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## GENERALIZED EIGENVALUE PROBLEMS OF NONHOMOGENEOUS ELLIPTIC OPERATORS AND THEIR APPLICATION


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# GENERALIZED EIGENVALUE PROBLEMS OF NONHOMOGENEOUS ELLIPTIC OPERATORS AND THEIR APPLICATION

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**We consider the equation $-\mathrm{div}(a(x,|\nabla u|)\nabla u)=\lambda|u|^{p-2}u$ (whose special case $a(x,t)=t^{p-2}$ is the $p$-Laplace equation) on a bounded domain $\Omega\subset\mathbb{R}^N$ with $C^2$ boundary, with null boundary condition. We prove that there are $\lambda\in\mathbb{R}$ for which the equation has a nontrivial solution. As an application, by variational methods, we present the existence of a positive solution to $-\mathrm{div}(a(x,|\nabla u|)\nabla u)=f(x,u)$ in $\Omega$, where $f$ is asymptotically $(p-1)$-linear near zero and $\infty$, considering the nonresonant, resonant, and doubly resonant cases. We show that, generally, the spectrum of the operator $-\mathrm{div}(a(x,|\nabla u|)\nabla u)$ on $W_0^{1,p}(\Omega)$ is not discrete.**

## 1. Introduction

Let $1<p<\infty$ and let $\Omega\subset\mathbb{R}^N$ be a bounded domain with $C^2$ boundary $\partial\Omega$. We are interested in values of $\lambda\in\mathbb{R}$ such that a nontrivial solution exists to the equation

$$(\mathrm{EV};\lambda)\qquad \begin{cases}-\mathrm{div}\,A(x,\nabla u)=\lambda|u|^{p-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega;\end{cases}$$

such a $\lambda$ is called an *eigenvalue* for $A$. Here $A\colon\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ is a map that is strictly monotone in the second variable and satisfies the regularity conditions in Assumption A below.

The $p$-Laplace equation is the special case of (EV; $\lambda$) with $A(x,y)=|y|^{p-2}y$, and in this case the eigenvalues for $A$ are the usual eigenvalues of the $p$-Laplacian. However, we do not suppose that $A$ is $(p-1)$-homogeneous in the second variable. Instead, these are the assumptions we make on the map $A$:

**Assumption A.** $A(x,y)=a(x,|y|)y$, where $a(x,t)>0$ for all $x\in\overline{\Omega}$ and all $t\in(0,+\infty)$; furthermore:

(i) $A\in C^0(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)\cap C^1(\overline{\Omega}\times(\mathbb{R}^N\setminus\{0\}),\mathbb{R}^N)$.

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(ii) There exists $C_1 > 0$ such that

$$|D_y A(x, y)| \le C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.$$

(iii) There exists $C_0 > 0$ such that

$$D_y A(x, y)\xi \cdot \xi \ge C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x, y)| \le C_2 (1 + |y|^{p-1}) \quad \text{for every } x \in \overline{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.$$

(v) There exist $C_3 > 0$ and a positive $t_0 \le 1$ such that

$$|D_x A(x, y)| \le C_3 |y|^{p-1} (-\log |y|)$$

for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

*From now on, we assume that* $C_0 \le p - 1 \le C_1$ which leads to no loss of generality, as can be seen from Assumption A(ii)–(iii).

A similar hypothesis to Assumption A is considered in the study of quasilinear elliptic problems; see [Motreanu and Papageorgiou 2011, Example 2.2; Damascelli 1998; Motreanu et al. 2011; Miyajima et al. 2012; Tanaka 2012a]. We also refer to [García-Huidobro et al. 1995; Kim 2009; Kim and Kim 2010; Fukagai and Narukawa 2007; Prado and Ubilla 1998; Robinson 2004] for generalized $p$-Laplace operators. In particular, when $A(x, y) = |y|^{p-2} y$ — that is, when $\operatorname{div} A(x, \nabla u)$ is the usual $p$-Laplacian $\Delta_p u$ — we can take $C_0 = C_1 = p - 1$ in Assumption A. Conversely, if $C_0 = C_1 = p - 1$ in Assumption A, the inequalities in Remark 1(ii)–(iii) below show that $a(x, t) = |t|^{p-2}$, whence $A(x, y) = |y|^{p-2} y$. In the $p$-Laplace case, the first eigenvalue $\lambda_1$ is obtained by the Rayleigh quotient: $\lambda_1 = \inf\bigl\{\int_\Omega |\nabla u|^p \, dx / \|u\|_p^p : u \ne 0\bigr\}$. But since our operator is nonhomogeneous, $\inf\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of } A\}$ is in general not obtained by such a Rayleigh quotient corresponding to $A$. In Section 3, since the Rayleigh quotient plays an important role, we study its behavior as $\|u\|_p \to 0$ or $\|u\|_p \to \infty$ under an additional condition describing an asymptotic $(p-1)$-homogeneity. For example, we can consider

$$\operatorname{div} A(x, \nabla u) = \operatorname{div}\bigl(\bigl(a_0(x)|\nabla u|^{p-2} + a_\infty(x)|\nabla u|^{q-2}\bigr)(1 + |\nabla u|^q)^{(p-q)/q} \nabla u\bigr)$$

for $1 < p \le q < \infty$, $a_0, a_\infty \in C^1(\overline{\Omega})$ with $\min_{\overline{\Omega}} a_0 > 0$ and $\min_{\overline{\Omega}} a_\infty > 0$. This satisfies

$$A(x, y) - a_0(x)|y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to 0,$$
$$A(x, y) - a_\infty(x)|y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to \infty.$$

Under these these conditions (see (AH0) and (AH) in Section 3), we shall prove that

$$\min\left\{\int_\Omega\int_0^{|\nabla u(x)|}\frac{a(x,t)t}{r^p}\,dt\,dx : \|u\|_p = r\right\}$$

approaches $\lambda_1(a_0)/p$ as $r \to +0$ and $\lambda_1(a_\infty)/p$ as $r \to +\infty$; here

$$\lambda_1(a_0) = \min\left\{\int_\Omega a_0(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\},$$
$$\lambda_1(a_\infty) = \min\left\{\int_\Omega a_\infty(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\}.$$

Concerning the eigenvalue problem for a nonhomogeneous operator, we can refer to [Robinson 2004; Tanaka 2012b] under the Neumann boundary condition.

In Section 4, as an application of Section 3, we present the existence of a positive solution for the quasilinear elliptic equation

$$\text{(P)}\qquad \begin{cases} -\operatorname{div} A(x,\nabla u) = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $f$ satisfies the following assumption.

**Assumption (*f*).** $f$ is a Carathéodory function on $\Omega\times\mathbb{R}$ with $f(x,0)=0$ for a.e. $x\in\Omega$, $f$ is bounded on bounded sets and $f$ is asymptotically $(p-1)$-linear near $+0$ and $+\infty$ in the following sense:

$$\text{(i)}\qquad \lim_{u\to+0}\frac{f(x,u)}{u^{p-1}} = \alpha_0 \quad \text{uniformly in a.e. } x\in\Omega,$$

$$\text{(ii)}\qquad \lim_{u\to+\infty}\frac{f(x,u)}{u^{p-1}} = \alpha \quad \text{uniformly in a.e. } x\in\Omega,$$

for some constants $\alpha_0$ and $\alpha$.

Regarding the existence of a positive solution under the Dirichlet boundary condition, we can refer to [Fukagai and Narukawa 2007; Prado and Ubilla 1998] for nonhomogeneous operators. However, we can not apply these results to our nonlinear term which is only asymptotically $(p-1)$-linear near $+0$ and $+\infty$, and furthermore with possibly different weights. In [García-Huidobro et al. 1995], it is proved the existence of a positive radial solution for nonhomogeneous operators.

For the $p$-Laplace equation, it is well known that if $(\alpha-\lambda_1)(\alpha_0-\lambda_1)<0$ (where $\lambda_1$ denotes the first eigenvalue of $-\Delta_p$ under a Dirichlet boundary condition),

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has a positive solution (see [Dancer and Perera 2001]). One of our main purposes is to extend this existence result from the $p$-Laplace equation to the corresponding problem involving our nonhomogeneous operator $A$. This is done in Theorem 25. We mention that in the special case of $A(x,y)=A(y)$, the result in [Kyritsi et al. 2010] provides the existence of a positive solution if $\alpha < \lambda_1 C_0/(p-1)$ and $\lambda_1 C_1/(p-1) < \alpha_0$ hold (note that we can apply this result only to the case where $\alpha < \alpha_0$). We emphasize that, for our general operator, the case $\lambda_1(a_0) \neq \lambda_1(a_1)$ can occur. Note that in such a situation, contrary to the $p$-Laplacian case, we can still apply our theorem when $\alpha_0 = \alpha$ provided this number is between $\lambda_1(a_0)$ and $\lambda_1(a_1)$. The known result for the $p$-Laplacian case is obtained from our theorem simply by setting $a_0 \equiv 1$ and $a_\infty \equiv 1$.

In particular, our theorem implies that if $\lambda_1(a_0) \neq \lambda_1(a_\infty)$, then every $\lambda$ between $\lambda_1(a_0)$ and $\lambda_1(a_\infty)$ is an eigenvalue of $A$ (see Corollary 26) and has a positive eigenfunction. This shows that, generally, the spectrum of the operator $-\mathrm{div}\, A(x, \nabla\cdot)$ on $W_0^{1,p}(\Omega)$ is not discrete.

In the final part of the paper, we treat the one side resonant and doubly resonant cases under additional conditions on $f$. For the $p$-Laplace equation, we refer to [Tanaka 2009] for the resonant and doubly resonant cases. Our Theorem 31 provides the existence of a positive solution in all cases of resonance for problem (P) with a nonhomogeneous operator in the principal part.

## 2. The properties of the map $A$

In what follows, the norm on $W_0^{1,p}(\Omega)$ is given by

$$\|u\|^p := \|\nabla u\|_p^p,$$

where $\|u\|_q$ denotes the usual norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \le q \le \infty)$. Setting

$$G(x, y) := \int_0^{|y|} a(x, t)t\, dt, \tag{1}$$

we can easily see that

$$\nabla_y G(x, y) = A(x, y) \quad \text{and} \quad G(x, 0) = 0$$

for every $x \in \overline{\Omega}$; see [Motreanu et al. 2011] for details.

**Remark 1.** The following assertions hold under Assumption A:

(i) For all $x \in \overline{\Omega}$, $A(x, y)$ is maximal monotone and strictly monotone in $y$.

(ii) $|A(x, y)| \le \dfrac{C_1}{p-1}|y|^{p-1}$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

(iii) $A(x, y)y \ge \dfrac{C_0}{p-1}|y|^p$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

(iv) $G(x, y)$ is strictly convex in $y$ for all $x$ and satisfies the inequalities

$$A(x, y)y \ge G(x, y) \ge \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x, y) \le \frac{C_1}{p(p-1)}|y|^p \tag{2}$$

for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

The following result is important for the proof of the Palais–Smale condition for the functionals related to our problem.

**Proposition 2** [Motreanu et al. 2011, Proposition 1]. *Let* $V : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ *be the map defined by*

$$\langle V(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v\, dx$$

*for* $u, v \in W_0^{1,p}(\Omega)$. *Then any sequence* $\{u_m\}$ *that converges weakly to* $u$ *and satisfies* $\limsup_{m\to\infty}\langle V(u_m), u_m - u\rangle \le 0$ *also converges strongly to* $u$.

