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We show that quasisymmetric homeomorphisms between (most) reducible Carnot groups are bilipschitz. This implies rigidity for quasi-isometries between certain negatively curved homogeneous manifolds. The proof uses Pansu's differentiability theorem for quasisymmetric homeomorphisms between Carnot groups.

1. Introduction

We study quasisymmetric homeomorphisms between reducible Carnot groups. The main result says that in most cases, the quasisymmetric homeomorphism must be bilipschitz.

A Carnot group is *reducible* if it is isomorphic to the direct product of two Carnot groups. Otherwise, a Carnot group is called *irreducible*. A reducible Carnot group G can be written as $G = G_0 \times G_1 \times \cdots \times G_m$, where G_0 is abelian (i.e., isomorphic to some $\mathbb{R}^n$), and G_j $(1 \leq j \leq m)$ is nonabelian irreducible. Such a decomposition is not unique in general; see Example 2.1.

All Carnot groups in this paper are equipped with the Carnot metric (see Section 3).

Theorem 1.1. *Let $F : G \to G'$ be a quasisymmetric map between two Carnot groups. Suppose G is reducible and admits a direct product decomposition of irreducible Carnot groups where at least two of the factors are not isomorphic. Then F is bilipschitz.*

The same claim remains open in the case when G is isomorphic to a direct product $N \times \cdots \times N$, where N is nonabelian irreducible.

Quasisymmetric homeomorphisms between general metric spaces are quasiconformal. In the case of Carnot groups (and of Loewner spaces more generally), a map is quasisymmetric if and only if it is quasiconformal (see [Heinonen and Koskela 1998]).

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Theorem 1.1 has consequences for the rigidity of quasi-isometries between certain negatively curved homogeneous manifolds. Recall that a quasi-isometry between two metric spaces is an almost isometry if it preserves distance up to an additive constant. A quasi-isometry between two negatively curved spaces induces a quasisymmetric homeomorphism between the ideal boundaries (of the negatively curved spaces), where the ideal boundaries are equipped with visual metrics. Conversely, under mild conditions on the negatively curved spaces, each quasisymmetric homeomorphism between the ideal boundaries is the boundary map of a quasi-isometry; see [Bonk and Schramm 2000]. Similarly, almost isometries between negatively curved spaces correspond to bilipschitz maps between the ideal boundaries [ibid.]. On the other hand, Carnot groups arise as the ideal boundary of certain negatively curved homogeneous manifolds (see below for more details). Hence a direct consequence of Theorem 1.1 is that each quasi-isometry between certain negatively curved homogeneous manifolds is an almost isometry.

Heintze [1974] characterized homogeneous manifolds with negative sectional curvature (HMNs): Each HMN is isometric to a simply connected solvable Lie group S equipped with a left invariant Riemannian metric, and furthermore $S = N \rtimes \mathbb{R}$ is a semidirect product of a nilpotent Lie group N with $\mathbb{R}$, where $\mathbb{R}$ acts on N by expanding (and contracting) automorphisms; conversely, every semidirect product as above admits a left invariant Riemannian metric with negative sectional curvature (hence is an HMN). The ideal boundary of an HMN $S = N \rtimes \mathbb{R}$ can be naturally identified with (the one-point compactification of) N. On the other hand, each Carnot group N is a simply connected nilpotent Lie group having a one-parameter family of dilations (see Section 3 for more details). These dilations induce an action of $\mathbb{R}$ on the Carnot group by expanding (and contracting) automorphisms, so there is an HMN $N \rtimes \mathbb{R}$ associated with each Carnot group N. It follows that each Carnot group can be identified with the ideal boundary of some HMN. Hence Theorem 1.1 implies that each quasi-isometry between these HMNs is an almost isometry.

We next make some comments about the proof of Theorem 1.1. A main step in the proof is to show that the quasisymmetric homeomorphism preserves a certain foliation. Then the arguments in [Shanmugalingam and Xie 2012] show that the quasisymmetric homeomorphism is bilipschitz. To show that the quasisymmetric homeomorphism preserves a foliation, one first proves that infinitesimally it preserves a foliation. The global result then follows by integration. Recall that Pansu's differentiability theorem (see [Pansu 1989] or Section 3) says that a quasisymmetric homeomorphism $F : G \to G'$ between Carnot groups is Pansu-differentiable a.e., and the Pansu differential is a.e. a graded isomorphism between the two Carnot groups. Under the assumption of Theorem 1.1, we show that there are (proper) connected and simply connected subgroups $N \subset G$ and $N' \subset G'$ such that $\phi(N) = N'$ for every graded isomorphism $\phi : G \to G'$; see Section 2.

