.3](#)) gives, uniformly in  $k$ ,

$$(3-20) \quad \left| \frac{1}{2\pi} \int_{I_2} e^{ir\Phi_{r,k}(\theta)} d\theta \right| \leq c_\delta r^{-1}.$$

When  $\theta \in I_3$ , we have  $|\Phi''_{r,k}(\theta)| \geq \sin \delta$ , and [Lemma 2.3](#) again yields that

$$(3-21) \quad \left| \frac{1}{2\pi} \int_{I_3} e^{ir\Phi_{r,k}(\theta)} (1 - \chi_\delta(\theta)) d\theta \right| \leq c_\delta r^{-\frac{1}{2}}$$

uniformly in  $k$ .

Using arguments similar to those above, it follows from (3-20) and (3-21) that

$$\begin{aligned} & \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left| \int_0^{\infty} e^{-it\rho} (\tilde{J}_{\nu(k)}^2(r\rho) + \tilde{J}_{\nu(k)}^3(r\rho)) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ & \lesssim R^{-\frac{1}{2}} \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^1(I)}. \end{aligned}$$

To establish (3-16) with  $J_{\nu(k)}$  replaced by  $\tilde{J}_{\nu(k)}^1$ , we need to use the oscillation of

$e^{it\rho}$  effectively. To this end, we write the Fourier series as  $b_{k,l}(\rho) = \sum_j b_{k,l}^j e^{i\frac{\pi}{2}\rho j}$ , where

$$(3-22) \quad b_{k,l}^j = \frac{1}{4} \int_0^4 e^{-i\frac{\pi}{2}\rho j} b_{k,l}(\rho) d\rho.$$

Then  $\sum_j |b_{k,l}^j|^2 = \|b_{k,l}(\rho)\|_{L^2_\rho(I)}^2$ . For simplicity, we use the scaling argument to reduce the problem by replacing  $t$  and  $r$  by  $2\pi t$  and  $2\pi r$  respectively, and define

$$(3-23) \quad \psi_{t-j/4}^k(r) = \frac{1}{2\pi} \int_0^\infty e^{-2\pi i(t-\frac{j}{4})\rho} \int_{\mathbb{R}} e^{2\pi i\rho r \sin\theta - i\nu(k)\theta} \chi_\delta(\theta) d\theta \varphi(\rho) d\rho.$$

Let  $m = t - j/4$ . Then we write

$$(3-24) \quad \begin{aligned} \psi_m^k(r) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{2\pi i\rho(r \sin\theta - m)} e^{-i\nu(k)\theta} \chi_\delta(\theta) \varphi(\rho) d\rho d\theta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \check{\varphi}(r \sin\theta - m) e^{-i\nu(k)\theta} \chi_\delta(\theta) d\theta. \end{aligned}$$

For our purpose, we need to investigate the asymptotic behavior of the function  $\psi_m^k(r)$ . We consider two subcases:

(a)  $4R \leq |m|$ .  $R \geq 1$ , hence  $|m| \geq 4$ . Since  $\check{\varphi}$  is a Schwartz function, we have

$$|\check{\varphi}(r \sin\theta - m)| \leq C_N (1 + |r \sin\theta - m|)^{-N} \quad \text{for all } N > 0.$$

On the other hand, we have

$$|r \sin\theta - m| \geq |m| - r|\sin\theta| \geq \frac{1}{100}|m|,$$

since  $r \leq 2R \leq |m|$  and  $|\theta| \leq 2\delta$ . Thus, (3-24) gives

$$(3-25) \quad |\psi_m^k(r)| \leq C_{\delta,N} (1 + |m|)^{-N}.$$

Keeping in mind that  $m = t - j/4$ , we have

$$\begin{aligned} & \left\| \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} \left| \sum_{\substack{j: \\ 4R \leq |t-j/4|}} b_{k,l}^j \psi_{t-j/4}^k(r) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_r^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\ & \leq C_{\delta,N} R^{-N} \left\| \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} \left| \sum_{\substack{j: \\ 4R \leq |t-j/4|}} |b_{k,l}^j| (1 + |t - \frac{j}{4}|)^{-N} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_r^\infty(\mathbb{R}; L_r^\infty([R, 2R])}. \end{aligned}$$