**Remark 3.** (i) *If* $u \in W_0^{1,p}(\Omega)$ *is a solution of* (P), *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$.

(ii) *If* $u \in W_0^{1,p}(\Omega)$ *is a nontrivial solution of* (P) *such that* $u \ge 0$, *then* $u > 0$ *in* $\Omega$ *and* $\partial u/\partial \nu < 0$ *on* $\partial\Omega$, *where* $\nu$ *denotes the outward unit normal vector on* $\partial\Omega$.

*Sketch of proof.* (i) Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P). Then, because $u \in L^\infty(\Omega)$ as shown by using the Moser iteration process (cf. [Miyajima et al. 2012, Appendix]), we see that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ by the regularity result in [Lieberman 1988].

(ii) Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P) satisfying $u \ge 0$ and $u \not\equiv 0$. Then, by Assumption $(f)$, we obtain a constant $\lambda > 0$ satisfying

$$-\mathrm{div}\, A(x, \nabla u) + \lambda u^{p-1} \ge 0 \quad \text{in } \Omega.$$

Noting that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ by (i), we have $u(x) > 0$ for every $x \in \Omega$ by [Miyajima et al. 2012, Appendix, Theorem B]. In addition, using the strong maximum principle [ibid., Appendix, Theorem A], we easily see that $\partial u(x)/\partial\nu < 0$ for every $x \in \partial\Omega$. □

**Proposition 4.** *Let* $f_n : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function satisfying*

$$|f_n(x, t)| \le D(1 + |t|^{r-1}) \quad \text{for every } x \in \Omega,\ t \in \mathbb{R}$$

*with some positive constant* $D$ *independent of* $n$ *and* $r \in [p, p^*)$, *where* $p^* = \infty$ *if* $N \le p$ *and* $p^* = pN/(N - p)$ *if* $N > p$. *Assume that* $A_n : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ *is a map satisfying parts* (i)–(iv) *of Assumption A with positive constants* $C_1'$, $C_0'$, *and* $C_2'$ *independent of* $n$. *If* $u_n$ *is a solution for*

$$-\mathrm{div}\, A_n(x, \nabla u) = f_n(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

*and* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$, *then there exist a subsequence* $\{u_{n_l}\}$ *of* $\{u_n\}$ *and* $u_0 \in C_0^1(\overline{\Omega})$ *such that* $u_{n_l} \to u_0$ *in* $C_0^1(\overline{\Omega})$ *as* $l \to \infty$.

*Proof.* Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we may assume that $u_n$ converges weakly to some $u_0$ in $W_0^{1,p}(\Omega)$ by choosing a subsequence. We can show that there exists a $C > 0$ depending only on $|\Omega|$, $p$, $N$, $D$, $C_0'$, $C_1'$, and the embedding constant of

$W_0^{1,p}(\Omega)$ into $L^{\bar{p}^*}(\Omega)$ such that $\|u_n\|_\infty \le C \max\{1, \|u_n\|^{(\bar{p}^*-p)/(\bar{p}^*-r)}\}$ by the Moser iteration process to [Miyajima et al. 2012, Theorem C], where $\bar{p}^* = p^*$ if $N > p$ and $\bar{p}^* > r$ is any constant if $N \le p$. Since $D$, $C_1'$, and $C_0'$ are independent of $n$, $\|u_n\|_\infty$ is bounded. Therefore, the regularity result in [Lieberman 1988] guarantees that there exist $\gamma \in (0,1)$ and $M > 0$ independent of $n$ such that $u_n \in C_0^{1,\gamma}(\overline{\Omega})$ and $\|u_n\|_{C_0^{1,\gamma}(\overline{\Omega})} \le M$ (where we use the fact that $C_2'$ is independent of $n$). Since the inclusion of $C_0^{1,\gamma}(\overline{\Omega})$ to $C_0^1(\overline{\Omega})$ is compact, $u_n$ converges to $u_0$ in $C_0^1(\overline{\Omega})$ (note that $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$). □

## 3. Eigenvalue problems

We introduce a function $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega G(x, \nabla u)\,dx \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3}$$

It is clear that $J$ is of class $C^1$. We also note that

$$rS := \{u \in W_0^{1,p}(\Omega) : \|u\|_p = r\} \quad \text{for } r > 0 \tag{4}$$

is a $C^1$ Finsler manifold (cf. [Deimling 1985, Sections 27.4 and 27.5]) because $r$ is a regular value of the function $u \mapsto \|u\|_p$ on $W_0^{1,p}(\Omega)$. Hence the norm of the derivative at $u \in (rS)$ of the restriction $\tilde{J}$ of $J$ to $rS$ is defined by

$$\begin{aligned} \|\tilde{J}'(u)\|_* &:= \min\{\|J'(u) - t\Phi'(u)\|_{W_0^{1,p}(\Omega)^*} : t \in \mathbb{R}\} \\ &= \sup\{\langle J'(u), v\rangle : v \in T_u(rS),\ \|v\| = 1\}, \end{aligned}$$

where $\Phi(u) := (1/p)\|u\|_p^p$ and $T_u(rS)$ denotes the tangent space of $rS$ at $u$, that is, $T_u(rS) = \{v \in W_0^{1,p}(\Omega) : \int_\Omega |u|^{p-2}uv\,dx = 0\}$. It follows that the restriction $\tilde{J} = J|_{(rS)}$ is a $C^1$-function on $rS$ in the sense of manifolds.

**Proposition 5.** *For $r > 0$, the infimum*

$$\mu_1(A, r) = \inf_{u \in (rS)} \int_\Omega G(x, \nabla u)\,dx \tag{5}$$

*is attained at points $\pm\hat{u}_r \in (rS)$ with $\hat{u}_r \in C^{1,\alpha}(\overline{\Omega})$ and $\hat{u}_r > 0$ in $\Omega$. Moreover, $\pm\hat{u}_r$ are solutions of* (EV; λ) *with $\lambda = \lambda_1(A, \hat{u}_r)/r^p$, where*

$$\lambda_1(A, \hat{u}_r) = \int_\Omega A(x, \nabla\hat{u}_r)\nabla\hat{u}_r\,dx \ge \frac{C_0}{p-1}\lambda_1 r^p. \tag{6}$$

*Proof.* Let $\{u_n\} \subset (rS)$ be a minimizing sequence for (5). Using (2), it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, so along a relabeled subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$, thus $u \in (rS)$. Since

$G(x,\cdot)$ is convex and continuous for all $x \in \Omega$, $J$ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ [Mawhin and Willem 1989, Theorem 1.2]. Therefore, we derive that

$$\mu_1(A,r) \le \int_\Omega G(x,\nabla u)\,dx \le \liminf_{n\to\infty} \int_\Omega G(x,\nabla u_n)\,dx,$$

which yields

$$\mu_1(A,r) = \int_\Omega G(x,\nabla u)\,dx.$$

The fact that the functional $J$ is even implies that $|u|$ is also a global minimizer of $\tilde{J}_r$. Consequently, we may assume that $u \ge 0$. On the other hand, the Lagrange multiplier rule leads to the existence of $t \in \mathbb{R}$ such that

$$\int_\Omega A(x,\nabla u)\nabla v\,dx = t\int_\Omega u^{p-1}v\,dx \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{7}$$

Inserting $v = u$ in (7) entails

$$\mathrm{t}r^p = \int_\Omega A(x,\nabla u)\nabla u\,dx \ge \frac{C_0}{p-1}\|\nabla u\|_p^p \ge \frac{C_0\lambda_1}{p-1}\|u\|_p^p = \frac{C_0\lambda_1}{p-1}r^p. \tag{8}$$

Therefore, we have

$$t = \frac{\lambda_1(A,u)}{r^p} \ge \frac{C_0\lambda_1}{p-1}.$$

From (7), it follows that $u$ is a solution of (EV; $\lambda$) with $\lambda = t = \lambda_1(A,u)/r^p$. According to Remark 3 with $f(x,u) = t|u|^{p-2}u$, it follows that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ and $u > 0$ in $\Omega$. Since $J$ is even and $\lambda_1(A,u) = \lambda_1(A,-u)$, we have that $J(-u) = J(u) = \mu_1(A,r)$ and $-u$ is a negative solution of (EV; $\lambda$) with $\lambda = t = \lambda_1(A,u)/r^p$. The result is thus established with $\hat{u}_r = u$. □

We define

$$K_1(A,r) := \{u \in (rS) : J(u) = \mu_1(A,r)\}.$$

Then it follows from Proposition 5 that $K_1(A,r)$ is not empty for each $r > 0$.

Because we do not know whether the minimizers of $\tilde{J}_r$ are only $\pm\hat{u}_r$, we introduce the following:

$$\underline{\lambda}_1(A,r) := \inf\left\{\int_\Omega A(x,\nabla u)\nabla u\,dx : u \in K_1(A,r)\right\},$$
$$\bar{\lambda}_1(A,r) := \sup\left\{\int_\Omega A(x,\nabla u)\nabla u\,dx : u \in K_1(A,r)\right\}.$$

**Lemma 6.** *For every $r > 0$, $\underline{\lambda}_1(A,r)$ and $\bar{\lambda}_1(A,r)$ are attained.*

*Proof.* We only deal with $\underline{\lambda}_1(A,r)$ because $\bar{\lambda}_1(A,r)$ can be treated similarly. Fix any $r > 0$. Let $u_n \in K_1(A,r)$ satisfy $\lambda_1(A,u_n) \to \underline{\lambda}_1(A,r)$ as $n \to \infty$. Then we see that $\|\nabla u_n\|_p$ is bounded from the inequality

$$\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \le \int_\Omega G(x, \nabla u_n)\,dx = \mu_1(A,r) \le \int_\Omega G(x,\nabla w)\,dx$$

for $w \in rS$, where we use the definition of $\mu_1(A,r)$ and (2). Recall that each $u_n$ is a solution of (EV; $\lambda$) with $\lambda = \lambda_1(A,u_n)/r^p$. Moreover, we have

$$\frac{C_0}{p-1}\lambda_1 r^p \le \lambda_1(A,u_n) \le \frac{C_1}{p-1}\|\nabla u_n\|_p^p$$

by Remark 1(ii) (see (6) for the first inequality), whence $\lambda_1(A,u_n)/r^p$ is bounded. As a result, due to Proposition 4, we may assume that there exists $u_0 \in W_0^{1,p}(\Omega)$ such that $u_n \to u_0$ in $C_0^1(\overline{\Omega})$ by choosing a subsequence if necessary. Since $J$ and $\lambda_1(A,\cdot)$ are continuous in $W_0^{1,p}(\Omega)$, we see that $J(u_0) = \lim_{n\to\infty} J(u_n) = \mu_1(A,r)$, $u_0 \in K_1(A,r)$, and $\lambda_1(A,u_0) = \lim_{n\to\infty}\lambda_1(A,u_n) = \underline{\lambda}_1(A,r)$. Thus, our conclusion holds. □

Define

$$\lambda_1(A) := \inf_{u\ne 0}\int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx \quad \text{and} \quad \mu_1(A) := \inf_{u\ne 0}\int_\Omega \frac{G(x,\nabla u)}{\|u\|_p^p}\,dx.$$

**Lemma 7.**

$$\frac{C_0}{p-1}\lambda_1 \le \lambda_1(A) \le \min\left\{\inf_{r>0}\frac{\underline{\lambda}_1(A,r)}{r^p}, \frac{C_1}{p-1}\lambda_1\right\} \quad \text{and} \quad \mu_1(A) = \inf_{r>0}\frac{\mu_1(A,r)}{r^p}.$$

*Proof.* First, we consider $\lambda_1(A)$. For every $0 \ne u \in W_0^{1,p}(\Omega)$, we have

$$\frac{C_0}{p-1}\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \le \int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx \le \frac{C_1}{p-1}\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \tag{9}$$

by Remark 1(ii)–(iii). Thus $(C_0/(p-1))\lambda_1 \le \lambda_1(A) \le (C_1/(p-1))\lambda_1$ by taking the infimum with respect to $u$.