In Section 2 we show that graded isomorphisms between reducible Carnot algebras preserve certain subalgebras, which implies that graded isomorphisms between reducible Carnot groups preserve certain subgroups (as indicated in the preceding paragraph). And in Section 3 we show that quasisymmetric homeomorphisms are bilipschitz.

2. Graded isomorphisms of Carnot algebras

In this section we show that graded isomorphisms between reducible Carnot algebras preserve certain subalgebras. This implies that graded isomorphisms between reducible Carnot groups preserve certain subgroups (see Section 3).

A *Carnot Lie algebra* is a finite-dimensional Lie algebra $\mathscr{G}$ together with a direct sum decomposition $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ of vector subspaces such that $[V_1, V_i] = V_{i+1}$ for all $1 \leq i \leq r$, where we set $V_{r+1} = \{0\}$. The integer r is called the degree of nilpotency of $\mathscr{G}$. Every Carnot algebra $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ admits a one-parameter family of automorphisms $\lambda_t : \mathscr{G} \to \mathscr{G}$ for $t \in (0, \infty)$, where $\lambda_t(x) = t^i x$ for $x \in V_i$. Let $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and $\mathscr{G}' = V_1' \oplus V_2' \oplus \cdots \oplus V_s'$ be two Carnot algebras. A Lie algebra homomorphism $\phi : \mathscr{G} \to \mathscr{G}'$ is graded if ϕ commutes with λ_t for all $t > 0$; that is, if $\phi \circ \lambda_t = \lambda_t \circ \phi$. We observe that $\phi(V_i) \subset V_i'$ for all $1 \leq i \leq r$.

A Carnot algebra $\mathscr{G}$ is called reducible if there exist two nontrivial Carnot algebras $\mathscr{G}_1$ and $\mathscr{G}_2$ and a graded isomorphism between $\mathscr{G}$ and $\mathscr{G}_1 \oplus \mathscr{G}_2$. It is called irreducible otherwise. The finite dimensionality implies that every reducible Carnot algebra $\mathscr{G}$ can be written as a direct sum of Carnot algebras $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$, where $\mathscr{G}_0$ is abelian, and each $\mathscr{G}_i$ with $i \geq 1$ is nonabelian and irreducible.

Let $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be a Carnot algebra. When $\mathscr{G}$ is reducible, it also has a decomposition $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$ as a direct sum of an abelian factor and irreducible nonabelian factors. We are interested in the question whether graded isomorphisms preserve such decompositions (after possibly permuting the factors). This question is equivalent to the uniqueness problem of such a decomposition. In general the decomposition is not unique, as the following example shows. The author thanks Bruce Kleiner for suggesting the example.

Example 2.1 (Kleiner). Let $\mathscr{G}$ be a nonabelian irreducible Carnot algebra, and let $f : \mathscr{G} \to \mathbb{R}^m$ be a nontrivial Lie algebra homomorphism into an abelian group. Let $G(f) \subset \mathscr{G} \oplus \mathbb{R}^m$ be the graph of f. Then $G(f)$ is a Carnot algebra (being isomorphic to $\mathscr{G}$), and $\mathscr{G} \oplus \mathbb{R}^m$ has two different decompositions $\mathscr{G} \oplus \mathbb{R}^m = G(f) \oplus \mathbb{R}^m$. Alternatively, let $g : \mathscr{G} \oplus \mathbb{R}^m \to \mathscr{G} \oplus \mathbb{R}^m$ be the map given by $g(x, a) = (x, a + f(x))$. Then g is a graded isomorphism and it does not preserve the factor $\mathscr{G}$.

Despite this example, we show that graded isomorphisms always preserve the abelian factor (Proposition 2.4) and, in the case of a trivial abelian factor, preserve the decomposition after possibly permuting the factors (Proposition 2.5).

Definition 2-1. Let $\mathscr{G}$ be a Lie algebra and $x \in \mathscr{G}$. Define $d(x) = \dim(\ker(\operatorname{ad} x))$, where $\operatorname{ad} x : \mathscr{G} \to \mathscr{G}$ is the linear map given by $\operatorname{ad} x(y) = [x, y]$.

