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We consider some of the complications that arise in attempting to generalize a version of Archimedes' principle concerning floating bodies to account for capillary effects. The main result provides a means to relate the floating position (depth in the liquid) of a symmetrically floating sphere in terms of other observable geometric quantities.

A similar result is obtained for an idealized case corresponding to a symmetrically floating infinite cylinder.

These results depend on a definition of equilibrium for capillary systems with floating objects which to our knowledge has not formally appeared in the literature. The definition, in turn, depends on a variational formula for floating bodies which was derived in a special case earlier (*Pacific J. Math.* 231:1 (2007), 167–191) and is here generalized to account for gravitational forces.

A formal application of our results is made to the problem of a ball floating in an infinite bath asymptotic to a prescribed level. We obtain existence and nonuniqueness results.

1. Introduction

Archimedes stated the principle that bears his name in a work titled *On floating bodies*. The principle is commonly stated as follows:

> *A body immersed in a fluid is buoyed up with a force equal to the weight of the displaced fluid.*

This is actually a reformulation of Archimedes' principle and, as Erlend Graf [2004] points out, it is deficient (and incorrect) in various respects.

Archimedes considered three distinct cases. The first case is that in which the density of the body is equal to the density of the liquid. The assertion is that the body, after it is deposited into the liquid and comes to rest, will not project above the surface of the liquid nor sink lower in the liquid (*On floating bodies*, part I, Proposition 3; see [Archimedes/Heath 1897, p. 255]).

MSC2010: 76B45.

The second case is that in which the density of the body is less than that of the liquid. The assertion is that the body, if left to interact freely with the liquid bath, will project above the surface of the bath and will displace a volume of liquid having the same weight as the object (Propositions 4 and 5; [ibid., pp. 256–257]). Furthermore, if the object is not allowed to float freely, but is manually pushed downward into the liquid from its floating position, then the object will experience an upward force equivalent to the difference of the weight of the object and the weight of the displaced liquid (Proposition 6; [ibid., p. 257]).

Finally, if the body is more dense than the liquid it will sink to the bottom and, if weighed while in the liquid will be found lighter than its true weight by the weight of the displaced liquid (Proposition 7; [ibid., p. 258]).

The reformulation is about the force experienced by a body deposited in a liquid bath (and nothing else). The original principle of Archimedes specifically addresses two additional questions:

(1) Will the body float[1] or sink?

(2) At what height will the object come to rest?

The first question is conditional; the second is geometric. The fact that the reformulation ignores these aspects of the problem is a deficiency of the reformulation and no reflection on the acuity of Archimedes.

An aspect of the problem that does seem to have escaped the notice of Archimedes involves the effect of surface tension or surface energy associated with wetting. Indeed, simple experiments show that it is possible, under certain circumstances, for even a convex[2] object with density greater than that of a given liquid bath to float (only) partially submerged on the surface of the bath, contradicting Archimedes' Proposition 7; see Figure 1.

Finn [2011] has recently given the first rigorous mathematical proof of this fact, at least in an idealized situation which we describe in Section 4 below. Finn and Vogel [2009] wrote: "One may assume that [Archimedes] was unaware of observations of Aristotles a century earlier" (concerning heavy floating objects). This may be true, or perhaps Archimedes restricted himself to a problem whose solution used the mathematical tools he had at hand. In either case, we find connections with the results of Archimedes, and derive from our new results what can be viewed as a generalization of results which follow from Archimedes' approach. Notice also

[1]That is, will the object project above the surface of the liquid?

[2]Convexity is mentioned here in contrast to something like a hollow boat hull often considered in connection with the density considerations of Archimedes. In fact, the possibility that objects with density greater than water might float on the surface of water was already considered by Aristotle a century before Archimedes, and it is surprising Archimedes makes no mention of it. The fact that a thin metal paper clip can float on water makes it clear convexity is not a necessary hypothesis. Nevertheless, we did not know if a sphere could float until we tried it (Figure 1).

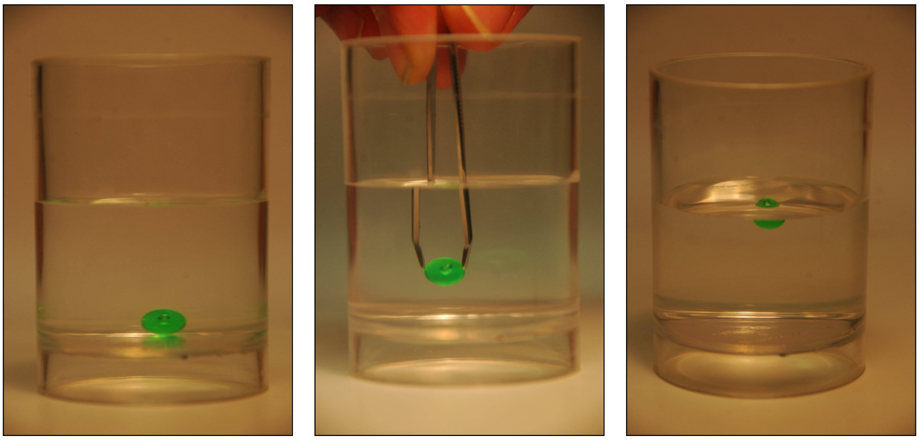


Figure 1. Photos of a plastic ball in a bath of water: sinking to the bottom (left), being raised to the surface (middle), floating (right)

that the results of [Finn 2011] and [Finn and Vogel 2009] initiate a return to the question addressed by Archimedes: Does the body sink or swim?

Our work below assumes the answer to the question of floating versus sinking is affirmative for floating and seeks to answer a version of Archimedes' second question: What is the geometry? More precisely: What is the height of the floating body and what is the geometry of the interface? We are able to give a partial answer under the assumption of rotational symmetry of the object and the interface. This symmetry appears to hold in the physical system of Figure 1, and similar symmetric interfaces have been shown to exist mathematically in [Treinen 2012] and [Elcrat et al. 2004b]. For further discussion of this point, see Section 6.

For purposes of comparison, we describe briefly this problem of a floating ball as we imagine Archimedes might have considered it.[3] Given the diagram in Figure 2, with an assumed planar interface meeting a floating sphere Σ along a circular contact line determined by an azimuthal angle $\bar{\phi}$, and assuming a density ρ of the ball less than the density ρ_l of the liquid, Archimedes' Proposition 5 then becomes

$$\rho_l V_d = \rho|\Sigma| \tag{1}$$

where V_d is the volume of displaced liquid. Equating this volume of liquid with the volume of the spherical cap below the plane of the interface,

$$V_d = \tfrac{1}{3}\pi a^3 \left(\sin^2\bar{\phi}\cos\bar{\phi} + 2 + 2\cos\bar{\phi}\right),$$

[3]The explanation of Vitruvius (in *De architectura*) is of particular interest for this discussion, as it provides some details not contained in Archimedes' work directly. In particular, Vitruvius identified the "displaced fluid" as that which overflows a vessel into which an object is deposited.

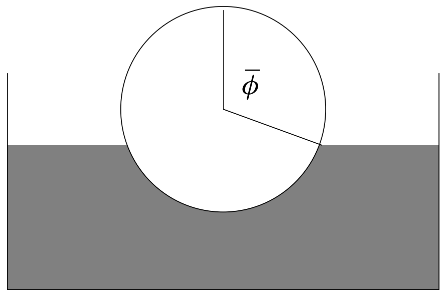


Figure 2. Azimuthal angle determined by a horizontal contact line.

we obtain this:

Theorem 1. *According to Archimedes' principle, a homogeneous sphere of density $\rho > \rho_l$ will sink to the bottom of a bath of density ρ_l, and a homogeneous sphere of density $\rho < \rho_l$ will float at a level determined by*

$$\cos^3 \bar{\phi} - 3\cos\bar{\phi} = 2\Big(1 - \frac{2\rho}{\rho_l}\Big). \tag{2}$$

It is easily checked that the function $F(\bar{\phi}) = \cos^3 \bar{\phi} - 3\cos\bar{\phi}$ is increasing from -2 to 2 on $[0, \pi]$, with zero derivative at the endpoints and strictly positive derivative interior to the interval. Thus, for each positive value $0 \le \rho \le \rho_l$, the condition (2) determines a unique azimuthal angle. See Figure 3.

Definitions of equilibria. From a more sophisticated point of view, liquid interfaces are rarely planar. Even without the introduction of a floating object, the interface of liquid in a cylinder is usually noticeably curved around the edges. With the introduction of a rigid floating object, one may assume the interface will be further deformed in possibly unexpected ways.

The modern theory of equilibrium capillary configurations developed by Young, Laplace, Gauss, and others (see [Finn 1986]) is now founded on the consideration of energies associated with the area of the outer surface of the liquid where it contacts the surrounding atmosphere and where it contacts the bounding container. This theory has been primarily pursued in the context of solid structures that are rigid and *fixed*. This has led to a commonly adopted definition of a capillary equilibrium [ibid.]:

> Up to the determination of a single real parameter (λ below) the problem of finding a capillary surface is a purely geometric one: *to find a surface whose mean curvature is a prescribed function of position and which meets prescribed* (*rigid*) *bounding walls in a prescribed angle* γ.

In terms of equations commonly used to model equilibrium capillary surfaces in a gravity field, we have

$$2H = \kappa z - \lambda \quad \text{and} \quad \cos\gamma = \beta, \tag{3}$$

where H denotes the mean curvature of the interface, z denotes the vertical height of a point on the interface, $\kappa = \rho_l g/\sigma$ is the *capillary constant*, constructed using the gravitational acceleration g and the *surface tension* σ, and λ is a single real (Lagrange) parameter related to the constraints of the problem; in the second equation one finds the *relative adhesion coefficient* β defined by the assumption that $\sigma\beta$ is the local energy density[4] associated with contact between the liquid volume and solid structures; one integrates $\sigma\beta$ over the area of contact, or *wetted area*, to obtain the total *energy of wetting*. The angle γ is assumed to be defined along a curve where the liquid, the container, and the surrounding atmosphere all meet. This curve is called the *contact line* and γ is referred to as the *contact angle*.