Here we fix any $\varepsilon > 0$. Then there exists an $r_\varepsilon > 0$ such that $\underline{\lambda}_1(A,r_\varepsilon)/r_\varepsilon^p \le \inf_{r>0}(\underline{\lambda}_1(A,r)/r^p) + \varepsilon$. By Lemma 6, we can choose $u_\varepsilon \in (r_\varepsilon S)$ such that $\lambda_1(A,u_\varepsilon) = \underline{\lambda}_1(A,r_\varepsilon)$, that is, $\int_\Omega A(x,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx = \underline{\lambda}_1(A,r_\varepsilon)$. By the definition of $\lambda_1(A)$, we obtain

$$\lambda_1(A) \le \int_\Omega \frac{A(x,\nabla u_\varepsilon)\nabla u_\varepsilon}{\|u_\varepsilon\|_p^p}\,dx = \frac{\underline{\lambda}_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \inf_{r>0}\frac{\underline{\lambda}_1(A,r)}{r^p} + \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we have $\lambda_1(A) \le \inf_{r>0}(\underline{\lambda}_1(A,r)/r^p)$.

Next we treat $\mu_1(A)$. Fix any $\varepsilon > 0$. Then there exists an $r_\varepsilon > 0$ such that $\mu_1(A,r_\varepsilon)/r_\varepsilon^p \le \inf_{r>0}(\mu_1(A,r)/r^p) + \varepsilon$. On the other hand, because $\mu_1(A,r_\varepsilon)$ is attained at some $u_\varepsilon \in (r_\varepsilon S)$, we have

$$\inf_{u\neq 0}\int_\Omega \frac{G(x,\nabla u)}{\|u\|_p^p}\,dx \le \int_\Omega \frac{G(x,\nabla u_\varepsilon)}{\|u_\varepsilon\|_p^p}\,dx = \frac{\mu_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \inf_{r>0}\frac{\mu_1(A,r)}{r^p}+\varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, this yields that $\mu_1(A) \le \inf_{r>0}(\mu_1(A,r)/r^p)$.

For any $\varepsilon > 0$, we take $v_\varepsilon \neq 0$ such that $\int_\Omega (G(x,\nabla v_\varepsilon)/\|v_\varepsilon\|_p^p)\,dx \le \mu_1(A)+\varepsilon$. Then $r_\varepsilon := \|v_\varepsilon\|_p > 0$ and so

$$\frac{\mu_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \int_\Omega \frac{G(x,\nabla v_\varepsilon)}{\|v_\varepsilon\|_p^p}\,dx \le \mu_1(A)+\varepsilon.$$

This leads to $\mu_1(A) \ge \inf_{r>0}(\mu_1(A,r)/r^p)$. □

**Proposition 8.** *If* $\lambda < \lambda_1(A)$, (EV; $\lambda$) *has no nontrivial solutions.*

*Proof.* Let $u$ be a nontrivial solution of (EV; $\lambda$) with $\lambda < \lambda_1(A)$. Then we have

$$\lambda_1(A) \le \int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx = \lambda$$

by the definition of $\lambda_1(A)$. This is a contradiction. □

Set

$$A_p := \frac{C_1}{p-1}\left(\frac{C_1}{C_0}\right)^{p-1} \ge 1, \tag{10}$$

which is equal to 1 exactly in the case of $A(x,y) = |y|^{p-2}y$ (that is, the special case of the $p$-Laplacian ) because we can choose $C_0 = C_1 = p-1$.

**Lemma 9** [Tanaka 2012a, Lemma 16]. *Let* $\varepsilon > 0$. *For every*

$$u,\varphi \in W^{1,p}(\Omega)\cap C^1(\Omega)\cap L^\infty(\Omega)$$

*with* $u \ge 0$ *and* $\varphi \ge 0$ *in* $\Omega$, *we have*

$$\int_\Omega A(x,\nabla u)\nabla\left(\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\right)dx \le A_p\|\nabla\varphi\|_p^p.$$

**Proposition 10.** *Any nontrivial solution of* (EV; $\lambda$) *with* $\lambda > A_p\lambda_1$ *changes sign.*

*Proof.* By way of contradiction, assume there is a solution $u$ that does not change sign. Then we may suppose that $u \ge 0$ because $A$ is odd. Due to the strong maximum principle and the regularity theorem (see Remark 3), it follows that $u \in C_0^1(\overline{\Omega})$ and $u > 0$ in $\Omega$. Let $\varphi_1$ be the positive eigenfunction of $-\Delta_p$ corresponding to $\lambda_1$ such that $\|\varphi_1\|_p = 1$. According to Lemma 9, we obtain

$$A_p\lambda_1 = A_p\|\nabla\varphi_1\|_p^p \ge \int_\Omega A(x,\nabla u)\nabla\left(\frac{\varphi_1^p}{(u+\varepsilon)^{p-1}}\right)dx = \lambda\int_\Omega\left(\frac{u}{u+\varepsilon}\right)^{p-1}\varphi_1^p\,dx$$

for every $\varepsilon > 0$. By taking $\varepsilon \downarrow 0$, we have $\lambda \le A_p\lambda_1$. This is a contradiction. □

**Proposition 11.** *Assume $A_p\lambda_1 < C_0\lambda_2/(p-1)$, where $\lambda_2 > \lambda_1$ is the second eigenvalue of $-\Delta_p$. If $A_p\lambda_1 < \lambda < C_0\lambda_2/(p-1)$, (EV; $\lambda$) has no nontrivial solutions.*

*Proof.* By way of contradiction, we assume that (EV; $\lambda$) has a nontrivial solution $u$. Then it follows from Proposition 10 that $u$ changes sign. Moreover, by taking $u_\pm$ as a test function in (EV; $\lambda$), we have

$$\frac{C_0}{p-1}\|\nabla u_\pm\|_p^p \le \int_\Omega A(x,\nabla u)(\pm\nabla u_\pm)\,dx = \lambda\|u_\pm\|_p^p,$$

whence

$$\|\nabla u_\pm\|_p^p < \lambda_2\|u_\pm\|_p^p. \tag{11}$$

This inequality guarantees the existence of a continuous path $\gamma_0$ on $S$ such that $\gamma_0(0) = \varphi_1$, $\gamma_0(1) = -\varphi_1$ and $\max_{t\in[0,1]}\|\nabla\gamma_0(t)\|_p^p < \lambda_2$; refer to [Cuesta et al. 1999, Lemma 5.3]. This contradicts the equality

$$\lambda_2 = \inf_{\gamma\in\Sigma}\max_{t\in[0,1]}\Phi(\gamma(t)),$$

where $\Phi(u) := \|\nabla u\|_p^p$ and $\Sigma := \{\gamma\in C([0,1],S) : \gamma(0)=\varphi_1,\ \gamma(1)=-\varphi_1\}$; see [Anane 1987; Cuesta et al. 1999]. This contradiction proves our result.

For the reader's convenience, we give the sketch of the construction of a path $\gamma_0$ as required above. Define paths as follows:

$$\gamma_1(t) := \frac{tu+(1-t)u_+}{\|tu+(1-t)u_+\|_p} = \frac{u_+-tu_-}{\|u_+-tu_-\|_p}, \qquad \gamma_2(t) := \frac{tu_+ + (1-t)u_-}{\|tu_+ + (1-t)u_-\|_p},$$

$$\gamma_3(t) := \frac{(1-t)u - tu_-}{\|(1-t)u-tu_-\|_p} = \frac{(1-t)u_+ - u_-}{\|(1-t)u_+-u_-\|_p}$$

for $t\in[0,1]$. Then, setting $\widetilde{\Phi} := \Phi|_S$, we obtain by (11)

$$\max_{t\in[0,1]}\widetilde{\Phi}(\gamma_i(t)) < \lambda_2, \quad \text{for } i = 1,2,3.$$

We recall that any component of $\mathcal{O}(r) := \{u\in S : \widetilde{\Phi}(u) < r\}$ contains at least one critical point of $\widetilde{\Phi}$, where $r > 0$ [Cuesta et al. 1999, Lemma 3.6]. Note that $\mathcal{O}(\lambda_2)$ contains just two critical points $\varphi_1$ and $-\varphi_1$ because a critical value $c$ of $\widetilde{\Phi}$ corresponds to the eigenvalue $c$ of the negative $p$-Laplacian. Since any component of $\mathcal{O}(\lambda_2)$ is path connected [ibid., Lemma 3.5], there exists a path $\gamma_4$ joining from $u_-/\|u_-\|_p$ to $\varphi_1$ or $-\varphi_1$ in $\mathcal{O}(\lambda_2)$. Thus, by noting that $\Phi$ is even, we can construct a path $\gamma_0\in\Sigma$ such that $\max_t\widetilde{\Phi}(\gamma_0(t)) < \lambda_2$ by considering $\gamma_4^{-1}\cdot\gamma_2\cdot\gamma_1\cdot\gamma_3\cdot(-\gamma_4)$ or its inverse, where $\gamma_i^{-1}(t) := \gamma_i(1-t)$ and $\gamma_i\cdot\gamma_j$ denotes the path defined by $\gamma_i(2t)$ if $0\le t\le\frac{1}{2}$ and $\gamma_j(2t-1)$ if $\frac{1}{2} < t \le 1$. ☐

**3.1.** ***Asymptotically homogeneous case near zero.*** We now consider the case where $A$ is asymptotically $(p-1)$-homogeneous near zero in the following sense.

(AH0) *There exist a positive function* $a_0 \in C^1(\overline{\Omega}, \mathbb{R})$ *and a continuous function* $\tilde{a}_0(x,t)$ *on* $\overline{\Omega} \times [0, +\infty)$ *such that*

$$A(x, y) = a_0(x)|y|^{p-2}y + \tilde{a}_0(x, |y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$

*where*

$$\lim_{t\to+0} \frac{\tilde{a}_0(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For this weight function $a_0$, we define

$$\lambda_1(a_0) := \inf\left\{\int_\Omega a_0(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\}. \tag{12}$$

Because $0 < \min_{x\in\overline{\Omega}} a_0(x) \le \max_{x\in\overline{\Omega}} a_0(x) < \infty$, by the same argument as the one for the first eigenvalue of the negative $p$-Laplacian, we can prove that $\lambda_1(a_0)$ is the first eigenvalue of

$$-\mathrm{div}\big(a_0(x)|\nabla u|^{p-2}\nabla u\big) = \lambda|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{13}$$

Moreover, $\lambda_1(a_0)$ has a positive eigenfunction $\varphi_{a_0} \in C^1(\overline{\Omega})$ and it is simple. It is proved that (13) has no constant sign solutions other than 0 provided $\lambda \ne \lambda_1(a_0)$.

**Theorem 12.** *Assume* (AH0). *For every* $\varepsilon > 0$ *there exists* $r_0 > 0$ *such that equation* (EV; $\lambda$) *has no nontrivial solutions in* $B_p(r_0) := \{v \in W_0^{1,p}(\Omega) : \|v\|_p < r_0\}$ *provided* $\lambda < \lambda_1(a_0) - \varepsilon$.