Lemma 2.2. *If $d(x) = \dim \mathscr{G} > 1$ for some $x \in V_1 \backslash \{0\}$, then $\mathscr{G}$ is reducible.*

Proof. Note $[x, y] = 0$ for all $y \in \mathscr{G}$. Let $\mathscr{G}_1$ be the one-dimensional subspace of V_1 spanned by x, and let W be a complementary subspace of $\mathscr{G}_1$ in V_1. Set $\mathscr{G}_2 = W \oplus V_2 \oplus \cdots \oplus V_r$. Then $\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_2$ is a direct sum of vector subspaces. The assumption on x now implies that $\mathscr{G}_1$ and $\mathscr{G}_2$ are ideals of $\mathscr{G}$ and that both $\mathscr{G}_1$ and $\mathscr{G}_2$ are Carnot algebras. Hence $\mathscr{G}$ is reducible. □

The next lemma provides an intrinsic characterization of the abelian factor $\mathscr{G}_0$.

Lemma 2.3. *Let $\mathscr{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be a Carnot algebra, and consider a direct sum decomposition $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$ of $\mathscr{G}$ into an abelian factor and irreducible nonabelian factors. Let $x \in V_1$. Then $x \in \mathscr{G}_0$ if and only if $d(x) = \dim \mathscr{G}$.*

Proof. It is clear that $x \in \mathscr{G}_0$ implies $d(x) = \dim \mathscr{G}$. We assume $d(x) = \dim \mathscr{G}$ and shall prove that $x \in \mathscr{G}_0$. Note $[x, y] = 0$ for all $y \in \mathscr{G}$. Write $x = x_0 + x_1 + \cdots + x_m$ with $x_i \in \mathscr{G}_i \cap V_1$. Suppose $x \notin \mathscr{G}_0$. Then $x_i \neq 0$ for some $i \geq 1$. Since $[x_i, y] = [x, y] = 0$ for all $y \in \mathscr{G}_i$, Lemma 2.2 implies $\mathscr{G}_i$ is reducible, contradicting the assumption. □

Recall that the goal of this section is to show that a graded isomorphism of reducible Carnot algebras preserves certain Lie subalgebras. The case when the abelian factor is nontrivial is covered by Proposition 2.4.

Proposition 2.4. *Let $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$ and $\mathscr{G}' = \mathscr{G}'_0 \oplus \mathscr{G}'_1 \oplus \cdots \oplus \mathscr{G}'_n$ be two reducible Carnot algebras written as direct sums of an abelian factor and irreducible nonabelian factors. Let $\phi : \mathscr{G} \to \mathscr{G}'$ be a graded isomorphism. Then $\phi(\mathscr{G}_0) = \mathscr{G}'_0$.*

Proof. By Lemma 2.3, $\phi(\mathscr{G}_0) \subset \mathscr{G}'_0$. Since ϕ is an isomorphism, the conclusion follows by considering ϕ^{-1}. □

Proposition 2.5 treats the case when the abelian factor is trivial.

Proposition 2.5. *Let $\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$ and $\mathscr{G}' = \mathscr{G}'_0 \oplus \mathscr{G}'_1 \oplus \cdots \oplus \mathscr{G}'_n$ be two reducible Carnot algebras written as direct sums of an abelian factor and irreducible nonabelian factors. Let $\phi : \mathscr{G} \to \mathscr{G}'$ be a graded isomorphism. Suppose $\mathscr{G}$ has no abelian factor (that is, $\mathscr{G}_0 = \{0\}$). Then $\mathscr{G}'_0 = \{0\}$, $m = n$ and after possibly permuting the factors $\mathscr{G}'_1, \ldots, \mathscr{G}'_m$, there exist graded isomorphisms $\phi_i : \mathscr{G}_i \to \mathscr{G}'_i$ such that $\phi = \phi_1 \oplus \cdots \oplus \phi_m$.*

Now we start the proof of Proposition 2.5. First observe that Proposition 2.4 implies $\mathscr{G}'_0 = \{0\}$. In the following proofs, we shall use both decompositions $\mathscr{G} = V_1 \oplus \cdots \oplus V_r = \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_m$ of $\mathscr{G}$, as well as those for $\mathscr{G}'$.