While the problem of a floating object considered here is still purely geometric, the conditions (3) are inadequate to characterize equilibria, even if the object is rigid and the Lagrange parameter λ is known. One still has recourse to the general principle of virtual work, that is, the energy is stationary with respect to variations compatible with the constraints of the problem. Nevertheless, attaining a collection of fundamental necessary conditions analogous to (3) that may be taken as a working definition of equilibrium in particular cases is of evident utility both for applications and the mathematical theory of capillarity. A preliminary discussion of the need for this development was suggested in [McCuan 2007] in the absence of external forces (i.e., zero gravity), and we provide here a general flux condition (13) to augment (3), thus providing a new definition of equilibrium in this context. A discussion of this formula for capillary surfaces is in Section 2.

From the flux formula we obtain the following result which may be compared to Theorem 1 and is proved in Section 3.

Theorem 2. *A sphere of radius a that floats in a centrally symmetric position as described above under the effects of surface tension and adhesion effects of an axially symmetric bath must float at a level determined by the azimuthal angle* $\overline{\phi}$ *satisfying*

$$\cos^3\overline{\phi} - 3\cos\overline{\phi} + \frac{6}{\kappa a}\left(\overline{H} + \frac{\cos\gamma}{a}\right)\sin^2\overline{\phi} - \frac{3\sin\gamma}{\kappa a^2}\sin(2\overline{\phi}) = 2\left(1 - \frac{2\rho}{\rho_l}\right), \tag{4}$$

[4]In [Finn 1986], the relative adhesion coefficient is given on page 6 as the difference $\beta^* - \hat{\beta}^*$ of energy densities associated with contact between one fluid and the container (β^*) and a complementary fluid and the container ($\hat{\beta}^*$). Using the approximation $\hat{\beta}^* \approx 0$, the formulation used here is equivalent. For simplicity, we will also assume σ and β are constants; the reasoning below extends in a straightforward manner to the general case.

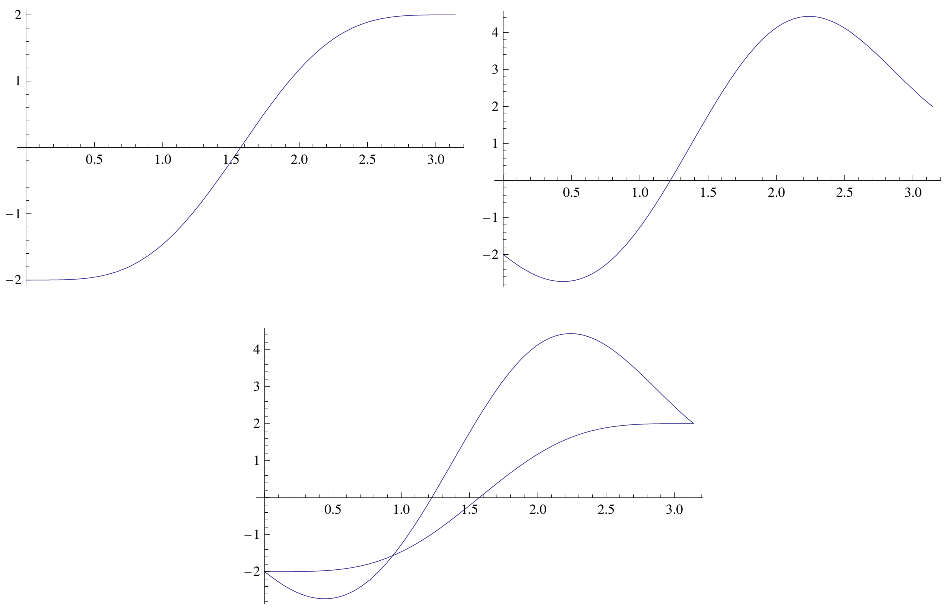


Figure 3. The azimuthal angles determined by Theorems 1 (top left) and 2 (top right); plotted together on the bottom.

where κ is the capillary constant described above, $\overline{H}$ is the mean curvature of the liquid interface at the contact line, and $\gamma \in (0, \pi)$ is the contact angle of the liquid interface with the floating sphere.

The function $F(\overline{\phi})$ appearing on the left side of (4) *takes the values -2 and 2 at the endpoints $\overline{\phi} = 0$ and π respectively. However, F is decreasing at $\overline{\phi} = 0$ and decreases to a unique local interior minimum at $\overline{\phi} = \overline{\phi}_1$. On the interval from $\overline{\phi} = \overline{\phi}_1$ to $\overline{\phi} = \pi$ the function F has a unique interior local maximum at $\overline{\phi} = \overline{\phi}_2$.*

If $\gamma = 0, \pi$, then the value of the azimuthal angle is uniquely determined by the same function F, which is increasing and satisfies $F'(0) = 0 = F'(\pi)$ but is distinct from the function appearing in Theorem 1.

The existence of the unique local interior minimum at $\overline{\phi} = \overline{\phi}_1$ allows values of $\rho > \rho_l$ and leads to the determination of a unique maximum density $\rho_{\max} = \rho_{\max}(a, \gamma, \kappa, \bar{H})$ for which $\rho > \rho_{\max}$ implies no floating is possible. It will be noted from the properties of F that a unique azimuthal angle $\overline{\phi}$ is determined for all values $0 < \rho < \rho_l$, and that two values are possible for certain values $\rho \geq \rho_l$ (as long as ρ is not too large). We presume by continuity that the physically observed value for heavy floating spheres is the larger one determined by (4). The physical relevance of the other value is discussed in Section 6 of the paper.

We note also that the graph of F takes values corresponding to negative densities ρ. This can be imagined to have physical relevance in a situation where a gravitational field acts on the floating object, but one with the opposite direction as that acting on the liquid. It is not readily apparent how such a physical situation would arise, but one can easily imagine a magnetic field producing an upward force on a floating object in a downward gravity field, which would be quite similar.

Further remarks. The quantity $\overline{H}$ appearing in the formula (4) of Theorem 2 is presumed to depend in some manner on other parameters, and perhaps globally imposed geometric constraints in the problem. Perhaps the quantity $\overline{H}$ and its appearance in (4) is best viewed in contrast to the following specific quantities: the enclosed volume of liquid (n.b., the Lagrange parameter λ), the outer radius R of the cylindrical vessel, and the contact angle γ_{out} between the interface and the outer wall, all of which are conspicuously absent from formula (4). As far as we know, this paper and [McCuan 2007] are the first to consider the global floating configuration for a floating ball including a finite outer bounding wall. Indeed, one might be tempted to dismiss the effects of the interface at the outer bounding wall. Several authors have considered floating objects in an infinite bath asymptotic to a plane (and we do so below in § 6 as well). Under certain assumptions, estimates have been derived [Siegel 1980] to establish the fact that such an interface converges to the planar asymptote exponentially with distance from a floating object.

We offer the following description of an experiment as a caution against assuming the influence of an outer wall is not important.

If a cylinder of water is partially filled, and a ball of density $\rho < \rho_l$ is deposited in the center of the resulting interface, it will move rapidly to the outer wall. See Figure 4. If the same cylinder is subsequently slightly overfilled so that the (roughly flat) interface curves downward at the edges, then the ball will move rapidly to the center of the interface and remain there in an apparently stable configuration; if the ball is manually moved away from the center it will return.

This experiment brings up a question that is fundamentally different from the one considered in this paper, but it indicates in broad terms that the question of

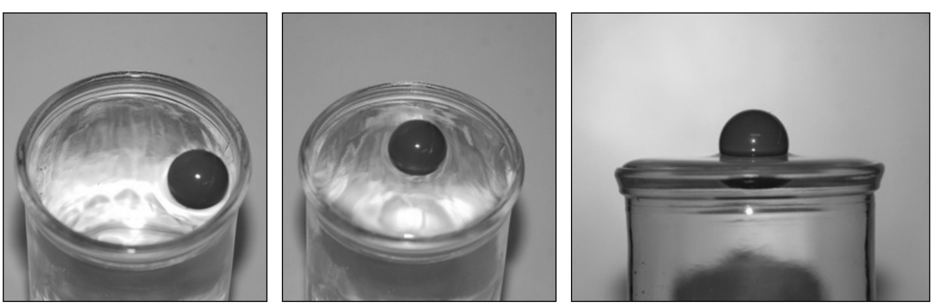


Figure 4. Photos of a plastic ball in a bath of water: tending to the edge (left), stable in the middle (center and right).

how an object floats on a liquid interface can have an answer depending strongly on nonlocal conditions involving the outer bounding wall.

Ideally one would like a formula for the azimuthal angle $\bar{\phi}$ in terms of the volume of liquid in the bath V, the radii a and R of the ball and the container respectively, and the contact angles γ between the liquid interface and the surface of the floating object and γ_{out} between the liquid interface and the surface of the container, and from the classical point of view, this is what one would expect. We were unable to attain such a result, and the result we obtain (4) may be viewed simply as a relation between $\overline{H}$ and $\bar{\phi}$ for any equilibrium. The interpretation we give in the context of Archimedes' geometric question may then be viewed as the most explicit currently available information arising from (13).

The barrier to getting a more definitive result lies in the complicated nature of the system of ordinary differential equations determining the rotationally symmetric interface. For a survey of recent progress in understanding the family of solutions to these equations, see [Finn 1986; Vogel 1982; Siegel 2006; Siegel 1980; Nickolov 2002; Elcrat et al. 2004a; Turkington 1980; Johnson and Perko 1968; Treinen 2012].