*Proof.* We argue by contradiction. Thus we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n < \lambda_1(a_0) - \varepsilon_0$, $u_n \in B_p(1/n)$ and $u_n$ is a nontrivial solution of (EV; $\lambda_n$). By taking $u_n$ as a test function in (EV; $\lambda_n$), we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\nabla u_n\,dx = \lambda_n\|u_n\|_p^p \le (\lambda_1(a_0) - \varepsilon_0)/n^p \to 0 \tag{14}$$

as $n \to \infty$. Therefore, $u_n \to 0$ in $W_0^{1,p}(\Omega)$. In addition, by noting that $u_n$ is a nontrivial solution of (EV; $\lambda_n$) and $0 \le \lambda_n < \lambda_1(a_0) - \varepsilon_0$, Proposition 4 yields that $u_n$ converges to 0 in $C^1(\overline{\Omega})$.

Set $v_n := u_n/\|u_n\|_p$. Then it follows from (14) and the boundedness of $\{\lambda_n\}$ that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may assume that $v_n$ converges to some $v_0$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Again by taking $u_n/\|u_n\|_p^p$ as a test function in (EV; $\lambda_n$), we obtain

$$\begin{aligned}\lambda_1(a_0)-\varepsilon_0>\lambda_n&=\int_\Omega\frac{a_0(x)|\nabla u_n|^p}{\|u_n\|_p^p}\,dx+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx\\&=\int_\Omega a_0(x)|\nabla v_n|^p\,dx+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\\&\ge\lambda_1(a_0)+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}=:\lambda_1(a_0)+I\end{aligned}$$

because of the characterization of $\lambda_1(a_0)$. Hypothesis (AH0) guarantees that for every $\delta>0$ there exists $\rho_0>0$ such that $|\tilde a_0(x,t)|\le\delta|t|^{p-2}$ if $|t|\le\rho_0$. Since $\|u_n\|_{C^1(\overline\Omega)}\to0$ and in view of (14), we can get

$$|I|\le\delta\int_\Omega|\nabla v_n|^p\,dx\le\frac{\delta(p-1)}{C_0}\lambda_n\le\frac{\delta(p-1)}{C_0}(\lambda_1(a_0)-\varepsilon_0)$$

for sufficiently large $n$. As a result, by taking a sufficiently small $\delta>0$, we have a contradiction for sufficiently large $n$. □

**Theorem 13.** *Assume* (AH0). *For every* $\varepsilon>0$ *there exists* $r_1>0$ *such that* (EV; $\lambda$) *has no constant sign solutions in* $B_p(r_1)\setminus\{0\}$ *provided* $\lambda>\lambda_1(a_0)+\varepsilon$.

*Proof.* By way of contradiction, we assume that there exist $\varepsilon_0>0$, $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n>\lambda_1(a_0)+\varepsilon_0$, $0\ne u_n\in B_p(1/n)$ and $u_n$ is a constant sign solution of (EV; $\lambda_n$). Because $A$ is odd, we may suppose that $u_n\ge0$ by considering $-u_n$ if necessary. Thus, by Remark 3(i)–(ii), $u_n\in C^1(\overline\Omega)$ and $u_n>0$ in $\Omega$. We note that $\lambda_n\le A_p\lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of $A_p$), and so $\{\lambda_n\}$ is bounded. Therefore, we may assume that $\lambda_n$ converges to some $\lambda_0$ by choosing a subsequence. In addition, by the same argument as in Theorem 12, we can show that $u_n\to0$ in $C^1(\overline\Omega)$.

Set $A_n(x,y):=A(x,\|u_n\|_py)/\|u_n\|_p^{p-1}$ and $f_n(x,t):=\lambda_n|t|^{p-2}t$. Then $A_n$ satisfies Assumption A(i)–(iv) with the same constants $C_0$, $C_1$, and $C_2$. Moreover, $|f_n(x,t)|\le\lambda_n|t|^{p-1}\le A_p\lambda_1(-\Delta_p)|t|^{p-1}$ for every $t\in\mathbb{R}$, a.e. $x\in\Omega$. Note also that we have the boundedness of $\|v_n\|$ due to the inequality $C_0\|\nabla u_n\|_p^p/(p-1)\le\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx=\lambda_n\|u_n\|_p^p$. Since $v_n:=u_n/\|u_n\|_p$ is a positive solution of

$$-\mathrm{div}(A_n(x,\nabla u))=f_n(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,$$

Proposition 4 guarantees that $\{v_n\}$ has a convergent subsequence in $C^1(\overline\Omega)$. By choosing a subsequence, we may suppose that $v_n\to v_0\ne0$ in $C^1(\overline\Omega)$ (note that $\|v_0\|_p=1$). Using that we obtain, for every $w\in W_0^{1,p}(\Omega)$, that

$$\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla w\,dx=\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)\nabla u_n}{|\nabla u_n|^{p-1}}\nabla w|\nabla v_n|^{p-1}\,dx\to0$$

as $n\to\infty$ in view of (AH0) and the convergence $u_n\to0$. As a result, letting

$n \to \infty$ in the equality

$$\int_\Omega a_0(x)|\nabla v_n|^{p-2}\nabla v_n \nabla w\,dx + \int_\Omega \frac{\tilde{a}_0(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla w\,dx = \lambda_n \int_\Omega |v_n|^{p-2}v_n w\,dx$$

for each $w \in W_0^{1,p}(\Omega)$, we see that $v_0 \neq 0$ is a positive solution of (13) with $\lambda = \lambda_0$ (see Remark 3(ii) for $v_0 > 0$). This yields that $\lambda_0 = \lambda_1(a_0)$, because (13) has no positive solutions other that $\lambda = \lambda_1(a_0)$. Therefore we have a contradiction, because $\lambda_0 = \lim_{n\to\infty} \lambda_n \geq \lambda_1(a_0) + \varepsilon_0$. □

**Proposition 14.** *Assume* (AH0). *Then, for every* $\varepsilon > 0$, *there exists* $r_0 > 0$ *such that*

$$\frac{\underline{\lambda}_1(A,r)}{r^p} \geq \lambda_1(a_0) - \varepsilon \quad \text{for every } 0 < r < r_0.$$

*Proof.* Assume that there exist $\varepsilon > 0$ and $r_n > 0$ such that $r_n \to 0$ as $n \to \infty$ and $\underline{\lambda}_1(A,r_n)/r_n^p < \lambda_1(a_0) - \varepsilon$ for every $n \in \mathbb{N}$. Because of Proposition 5 and Lemma 6 (note that $A$ is odd in the second variable), we can choose a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n), \qquad \min_{v\in r_n S}\int_\Omega G(x,\nabla v)\,dx = \int_\Omega G(x,\nabla u_n)\,dx.$$

Note that

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \leq \int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n) < (\lambda_1(a_0)-\varepsilon)r_n^p \to 0, \tag{15}$$

and so $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Because $u_n$ is a solution of (EV; $\lambda$) with $\lambda = \underline{\lambda}_1(A,r_n)/r_n^p$ (see Proposition 5), by combining the inequality

$$\lambda_1(a_0) - \varepsilon > \frac{\underline{\lambda}_1(A,r_n)}{r_n^p} = \int_\Omega a_0(x)|\nabla v_n|^p\,dx + \int_\Omega \frac{\tilde{a}_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx$$

and an argument as in Theorem 12 with $\lambda_n = \underline{\lambda}_1(A,r_n)/r_n^p$, we have a contradiction. □

**Proposition 15.** *Assume* (AH0). *Then, for every* $\varepsilon > 0$, *there exists* $r_1 > 0$ *such that*

$$\frac{\bar{\lambda}_1(A,r)}{r^p} \leq \lambda_1(a_0) + \varepsilon \quad \text{for every } 0 < r < r_1.$$

*Proof.* Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to 0$ as $n \to \infty$ and $\bar{\lambda}_1(A,r_n)/r_n^p > \lambda_1(a_0) + \varepsilon_0$ for every $n \in \mathbb{N}$. According to Lemma 6 and Proposition 5, we can take a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \bar{\lambda}_1(A,r_n), \qquad \min_{v\in r_n S}\int_\Omega G(x,\nabla v)\,dx = \int_\Omega G(x,\nabla u_n)\,dx.$$

Noting that, with $\varphi_{a_0}$ the positive eigenfunction corresponding to $\lambda_1(a_0)$ satisfying

$\|\varphi_{a_0}\|_p = 1$, we have

$$\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \le \int_\Omega G(x,\nabla u_n)\,dx \le \int_\Omega G(x, r_n\nabla\varphi_{a_0})\,dx \le \frac{C_1 r_n^p}{p(p-1)}\|\nabla\varphi_{a_0}\|_p^p,$$

we see that $u_n \to 0$ in $C^1(\overline{\Omega})$ due to Proposition 4, because $u_n$ is a positive solution of (EV; $\lambda$) with $\lambda = \bar{\lambda}_1(A, r_n)/r_n^p$ and $(\lambda_1(a_0)+\varepsilon_0 <)\bar{\lambda}_1(A, r_n)/r_n^p \le A_p\lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of $A_p$). Therefore, by the same argument as in Theorem 13 with $\lambda_n = \bar{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

The following result follows from Propositions 14 and 15, (note $\underline{\lambda}_1(A, r) \le \bar{\lambda}_1(A, r)$ for every $r > 0$).

**Corollary 16.** *Under* (AH0), *we have*

$$\lim_{r\to+0}\frac{\bar{\lambda}_1(A, r)}{r^p} = \lim_{r\to+0}\frac{\underline{\lambda}_1(A, r)}{r^p} = \lambda_1(a_0).$$

**Proposition 17.** *Under* (AH0), *we have*

$$\lim_{r\to+0}\frac{\mu_1(A, r)}{r^p} = \frac{\lambda_1(a_0)}{p}.$$

*Proof.* Due to Proposition 5, for every $r > 0$, there exists a positive solution $u_r \in (rS)\cap C^1(\overline{\Omega})$ of (EV; $\lambda$) with $\lambda = \lambda_1(A, u_r)/r^p$ and $\mu_1(A, r) = J(u_r)$. Then we can prove that $u_r \to 0$ in $C^1(\overline{\Omega})$ as $r \to +0$ and $u_r/\|u_r\|_p$ is bounded in $W_0^{1,p}(\Omega)$ as $r \to +0$ by a similar reason to the one in Proposition 15 (note that $\lambda_1(A, u_r)/r^p$ is bounded as $r \to +0$ by the inequality below and Corollary 16).