Lemma 2.6. *Let $x \in V_1$. Write $x = x_{i_1} + \cdots + x_{i_k}$ $(1 \le i_1 < \cdots < i_k \le m)$ with $x_{i_j} \in (\mathcal{G}_{i_j} \cap V_1)\backslash\{0\}$. If $k \ge 2$, then $d(x_{i_j}) > d(x)$.*

Proof. We first show that $\ker(\operatorname{ad} x) \subset \ker(\operatorname{ad} x_{i_j})$ for all $1 \le j \le k$. Let $y \in \ker(\operatorname{ad} x)$. Write $y = y_1 + \cdots + y_m$ with $y_i \in \mathcal{G}_i$. Then $0 = [x, y] = [x_{i_1}, y_{i_1}] + \cdots + [x_{i_k}, y_{i_k}]$. Since $[x_{i_j}, y_{i_j}] \in \mathcal{G}_{i_j}$, we have $[x_{i_j}, y_{i_j}] = 0$. Hence $[x_{i_j}, y] = [x_{i_j}, y_{i_j}] = 0$; that is, $y \in \ker(\operatorname{ad} x_{i_j})$.

Next we shall find an element $y \in \ker(\operatorname{ad} x_{i_j})\backslash \ker(\operatorname{ad} x)$. Since $k \ge 2$, there is some $1 \le l \le k$ with $l \ne j$. By Lemma 2.2, since $\mathcal{G}_{i_l}$ is nonabelian and irreducible, there is some $y \in \mathcal{G}_{i_l}$ such that $[x_{i_l}, y] \ne 0$. Now notice that $[x_{i_j}, y] = 0$ and $[x, y] = [x_{i_l}, y] \ne 0$. □

For each $1 \le i \le m$, set

$$A_i = \{x \in \mathcal{G}_i \cap V_1 : \phi(x) \in \mathcal{G}'_j \text{ for some } j\}.$$

Let $N_i \subset \mathcal{G}_i \cap V_1$ be the vector subspace spanned by A_i. Similarly, for each $1 \le j \le n$ set

$$A'_j = \{y \in \mathcal{G}'_j \cap V'_1 : \phi^{-1}(y) \in \mathcal{G}_i \text{ for some } i\}.$$

Let $N'_j \subset \mathcal{G}'_j \cap V'_1$ be the vector subspace spanned by A'_j.

Lemma 2.7. *We have $N_i = \mathcal{G}_i \cap V_1$ for each i and $N'_j = \mathcal{G}'_j \cap V'_1$ for each j.*

Proof. We prove by contradiction. Suppose $N_i \ne \mathcal{G}_i \cap V_1$ for some i or $N'_j \ne \mathcal{G}'_j \cap V'_1$ for some j. Let $d_1 = 0$ if $N_i = \mathcal{G}_i \cap V_1$ for all i; otherwise, let

$$d_1 = \max\{d(x) : x \in (\mathcal{G}_i \cap V_1)\backslash N_i \text{ for some } i\}.$$

Similarly, let $d_2 = 0$ if $N'_j = \mathcal{G}'_j \cap V'_1$ for all j; otherwise, let

$$d_2 = \max\{d(y) : y \in (\mathcal{G}'_j \cap V'_1)\backslash N'_j \text{ for some } j\}.$$

Let $d_0 = \max\{d_1, d_2\}$. After possibly replacing ϕ with ϕ^{-1}, we may assume $d_0 = d_1$. Pick $x \in (\mathcal{G}_i \cap V_1)\backslash N_i$ (for some i) with $d(x) = d_0$. By the definition of N_i we have $x \notin A_i$. Hence $\phi(x)$ can be written as

$$\phi(x) = y_1 + \cdots + y_k, \tag{2-2}$$

where $k \ge 2$ and $y_s \in (\mathcal{G}'_{j_s} \cap V'_1)\backslash\{0\}$ for each $1 \le s \le k$, and $1 \le j_1 < \cdots < j_k \le n$. By Lemma 2.6, $d(y_s) > d(\phi(x)) = d(x) = d_0$. It follows from the definition of d_0 that $y_s \in N'_{j_s}$. Hence there is an expression

$$y_s = z_{s,1} + \cdots + z_{s,u_s} + w_{s,1} + \cdots + w_{s,v_s} \tag{2-3}$$

with $z_{s,p}, w_{s,q} \in A'_{j_s}$ such that $\phi^{-1}(z_{s,p}) \in \mathcal{G}_i \cap V_1$ and $\phi^{-1}(w_{s,q}) \in \mathcal{G}_t \cap V_1$ for some $t \neq i$ (here t may depend on q). Notice that (2-2) and (2-3) imply

$$x = \sum_{s,p} \phi^{-1}(z_{s,p}) + \sum_{s,q} \phi^{-1}(w_{s,q}).$$

Since $x \in \mathcal{G}_i$ and $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ is a direct sum decomposition, we obtain $x = \sum_{s,p} \phi^{-1}(z_{s,p})$. Notice that each $\phi^{-1}(z_{s,p}) \in A_i$. It follows that $x \in N_i$, contradicting the assumption. □