By the Cauchy–Schwarz inequality, and choosing  $N$  large enough, the above is bounded by

$$\begin{aligned}
 C_{\delta,N} R^{-N} & \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \sum_j |b_{k,l}^j|^2 (1 + |t - \frac{j}{4}|)^{-N} \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\
 & \leq C_{\delta,N} R^{-N} \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \sum_j |b_{k,l}^j|^2 \right)^{\frac{1}{2}} \lesssim R^{-N} \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2(I)}.
 \end{aligned}$$

(b)  $|m| < 4R$ . Again, since  $\check{\varphi}$  is a Schwartz function,

$$(3-26) \quad |\check{\varphi}(r \sin \theta - m)| \leq C_N (1 + |r \sin \theta - m|)^{-N} \quad \text{for all } N > 0.$$

By (3-24), this gives

$$|\psi_m^k(r)| \leq \frac{C_N}{2\pi} \left( \int_{\substack{\{\theta: |\theta| < 2\delta, \\ |r \sin \theta - m| \geq 1\}}} d\theta + \int_{\substack{\{\theta: |\theta| < 2\delta, \\ |r \sin \theta - m| \geq 1\}}} (1 + |r \sin \theta - m|)^{-N} d\theta \right).$$

Let  $y = r \sin \theta - m$ ; then

$$(3-27) \quad |\psi_m^k(r)| \leq \frac{C_N}{2\pi r} \left( \int_{\{|y| \leq 1\}} dy + \int_{\{|y| \geq 1\}} (1 + |y|)^{-N} dy \right) \lesssim \frac{1}{r}.$$

For fixed  $t, R$ , we define the set  $A = \{j \in \mathbb{Z} : |t - j/4| \leq 4R\}$ . It is easy to see the cardinality of  $A$  is  $O(R)$ . Thus, it follows from (3-27) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
 & \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \left| \sum_{j \in A} b_{k,l}^j \psi_{t-j/4}^k(r) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty(\mathbb{R}; L_r^\infty([R, 2R])} \\
 & \leq C_{\delta,N} R^{-\frac{1}{2}} \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} \sum_j |b_{k,l}^j|^2 \right)^{\frac{1}{2}} \lesssim R^{-\frac{1}{2}} \left\| \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2(I)} \quad \square
 \end{aligned}$$

**Proposition 3.3.** *Let  $\varphi$  be a smooth function supported on  $I = [\frac{1}{2}, 1]$  and taking values in  $[0, 1]$ , and let  $R$  be a positive real number. Assume (1-5) is satisfied, and consider the quantity*

$$\begin{aligned}
 Q = & \left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty e^{-it\rho} J_{\nu(k)}(r\rho) b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q(\mathbb{R}; L_{r,n-1,dr}^q([R, 2R])}.
 \end{aligned}$$

(Recall the definition of  $\nu = \nu(k)$  in (2-11).)

(i) *When  $K$  is finite, there exists a constant  $C_K$  independent of  $R$  such that*

$$(3-28) \quad Q \leq C_K \min \left\{ R^{\frac{n}{q}}, R^{-\frac{n-1}{2} \left( 1 - \frac{2n-1}{q(n-1)} \right)} \right\} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^p(I)}.$$

(ii) When  $K$  is infinite, there exists a constant  $C$  independent of  $R$  such that

$$(3-29) \quad Q \leq C \min \left\{ R^{\frac{n}{q}}, R^{-\frac{n-1}{2} \left(1 - \frac{2n}{q(n-1)}\right)} \right\} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2(I)}.$$

*Proof.* We first consider the case  $R \lesssim 1$ . The Minkowski inequality and the Hausdorff–Young inequality in  $t$  show that

$$Q \lesssim \left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \|J_{\nu(k)}(r\rho)b_{k,l}(\rho)\varphi(\rho)\|_{L_\rho^{q'}}^2 \right)^{\frac{1}{2}} \right\|_{L_{r^{n-1} dr}^q([R, 2R])}.$$