Set $\widetilde{G}_0(x, y) := \int_0^{|y|} \tilde{a}_0(x, t)t\,dt$ for $y \in \mathbb{R}^N$. We point out that

$$\underline{\lambda}_1(A, r) \le \lambda_1(A, u_r) \le \bar{\lambda}_1(A, r)$$

and

$$\begin{aligned}\mu_1(A, r) &= \int_\Omega G(x,\nabla u_r)\,dx = \frac{1}{p}\int_\Omega a_0(x)|\nabla u_r|^p\,dx + \int_\Omega \widetilde{G}_0(x,\nabla u_r)\,dx\\ &= \frac{\lambda_1(A, u_r)}{p} - \frac{1}{p}\int_\Omega \tilde{a}_0(x, |\nabla u|)|\nabla u_r|^2\,dx + \int_\Omega \widetilde{G}_0(x,\nabla u_r)\,dx.\end{aligned}$$

Thus, by Corollary 16 and $r = \|u_r\|_p$, it suffices to prove

$$\lim_{r\to+0}\int_\Omega \frac{\tilde{a}_0(x, |\nabla u|)|\nabla u_r|^2}{\|u_r\|_p^p}\,dx = 0 \quad\text{and}\quad \lim_{r\to+0}\int_\Omega \frac{\widetilde{G}_0(x,\nabla u_r)}{\|u_r\|_p^p}\,dx = 0.$$

Now we fix any $\varepsilon > 0$. Then, by (AH0), there exists $\delta > 0$ such that

$$|\tilde{a}_0(x, t)| \le \varepsilon t^{p-2} \quad\text{and}\quad |\widetilde{G}_0(x, y)| \le \varepsilon|y|^p/p \quad\text{for every } 0 < t \le \delta,\ |y| \le \delta.$$

Because $u_r \to 0$ in $C^1(\overline{\Omega})$ as $r \to +0$, we may assume that $\|u_r\|_{C^1(\overline{\Omega})} \le \delta$ for sufficiently small $r > 0$. Therefore, we obtain

$$\left| \int_\Omega \frac{\tilde{a}_0(x, |\nabla u|)|\nabla u_r|^2}{\|u_r\|_p^p} \, dx \right| \le \varepsilon \frac{\|\nabla u_r\|_p^p}{\|u_r\|_p^p}, \quad \left| \int_\Omega \frac{\widetilde{G}_0(x, \nabla u_r)}{\|u_r\|_p^p} \, dx \right| \le \varepsilon \frac{\|\nabla u_r\|_p^p}{p\|u_r\|_p^p}.$$

Since $\|\nabla u_r\|_p / \|u_r\|_p$ is bounded as $r \to +0$ and $\varepsilon > 0$ is arbitrary, our conclusion holds. □

**3.2.** ***Asymptotically homogeneous case near $\infty$.*** In this subsection, we consider the case where $A$ is asymptotically $(p-1)$-homogeneous near $\infty$ in the following sense.

(AH) *There exist a positive function $a_\infty \in C^1(\overline{\Omega}, \mathbb{R})$ and a continuous function $\tilde{a}(x,t)$ on $\overline{\Omega} \times \mathbb{R}$ such that*

$$A(x, y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x, |y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$

*where*

$$\lim_{t \to +\infty} \frac{\tilde{a}(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For the weight function $a_\infty$, we define

$$\lambda_1(a_\infty) := \inf\left\{ \int_\Omega a_\infty(x)|\nabla u|^p \, dx : \|u\|_p = 1 \right\}. \tag{16}$$

Because $0 < \min_{x \in \overline{\Omega}} a_\infty(x) \le \max_{x \in \overline{\Omega}} a_\infty(x) < \infty$, by the same argument as for the first eigenvalue of $-\Delta_p$, we can prove the following elementary results:

(i) $\lambda_1(a_\infty)$ is the first eigenvalue of

$$-\mathrm{div}(a_\infty(x)|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{17}$$

(ii) $\lambda_1(a_\infty)$ has a positive eigenfunction $\varphi_{a_\infty} \in C^1(\overline{\Omega})$ with $\|\varphi_{a_\infty}\|_p = 1$ and it is simple.

(iii) If $\lambda \ne \lambda_1(a_\infty)$, then (17) has no constant sign solutions other than 0.

**Theorem 18.** *Assume* (AH). *For every $\varepsilon > 0$ there exists $R_0 > 0$ such that equation* (EV; $\lambda$) *has no solutions in $W_0^{1,p}(\Omega) \setminus B_p(R_0)$ provided $\lambda < \lambda_1(a_\infty) - \varepsilon$.*

To prove the theorem, we need the following result.

**Lemma 19.** *Assume* (AH) *and let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a sequence satisfying $\|u_n\|_p \to \infty$ as $n \to \infty$. If $v_n := u_n / \|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$, the following assertions hold*:

(i) $\displaystyle \lim_{n \to \infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p} \, dx = 0.$

(ii) *For every* $w \in W_0^{1,p}(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n \nabla w}{\|u_n\|_p^{p-1}}\,dx = 0.$$

(iii) $\displaystyle\lim_{n\to\infty}\int_\Omega \frac{\widetilde{G}(x,\nabla u_n)}{\|u_n\|_p^p}\,dx = 0,$ *where* $\widetilde{G}(x,y) := \int_0^{|y|}\tilde{a}(x,t)t\,dt$ *for* $y \in \mathbb{R}^N$.

*Proof.* (i) Fix any $\varepsilon > 0$. By the property of the function $\tilde{a}$, there exist $R > 0$ and $C > 0$ such that

$$(18) \qquad |\tilde{a}(x,t)| \le \varepsilon|t|^{p-2} \text{ if } t \ge R \quad\text{and}\quad |\tilde{a}(x,t)| \le C \text{ if } 0 \le t \le R.$$

Therefore, we obtain

$$\left|\int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx\right| \le \int_{|\nabla u_n|>R}\varepsilon|\nabla v_n|^p\,dx + \int_{|\nabla u_n|\le R}\frac{C|\nabla u_n|^2}{\|u_n\|_p^p}\,dx$$
$$\le \varepsilon\|\nabla v_n\|_p^p + \frac{CR^2|\Omega|}{\|u_n\|_p^p} \le \varepsilon D^p + \frac{CR^2|\Omega|}{\|u_n\|_p^p}$$

by (18), where $D := \sup_n \|\nabla v_n\|_p$. Letting $n \to \infty$, we have

$$\limsup_{n\to\infty}\left|\int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx\right| \le \varepsilon D^p,$$

because $\|u_n\|_p \to \infty$ as $n \to \infty$. Thus, since $\varepsilon > 0$ is arbitrary, our conclusion holds.

(ii) For any $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$, we have

$$\left|\int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n\nabla w}{\|u_n\|_p^{p-1}}\,dx\right|$$
$$\le \int_{|\nabla u_n|>R}\varepsilon|\nabla v_n|^{p-1}|\nabla w|\,dx + \int_{|\nabla u_n|\le R}\frac{C|\nabla u_n||\nabla w|}{\|u_n\|_p^{p-1}}\,dx$$
$$\le \varepsilon\|\nabla v_n\|_p^{p-1}\|\nabla w\|_p + \frac{CR\|\nabla w\|_p|\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}$$

by Hölder's inequality and (18). By combining this inequality and a similar argument to that used in (i), our conclusion is shown.

(iii) It is easily shown that, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$|\widetilde{G}(x,y)| \le \varepsilon|y|^p + C \quad \text{for every } y \in \mathbb{R}^N.$$

Therefore, $\left|\int_\Omega \widetilde{G}(x,\nabla u_n)\,dx\right| \le \varepsilon\|\nabla u_n\|_p^p + C|\Omega|$. This implies our conclusion. □

*Proof of Theorem 18.* By way of contradiction, we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$, and $\{u_n\}$ such that $\lambda_n < \lambda_1(a_\infty) - \varepsilon_0$, $\lim_{n\to\infty} \|u_n\|_p = \infty$, and $u_n$ is a solution of (EV; $\lambda_n$). By taking $u_n$ as a test function in (EV; $\lambda_n$), we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\nabla u_n\, dx = \lambda_n \|u_n\|_p^p \le (\lambda_1(a_\infty) - \varepsilon_0)\|u_n\|_p^p;$$

refer to Remark 1(iii). Therefore, $v_n := u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$.

Again by taking $u_n/\|u_n\|_p^p$ as a test function in (EV; $\lambda_n$), we obtain

$$\begin{aligned}\lambda_1(a_\infty) - \varepsilon_0 > \lambda_n &= \int_\Omega \frac{a_\infty(x)|\nabla u_n|^p}{\|u_n\|_p^p}\, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx \\ &= \int_\Omega a_\infty(x)|\nabla v_n|^p\, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx \\ &\ge \lambda_1(a_\infty) + o(1),\end{aligned}$$

using the definition of $\lambda_1(a_\infty)$ and Lemma 19(i). This is a contradiction. □

**Theorem 20.** *Assume* (AH). *For every $\varepsilon > 0$ there exists $R_1 > 0$ such that* (EV; $\lambda$) *has no constant sign solutions in $W_0^{1,p}(\Omega) \setminus B_p(R_1)$ provided $\lambda > \lambda_1(a_\infty) + \varepsilon$.*

*Proof.* By way of contradiction, we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$, and $\{u_n\}$ such that $\lambda_n > \lambda_1(a_\infty) + \varepsilon_0$, $\lim_{n\to\infty} \|u_n\|_p = \infty$, and $u_n$ is a constant sign solution of (EV; $\lambda_n$). Because $A$ is odd, we may suppose that $u_n \ge 0$ by considering $-u_n$ if necessary. Thus, by Remark 3, $u_n \in C^1(\overline{\Omega})$ and $u_n > 0$ in $\Omega$. Here we note that $\lambda_n \le A_p \lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of $A_p$), and so $\{\lambda_n\}$ is bounded. Hence we may assume, by taking a subsequence, that $\lambda_n$ converges to some

$$\lambda_0 \in \big[\lambda_1(a_\infty) + \varepsilon_0,\ A_p\lambda_1(-\Delta_p)\big].$$

In addition, we know that $v_n := u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\, dx = \lambda_n \|u_n\|_p^p,$$

where we take $u_n$ as a test function in (EV; $\lambda_n$). Thus, by choosing a subsequence, we may suppose that $v_n$ converges to some $v$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$.

We claim that $v$ is a positive solution of

$$(19)\qquad -\mathrm{div}(a_\infty(x)|\nabla v|^{p-2}\nabla v) = \lambda_0 |v|^{p-2} v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

that is, $v$ is a positive eigenfunction corresponding to $\lambda_0$. If our claim holds, then $\lambda_0 = \lambda_1(a_\infty)$ occurs because (17) has no positive solutions in the case of $\lambda \ne \lambda_1(a_\infty)$. Hence this contradicts $\lambda_1(a_\infty) + \varepsilon_0 \le \lim_{n\to\infty} \lambda_n = \lambda_0$.