Lemma 2.8. *For each i, there is some j such that $\phi(\mathcal{G}_i \cap V_1) \subset \mathcal{G}'_j \cap V'_1$.*

Proof. Fix i. By Lemma 2.7, $\mathcal{G}_i \cap V_1 = N_i$. Hence there is a vector space basis B of $\mathcal{G}_i \cap V_1$ consisting of elements of A_i. Write B as a disjoint union $B = \sqcup B_j$, where B_j consists of those elements of B that are mapped into $\mathcal{G}'_j$ under ϕ. Since ϕ is an isomorphism and $\mathcal{G}'_{j_1}$ and $\mathcal{G}'_{j_2}$ commute for $j_1 \neq j_2$, we see that $[X, Y] = 0$ for $X \in B_{j_1}$ and $Y \in B_{j_2}$. Let $E_j \subset \mathcal{G}_i$ be the subalgebra of $\mathcal{G}_i$ generated by B_j. Observe that E_j is an ideal of $\mathcal{G}_i$ and $\mathcal{G}_i$ admits the direct sum decomposition $\mathcal{G}_i = E_1 \oplus \cdots \oplus E_n$. Since $\mathcal{G}_i$ is irreducible, $E_j = \{0\}$ for all j except exactly one. It follows that for some j, all the elements in B are mapped into $\mathcal{G}'_j$. Since B is a basis of $\mathcal{G}_i \cap V_1$, we have $\phi(\mathcal{G}_i \cap V_1) \subset \mathcal{G}'_j$. □

Applying Lemma 2.8 to ϕ^{-1}, we see that for each j, there is some i such that $\phi^{-1}(\mathcal{G}'_j \cap V'_1) \subset \mathcal{G}_i \cap V_1$. From this it is easy to see that $m = n$, and after possibly permuting the factors $\mathcal{G}'_j$ we have $\phi(\mathcal{G}_i) = \mathcal{G}'_i$. Proposition 2.5 follows.

3. Quasisymmetric homeomorphisms are bilipschitz

In this section we show that in most cases quasisymmetric homeomorphisms between reducible Carnot groups are bilipschitz.

A simply connected nilpotent Lie group is a *Carnot group* if its Lie algebra is a Carnot algebra. Let G be a Carnot group with Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$. The subspace V_1 defines a left invariant distribution $HG \subset TG$ on G. We fix a left invariant inner product on HG. An absolutely continuous curve γ in G whose velocity vector $\gamma'(t)$ is contained in $H_{\gamma(t)}G$ for a.e. t is called a horizontal curve. By Chow's theorem ([Bellaïche and Risler 1996], Theorem 2.4), any two points of G can be connected by horizontal curves. Let $p, q \in G$; the *Carnot distance* $d(p, q)$ between them is defined as the infimum of length of horizontal curves that join p and q.

Since the inner product on HG is left invariant, the Carnot metric on G is also left invariant. Different choices of inner product on HG result in Carnot metrics that are bilipschitz equivalent. The Hausdorff dimension of G with respect to a Carnot metric is given by $\sum_{i=1}^{r} i \cdot \dim V_i$. We use the corresponding Hausdorff measure on G. When $G = G_1 \times G_2$ is a direct product of two Carnot groups (with a suitable choice of inner product on HG), the Carnot metric on G is the product of the Carnot metrics on G_1 and G_2, and the Hausdorff measure on G is the product of the Hausdorff measures on G_1 and G_2.

Recall that, for a simply connected nilpotent Lie group G with Lie algebra $\mathcal{G}$, the exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism. Furthermore, the exponential map induces a one-to-one correspondence between Lie subalgebras of $\mathcal{G}$ and connected Lie subgroups of G.