Hence, by (2-4), there exists a constant  $C$  independent of  $K$  such that

$$Q \leq C \left( \int_R^{2R} r^{-\frac{(n-2)q}{2}} \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \frac{(8\pi r)^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \right|^2 \|\varphi(\rho)b_{k,l}(\rho)\|_{L_\rho^{q'}}^2 \right)^{\frac{q}{2}} r^{n-1} dr \right)^{\frac{1}{q}}$$

and

$$Q \leq CR^{\frac{n}{q}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |\varphi(\rho)b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^{q'}(I)} r.$$

Secondly, we consider the case  $R \gg 1$ . By Propositions 3.1 and 3.2, we use interpolation to obtain

$$\begin{aligned} & \left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \left| \int_0^\infty J_\nu(r\rho) e^{it\rho} b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q(\mathbb{R}; L_{r^{n-1} dr}^q([R, 2R]))} \\ & \leq C_K R^{-\frac{n-1}{2} \left(1 - \frac{2n}{q(n-1)}\right)} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |\varphi(\rho)b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^{q'}(I)}. \end{aligned}$$

When  $K$  is infinite,

$$\begin{aligned} & \left\| r^{-\frac{n-2}{2}} \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} \left| \int_0^\infty J_\nu(r\rho) e^{it\rho} b_{k,l}(\rho) \varphi(\rho) d\rho \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q(\mathbb{R}; L_{r^{n-1} dr}^q([R, 2R]))} \\ & \leq CR^{-\frac{n-1}{2} \left(1 - \frac{2n}{q(n-1)}\right)} \left\| \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} |\varphi(\rho)b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2(I)}. \end{aligned}$$

In view of (1-5) and since  $\text{supp } \varphi \subset [\frac{1}{2}, 1]$ , this shows (3-28) and (3-29).  $\square$

*Conclusion of the proof of the Main Theorem.* We know that if  $f$  is radial, so is  $u$  in (3-9). To prove the Main Theorem, we need to estimate the following by (3-9):

$$(3-30) \quad \left\| \sum_{k=0}^K \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \mathfrak{H}_\nu(\rho^{-1} \sin(t\rho)b_{k,l}(\rho))(r) \right\|_{L_t^q(\mathbb{R}; L_{r^{n-1} dr}^q L_\theta^2(\mathbb{S}^{n-1}))}$$

in the cases of  $K = 0$  and  $K = \infty$ , which correspond to the radial case and the general case respectively. To this end, we use orthogonality and apply a dyadic decomposition to (3-30) to obtain the estimate

(3-30)

$$\begin{aligned} &\leq C \left( \sum_R \left( \sum_M \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} J_\nu(r\rho) e^{it\rho} \chi\left(\frac{\rho}{M}\right) \right. \right. \right. \right. \\ &\quad \left. \left. \left. \times \left| \int_0^\infty (r\rho)^{-\frac{n-2}{2}} b_{k,l}(\rho) \rho^{n-2} d\rho \right|^2 \right\| \right\|_{L_t^q(\mathbb{R}; L_{r^{n-1} dr}^q([R, 2R]))} \right)^q \right)^{\frac{1}{q}}, \\ &=: \Sigma \end{aligned}$$

where  $R$  and  $M$  are dyadic numbers and  $\chi$  is a smooth function supported on  $[\frac{1}{2}, 1]$  and taking values in  $[0, 1]$ . By a scaling argument, we have

$$\begin{aligned} \Sigma &\leq C \left( \sum_R \left( \sum_M M^{(n-1)-\frac{n+1}{q}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \left| \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho) e^{it\rho} \chi(\rho) b_{k,l}(M\rho) \rho^{n-2} d\rho \right|^2 \right\| \right\|_{L_t^q(\mathbb{R}; L_{r^{n-1} dr}^q([RM, 2RM]))} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Applying Proposition 3.3 with  $\varphi(\rho) = \chi(\rho) \rho^{\frac{n}{2}-1}$  to the above, one can see that when  $K$  is finite,