We now prove our claim. First, we show that $v_n$ converges to $v$ strongly in $W_0^{1,p}(\Omega)$. Indeed, by taking $(v_n - v)/\|u_n\|_p^{p-1}$ as a test function in (EV; $\lambda_n$), we have

$$\begin{aligned}\lambda_n \int_\Omega v_n^{p-1}(v_n - v)\,dx &= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx + \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla(v_n - v)\,dx \\ &= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx + o(1)\end{aligned}$$

as $n \to \infty$ due to Lemma 19(i)–(ii). Since $v_n \to v$ in $L^p(\Omega)$, this implies that $\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx$ converges to 0 as $n \to \infty$. Noting that

$$\begin{aligned}o(1) &= \int_\Omega a_\infty(x)(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla(v_n - v)\,dx \\ &\ge \min_{\overline{\Omega}} a_\infty \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla(v_n - v)\,dx \\ &\ge \min_{\overline{\Omega}} a_\infty(\|\nabla v_n\|_p^{p-1} - \|\nabla v\|_p^{p-1})(\|\nabla v_n\|_p - \|\nabla v\|_p) \ge 0,\end{aligned}$$

we have $v_n \to v$ in $W_0^{1,p}(\Omega)$ (note $0 < \min_{\overline{\Omega}} a_\infty \le \max_{\overline{\Omega}} a_\infty < \infty$). As a result, $v$ is a solution of (19) by letting $n \to \infty$ in the equality

$$\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla w\,dx + \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n \nabla w}{\|u_n\|_p^{p-1}}\,dx = \lambda_n \int_\Omega v_n^{p-1} w\,dx$$

for every $w \in W_0^{1,p}(\Omega)$; note that, by Lemma 19(ii), the second term converges to zero. Since $v_n = u_n/\|u_n\|_p > 0$ in $\Omega$, $v$ is nonnegative, and so $v$ is positive by Remark 3(i) and $\|v\|_p = 1$. Thus our claim is shown. □

**Proposition 21.** *Assume* (AH). *Then, for every* $\varepsilon > 0$, *there exists* $R_0 > 0$ *such that*

$$\frac{\underline{\lambda}_1(A,r)}{r^p} \ge \lambda_1(a_\infty) - \varepsilon \quad \text{for every } r > R_0.$$

*Proof.* Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ and $\underline{\lambda}_1(A, r_n)/r_n^p < \lambda_1(a_\infty) - \varepsilon_0$ for every $n \in \mathbb{N}$. Because of Proposition 5 and Lemma 6, we can choose a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n), \quad \min_{v \in r_n S}\int_\Omega G(x,\nabla v)\,dx = \int_\Omega G(x,\nabla u_n)\,dx.$$

Note that

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n) < (\lambda_1(a_\infty) - \varepsilon_0)r_n^p,$$

and so $u_n/r_n = u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$. Because $u_n$ is a solution of (EV; $\lambda$) with $\lambda = \underline{\lambda}_1(A, r_n)/r_n^p$ (see Proposition 5), by the same argument as in Theorem 18 with $\lambda_n = \underline{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

**Proposition 22.** *Assume* (AH). *Then, for every* $\varepsilon > 0$, *there exists* $R_1 > 0$ *such that*

$$\frac{\bar{\lambda}_1(A, r)}{r^p} \leq \lambda_1(a_\infty) + \varepsilon \quad \text{for every } r > R_1.$$

*Proof.* Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ and $\bar{\lambda}_1(A, r_n)/r_n^p > \lambda_1(a_\infty) + \varepsilon_0$ for every $n \in \mathbb{N}$. According to Lemma 6 and Proposition 5, we can take a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x, \nabla u_n) \nabla u_n \, dx = \bar{\lambda}_1(A, r_n), \qquad \min_{v \in r_n S} \int_\Omega G(x, \nabla v) \, dx = \int_\Omega G(x, \nabla u_n) \, dx.$$

Note that, with $\varphi_{a_\infty}$ as in item (ii) of page 165, we have

$$\frac{C_0}{p(p-1)} \|\nabla u_n\|_p^p \leq \int_\Omega G(x, \nabla u_n) \, dx \leq \int_\Omega G(x, r_n \nabla \varphi_{a_\infty}) \, dx \leq \frac{C_1 r_n^p}{p(p-1)} \|\nabla \varphi_{a_\infty}\|_p^p.$$

Hence $u_n/r_n = u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$. Since $u_n$ is a positive solution of (EV; $\lambda$) with $\lambda = \bar{\lambda}_1(A, r_n)/r_n^p$, by the same argument as in Theorem 20 with $\lambda_n = \bar{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

By Propositions 21 and 22, we have the following result.

**Corollary 23.** *Under* (AH), *we have*

$$\lim_{r \to +\infty} \frac{\bar{\lambda}_1(A, r)}{r^p} = \lim_{r \to +\infty} \frac{\underline{\lambda}_1(A, r)}{r^p} = \lambda_1(a_\infty).$$

**Proposition 24.** *Under* (AH), *we have*

$$\lim_{r \to +\infty} \frac{\mu_1(A, r)}{r^p} = \frac{\lambda_1(a_\infty)}{p}.$$

*Proof.* Due to Proposition 5, for every $r > 0$, there exists a positive solution $u_r \in (rS) \cap C^1(\overline{\Omega})$ of (EV; $\lambda$) with $\lambda = \lambda_1(A, u_r)/r^p$ and $\mu_1(A, r) = J(u_r)$. Then $u_r/\|u_r\|_p = u_r/r$ is bounded in $W_0^{1,p}(\Omega)$, as seen from

$$\frac{C_0}{p(p-1)} \|\nabla u_r\|_p^p \leq \int_\Omega G(x, \nabla u_r) \, dx \leq \int_\Omega G(x, r \nabla w) \, dx \leq \frac{r^p C_1}{p(p-1)} \|\nabla w\|_p^p$$

for any $w \in W_0^{1,p}(\Omega)$ with $\|w\|_p = 1$.

Set

$$\widetilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t) t \, dx \quad \text{for } y \in \mathbb{R}^N.$$

Note that

$$\underline{\lambda}_1(A, r) \leq \lambda_1(A, u_r) \leq \bar{\lambda}_1(A, r)$$

and

$$\begin{aligned}\mu_1(A,r)&=\int_\Omega G(x,\nabla u_r)\,dx=\frac{1}{p}\int_\Omega a_\infty(x)|\nabla u_r|^p\,dx+\int_\Omega \widetilde{G}(x,\nabla u_r)\,dx\\&=\frac{\lambda_1(A,u_r)}{p}-\frac{1}{p}\int_\Omega \tilde{a}(x,|\nabla u|)|\nabla u_r|^2\,dx+\int_\Omega \widetilde{G}(x,\nabla u_r)\,dx.\end{aligned}$$

According to Corollary 23 and Lemma 19(i) and (iii) (note $\|u_r\|_p=r\to+\infty$), our conclusion is achieved. □

## 4. Existence of a positive solution

In this section, we provide the existence of a positive solution to the equation

$$\text{(P)}\qquad \begin{cases}-\operatorname{div} A(x,\nabla u)=f(x,u) & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}$$

where the nonlinear term $f$ satisfies Assumption ($f$).

**Theorem 25.** *Assume* (AH0), (AH), *and* ($f$). *Let* $\lambda_1(a_0)$ *and* $\lambda_1(a_\infty)$ *be the first eigenvalues of*, *respectively*, (13) *and* (17) (*see the discussion there*). *If one of the following conditions holds*, (P) *has at least one positive solution.*

(i) $\alpha_0>\lambda_1(a_0)$ *and* $\alpha<\lambda_1(a_\infty)$.

(ii) $\alpha_0<\lambda_1(a_0)$ *and* $\alpha>\lambda_1(a_\infty)$.

This addresses the existence of an eigenvalue for our operator because we can apply Theorem 25 to $f(x,u)=\lambda|u|^{p-2}u$.

**Corollary 26.** *Assume* (AH0), (AH), *and* $\lambda_1(a_0)\neq\lambda_1(a_\infty)$. *Then*, *for every* $\lambda$ *between* $\lambda_1(a_0)$ *and* $\lambda_1(a_\infty)$, (EV; $\lambda$) *has a nontrivial* (*positive*) *solution. Therefore* $\lambda$ *is an eigenvalue of* $A$

To show the existence of a positive solution, we define a $C^1$ functional $I$ on $W_0^{1,p}(\Omega)$ by

$$I(u):=\int_\Omega G(x,\nabla u)\,dx-\int_\Omega F_+(x,u)\,dx\quad\text{for }u\in W_0^{1,p}(\Omega),$$

where $F_+(x,u):=\int_0^u f_+(x,u)\,dx$, with $f_+(x,t)$ given by $f(x,t)$ if $t\ge 0$ and 0 if $t\le 0$.

**Remark 27.** If $u\in W_0^{1,p}(\Omega)$ is a nontrivial critical point of $I$, then $u$ is a positive solution of (P).

Indeed, by taking $-u_-$ as a test function, we obtain

$$\begin{aligned}0=\langle I'(u),-u_-\rangle&=\int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx-\int_\Omega f_+(x,u)(-u_-)\,dx\\&=\int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx\ge\frac{C_0}{p-1}\|\nabla u_-\|_p^p.\end{aligned}$$

Thus $u \geq 0$. By Remark 3(ii) (note that $u \not\equiv 0$), we see that $u$ is a positive solution of (P) (note that $f_+(x, u) = f(x, u)$).

**Convention.** From now on, let Assumption ($f$) be satisfied.

**Lemma 28.** *If $\alpha \neq \lambda_1(a_\infty)$, then $I$ satisfies the Palais–Smale condition.*

*Proof.* Let $\{u_n\}$ be a Palais–Smale sequence of $I$, which means that

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{W_0^{1,p}(\Omega)^*} \to 0 \quad \text{as } n \to \infty$$

for some $c \in \mathbb{R}$. In view of Proposition 2 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, it is sufficient to prove the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Then, in view of the inequality

$$\begin{aligned} \text{(20)} \qquad \frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \leq \int_\Omega G(x, \nabla u_n)\,dx &= I(u_n) + \int_\Omega F_+(x, u_n)\,dx \\ &\leq I(u_n) + C\|u_n\|_p^p, \end{aligned}$$

it is sufficient to prove the boundedness of $\{u_n\}$ in $L^p(\Omega)$. By way of contradiction we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Set $v_n := u_n/\|u_n\|_p$. The inequality (20) ensures that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may suppose that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ for some $v_0$.

First, we see that $v_0 \geq 0$ for a.e. $x \in \Omega$. Indeed, by taking $-(u_n)_-$ as a test function, we have

$$\begin{aligned} o(1)\|\nabla(u_n)_-\|_p &= \langle I'(u_n), -(u_n)_-\rangle \\ &= \int_\Omega A(x, \nabla u_n)(-\nabla(u_n)_-)\,dx \geq \frac{C_0}{p-1}\|\nabla(u_n)_-\|_p^p. \end{aligned}$$

Because $p > 1$, we have $\|\nabla(u_n)_-\|_p \to 0$ as $n \to \infty$. Thus $(v_n)_- \to 0$ in $W_0^{1,p}(\Omega)$, and hence $(v_0)_- = 0$ for a.e. $x \in \Omega$.

Now we prove that

$$\text{(21)} \qquad \lim_{n\to\infty} \frac{\|f_+(\cdot\,, u_n) - \alpha(u_n)_-^{p-1}\|_{p'}}{\|u_n\|_p^{p-1}} = 0,$$

where $p' = p/(p-1)$. Fix an arbitrary $\varepsilon > 0$. It follows from condition (ii) of Assumption ($f$) that there exists a $C_\varepsilon > 0$ such that

$$|f(x, u) - \alpha u^{p-1}| \leq \varepsilon|u|^{p-1} + C_\varepsilon \quad \text{for every } u \geq 0, \text{ a.e. } x \in \Omega.$$

Then we obtain

$$\int_\Omega |f_+(x, u_n) - \alpha(u_n)_+^{p-1}|^{p'}\,dx \leq 2^{p'-1}(\varepsilon^{p'-1}\|(u_n)_+\|_p^p + C_\varepsilon^{p'-1}|\Omega|).$$

Since we are assuming that $\|u_n\|_p \to \infty$ as $n \to \infty$, this shows that

$$\lim_{n\to\infty} \left\| f_+(\,\cdot\,, u_n) - \alpha(u_n)_+^{p-1} \right\|_{p'} / \|u_n\|_p^{p-1} = 0,$$

because $\varepsilon > 0$ is arbitrary.