Let G be a Carnot group with Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$. Since $\lambda_t : \mathcal{G} \to \mathcal{G}$ ($t > 0$) is a Lie algebra automorphism and G is simply connected, there is a unique Lie group automorphism $\Lambda_t : G \to G$ whose differential at the identity is λ_t. For each $t > 0$, Λ_t is a similarity with respect to the Carnot metric: $d(\Lambda_t(p), \Lambda_t(q)) = t\, d(p, q)$ for any two points $p, q \in G$. A Lie group homomorphism $f : G \to G'$ between two Carnot groups is a graded homomorphism if it commutes with Λ_t for all $t > 0$; that is, if $f \circ \Lambda_t = \Lambda_t \circ f$. Notice that a Lie group homomorphism $f : G \to G'$ between two Carnot groups is graded if and only if the corresponding Lie algebra homomorphism is graded.

A Carnot group is reducible if its Lie algebra is reducible. Equivalently, a Carnot group is reducible if it is isomorphic to the direct product of two Carnot groups. A Carnot group is called irreducible otherwise.

Proposition 2.4 and Proposition 2.5 respectively immediately imply Corollary 3.1 and Corollary 3.2, which say that a graded isomorphism of reducible Carnot groups preserves certain Lie subgroups.

Corollary 3.1. *Let $G = G_0 \times G_1 \times \cdots \times G_m$ and $G' = G'_0 \times G'_1 \times \cdots \times G'_n$ be two reducible Carnot groups written as direct products of an abelian factor and irreducible nonabelian factors. Let $f : G \to G'$ be a graded isomorphism. Then $f(G_0) = G'_0$.*

Corollary 3.2. *Let $G = G_0 \times G_1 \times \cdots \times G_m$ and $G' = G'_0 \times G'_1 \times \cdots \times G'_n$ be two reducible Carnot groups written as direct products of an abelian factor and irreducible nonabelian factors. Let $f : G \to G'$ be a graded isomorphism. Suppose G has no abelian factor (that is, $G_0 = \{e\}$). Then $G'_0 = \{e\}$, $m = n$ and after possibly permuting the factors $G'_1, \ldots, G'_m$, there exist graded isomorphisms $f_i : G_i \to G'_i$ such that $f = f_1 \times \cdots \times f_m$.*

Definition 3-1. Let G and G' be two Carnot groups endowed with Carnot metrics. A map $F : G \to G'$ is *Pansu-differentiable* at $x \in G$ if there exists a graded homomorphism $L : G \to G'$ such that

$$\lim_{y \to x} \frac{d\big(F(x)^{-1}F(y), L(x^{-1}y)\big)}{d(x, y)} = 0.$$

In this case, the graded homomorphism $L : G \to G'$ is called the *Pansu differential* of F at x, and is denoted by $dF(x)$.

Definition 3-2. Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism of metric spaces $F : X \to Y$ is an η-*quasisymmetric* homeomorphism if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(F(x), F(y))}{d(F(x), F(z))} \le \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

A map $F : X \to Y$ is a quasisymmetric homeomorphism if it is an η-quasisymmetric homeomorphism for some η.

The following result (except the terminology) is due to Pansu [1989].

Theorem 3.3. *Let $F : G \to G'$ be a quasisymmetric homeomorphism between two Carnot groups. Then F is a.e. Pansu-differentiable. Furthermore, at a.e. $x \in G$, the Pansu differential $dF(x) : G \to G'$ is a graded isomorphism.*

In Theorem 3.3 and the proofs below, “a.e.” is with respect to the Hausdorff measure on G.

For the proof of Theorem 1.1, we need the following:

Proposition 3.4. *Let G and G' be two Carnot groups, $W \subset V_1$, $W' \subset V_1'$ be subspaces. Denote by $\mathscr{G}_W \subset \mathscr{G}$ and $\mathscr{G}'_{W'} \subset \mathscr{G}'$, respectively, the Lie subalgebras generated by W and W'. Let $H \subset G$ and $H' \subset G'$, respectively, be the connected Lie subgroups of G and G' corresponding to $\mathscr{G}_W$ and $\mathscr{G}'_{W'}$. Let $F : G \to G'$ be a quasisymmetric homeomorphism. If $dF(x)(W) \subset W'$ for a.e. $x \in G$, then F sends left cosets of H into left cosets of H'.*