$$\begin{aligned} \Sigma &\leq C_K \left( \sum_R \left( \sum_M \min \left\{ (RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}} \left(1 - \frac{2n}{q(n-1)}\right) \right\} \right. \right. \\ &\quad \left. \left. \times M^{(n-1)-\frac{n+1}{q}} \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |\chi(\rho) \rho^{\frac{n}{2}-1} b_{k,l}(M\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^p} \right)^q \right)^{\frac{1}{q}}, \end{aligned}$$

and when  $K$  is infinite,

$$\begin{aligned} \Sigma &\leq C \left( \sum_R \left( \sum_M \min \left\{ (RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}} \left(1 - \frac{2n}{q(n-1)}\right) \right\} \right. \right. \\ &\quad \left. \left. \times M^{(n-1)-\frac{n+1}{q}} \left\| \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} |\chi(\rho) \rho^{\frac{n}{2}-1} b_{k,l}(M\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^2} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Since  $q > 2n/(n-1)$ , one has

$$\begin{aligned} &\sup_R \sum_M \min \left\{ (RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}} \left(1 - \frac{2n}{q(n-1)}\right) \right\} < \infty, \\ &\sup_M \sum_R \min \left\{ (RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}} \left(1 - \frac{2n}{q(n-1)}\right) \right\} < \infty. \end{aligned}$$



Then by Schur's test lemma and the embedding  $l^p \hookrightarrow l^q$  with  $q > \frac{2n}{n-1} > p$ , we have, in the case when  $K$  is finite,

$$\begin{aligned} \Sigma &\leq C_K \left( \sum_M M^{((n-1) - \frac{n+1}{q})p} \left\| \chi(\rho) \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(M\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_\rho^p}^p \right)^{\frac{1}{p}} \\ &\leq C_K \left( \sum_M M^{(\frac{n-1}{p'} - \frac{n+1}{q})p} \left\| \chi\left(\frac{\rho}{M}\right) \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \rho^{\frac{n-2}{p}} \right\|_{L_\rho^p}^p \right)^{\frac{1}{p}} \\ &\leq C_K \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_{\rho^{n-2}d\rho}^p(\mathbb{R}^+)}, \end{aligned}$$

and in the case when  $K$  is infinite and  $p \geq 2$ ,

$$\begin{aligned} \Sigma &\leq C \left( \sum_M M^{(\frac{n-1}{p'} - \frac{n+1}{q})q} \left\| \chi\left(\frac{\rho}{M}\right) \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \rho^{\frac{n-2}{2}} \right\|_{L_\rho^p}^q \right)^{\frac{1}{q}} \\ &\leq C \left\| \left( \sum_{k=0}^\infty \sum_{l=1}^{d(k)} |b_{k,l}(\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_{\rho^{n-2}d\rho}^p(\mathbb{R}^+)}. \end{aligned}$$

By [Lemma 2.2](#), we have  $b_{k,l}(\rho) = \mathcal{H}_{\nu(k)} \mathcal{H}_{\mu(k)} [\mathcal{H}_{\mu(k)} a_{k,l}](\rho)$ . To proceed we make use of the following fact, whose proof we defer to the [Appendix](#):

**Claim.** For the measure space  $(\mathbb{R}^+, dw(\rho))$ , where  $dw(\rho) = \rho^{n-2} d\rho$ , and for  $1 < p < \infty$ , denote by  $L^p(w)$  the corresponding Lebesgue space equipped with the norm

$$\|f\|_p = \left( \int_0^\infty |f|^p dw \right)^{\frac{1}{p}}.$$

Let  $\mathcal{H}_{\nu(k)}, \mathcal{H}_{\mu(k)}$  be the Hankel transforms defined above and suppose that

$$(3-31) \quad \frac{n-2}{2} - \nu(0) < \frac{n-1}{p} < \frac{n-2}{2} + \mu(0) + 2.$$

Then there exists a constant  $C$  such that, for any  $\{f_k\}_{k=0}^\infty \in L^p(w; l^2)$ , we have

$$(3-32) \quad \left\| \left( \sum_k |\mathcal{H}_{\nu(k)} \mathcal{H}_{\mu(k)} f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \leq C \left\| \left( \sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}.$$