2. Variational formulation

The general assumptions of our model are outlined in [McCuan 2007] though the derivation given there was aimed at the zero gravity case in which buoyancy plays no role, and the effects of gravity were not properly considered. For the sake of making this paper somewhat more self-contained we include a short review/summary of the model and amend the deficiencies in the former derivation.

Quite generally, we consider a solid structure

$$\Sigma = \Sigma_s \cup \Sigma_m$$

consisting of a stationary part Σ_s and a movable, or floating, part Σ_m. In addition, we hypothesize an equilibrium liquid interface Λ with corresponding wetted region $\mathcal{W} = \mathcal{W}_s \cup \mathcal{W}_m$, so that the liquid volume $\mathcal{V}$ satisfies $\partial\mathcal{V} = \Lambda \cup \mathcal{W}$ and the contact line/triple interface is given by $\partial\Lambda = \partial\mathcal{W}$. Under these assumptions, we consider the variational problem associated with

$$\mathcal{E} = \sigma|\Lambda| - \sigma\beta|\mathcal{W}| + \mathcal{G} \tag{5}$$

where $\mathcal{G} = \int_{\mathcal{V}\cup\Sigma_m} G$ and G is a position dependent function representing field forces such as gravity.[5]

One specific application of the discussion which now follows is that it justifies the following fundamental definition:

[5]We included only $\int_{\mathcal{V}} G$ in [McCuan 2007].

Definition 3. A floating configuration Σ_s, Σ_m, $\mathcal{V}$ as described above is said to be in *free-floating equilibrium* for the functional (5) if

1. $2H = G/\sigma - \lambda$, where H is the mean curvature of the free surface interface Λ and λ is some constant,
2. $\cos\gamma = \beta$ where γ is the angle at which the free surface interface meets the surface of the solid structures measured within $\mathcal{V}$ and β is the (possibly location dependent) adhesion coefficient, and
3. $$\int_{\partial\mathcal{W}_m} \vec{n} + \int_{\mathcal{W}_m} (G/\sigma - \lambda)N - \int_{\partial\Sigma_m} (G/\sigma)N = 0,$$
   where n is the outward pointing unit conormal along $\partial\Lambda$, and N is the unit normal to $\partial\mathcal{V}$ pointing out of $\mathcal{V}$.

Under rather general hypotheses, as described in [McCuan 2007], a family of variations leaving Σ_m fixed leads to the (standard) variational formulas (6)–(8) below:

$$|\dot{\Lambda}| = -\int_\Lambda 2H\dot{X}\cdot N + \int_{\partial\Lambda} \dot{X}\cdot\vec{n}, \tag{6}$$

where H is the mean curvature defined on Λ, $\dot{X}$ is the variation vector, N is the unit normal pointing out of the liquid volume $\mathcal{V}$, and $\vec{n}$ is the unit conormal to N and $\partial\Lambda$ pointing out of Λ;

$$|\dot{\mathcal{W}}| = \int_{\partial\Lambda} \dot{X}\cdot\vec{\nu}, \tag{7}$$

where $\vec{\nu}$ is the unit conormal to $N^{\mathcal{W}}$ and $\partial\mathcal{W}$ pointing out of $\mathcal{W}$; note that $N^{\mathcal{W}}$ denotes the unit normal to $\mathcal{W}$ pointing out of $\mathcal{V}$ and may also be denoted by N on the interior of $\mathcal{W}$ where no ambiguity arises;

$$\dot{\mathcal{G}} = \int_\Lambda G\dot{X}\cdot N \quad\text{and}\quad |\dot{\mathcal{V}}| = \int_\Lambda \dot{X}\cdot N. \tag{8}$$

These last two formulas apparently require an interesting and somewhat delicate application of more general mathematical principles of fluid mechanics, and we outline their derivation under more general assumptions below.

For now, we assemble $\dot{\mathcal{E}}/\sigma - \lambda|\dot{\mathcal{V}}|$ from the constituent parts above where λ is a Lagrange multiplier associated with the volume constraint:

$$\dot{\mathcal{E}}/\sigma - \lambda|\dot{\mathcal{V}}| = \int_\Lambda (-2H + G/\sigma - \lambda)\dot{X}\cdot N + \int_{\partial\Lambda} (\dot{X}\cdot\vec{n} - \beta\dot{X}\cdot\vec{\nu}).$$

The vanishing of this quantity for all variation vectors $\dot{X}$ results in the well known geometric boundary value problem

$$\begin{cases} 2H = G/\sigma - \lambda & \text{on } \Lambda, \\ \cos\gamma = \beta & \text{on } \partial\Lambda, \end{cases} \tag{9}$$

since

$$\vec{n} = (\vec{n} \cdot N^{\mathcal{W}}) N^{\mathcal{W}} + \cos\gamma\, \vec{\nu}.$$

In the special case under consideration in this paper, G represents the limiting value $\rho_l g z$ taken as a limit from inside the liquid, so that

$$2H = \kappa z - \lambda$$

where $\kappa = \rho_l g/\sigma$ is a capillary constant for the problem. Furthermore, we restrict attention in this paper to cases in which the adhesion coefficient satisfies $-1 < \beta < 1$ or equivalently, the contact angle γ is strictly between 0 and π.

A more general variation allowing rigid motion of Σ_m takes the form

$$X = X(\boldsymbol{p}; t, h) : M \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to \mathbb{R}^3,$$

where $M = \overline{\Sigma \cup \mathcal{V}}$ is considered as an abstract manifold; see Figure 5.

It is assumed here, as indicated in the figure, that h parametrizes a family of rigid motions $w = w(x; h)$ to which Σ_m is subject. Denoting derivatives with respect to h by an acute accent, we find

$$|\acute{\Lambda}| = -\int_{\Lambda} 2H \acute{X} \cdot N + \int_{\partial\Lambda} \acute{X} \cdot \vec{n}, \tag{10}$$

$$|\acute{\mathcal{W}}| = -\int_{\mathcal{W}_m} 2H^{\mathcal{W}} \acute{X} \cdot N + \int_{\partial\mathcal{W}_m} \acute{X} \cdot \vec{\nu}, \tag{11}$$

$$\acute{\mathcal{G}} = \int_{\Lambda} G \acute{X} \cdot N + \int_{\mathcal{W}_m} G \acute{X} \cdot N^{\mathcal{W}} + \int_{\partial\Sigma_m} G_m \acute{X} \cdot N^m. \tag{12}$$

This last term requires some explanation. The quantity G_m denotes the value of the volumetric force field potential taken as a limit from inside the movable solid

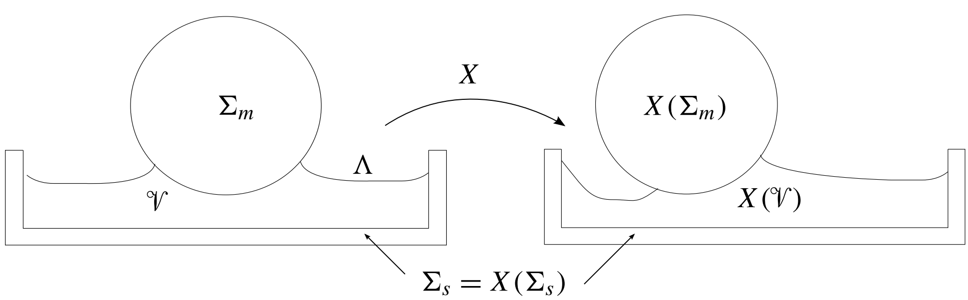


Figure 5. The variation map and its notation.

structure Σ_m. In the special case of a floating object of density ρ, we typically take $G_m = \rho g z$. Also in this last identity N^m denotes the unit normal to the boundary $\partial\Sigma_m$ of the movable/floating solid structure and points out of Σ_m, so that $N^m = -N^{\mathscr{W}}$ on their common domain of definition $\mathscr{W}_m$. Finally, we include a brief derivation.

Up until this point, we have stated all variational formulae in their final form, that is to say with the parameters of the variation set to zero so that $\dot{X}$ represents

$$\frac{d}{dt} X(\boldsymbol{p}; t)\Big|_{t=0},$$

where $X = X(\boldsymbol{p}; t) : M \times (-\epsilon, \epsilon) \to \mathbb{R}^3$. For this calculation, we must temporarily assume the parameters t and h are not evaluated at zero. Notationally, this is conveniently indicated by a tilde so that $\tilde{\Sigma}_m = X(\Sigma_m) = X(\Sigma_m; t; h)$, and we will evaluate at $t = h = 0$ at the end.

Consideration of the second term should suffice. Setting

$$\mathscr{G}_m = \int_{\tilde{\Sigma}_m} G_m,$$

we have

$$\mathscr{G}_m = \int_{\Sigma_m} G_m \circ X \det DX,$$

where X represents the restriction of the variation to Σ_m and the derivative is taken in $M \subset \mathbb{R}^3$ with respect to $\boldsymbol{p}$. Euler's kinematical formula [Serrin 1959] tells us how a material integral changes with the flow of a region of fluid. We can cast our present situation into this framework starting with the preliminary identity

$$\frac{\partial}{\partial h} \det DX = (\operatorname{div}_{\mathbb{R}^3} \boldsymbol{v}) \circ X \det DX$$

where $\boldsymbol{v}(x; h) = \acute{X}(X^{-1}(x; h); h)$ is the spatial velocity associated with the flow $X = X(\boldsymbol{p}; h)$ and we have simply suppressed the t dependence. It might be expected (or hoped) that in our situation the motion/flow associated with the variation should be particularly simple, at least on the solid movable object Σ_m, and that we might have, for example, $X(\boldsymbol{p}; h) \equiv w(\boldsymbol{p}; h)$ there. However, taking into account the motion of the liquid and that of the contact line of the liquid interface Λ in particular, it is clear that this would violate the continuity assumption on the variation $X : M \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to \mathbb{R}^3$. Having made this concession and subjected ourselves to the added complication that other authors seem to have avoided, it is some consolation, as pointed out in [Finn 2005], that the internal motion of the liquid under a variation of the free surface interface could be very complicated, and we are taking account of such possibilities.