Proof. For each nonzero vector $u \in W$, the set $\{\exp(tu) : t \in \mathbb{R}\}$ is a subgroup of G. It is a geodesic with respect to the Carnot metric and shall be called a horizontal line. For each nonzero vector $u \in W$, let $\mathscr{F}_u$ be the set of left cosets of $\{\exp(tu) : t \in \mathbb{R}\}$ in G. By the main result in [Balogh et al. 2007], $F : G \to G'$ is absolutely continuous on almost every curve. It follows that for almost every $L \in \mathscr{F}_u$, the map $F|_L : L \to G'$ is an absolutely continuous curve in G'. On the other hand, by Pansu's theorem, F is a.e. Pansu-differentiable and the Pansu differential $dF(x) : G \to G'$ is a graded isomorphism for a.e. $x \in G$. Also by assumption, $dF(x)(W) \subset W'$ for a.e. $x \in G$. It follows from Fubini's theorem that, for almost every $L \in \mathscr{F}_u$, the Pansu differential $dF(x) : G \to G'$ exists, is a graded isomorphism and satisfies $dF(x)(W) \subset W'$ for a.e. $x \in L$. Hence, the tangent vectors of the curve $F|_L$ lie in W' almost everywhere. It follows that for almost every $L \in \mathscr{F}_u$, $F(L)$ lies in a left coset of H'. Now the continuity of F and a limiting argument show that the same is true for all $L \in \mathscr{F}_u$. Conceivably, it might be possible for distinct $L_1, L_2 \in \mathscr{F}_u$ to lie in the same left coset of H, while their images $F(L_1)$ and $F(L_2)$ lie in distinct cosets of H'. We next show that this cannot happen.

In a Carnot group, every two points can be joined by a piecewise geodesic, where each piece is a left translation of a segment in a horizontal line. The preceding paragraph shows that the image under F of each piece lies in a left coset of H'. It follows that the image of the entire piecewise geodesic lies in a left coset of H'. Hence F sends left cosets of H into left cosets of H'. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $F : G \to G'$ be a quasisymmetric homeomorphism between two Carnot groups. Suppose G is reducible and admits a direct product decomposition of irreducible Carnot groups where at least two of the factors are not isomorphic. We first use Proposition 3.4 to show that F preserves a certain foliation. The arguments in [Shanmugalingam and Xie 2012] then show that F is bilipschitz.

Write $G = G_0 \times G_1 \times \cdots \times G_m$ and $G' = G'_0 \times G'_1 \times \cdots \times G'_n$, where G_0, G'_0 are abelian and G_i, G'_j are irreducible nonabelian factors.

First consider the case when G_0 is nontrivial. Let $\mathcal{F}$ be the foliation of G consisting of the cosets of G_0, and similarly let $\mathcal{F}'$ be the foliation of G' consisting of the cosets of G'_0. The leaf space of $\mathcal{F}$ can be naturally identified with $N := G_1 \times \cdots \times G_m$, and that of $\mathcal{F}'$ with $N' := G'_1 \times \cdots \times G'_n$. By Corollary 3.1 and Proposition 3.4, the map F sends the leafs of $\mathcal{F}$ to the leafs of $\mathcal{F}'$. Hence F induces a map $F_1 : N \to N'$. Notice that $G = G_0 \times N$ with the Carnot metric is isometric to the product of G_0 and N (also equipped with the Carnot metric); similarly for G'. The arguments in [Shanmugalingam and Xie 2012] go through and imply that F_1 is also quasisymmetric. Since both the leafs and the leaf spaces are geodesic metric spaces, the arguments further show that F is bilipschitz.

Next we consider the case when G_0 is trivial. Then $G = G_1 \times \cdots \times G_m$ is a direct product of nonabelian irreducible Carnot groups. We combine all isomorphic factors in the above decomposition to obtain $G = N_1 \times \cdots \times N_s$. Each N_j is a direct product of isomorphic nonabelian irreducible Carnot groups, and the factors in N_i and N_j are not isomorphic for $i \neq j$. Similarly, G' can also be written as such a product $G' = N'_1 \times \cdots \times N'_t$. Notice that the assumption of Theorem 1.1 implies that $s \geq 2$. Corollary 3.2 implies that $s = t$, and after possibly permuting the factors N'_i, the Pansu differential satisfies $dF(x)(N_i) = N'_i$ for all i and a.e. $x \in N$. Now the arguments in the preceding paragraph show that F is bilipschitz. The proof of Theorem 1.1 is now complete. □

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