Condition (3-31) is satisfied because  $1 < p < \frac{2n}{n-1}$  by the [Main Theorem's](#) assumptions. Thus, applying the [Claim](#) we get

$$\Sigma \leq C \left\| \left( \sum_{k=0}^K \sum_{l=1}^{d(k)} |[\mathcal{H}_{\mu(k)} a_{k,l}](\rho)|^2 \right)^{\frac{1}{2}} \right\|_{L_{\rho^{n-2}d\rho}^p(\mathbb{R}^+)}.$$

By (2-2) and (2-13), under the conditions of the [Main Theorem](#), we further have

$$\Sigma \leq C \left\| |\xi|^{-\frac{1}{p}} \hat{f}(\xi) \right\|_{L^p_{\rho^{n-1} d\rho}(\mathbb{R}^+; L^2_{\omega}(\mathbb{S}^{n-1}))} \leq C \left\| |\xi|^{-\frac{1}{p}} \hat{f}(\xi) \right\|_{L^p_{\xi}(\mathbb{R}^n)}.$$

This completes the proof.  $\square$

### Appendix: Proof of (3-32)

Let  $T_k = \mathcal{H}_{\mu(k)} \mathcal{H}_{\nu(k)}$ , and set  $\lambda = \frac{n-2}{2}$ . We first show that

$$(A-1) \quad \|T_k f_k\|_{L^p(w)} \leq C \|f_k\|_{L^p(w)},$$

by following the argument used to prove Theorem 3.1 in [\[Planchon et al. 2003b\]](#). By that argument, we can write

$$(T_k f_k)(\rho) = \int_0^{\infty} k_{\nu, \mu}^0(\rho, s) f_k(s) s^{n-1} ds,$$

where the kernel is given by

$$k_{\alpha, \beta}^0(\rho, s) = \begin{cases} A_{\alpha, \beta} \frac{s^{\beta-\lambda}}{\rho^{\lambda+\beta+2}} F\left(\frac{\alpha+\beta}{2} + 1, \frac{\beta-\alpha}{2} + 1; \beta + 1; \left(\frac{s}{\rho}\right)^2\right) & \text{for } s < \rho, \\ A_{\beta, \alpha} \frac{s^{\alpha-\lambda}}{\rho^{\lambda+\alpha+2}} F\left(\frac{\beta+\alpha}{2} + 1, \frac{\alpha-\beta}{2} + 1; \alpha + 1; \left(\frac{\rho}{s}\right)^2\right) & \text{for } s > \rho, \end{cases}$$

where  $F(a, b; c; d)$  is the hypergeometric function and

$$A_{\alpha, \beta} = \frac{2\Gamma(\frac{\alpha+\beta}{2} + 1)}{\Gamma(\frac{\beta-\alpha}{2})\Gamma(\beta + 1)}.$$

When  $s$  is near  $\rho$ , the kernel  $k_{\nu, \mu}^0(\rho, s)$  behaves like  $c(\rho - s)^{-1} + O(-\log |\rho - s|)$ . Define

$$(\tilde{T}_k [s^{\frac{n-1}{p}} f_k(s)])(\rho) := \int_0^{\infty} \tilde{k}_{\nu, \mu}^0(\rho, s) [s^{\frac{n-1}{p}} f_k(s)] \frac{ds}{s},$$

where

$$\tilde{k}_{\nu, \mu}^0(\rho, s) = \rho^{\frac{n-1}{p}} k_{\nu, \mu}^0(\rho, s) s^{n-\frac{n-1}{p}}.$$

Then

$$(\tilde{T}_k [s^{\frac{n-1}{p}} f_k(s)])(\rho) = \int_0^{\infty} \rho^{\frac{n-1}{p}} k_{\nu, \mu}^0(\rho, s) s^{n-\frac{n-1}{p}} [s^{\frac{n-1}{p}} f_k(s)] \frac{ds}{s} = \rho^{\frac{n-1}{p}} (T_k f_k)(\rho).$$

Note that

$$\|(T_k f_k)(\rho)\|_{L^p(w)} = \left\| \rho^{\frac{n-1}{p}} (T_k f_k)(\rho) \right\|_{L^p_{\rho^{-1} d\rho}}.$$