In any case, we continue to obtain

$$\acute{\mathscr{G}}_m = \int_{\tilde{\Sigma}_m} DG_m \cdot \boldsymbol{v} + \operatorname{div}_{\mathbb{R}^3} \boldsymbol{v} = \int_{\tilde{\Sigma}_m} \operatorname{div}_{\mathbb{R}^3}(G_m \boldsymbol{v}) = \int_{\partial \tilde{\Sigma}_m} G_m \boldsymbol{v} \cdot N^m,$$

so that

$$\acute{\mathscr{G}}_m\big|_{h=0} = \int_{\partial \Sigma_m} G_m \acute{X} \cdot N^m.$$

A similar argument applies to the integral over $\mathscr{V}$ appearing in $\mathscr{G}$ and also yields

$$|\acute{\mathscr{V}}| = \int_{\Lambda} \acute{X} \cdot N + \int_{\mathscr{W}_m} \acute{X} \cdot N,$$

where we have returned to the general assumption on evaluation, that $t = h = 0$. Combining this with (10)–(12), we have

$$\begin{aligned}
\acute{\mathscr{E}}/\sigma - \lambda|\acute{\mathscr{V}}| &= \int_{\Lambda} (-2H + G/\sigma - \lambda)\acute{X} \cdot N + \int_{\partial\Lambda} (\acute{X} \cdot \vec{n} - \beta \acute{X} \cdot \vec{\nu}) \\
&\quad + \beta \int_{\mathscr{W}_m} 2H^{\mathscr{W}} \acute{X} \cdot N + \int_{\mathscr{W}_m} (G/\sigma - \lambda)\acute{X} \cdot N + \int_{\partial \Sigma_m} (G_m/\sigma)\acute{X} \cdot N^m \\
&= \int_{\partial \mathscr{W}_m} \acute{X} \cdot \vec{n} - \cos\gamma \int_{\partial \mathscr{W}_m} \acute{X} \cdot \vec{\nu} \\
&\quad + \cos\gamma \int_{\mathscr{W}_m} 2H^{\mathscr{W}} \acute{X} \cdot N + \int_{\mathscr{W}_m} (G/\sigma - \lambda)\acute{X} \cdot N + \int_{\partial \Sigma_m} (G_m/\sigma)\acute{X} \cdot N^m.
\end{aligned}$$

Next we refer to a calculation from [McCuan 2007] that uses the fact that

$$w^{-1}(X; h) \in \Sigma_m$$

when $X = X(\boldsymbol{p}; h) \in w(\Sigma_m; h)$ to show that

$$\acute{X} - \acute{w} \in T_X \Sigma_m.$$

It follows that $\acute{X}$ may be replaced with $\acute{w}$ in the formula above. A second calculation involving an explicit auxiliary variation shows

$$\int_{\mathscr{W}_m} 2H^{\mathscr{W}} \acute{w} \cdot N = \int_{\partial \mathscr{W}_m} \acute{w} \cdot \vec{\nu}.$$

Making the indicated substitutions, we arrive at our new necessary condition for equilibrium of a floating object:

Theorem 4. *If a floating configuration $\Sigma_m, \mathscr{V}$ subject to forces (having volumetric potentials denoted by G and G_m as described above) locally minimizes energy among liquid interface configurations compatible with a smooth family of rigid* *motions $w = w(x;h)$ with $w(x;0) = \mathrm{id}_{\mathbb{R}^3}$ and the wetted region on the floating object is denoted by $\mathcal{W}_m$, then the configuration must satisfy*

$$\int_{\partial \mathcal{W}_m} \acute{w} \cdot \vec{n} + \int_{\mathcal{W}_m} (G/\sigma - \lambda)\acute{w} \cdot N^{\mathcal{W}} + \int_{\partial \Sigma_m} (G_m/\sigma)\acute{w} \cdot N^m = 0, \tag{13}$$

where $\vec{n}$ is the outward pointing unit conormal along the boundary of the liquid interface Λ, $N^{\mathcal{W}}$ is the unit normal to Σ_m pointing out of the liquid, $N^m = -N^{\mathcal{W}}$, and $\acute{w}$ represents the derivative with respect to h evaluated at $h = 0$.

The condition of the theorem must hold for all $\acute{w} \in \mathbb{R}^3$ for free floating, or more generally for any collection of directions in which Σ_m is free to move. In the case in which all directions $\acute{w}$ are possible, the condition (13) simplifies to

$$\int_{\partial \mathcal{W}_m} \vec{n} + \int_{\mathcal{W}_m} (G/\sigma - \lambda)N^{\mathcal{W}} + \int_{\partial \Sigma_m} (G_m/\sigma)N^m = 0.$$

One immediately notes the integral over the boundary of the movable wetted surface of the conormal to the free surface interface (the first term) as marking this as a kind of *flux formula* or *force balance formula* as is well known from the work of A. Ros [1996] in minimal surfaces. It is tempting to interpret the other two integrals appearing in the formula as force vectors, and without doubt they are such. We are indebted to a referee for explaining how to do this for a constant vertical gravity field. Similar calculations for that case are also contained in [Bhatnagar and Finn 2006] where a somewhat different problem is considered; see Sections 4 and 6 for further remarks. With this help, we were able to see the following general interpretation.

In order to be dimensionally correct, multiply the equation by the surface tension σ. The first term is then the negative of the force exerted on the object by the interface itself — the surface tension force.

The integrand of the second term $G - \lambda\sigma$ will be recognized from (9) as the quantity $2\sigma H$ at the interface and, according to the insight of Thomas Young, the difference in *pressure* across the interface. It is natural to assume that $G - \lambda\sigma$ gives a pressure field extending throughout the volume of liquid, up to a sign. Since the mean curvature is calculated with respect to the normal N pointing out of the liquid, we see that the second integral represents the negation of the force this pressure exerts on the floating object, i.e., the buoyancy force.

Let us consider the third term componentwise. If $\boldsymbol{e}_j$ is the j-th standard unit vector, then the j-th component of the third integral is

$$\int_{\partial \Sigma_m} G_m \boldsymbol{e}_j \cdot N^m = \int_{\Sigma_m} \operatorname{div}(G_m \boldsymbol{e}_j) = \int_{\Sigma_m} DG_m \cdot \boldsymbol{e}_j,$$

where the first equality is by the divergence theorem, and we recognize the negation of the volumetric force density in the gradient of the potential appearing in the last expression. Recombining the components, the third term

$$\int_{\Sigma_m} DG_m$$

evidently lends itself to being interpreted as (minus) the "weight" of the floating object with respect to the potential field G_m.

In summary, our third equilibrium condition may be read (without the slightest ambiguity in the case of a constant downward gravitational field $G_m = \rho g z$) thus:

The weight, the pressure/buoyancy force, and the surface tension force on the floating object must sum to zero.

We next proceed to examine the consequences of (13) for the simple cases of floating suggested in the introduction.

3. Floating in three dimensions

Here we assume a vertical circular cylindrical vessel is observed with a sphere Σ_m floating symmetrically along the axis of the vessel and having symmetric circular contact line at azimuthal angle $\phi = \bar{\phi}$. Assuming the surface of the liquid is also rotationally symmetric with respect to the same axis, the meridian of the surface with vertical component u and radial component r considered as functions of arclength along the meridian must satisfy the boundary value problem

$$(14) \qquad \begin{cases} \dot{r} = \cos\psi, \\ \dot{u} = \sin\psi, \\ \dot{\psi} = \kappa u - \lambda - \sin\psi/r, \\ \psi = \gamma - \bar{\phi} \text{ and } u = d + a\cos\bar{\phi} & \text{when } r = r(0) = a\sin\bar{\phi}, \\ \psi = \pi/2 - \gamma_{\text{out}} & \text{when } r = r(l) = R, \end{cases}$$

where we have chosen coordinates so that the center of the floating sphere is $(0, 0, d)$, and we have denoted by l the total length and by ψ the inclination angle of the meridian.

It would be desirable to preface our discussion of the geometry of the floating ball in Figure 1 with an existence result, but we are unable to obtain such a result for essentially the same reason that our geometric result is somewhat suboptimal: The system of ordinary differential equations appearing in the problem above has been studied extensively, but the structure of the family of all solutions is not well enough understood. Thus, we turn directly to the auxiliary condition (13).

The following formulae, valid in the plane $y = x_2 = 0$, are useful in simplifying the integrals in (13):

$$
\begin{aligned}
N^m[\phi] &= \sin\bar{\phi}\,\boldsymbol{e}_1 + \cos\bar{\phi}\,\boldsymbol{e}_3,\\
N^{\mathcal{W}}[\phi] &= -N^m\\
&= -\sin\bar{\phi}\,\boldsymbol{e}_1 - \cos\bar{\phi}\,\boldsymbol{e}_3,\\
\vec{\nu}[\phi] &= (N^m)^{\perp}\\
&= -\cos\bar{\phi}\,\boldsymbol{e}_1 + \sin\bar{\phi}\,\boldsymbol{e}_3,\\
\vec{n} &= \cos\gamma\,\vec{\nu} + \sin\gamma\, N^{\mathcal{W}}\\
&= -\cos(\bar{\phi}-\gamma)\,\boldsymbol{e}_1 + \sin(\bar{\phi}-\gamma)\,\boldsymbol{e}_3,\\
N^{\Lambda} &= (-\vec{n})^{\perp}\\
&= \sin(\bar{\phi}-\gamma)\,\boldsymbol{e}_1 + \cos(\bar{\phi}-\gamma)\,\boldsymbol{e}_3.
\end{aligned} \tag{15}
$$

In these formulae, the bracketed ϕ indicates validity in the form of the result for an arbitrary azimuthal angle on $\partial\Sigma_m$ though the main interest is on $\partial\mathcal{W}_m$; $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ are the standard orthonormal unit vectors in $\mathbb{R}^3$.