Here we recall the following result proved in Lemma 19:

$$(22)\quad \lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla(v_n - v_0)\, dx = \lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla\varphi\, dx = 0$$

for every $\varphi \in W_0^{1,p}(\Omega)$. Thus, by considering

$$o(1) = \frac{\langle I'(u_n), v_n - v_0\rangle}{\|u_n\|_p^{p-1}} = \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v_0)\, dx + o(1),$$

and using Proposition 2, we see that $v_n$ converges strongly to $v_0$ in $W_0^{1,p}(\Omega)$. Hence, by passing to the limit in $o(1) = \langle I'(u_n), \varphi\rangle / \|u_n\|_p^{p-1}$ for any $\varphi \in W_0^{1,p}(\Omega)$ and by noting (21) and (22), we infer that $v_0$ is a nontrivial solution of

$$-\mathrm{div}(a_\infty |\nabla u|^{p-2}\nabla u) = \alpha |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

(note that $\|v_0\|_p = 1$ and $v_0 \ge 0$ for a.e. $x \in \Omega$). Since $v_0 \ge 0$ for a.e. $x \in \Omega$, $v$ is a positive solution of (17) with $\lambda = \alpha$ (see Remark 3). This implies that $\alpha = \lambda_1(a_\infty)$, because (17) has no positive solutions if $\lambda \ne \lambda_1(a_\infty)$. It contradicts the hypothesis $\alpha \ne \lambda_1(a_\infty)$. Hence $\|u_n\|_p$ is bounded, which completes the proof. □

**Lemma 29.** *Assume* (AH) *and* $\alpha < \lambda_1(a_\infty)$. *Then $I$ is coercive*, *bounded from below and weakly lower semicontinuous* (*wlsc*) *on* $W_0^{1,p}(\Omega)$.

*Proof.* Because $\alpha < \lambda_1(a_\infty)$, we can take sufficiently small constants $\varepsilon > 0$ and $0 < \delta < 1$ satisfying

$$(23)\qquad (1 - \delta)(\lambda_1(a_\infty) - \varepsilon) > \alpha + \varepsilon.$$

By condition (ii) of Assumption ($f$), there exists a $C > 0$ such that

$$|F_+(x, u)| \le (\alpha + \varepsilon)\frac{u^p}{p} + C$$

for every $u \ge 0$ and a.e. $x \in \Omega$. Due to Proposition 24 and the definition of $\mu_1(A, r)$, there exists an $R > 0$ such that, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_p \ge R$,

$$(24)\qquad \int_\Omega G(x, \nabla u)\, dx \ge \mu_1(A, \|u\|_p) \ge \frac{\lambda_1(a_\infty) - \varepsilon}{p} \|u\|_p^p.$$

Hence, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_p \ge R$, we obtain

$$\begin{aligned} I(u) &\geq \frac{(1-\delta)(\lambda_1(a_\infty)-\varepsilon)}{p}\|u\|_p^p + \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\alpha+\varepsilon}{p}\|u_+\|_p^p - C|\Omega| \\ &\geq \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - C|\Omega| \end{aligned}$$

by (2), (23), and (24), where $u_+ := \max\{0, u\}$. This yields that $I$ is coercive. Moreover, because $I$ is bounded from below on $B_p(R)$, we see that $I$ is bounded from below on $W_0^{1,p}(\Omega)$. Since $J$ is wlsc (see the proof of Proposition 5) and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, $I$ is wlsc on $W_0^{1,p}(\Omega)$. □

**Lemma 30.** *Assume* (AH0) *and* $\alpha_0 < \lambda_1(a_0)$. *Let* $p < q \leq p^*$, *where* $p^* = Np/(N-p)$ *if* $N > p$ *and* $p^* = +\infty$ *if* $N \leq p$. *Then there exists* $\rho_0 > 0$ *such that*

$$\inf\{I(u) : \|u\|_q = \rho\} > 0 \quad \text{for every } 0 < \rho < \rho_0.$$

*Proof.* Because $\alpha_0 < \lambda_1(a_0)$, we can take some sufficiently small $\varepsilon > 0$ and $0 < \delta < 1$ satisfying

$$(1-\delta)(\lambda_1(a_0)-\varepsilon) > \alpha_0 + \varepsilon. \tag{25}$$

According to Proposition 17, there exists an $r_0 > 0$ such that

$$\frac{\mu_1(A, r)}{r^p} \geq \frac{\lambda_1(a_0)-\varepsilon}{p} \quad \text{for every } 0 < r < r_0. \tag{26}$$

In addition, Assumption $(f)$ guarantees the existence of $D_q > 0$ satisfying

$$F_+(x, u) \leq \frac{\alpha_0+\varepsilon}{p}u^p + D_q u^q \quad \text{for every } u \geq 0, \text{ a.e. } x \in \Omega. \tag{27}$$

Because $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, we can take a positive constant $C_q$ such that $\|u\|_q \leq C_q\|\nabla u\|_p$ for every $W_0^{1,p}(\Omega)$. We choose a positive constant $\rho$ satisfying

$$\rho < \min\left\{ r_0|\Omega|^{1/q-1/p}, \left(\frac{\delta C_0}{2p(p-1)D_q C_q^p}\right)^{1/(q-p)} \right\} =: \rho_0. \tag{28}$$

Note that $\|u\|_p < r_0$ if $\|u\|_q = \rho$, by Hölder's inequality and (28). Therefore, for every $\|u\|_q = \rho$, we have

$$\begin{aligned} I(u) &= (1-\delta)\int_\Omega G(x, \nabla u)\,dx + \delta\int_\Omega G(x, \nabla u)\,dx - \int_\Omega F_+(x, u)\,dx \\ &\geq (1-\delta)\frac{\mu_1(A, \|u\|_p)}{\|u\|_p^p}\|u\|_p^p + \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\alpha_0+\varepsilon}{p}\|u_+\|_p^p - D_q\|u_+\|_q^q \\ &\geq \frac{1}{p}\{(1-\delta)(\lambda_1(a_0)-\varepsilon) - \alpha_0 - \varepsilon\}\|u\|_p^p + \left(\frac{\delta C_0}{p(p-1)C_q^p} - D_q\|u\|_q^{q-p}\right)\|u\|_q^p \end{aligned}$$

$$\geq \frac{\delta C_0}{2p(p-1)C_q^p}\rho^p,$$

by the definition of $\mu_1(A,r)$, (2), (27), (26), (25), and (28). This ensures our conclusion. □

*Proof of Theorem 25.* (i) Lemma 29 guarantees the existence of a global minimizer of $I$. Thus it suffices to prove that $\min_{W_0^{1,p}(\Omega)} I < 0$ to show the existence of a nontrivial critical point of $I$. Choose a positive constant $\varepsilon > 0$ such that $\alpha_0 > \lambda_1(a_0) + 2\varepsilon$. Let $\varphi_{a_0} \in C^1(\overline{\Omega})$ be a positive eigenfunction corresponding to $\lambda_1(a_0)$ with $\|\varphi_{a_0}\|_p = 1$ (refer to the text below (13) and note that (13) is a homogeneous equation). It is easily seen that $\int_\Omega \widetilde{G}_0(x, r\nabla\varphi_{a_0})\,dx/r^p \to 0$ as $r \to +0$ (refer to the proof of Proposition 17 with $\|r\varphi_{a_0}\|_p = r$). Hence there exists $r_0 > 0$ such that

$$\begin{aligned}(29)\qquad \int_\Omega G(x, r\nabla\varphi_{a_0})\,dx &= \frac{r^p}{p}\int_\Omega a_0(x)|\nabla\varphi_{a_0}|^p\,dx + r^p\int_\Omega \frac{\widetilde{G}_0(x, r\nabla\varphi_{a_0})}{r^p}\,dx \\ &\leq \frac{\lambda_1(a_0)+\varepsilon}{p}r^p = \frac{\lambda_1(a_0)+\varepsilon}{p}\|r\varphi_{a_0}\|_p^p\end{aligned}$$

for every $0 < r < r_0$. On the other hand, it follows from part (i) of Assumption $(f)$ that there exists a $\delta > 0$ such that

$$(30)\qquad F_+(x,u) \geq \frac{\alpha_0 - \varepsilon}{p}u^p \quad \text{for every } u \in [0,\delta],\ \text{a.e. } x \in \Omega.$$

Therefore, for every $0 < r < \min\{r_0, \delta/\|\varphi_{a_0}\|_\infty\}$, we have

$$I(ru_0) \leq \frac{r^p}{p}(\lambda_1(a_0) + 2\varepsilon - \alpha_0)\|\varphi_{a_0}\|_p^p < 0,$$

by (29) and (30) (note $\lambda_1(a_0) + 2\varepsilon - \alpha_0 < 0$), whence $\min_{W_0^{1,p}(\Omega)} I < 0$.

(ii) Let $p < q \leq p^*$. Then, by Lemma 30, we obtain $\rho > 0$ satisfying

$$\delta_0 := \inf\{I(u) : \|u\|_q = \rho\} > 0.$$

Now we claim the existence of $w \in W_0^{1,p}(\Omega)$ such that

$$(31)\qquad \|w\|_q > \rho \quad \text{and} \quad I(w) < \delta_0.$$

Admitting this claim, we define

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \qquad \Gamma := \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = w\}.$$

It is obvious that $\Gamma \neq \varnothing$ and $\gamma([0,1]) \cap \{u \in W_0^{1,p}(\Omega) : \|u\|_q = \rho\} \neq \varnothing$ for every $\gamma \in \Gamma$, since $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Thus the mountain pass theorem guarantees that $c(\geq \delta_0)$ is a nontrivial critical value of $I$ because $I$ satisfies the Palais–Smale condition by Lemma 28.

Finally, we prove the existence of $w$ satisfying (31). Because $\alpha > \lambda_1(a_\infty)$, we can choose a positive constant $\varepsilon_0 > 0$ such that

$$\alpha > \lambda_1(a_\infty) + 2\varepsilon_0. \tag{32}$$

Using item (ii) on page 165, we can take $\varphi_{a_\infty} \in C^1(\overline{\Omega})$ be a positive eigenfunction corresponding to $\lambda_1(a_\infty)$ with $\|\varphi_{a_\infty}\|_p = 1$. It follows from Lemma 19(iii) that

$$\int_\Omega \widetilde{G}(x, r\nabla\varphi_{a_\infty})\, dx / r^p \to 0$$

as $r \to +\infty$ (note that $\|r\varphi_{a_\infty}\|_p = r$). Hence there exists $R_0 > 0$ such that

$$\begin{aligned} \int_\Omega G(x, r\nabla\varphi_{a_\infty})\, dx &= \frac{r^p}{p}\int_\Omega a_\infty(x)|\nabla\varphi_{a_\infty}|^p\, dx + r^p \int_\Omega \frac{\widetilde{G}_0(x, r\nabla\varphi_{a_\infty})}{r^p}\, dx \\ &\le \frac{\lambda_1(a_\infty)+\varepsilon_0}{p} r^p = \frac{\lambda_1(a_\infty)+\varepsilon_0}{p}\|r\varphi_{a_\infty}\|_p^p \end{aligned} \tag{33}$$

for every $r \ge R_0$. In addition, it follows from condition (ii) of Assumption ($f$) that there exists $D > 0$ such that

$$F_+(x, u) \ge \frac{\alpha - \varepsilon_0}{p} u^p - D \quad \text{for every } u \ge 0, \ \text{a.e. } x \in \Omega. \tag{34}$$

Consequently, by (32), (33), and (34), we obtain

$$I(r\varphi_{a_0}) \le \frac{r^p}{p}(\lambda_1(a_\infty) + 2\varepsilon_0 - \alpha)\|\varphi_{a_0}\|_p^p + D|\Omega| \to -\infty$$

as $t \to +\infty$. This implies the existence of $w$ satisfying (31). □

**4.1.** ***Resonant cases.*** To consider the resonant cases, we introduce the following hypotheses for

$$\widetilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t)t\, dt \quad \text{and} \quad \widetilde{G}_0(x, y) := \int_0^{|y|} \tilde{a}_0(x, t)t\, dt,$$

where $\tilde{a}$ and $\tilde{a}_0$ are as in (AH) and (AH0).