To prove (A-1), it suffices to show

$$(A-2) \quad \|\tilde{T}_k f_k\|_{L^p(\rho^{-1} d\rho)} \leq C \|f_k\|_{L^p(\rho^{-1} d\rho)}.$$

Again by the argument in [Planchon et al. 2003b], one has

$$|k_{\nu,\mu}^0(\rho, s)| = \begin{cases} O(\rho^{-\lambda-\mu-2+\epsilon} s^{-\lambda+\mu-\epsilon}) & \text{for } s < \rho, \\ O(\rho^{\nu-\lambda-\epsilon} s^{-\lambda-\nu-2+\epsilon}) & \text{for } s > \rho. \end{cases}$$

Then

$$|\tilde{k}_{\nu,\mu}^0(\rho, s)| = \begin{cases} O((s/\rho)^{\lambda+\mu+2-\frac{n-1}{p}-\epsilon}) & \text{for } s < \rho, \\ O((s/\rho)^{\lambda-\nu-\frac{n-1}{p}+\epsilon}) & \text{for } s > \rho. \end{cases}$$

Since  $\lambda - \nu < (n - 1)/p < \lambda + \mu + 2$ , the kernel  $\tilde{k}_{\nu,\mu}^0$  is bounded in  $L^1(d\rho/\rho)$ . Using logarithmic coordinates, we express the operator  $T_k$  as a convolution operator with the kernel  $\tilde{k}_{\nu,\mu}^0$ . When  $s \sim \rho$ , we recall that  $\tilde{k}_{\nu,\mu}^0$  is a Calderón–Zygmund kernel behaving like

$$c(\rho - s)^{-1} + O(-\log|\rho - s|).$$

Applying Young’s inequality to the region away from  $\rho \sim s$  and Calderón–Zygmund theory to the region  $\rho \sim s$ , we obtain (A-2), and so (A-1).

By a similar argument, we show the adjoint operator  $T_k^*$  is also bounded in  $L^{p'}(w)$ , provided  $\lambda - \mu < (n - 1)/p' < \lambda + \nu + 2$ , which is true for  $1 < p < \infty$ .

Now we are ready to show (3-32). We consider two cases:

If  $1 < p \leq 2$ , since the adjoint operator  $T_k^* = \mathcal{H}_{\nu(k)}\mathcal{H}_{\mu(k)}$  is bounded in  $L^{p'}(w)$ , we get (3-32) by duality.

If  $2 \leq p < \infty$ , we set  $q := p/2$  (forgetting the earlier value of  $q$ ). Then  $q \geq 1$ , and we have

$$\begin{aligned} \left\| \left( \sum_k |T_k f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}^2 &= \left\| \sum_k |T_k f_k|^2 \right\|_{L^q(w)} \\ &= \sup_{\substack{g \geq 0 \\ g \in L^{q'}(w)}} \left| \int_0^\infty \sum_k |T_k f_k|^2 g(\rho) \rho^{n-2} d\rho \right| \\ &= \sum_k \sup_{\substack{g \geq 0 \\ g \in L^{q'}(w)}} \int_0^\infty |T_k f_k|^2 g(\rho) \rho^{n-2} d\rho \\ &= \sum_k \|T_k f_k\|_{L^p(w)}^2. \end{aligned}$$

By (A-1), we see that

$$\begin{aligned} \left\| \left( \sum_k |T_k f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}^2 &\leq C \sum_k \|f_k\|_{L^p(w)}^2 \\ &= C \sum_k \sup_{\substack{g \geq 0 \\ g \in L^{q'}(w)}} \left| \int_0^\infty |f_k|^2 g(\rho) \rho^{n-2} d\rho \right| \end{aligned}$$

$$\begin{aligned} &\leq C \sup_{\substack{g \geq 0 \\ g \in L^{q'}(w)}} \int_0^\infty \sum_k |f_k|^2 g(\rho) \rho^{n-2} d\rho \\ &\leq C \left\| \sum_k |f_k|^2 \right\|_{L^q(w)}^2 \leq C \left\| \left( \sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}^2. \end{aligned}$$

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