Taking a vertical translation for the rigid motion of Σ_m so that $\acute{w} = \boldsymbol{e}_3$, the three terms of (13) are as follows:

$$
\begin{gathered}
\int_{\partial\mathcal{W}_m} \boldsymbol{e}_3\cdot\vec{n} = 2\pi a \sin\bar{\phi}\sin(\bar{\phi}-\gamma),\\
\int_{\mathcal{W}_m} (\kappa z - \lambda)\boldsymbol{e}_3\cdot N = \pi a^2\left((\kappa d - \lambda)\sin^2\bar{\phi} - \tfrac{2}{3}\kappa a(1+\cos^3\bar{\phi})\right),\\
\int_{\partial\Sigma_m} \kappa\frac{\rho}{\rho_l} z\boldsymbol{e}_3\cdot N^m = \tfrac{4}{3}\pi\kappa a^3\frac{\rho}{\rho_l}.
\end{gathered}
$$

Combining these terms and rearranging:

$$
\frac{6\sin\bar{\phi}\sin(\bar{\phi}-\gamma)}{\kappa a^2} + \frac{3(\kappa d-\lambda)\sin^2\bar{\phi}}{\kappa a} - 2\cos^3\bar{\phi} = 2\Big(1-\frac{2\rho}{\rho_l}\Big). \tag{16}
$$

Next, we make the substitution

$$
2\bar{H} = \kappa(d + a\cos\bar{\phi}) - \lambda,
$$

which follows directly from (9). This leads to

$$
\frac{6\sin\bar{\phi}\sin(\bar{\phi}-\gamma)}{\kappa a^2} + \frac{3(2\bar{H}-\kappa a\cos\bar{\phi})\sin^2\bar{\phi}}{\kappa a} - 2\cos^3\bar{\phi} = 2\Big(1-\frac{2\rho}{\rho_l}\Big).
$$

This last condition simplifies directly into condition (4) of Theorem 2. It remains to verify the description of the function

$$
F(\bar{\phi}) = \cos^3\bar{\phi} - 3\cos\bar{\phi} + \frac{6}{\kappa a}\left(\bar{H} + \frac{\cos\gamma}{a}\right)\sin^2\bar{\phi} - \frac{3\sin\gamma}{\kappa a^2}\sin(2\bar{\phi}),
$$

where $\overline{H}$ is taken to be a given constant. The values at the endpoints are immediate. We find also that

$$\begin{aligned}\frac{F'(\overline{\phi})}{3} &= -\cos^2\overline{\phi}\sin\overline{\phi} + \sin\overline{\phi} + \frac{4}{\kappa a}\Big(\overline{H} + \frac{\cos\gamma}{a}\Big)\sin\overline{\phi}\cos\overline{\phi} - \frac{2\sin\gamma}{\kappa a^2}\cos(2\overline{\phi})\\ &= \sin^3\overline{\phi} + \frac{2}{\kappa a}\Big(\overline{H} + \frac{\cos\gamma}{a}\Big)\sin(2\overline{\phi}) - \frac{2\sin\gamma}{\kappa a^2}\cos(2\overline{\phi}).\end{aligned}$$

Thus, $F'(0) = F'(\pi) = -(6/\kappa a^2)\sin\gamma < 0$. From this it is clear that F must attain an absolute min at some value less than -2 and an absolute max greater than 2. At these points, F' must vanish, and it only remains to show these are the only zeros of F' on $[0, \pi]$. In fact, we see that

$$\tfrac{1}{3}F'(\overline{\phi}) = \sin^3\overline{\phi} + A\sin(2\overline{\phi} - B)$$

for some quantities $A > 0$ and B independent of $\overline{\phi}$. The fact that $F'(0) < 0$ tells us that we may assume $0 < B < \pi$. Clearly, since $0 \le \overline{\phi} \le \pi$, we have $\sin^3\overline{\phi} \ge 0$ and there can be no zero of F' on the interval $[B/2, \pi/2 + B/2]$. For the rest, we consider two cases.

Case I: $0 < B \le \pi/2$, i.e., $F''(0) \ge 0$. In this case, both terms in the expression for F' are increasing on the interval $0 < \overline{\phi} < B/2$, so F' can have at most one zero there. (And since $F'(B/2) > 0$ it does have exactly one.)

F' must also have a zero on $[\pi/2 + B/2, \pi]$. We note that

$$\begin{aligned}\tfrac{1}{3}F''(\overline{\phi}) &= 3\cos\overline{\phi}\sin^2\overline{\phi} + 2A\cos(2\overline{\phi} - B)\\ &= \tfrac{3}{2}\sin(2\overline{\phi})\sin\overline{\phi} + 2A\cos(2\overline{\phi} - B)\end{aligned}$$

and consider two subcases, depending on the sign of $A - \sin^3(3\pi/4 + B/2)$.

- First assume that $A \ge \sin^3(3\pi/4 + B/2)$.

Since $F'(\pi/2 + B/2)/3 = \sin^3(\pi/2 + B/2) + A > 0$, and $F'(3\pi/4 + B/2)/3 = \sin^3(3\pi/4 + B/2) - A \le 0$, there is some zero of F' on the interval

$$(\pi/2 + B/2, 3\pi/4 + B/2].$$

Since both $\sin^3\overline{\phi}$ and $A\sin(2\overline{\phi} - B)$ are decreasing on this interval, there is exactly one zero of F' there.

Let us assume there is another zero $\overline{\phi}_0$ of F' with $3\pi/4 + B/2 < \overline{\phi}_0 < \pi$. Since $F''(3\pi/4 + B/2) < 0$ and $F'(3\pi/4 + B/2) \le 0$, we conclude that F'' must have a zero $\overline{\phi}_1$ on the interval $(3\pi/4 + B/2, \overline{\phi}_0)$ at a negative local minimum of F'. Furthermore, since $F'(\pi) < 0$, it must be the case that F'' has another zero $\overline{\phi}_2$ on the interval $(\overline{\phi}_1, \pi)$ at a nonnegative local maximum of F'.

We now show this situation leads to a contradiction by establishing that F'' has exactly one zero on the interval $(3\pi/4 + B/2, \pi]$. In fact, we will show more:

Lemma 5. *If* $0 < B < \pi/2$, *then* F'' *has exactly one zero in* $[\pi/2 + B/2, \pi]$, *and it occurs on the interval* $(3\pi/4 + B/2, \pi)$ *at a local minimum of* F'.

This is because $3\sin^2\overline{\phi}\cos\overline{\phi}$ is increasing on the interval $\left(\pi - \arccos\frac{1}{\sqrt{3}}, \pi\right)$. Indeed,

$$\frac{d}{d\overline{\phi}}(\sin^2\overline{\phi}\cos\overline{\phi}) = 2\sin\overline{\phi}\cos^2\overline{\phi} - \sin^3\overline{\phi} = \sin\overline{\phi}(3\cos^2\overline{\phi} - 1).$$

Furthermore, it is easily checked that $\pi - \arccos(1/\sqrt{3}) < 3\pi/4$. Thus, F'' is increasing on the interval $[3\pi/4 + B/2, \pi]$ and has exactly one zero there. Finally, F'' is negative on the interval $[\pi/2 + B/2, 3\pi/4 + B/2]$, so we have established the lemma and finished this subcase.

- Still under the assumption $0 < B \leq \pi/2$ (Case I), we now suppose instead that $A < \sin^3(3\pi/4 + B/2)$.

In this case F' is positive throughout the interval $[\pi/2 + B/2, 3\pi/4 + B/2]$. Thus, the first zero $\overline{\phi}_0$ of F' on $[\pi/2 + B/2, \pi]$ must occur inside $(3\pi/4 + B/2, \pi)$. Since $F'(\pi) < 0$, and $F''(\pi) > 0$, the unique zero ϕ_1 of F'' given by Lemma 5 must satisfy

$$\max\{\overline{\phi}_0, 3\pi/4 + B/2\} < \overline{\phi}_1.$$

If we assume the existence of a second zero of F' on the interval $(\overline{\phi}_0, \pi)$, we obtain a zero of F'' at a local maximum of F' (and a contradiction) as before.

Case II: $\pi/2 \leq B < \pi$, i.e., $F''(0) \leq 0$. The reflection $\overline{\phi} \to \pi - \overline{\phi}$ transforms this case into the first one with $B \to \pi - B$. □

The reader will have no trouble verifying that under Archimedes' assumptions $\overline{H} = 0$ (a planar interface) and $\overline{\phi} = \gamma$ (the appropriate azimuthal angle for a horizontal plane to meet the sphere at the correct contact angle) the formula in Theorem 2 reduces to the condition of Archimedes.

4. Floating in two dimensions

The result of [Finn 2011] referred to in the introduction and termed by the author a "criterion for floating" concerns a variational problem considered earlier in [Bhatnagar and Finn 2006] for the energy

$$\mathscr{E} = -\sigma|\hat{\Lambda}| - \sigma\beta|\mathscr{W}| + \mathscr{G}, \tag{17}$$

where $\hat{\Lambda}$ is the linear segment of intersection of a planar/linear interface with a two-dimensional convex body and $\mathscr{G}$ is the specific gravitational energy we have considered above. The measures appearing in the first two terms in this functional are one-dimensional (length) and the integral is an area integral. There is no volume constraint in Bhatnagar and Finn's problem, nor outer container. With certain other

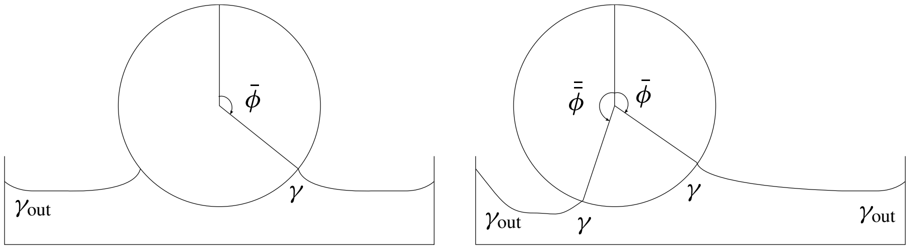


Figure 6. Azimuthal angles determined by a horizontal contact line (left) and differing azimuthal angles in the two-dimensional case (right).

assumptions, they also find that the interface always lies along a fixed line. From this point, Finn goes on to obtain the striking result that for some values of $\rho > \rho_l$, σ and β there will be an equilibrium which is a local minimum for energy in which the convex body contacts the interface, i.e., floats. We now formulate and extend our results to a problem dimensionally similar to the problem of Bhatnagar and Finn.

Physically, we envision a trough consisting of two vertical walls and a horizontal bottom. The trough is assumed to extend infinitely in the $y = x_2$ direction and to be filled with a sea of liquid. Into this sea is introduced a horizontal floating circular cylinder (an infinitely long log) with axis parallel to $\boldsymbol{e}_2$. Let us assume that the free surface interface Λ also is always of cylindrical form with generator parallel to $\boldsymbol{e}_2$, so that if the log is centrally located between the walls and the interface shares the same midplane symmetry, then the projection of the system onto the x, z-plane resembles that of the system considered in the previous section (Figure 6, left), though the equation satisfied by the generating curve (and hence its shape) will be different from that of the meridian previously considered.

The energy of such a system can be taken to have the form of (5):

$$\mathscr{E} = \sigma|\Lambda| - \sigma\beta|\mathscr{W}| + \mathscr{G},$$

where the dimensions of the measures have been lowered by one and $\mathscr{G} = \int_{\mathscr{V}\cup\Sigma_m} G$ is an area integral. The first-order necessary conditions take the form

$$k = G/\sigma - \lambda \quad \text{on the curve } \Lambda,$$
$$\cos\gamma = \beta \quad \text{at the endpoints of } \Lambda,$$

and

$$(18) \qquad \acute{w}\cdot\vec{n}\,\big|_{\partial\Lambda} + \int_{\mathscr{W}_m} (G/\sigma - \lambda)\acute{w}\cdot N^{\mathscr{W}} + \int_{\partial\Sigma_m} (G_m/\sigma)\acute{w}\cdot N^m = 0,$$

where k is the curvature of Λ and λ arises from an area constraint on the cross section of liquid in the trough. In analogy to the three-dimensional case, we assume an area density ρ for the object, that the object floats in a liquid of area density ρ_l, a capillary constant $\kappa = \rho_l g/\sigma$, and that the radius of the log is a.

Before we begin an analysis of this variational problem in earnest, let us pause to note what Archimedes' principle would state in this lower-dimensional case (because it will appear in a surprising way later):

Theorem 6. *According to Archimedes' principle in one lower dimension, a homogeneous disk/log of density $\rho > \rho_l$ will sink to the bottom of a bath of density ρ_l, and a homogeneous disk/log of density $\rho < \rho_l$ will float at a level determined by*

$$2\bar{\phi} - \sin(2\bar{\phi}) = 2\pi\Big(1 - \frac{\rho}{\rho_l}\Big). \tag{19}$$

We assume initially the contact line (i.e., the two points where Λ meets Σ_m) is determined by two azimuthal angles, one $\bar{\phi}$ as before and a second $\bar{\bar{\phi}}$ measured counterclockwise from the vertical $\boldsymbol{e}_3$; see Figure 6, right. In addition to (15), the following identities have been found useful.

$$\begin{aligned} N^m[\phi] &= -\sin\bar{\bar{\phi}}\boldsymbol{e}_1 + \cos\bar{\bar{\phi}}\boldsymbol{e}_3, \\ N^{\mathcal{W}}[\phi] &= -N^m \\ &= \sin\bar{\bar{\phi}}\boldsymbol{e}_1 - \cos\bar{\bar{\phi}}\boldsymbol{e}_3, \\ \vec{\nu}[\phi] &= (N^{\mathcal{W}})^{\perp} \\ &= \cos\bar{\bar{\phi}}\boldsymbol{e}_1 + \sin\bar{\bar{\phi}}\boldsymbol{e}_3, \\ \vec{n} &= \cos\gamma\,\vec{\nu} + \sin\gamma\, N^{\mathcal{W}} \\ &= \cos(\bar{\bar{\phi}} - \gamma)\boldsymbol{e}_1 + \sin(\bar{\bar{\phi}} - \gamma)\boldsymbol{e}_3, \\ N^{\Lambda} &= (\vec{n})^{\perp} \\ &= -\sin(\bar{\bar{\phi}} - \gamma)\boldsymbol{e}_1 + \cos(\bar{\bar{\phi}} - \gamma)\boldsymbol{e}_3. \end{aligned} \tag{20}$$

Taking first a horizontal motion of the floating sphere, so that $\acute{w} = \boldsymbol{e}_1$, we find

$$\boldsymbol{e}_1 \cdot \vec{n}\,\big|_{\partial\mathcal{W}_m} = \cos(\bar{\bar{\phi}} - \gamma) - \cos(\bar{\phi} - \gamma) = -2\sin B\sin(A - \gamma),$$

where $A = \frac{1}{2}(\bar{\bar{\phi}} + \bar{\phi}),\ B = \frac{1}{2}(\bar{\bar{\phi}} - \bar{\phi})$,

$$\begin{aligned} \int_{\mathcal{W}_m} (\kappa z - \lambda)\boldsymbol{e}_1 \cdot N^{\mathcal{W}} &= a(\kappa d - \lambda)(\cos\bar{\bar{\phi}} - \cos\bar{\phi}) + \tfrac{1}{2}\kappa a^2(\cos^2\bar{\bar{\phi}} - \cos^2\bar{\phi}) \\ &= -2a\sin B\sin A(\kappa d - \lambda + \kappa a\cos A\cos B), \end{aligned}$$

and

$$\int_{\partial\Sigma_m} \Big(\kappa\frac{\rho}{\rho_l}z - \lambda\Big)\,\boldsymbol{e}_1 \cdot N^m = 0.$$

Since each of these terms has a factor $\sin B$, we see from condition (18), that one possibility is $\sin B = 0$. If this holds, it can readily be determined that $\bar{\bar{\phi}} = \bar{\phi}$.

Once this occurs, then since the left and right interfaces must start from the same height and with the same inclination angle, we have a proof that the axis of the floating cylinder must lie on the midplane between the vertical walls. This is the conclusion we would like to make. The other alternative is that

$$\sin(A-\gamma)+a\sin A(\kappa d-\lambda+\kappa a\cos A\cos B)=0,$$

which we rewrite as

$$\big(\cos\gamma+a(\kappa d-\lambda)\big)\sin A+\tfrac{1}{2}\kappa a^2\sin(2A)\cos B-\sin\gamma\cos A=0. \tag{21}$$

Leaving this open as a possibility for the moment, we turn to an independent vertical translation of Σ_m with $\acute{w}=\boldsymbol{e}_3$. In this case

$$\boldsymbol{e}_3\cdot\vec{n}\,\big|_{\partial\mathcal{W}_m}=\sin(\bar{\bar{\phi}}-\gamma)+\sin(\bar{\phi}-\gamma)=2\cos B\sin(A-\gamma);$$

moreover

$$\begin{aligned}\int_{\mathcal{W}_m}&(\kappa z-\lambda)\boldsymbol{e}_3\cdot N^{\mathcal{W}}\\ &=a(\kappa d-\lambda)(\sin\bar{\bar{\phi}}+\sin\bar{\phi})+\tfrac{1}{4}\kappa a^2(\sin(2\bar{\bar{\phi}})+\sin(2\bar{\phi}))+\tfrac{1}{2}\kappa a^2(\bar{\bar{\phi}}+\bar{\phi})-\kappa a^2\pi\\ &=2a\cos B\sin A(\kappa d-\lambda)+\tfrac{1}{2}\kappa a^2\sin(2A)\cos(2B)+\tfrac{1}{2}\kappa a^2(\bar{\bar{\phi}}+\bar{\phi})-\kappa a^2\pi\end{aligned}$$

and

$$\int_{\partial\Sigma_m}\kappa\frac{\rho}{\rho_l}z\,\boldsymbol{e}_3\cdot N^m=\kappa a^2\pi\frac{\rho}{\rho_l}.$$

Combining these terms to form the expression in (18), we arrive at a second necessary condition,

$$\begin{aligned}&\big(\cos\gamma+a(\kappa d-\lambda)\big)\sin A\cos B-\sin\gamma\cos A\cos B\\ &\qquad+\tfrac{1}{2}\kappa a^2\sin A\cos A(1-2\sin^2 B)+\tfrac{1}{4}\kappa a^2(\bar{\bar{\phi}}+\bar{\phi})=\frac{\kappa a^2\pi}{2}\Big(1-\frac{\rho}{\rho_l}\Big).\end{aligned} \tag{22}$$

Multiplying the equation in (21) by $\cos B$ and subtracting the result from (22) and simplifying, we obtain the surprising condition

$$\bar{\bar{\phi}}+\bar{\phi}-\sin(\bar{\bar{\phi}}+\bar{\phi})=2\pi\Big(1-\frac{\rho}{\rho_l}\Big). \tag{23}$$

This is surprising because it says that if the log floats anywhere but in the middle between the vertical walls of the trough, then the wetted region must match the wetted region predicted by (19) of Theorem 6, which is based on Archimedes' assumptions, including that of a flat interface. In particular, the portion that is wetted is independent of all parameters except the density fraction! We view this scenario as highly unlikely. The fact that we cannot rule out this possibility leads to the following curious result.

Theorem 7. *In the two-dimensional floating log problem, either the axis of the log lies in the vertical midplane determined by the sides of the vessel, or the wetted/nonwetted region is determined by the generalized version of Archimedes' condition given in* (23).

At this point, we proceed as in the three-dimensional case by assuming symmetry of the interface with respect to the midplane. When $\bar{\bar{\phi}} = \bar{\phi}$, condition (22) associated with the vertical translation is still nonvacuous and becomes

$$F(\bar{\phi}) = 2\bar{\phi} + \sin(2\bar{\phi}) + \frac{4}{\kappa a^2}\sin(\bar{\phi} - \gamma) + \frac{4}{\kappa a}(\kappa d - \lambda)\sin\bar{\phi} = 2\pi\Big(1 - \frac{\rho}{\rho_l}\Big).$$

Again following the three-dimensional case, we let

$$\bar{k} = \kappa(d + a\cos\bar{\phi}) - \lambda$$

denote the curvature of the interface at the contact line on the object. Substitution yields

Theorem 8. *A log that floats in a centrally symmetric position under the effects of surface tension and adhesion must float at a level determined by the azimuthal angle $\bar{\phi}$ satisfying*

$$(24)\qquad 2\bar{\phi} - \sin(2\bar{\phi}) + \frac{4}{\kappa a^2}\sin(\bar{\phi} - \gamma) + \frac{4\bar{k}}{\kappa a}\sin\bar{\phi} = 2\pi\Big(1 - \frac{\rho}{\rho_l}\Big),$$

where $\bar{k}$ is the curvature of the interface at the contact line, and γ is the contact angle of the interface with the floating log.

We emphasize that $\bar{k}$ is assumed to be given and constant. The behavior of the function

$$F(\bar{\phi}) = 2\bar{\phi} - \sin(2\bar{\phi}) + \frac{4}{\kappa a^2}\sin(\bar{\phi} - \gamma) + \frac{4\bar{k}}{\kappa a}\sin\bar{\phi}$$

is somewhat different than that in the three-dimensional case; see Figure 7. One sees first of all that

$$F(0) = -\frac{4}{\kappa a^2}\sin\gamma < 0 \quad\text{and}\quad F(\pi) = 2\pi + \frac{4}{\kappa a^2}\sin\gamma > 2\pi.$$

Thus, the endpoint values do not coincide with the extremes of the expression on the right in (24) associated with $\rho = 0$ and $\rho = \rho_l$. Nevertheless, the interval between 0 and 2π is clearly covered by the values of $F(\bar{\phi})$ and, in fact, each value is taken exactly once. To see this we compute

$$\frac{F'(\bar{\phi})}{2} = 1 - \cos(2\bar{\phi}) + \frac{2}{\kappa a^2}\cos(\bar{\phi} - \gamma) + \frac{2\bar{k}}{\kappa a}\cos\bar{\phi}$$

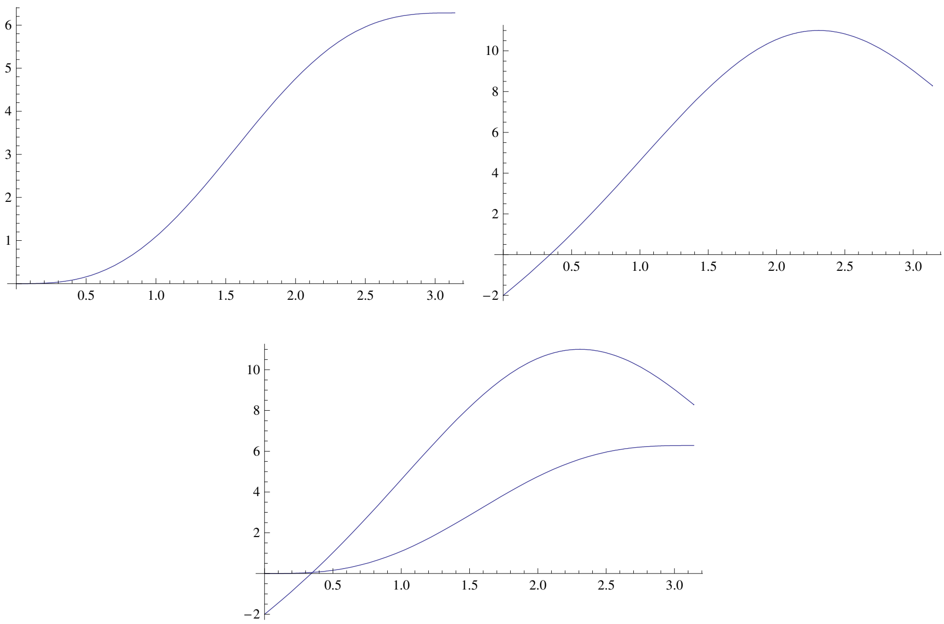


Figure 7. The azimuthal angles determined by Theorems 6 (top left) and 8 (top right); plotted together on the bottom.

and observe first that

$$\frac{F'(0)}{2} = \frac{2}{\kappa a^2}\cos(\gamma) + \frac{2\bar{k}}{\kappa a} = -\frac{F'(\pi)}{2}.$$

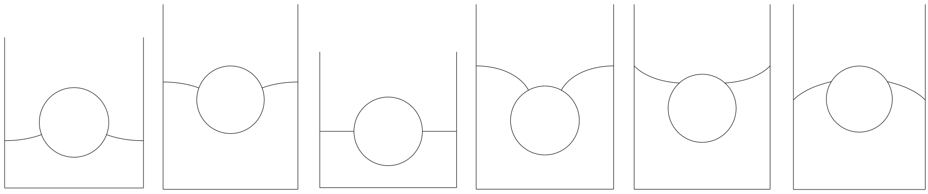
It follows that F' is nonpositive at one of the endpoints and has the opposite sign at the other. Using this, reasoning similar to that found in Section 3 shows F' can have at most one zero on $[0, \pi]$.

Thus, some salient features of Theorem 2 hold also in this lower-dimensional case. For fixed $\bar{k}$ and γ, if $\rho \leq \rho_l$, there is a unique height at which the disk/log can float; there is an interval $\rho_l < \rho < \rho_{\max}$ on which there is at least one (and sometimes two) possible heights at which floating can occur. One expects that if two azimuthal angles are determined by (24), the larger is the physically relevant one.

5. Global solutions numerically computed

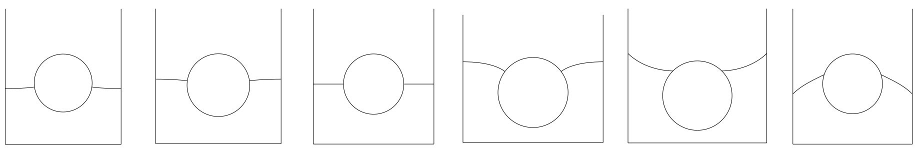
We have obtained global configurations of floating numerically for the problems considered above both in two and three dimensions. The stability and uniqueness of most of these configurations is not presently known.

In Figures 1 and 2 we give representative global configurations which have been obtained and a list of the relevant parameters.



	ρ/ρ_l	γ	γ_{out}	d	λ	$\bar{\phi}$
(1) Lightest	0.0	$\pi/2$	$\pi/2$	1.8850	1.0860	1.9284
(2) Heavy	1.0	$\pi/2$	$\pi/2$	2.6504	3.4494	1.2132
(3) Flat	0.5	$\pi/2$	$\pi/2$	1.6427	1.6427	1.5708
(4) Denser	1.6	$\pi/2$	$\pi/2$	1.9934	3.9663	0.4973
(5) Unstable (?)	0.5	$\pi/4$	$\pi/4$	2.3777	2.7878	0.7328
(6) Stable (?)	0.5	$\pi/4$	$3\pi/4$	2.7382	3.4704	1.0150

Table 1. Two-dimensional case (floating logs). Parameters for each configuration on the top, from left to right. In all cases $a = 1$, $\kappa = 1$, $R = 2$, and the cross-sectional area of liquid is 10.



	ρ/ρ_l	γ	γ_{out}	d	λ	$\bar{\phi}$
(1) Lightest	0.0	$\pi/2$	$\pi/2$	2.1174	1.8486	1.7086
(2) Heavy	1.0	$\pi/2$	$\pi/2$	1.8689	2.1377	1.4330
(3) Flat	0.5	$\pi/2$	$\pi/2$	1.9902	1.9902	1.5708
(4) Denser	2.1	$\pi/2$	$\pi/2$	1.4293	2.5321	0.9293
(5) Unstable (?)	0.5	$\pi/4$	$\pi/4$	1.3646	1.4679	0.7790
(6) Stable (?)	0.5	$\pi/4$	$3\pi/4$	2.0192	2.6403	1.2570

Table 2. Three-dimensional case (floating balls). Parameters for each configuration on the top, from left to right. In all cases $a = 1$, $\kappa = 1$, $R = 2$, and the volume of liquid is 25.

6. Existence and uniqueness

A referee has requested that we provide an existence and uniqueness result for some floating configurations at least superficially like those to which our main result applies. As the referee suggests, we provide in this section an existence result for a ball floating symmetrically in an infinite three-dimensional bath. We also prove that uniqueness does not hold in that case in general, and provide some remarks suggesting that uniqueness does not hold in the problem we consider either.

This problem has been considered in [Keller 1998; Vella and Mahadevan 2005] though not from a fundamentally variational point of view and with existence (and presumably some statement of uniqueness) assumed. Various aspects of the problem make it fundamentally simpler than the physical problem of floating in a finite container and, as we shall see, we can say much more in this case.

Analogues of the results below are shown numerically in the lower-dimensional case of Bhatnagar and Finn's problem [2006]. Also, a partial existence result is given in [Finn 2011] in the two-dimensional case and in [Finn and Vogel 2009] in the three-dimensional case. The methods below may be adapted to give versions of our results in this section for the two-dimensional problem.

As is customary for this kind of problem, we assume a prescribed zero level to which our symmetric interface, satisfying the first four requirements of the boundary value problem (14), is asymptotic. The requirement that the interface be asymptotic (to first order) to the zero level plane necessitates the additional conditions

$$\lim_{r\to\infty} u = \lim_{r\to\infty} \psi = 0.$$

These conditions along with the third equation in (14) imply that the constant λ is zero. In order to show existence, we must obtain a solution to this system which satisfies the additional requirement of Theorem 2. We stress that our application of Theorem 2 to this situation in which the energies we considered in the proof are infinite is somewhat formal, though under a suitable modification of the energies, it is fairly clear that condition (4) is the correct equilibrium condition for floating in this situation as well. With the aforementioned modifications, our problem becomes one of finding a height d for the center of the sphere of radius a, an azimuthal angle $\overline{\phi}$, and a meridian (r, u) with inclination angle ψ such that

$$(25)\qquad \begin{cases} \dot{r} = \cos\psi, \\ \dot{u} = \sin\psi, \\ \dot{\psi} = \kappa u - \sin\psi/r, \\ \psi = \gamma - \overline{\phi} \text{ and } u = d + a\cos\overline{\phi} \quad \text{when } r = r(0) = a\sin\overline{\phi}, \\ \lim_{r\to\infty} u = \lim_{r\to\infty} \psi = 0, \end{cases}$$

and

$$(26)\quad \cos^3\overline{\phi} - 3\cos\overline{\phi} + \frac{6}{\kappa a}\left(\overline{H} + \frac{\cos\gamma}{a}\right)\sin^2\overline{\phi} - \frac{3\sin\gamma}{\kappa a^2}\sin(2\overline{\phi}) = 2\left(1 - \frac{2\rho}{\rho_l}\right),$$

where $\overline{H} = \kappa(d + a\cos\overline{\phi})/2$.

It has been shown by Elcrat, Neel, and Siegel [Elcrat et al. 2004b] that given any $\bar{r} = a \sin \bar{\phi} > 0$ and any inclination angle $\bar{\psi} = \gamma - \bar{\phi}$, there is a unique solution (r, u) of the system (25) *except* for the condition $u = d + a \cos \bar{\phi}$ on the contact line. Since d has not been specified, we can obviously take the sphere of center height $d = u(0) - a \cos \bar{\phi}$ to get this condition as well. In this way, everything becomes a function of $\bar{\phi}$, and we have only to find $\bar{\phi}$ satisfying the following simplified version of (26):

$$\text{(27)} \quad \cos^3 \bar{\phi} - 3 \cos \bar{\phi} + \frac{3}{a}\Big(u(0) + \frac{2 \cos \gamma}{\kappa a}\Big) \sin^2 \bar{\phi} - \frac{3 \sin \gamma}{\kappa a^2} \sin(2\bar{\phi}) = 2\Big(1 - \frac{2\rho}{\rho_l}\Big).$$

Unfortunately, the dependence of $u(0) = u(0; \bar{\phi})$ on $\bar{\phi}$ is not explicit and not well understood. This fact prevents us from giving a full analysis of the solutions of (27). Nevertheless, we can set

$$G(\bar{\phi}) = \cos^3 \bar{\phi} - 3 \cos \bar{\phi} + \frac{3}{a}\Big(u(0) + \frac{2 \cos \gamma}{\kappa a}\Big) \sin^2 \bar{\phi} - \frac{3 \sin \gamma}{\kappa a^2} \sin(2\bar{\phi}),$$

which is a well defined smooth function of $\bar{\phi}$.

When $\bar{\phi}$ tends to zero (a sinking ball), we have that $r = a \sin \bar{\phi}$ tends to zero and necessarily $u(0; \bar{\phi})$ tends to zero as well. Thus,

$$\lim_{\bar{\phi} \to 0} G(\bar{\phi}) = -2.$$

Similarly,

$$\lim_{\bar{\phi} \to \pi} G(\bar{\phi}) = 2.$$

We draw attention to the fact that these values are shared by the function F considered in Section 3. In fact, we can numerically graph the function G for specific choices of κ and γ to see that G shares the qualitative properties of the function F analyzed in Section 3, initially decreasing to a unique minimum, then increasing to a unique maximum greater than 2, and decreasing on the remainder of the interval; see Figure 8. We expect that these qualitative features are always shared, but we are unable to prove that.

We are able to compute the following:

$$\lim_{\bar{\phi} \to 0, \pi} G'(\bar{\phi}) = -\frac{6 \sin \gamma}{\kappa a^2}.$$

This means that G is always decreasing at $\bar{\phi} = 0$ and $\bar{\phi} = \pi$. By continuity we obviously have enough to obtain existence for any density ρ between zero and the density of the liquid ρ_l. The last computation also gives existence for some range of densities greater than ρ_l and some "negative densities" as described in the discussion of the main result in Section 1. We have shown the following:

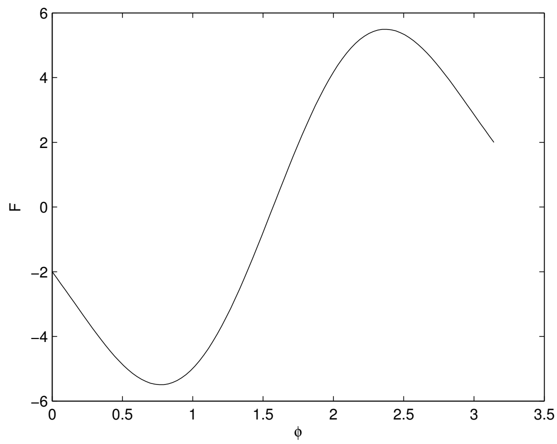


Figure 8. Numerical plot of the function G for $a = 1$, $\kappa = 1$ and $\gamma = \pi/2$.

Theorem 9. *There are positive numbers ϵ and δ depending on the capillarity constant κ, the radius of the sphere a, the contact angle γ, and the density of the liquid ρ_l, such that the floating ball problem for an infinite bath has a well defined equilibrium configuration (satisfying the flux condition obtained in this paper) for each density ρ with $-\epsilon < \rho < \rho_l + \delta$.*

It follows also that there is some $\bar{\phi} = \bar{\phi}_{\min}$ where G takes a minimum value $m < -2$. If we take a density ρ with

$$\rho_l < \rho < \rho_l(1 - m/2),$$

then we see there are at least two values $\bar{\phi}_1$ and $\bar{\phi}_2$ with $\bar{\phi}_1 < \bar{\phi}_{\min} < \bar{\phi}_2$ which correspond to distinct equilibrium configurations for different heights d of the ball.

Theorem 10. *The problem of a floating ball in an infinite bath with capillarity taken into account and $\gamma \in (0, \pi)$ does not have a unique equilibrium solution in general. More precisely, there is an interval $(\rho_m, \rho_M) \supset\supset (0, \rho_l)$ and for any density ρ in $(\rho_m, \rho_M)\backslash(0, \rho_l)$, there exist at least two equilibria.*

If we calculate a modified energy for the specific choices of parameters considered above in Figure 8 and the two distinct equilibria shown in Figure 9, we find that the one of smaller azimuthal angle and lower height d has greater energy. This strongly suggests that when a heavy ball is floating, the energy increases to a maximum (at another equilibrium) as the ball is pushed down. After the ball is pushed below the second equilibrium height (maximum energy), it will sink. These qualitative observations are consistent with experiments.

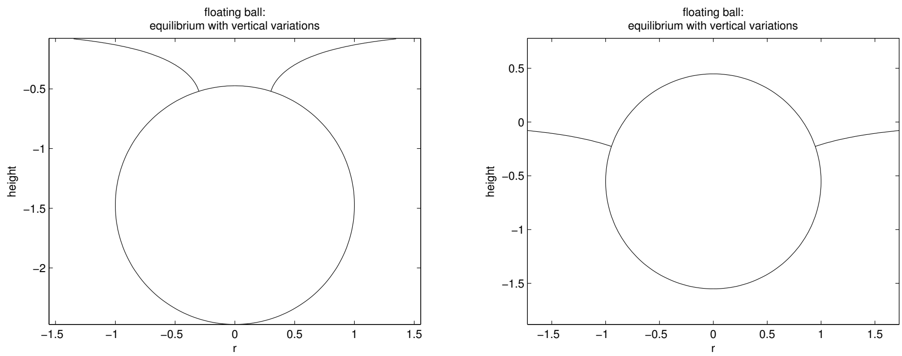


Figure 9. Distinct equilibria showing nonuniqueness for $\rho = 3/2 > \rho_l = 1$, $a = 1$, $\kappa = 1$ and $\gamma = \pi/2$.

Comparison to the graphs shown in Figure 7, suggests that the same situation holds in finite containers. It should be noted, however, that Figure 7 does not show this is the case, because $\overline{H}$ is considered constant there, and the value of $\overline{H}$ will undoubtedly be different in the two distinct equilibrium configurations.

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