(H+) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$\begin{aligned} &\lim_{|y|\to\infty} \frac{p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2}{|y|^q} = +\infty && \text{for a.e. } x \in \Omega, \\ &p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \ge -H_0(1 + |y|^q) && \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N, \\ &f(x, t)t - pF(x, t) \ge -H_0(1 + t^q) && \text{for a.e. } x \in \Omega, \text{ every } t \ge 0. \end{aligned}$$

(H−) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$\lim_{|y|\to\infty} \frac{p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2}{|y|^q} = -\infty \qquad \text{for a.e. } x \in \Omega,$$

$$p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2 \le H_0(1+|y|^q) \qquad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x,t)t - pF(x,t) \le H_0(t^q+1) \qquad \text{for a.e. } x \in \Omega, \text{ every } t \ge 0.$$

(HF+) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2 \ge -H_0(1+|y|^q) \qquad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x,t)t - pF(x,t) \ge -H_0(1+t^q) \qquad \text{for every } t \ge 0, \text{ a.e. } x \in \Omega,$$

$$\lim_{t\to+\infty} \frac{f(x,t)t - pF(x,t)}{t^q} = +\infty \qquad \text{for a.e. } x \in \Omega.$$

(HF−) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2 \le H_0(1+|y|^q) \qquad \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N,$$

$$f(x,t)t - pF(x,t) \le H_0(1+t^q) \qquad \text{for every } t \ge 0, \text{ a.e. } x \in \Omega,$$

$$\lim_{t\to+\infty} \frac{f(x,t)t - pF(x,t)}{t^q} = -\infty \qquad \text{for a.e. } x \in \Omega.$$

(H0+) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y|\to 0} \frac{p\widetilde{G}_0(x,y) - \tilde{a}_0(x,|y|)|y|^2}{|y|^r} = +\infty \qquad \text{for a.e. } x \in \Omega,$$

$$p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2 \ge -H_0|y|^r \qquad \text{for a.e. } x \in \Omega, \text{ every } |y| \le 1,$$

$$f(x,t)t - pF(x,t) \ge -H_0 t^r \qquad \text{for a.e. } x \in \Omega, \text{ every } t \in [0,1].$$

(H0−) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y|\to 0} \frac{p\widetilde{G}_0(x,y) - \tilde{a}_0(x,|y|)|y|^2}{|y|^r} = -\infty \qquad \text{for a.e. } x \in \Omega,$$

$$p\widetilde{G}(x,y) - \tilde{a}(x,|y|)|y|^2 \le H_0|y|^r \qquad \text{for a.e. } x \in \Omega, \text{ every } |y| \le 1,$$

$$f(x,t)t - pF(x,t) \le H_0 t^r \qquad \text{for a.e. } x \in \Omega, \text{ every } t \in [0,1].$$

(HF0+) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$p\widetilde{G}_0(x,y) - \tilde{a}_0(x,|y|)|y|^2 \ge -H_0|y|^r \qquad \text{for a.e. } x \in \Omega, \text{ every } |y| \le 1,$$

$$f(x,t)t - pF(x,t) \ge -H_0 t^r \qquad \text{for every } t \in [0,1], \text{ a.e. } x \in \Omega,$$

$$\lim_{t\to+0} \frac{f(x,t)t - pF(x,t)}{t^r} = +\infty \qquad \text{for a.e. } x \in \Omega.$$

(HF0−) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\begin{aligned} p\widetilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2 &\le H_0|y|^r \quad \text{for a.e. } x \in \Omega,\ \text{every } |y| \le 1, \\ f(x,t)t - pF(x,t) &\le H_0 t^r \quad \text{for every } t \in [0,1],\ \text{a.e. } x \in \Omega, \\ \lim_{t \to +0} \frac{f(x,t)t - pF(x,t)}{t^r} &= -\infty \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

**Theorem 31.** *Let* Assumption ($f$), (AH0), *and* (AH) *hold. If any of the following conditions is satisfied*, (P) *has at least one positive solution.*

(i) $\alpha_0 > \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *and* (HF+) *or* (H+).

(ii) $\alpha_0 < \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *and* (HF−) *or* (H−).

(iii) $\alpha_0 = \lambda_1(a_0)$, $\alpha < \lambda_1(a_\infty)$, *and* (HF0+) *or* (H0+).

(iv) $\alpha_0 = \lambda_1(a_0)$, $\alpha > \lambda_1(a_\infty)$, *and* (HF0−) *or* (H0−).

(v) $\alpha_0 = \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, (HF0+) *or* (H0+), *and* (HF+) *or* (H+).

(vi) $\alpha_0 = \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, (HF0−) *or* (H0−), *and* (HF−) *or* (H−).

The rest of this section is devoted to the proof of this theorem, which involves some preparatory steps.

*The singly resonant case.* Set $f_{\pm n}(x,t) := f(x,t) \pm \dfrac{p}{n}|t|^{p-2}t$ and define approximate functionals on $W_0^{1,p}(\Omega)$ by

$$I_{\pm n}(u) := \int_\Omega G(x, \nabla u)\,dx - \int_\Omega (F_{\pm n})_+(x,u)\,dx = I(u) \mp \frac{1}{n}\|u_+\|_p^p.$$

From now on, assume $f$ satisfies Assumption ($f$). Take first the case $\alpha = \lambda_1(a_\infty)$.

**Lemma 32.** *If either* (H+) *or* (HF+) (*resp. either* (H−) *or* (HF−)) *hold and* $\{u_n\}$ *satisfies*

$$\sup_{n \in \mathbb{N}} I_{\pm n}(u_n) < +\infty \quad \text{and} \quad \lim_{n\to\infty} \|I'_{\pm n}(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0$$

$$\Big(\textit{resp.}\ \inf_{n \in \mathbb{N}} I_{\pm n}(u_n) > -\infty \quad \text{and} \quad \lim_{n\to\infty} \|I'_{\pm n}(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0\Big),$$

*then* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

*Proof.* The boundedness of $\|u_n\|_p$ guarantees that $\|u_n\|$ is bounded, since

$$o(1)\|u_n\| = \langle I'_{\pm n}(u_n), u_n\rangle \ge \frac{C_0}{p-1}\|u_n\|^p - C(1 + \|u_n\|_p^p) \mp \frac{1}{n}\|(u_n)_+\|_p^p$$

for some $C > 0$ independent of $n$. So, by way of contradiction, we assume that $\|u_n\|_p \to \infty$ as $n \to \infty$. Then, by the same argument as in Lemma 28, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence strongly converging to a positive solution $v_0$ of

$$-\mathrm{div}(a_\infty|\nabla u|^{p-2}\nabla u) = \alpha|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{35}$$

If $\alpha \neq \lambda_1(a_\infty)$, we have a contradiction, because (35) does not have a positive solution except when $\lambda = \lambda_1(a_\infty)$. So we may assume that $\alpha = \lambda_1(a_\infty)$ and $v_0 = \varphi_{a_\infty}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$. Thus $u_n(x) \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega$ (note $v_0 = \varphi_{a_\infty} > 0$ in $\Omega$).

Assume (HF+) or (HF−). We show that

$$I := \int_\Omega \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{\|u_n\|_p^q}\, dx \to \pm\infty, \tag{36}$$

where the sign on $\infty$ matches (HF±) and $q$ is a constant as in (HF±). Indeed, it follows from (HF+) that $(f_+(x,t)t - pF_+(x,t))/t^q$ is bounded from below on $\Omega \times [1, +\infty)$. Therefore, since $u_n(x) \to \infty$ for a.e. $x \in \Omega$, we have (36) if (HF+) holds, by applying Fatou's lemma to the inequality

$$I \geq \int_{u_n(x)\geq 1} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{u_n^q} v_n^q\, dx - \frac{2H_0}{\|u_n\|_p^p}|\Omega|,$$

where $H_0 > 0$ is a constant as in (HF+). The case of (HF−) is handled by the same argument, with $-f$ instead of $f$. This shows (36).

Furthermore, by Hölder's inequality, we have

$$\begin{aligned} II &:= \int_\Omega \frac{p\widetilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q}\, dx \\ &\leq H_0 \int_\Omega (|\nabla v_n|^q + \frac{1}{\|u_n\|_p^q})\, dx \leq H_0 \|\nabla v_n\|_p^q |\Omega|^{(p-q)/p} + o(1) \\ &\leq H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1) \end{aligned} \tag{37}$$

in the case of (HF−), because $v_n \to v_0$ in $W_0^{1,p}(\Omega)$, where $q \in [1, p)$ and $H_0 > 0$ are constants as in (HF−). Similarly, we obtain

$$II \geq -H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1) \tag{38}$$

in the case of (HF+).

Hence we have a contradiction because of (36), (37) or (38) by taking the limit inferior or superior in the equality

$$\frac{pI_{\pm n}(u_n) - \langle I'_{\pm n}(u_n), u_n\rangle}{\|u_n\|_p^q} = II + I.$$

Assume (H+) or (H−). Because $v_0$ is a positive solution of (35), we have $|\nabla u_n(x)| \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega_0 := \{x' \in \Omega : |\nabla v_0(x')| \neq 0\}$. Because $|\Omega_0| > 0$, we can show, by an argument similar to the one used for $f$, that

$$\int_\Omega \frac{p\widetilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q}\, dx \to \pm\infty,$$

where again the sign matches that of (H±). In addition, we easily obtain that

$$\pm \int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^q}\,dx \geq -H_0\|v_n\|_q^q + o(1) = -H_0\|v_0\|_q^q + o(1)$$

(again, the sign matches). Hence we have a contradiction by considering the limit of $(pI_{\pm n}(u_n) - \langle I'_{\pm n}(u_n), u_n\rangle)/\|u_n\|_p^q$. □

*Proof of Theorem 31*(i). Because $\alpha_0 > \lambda_1(a_0)$, there exists an $n_0 \in \mathbb{N}$ such that $\alpha_0 - p/n_0 > \lambda_1(a_0)$. Note that $f_{-n}(x,t)/t^{p-1} \to \alpha_0 - p/n > \lambda_1(a_0)$ as $t \to +0$ for $n \geq n_0$ and $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n = \lambda_1(a_\infty) - p/n < \lambda_1(a_\infty)$ as $t \to +\infty$. Hence, by using the proof of Theorem 25(i) to $f_{-n}$, we can find a global minimizer $u_n$ of $I_{-n}$ with $I_{-n}(u_n) < 0$ for each $n \geq n_0$. Here we remark that $\sup_{n\geq n_0} I_{-n}(u_n) < 0$. In fact, for every $n \geq n_0$, we have

$$I_{-n}(u_n) \leq I_{-n}(u_{n_0}) = I(u_{n_0}) + \frac{1}{n}\|u_{n_0}\|_p^p \leq I(u_{n_0}) + \frac{1}{n_0}\|u_{n_0}\|_p^p = I_{-n_0}(u_{n_0}) < 0,$$

where, in the first inequality, we use the fact that $u_n$ is a global minimizer of $I_{-n}$. Now, due to Lemma 32, we see that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore,

$$\|I'(u_n)\|_{W_0^{1,p}(\Omega)^*} = \|I'(u_n) - I'_{-n}(u_n)\|_{W_0^{1,p}(\Omega)^*} \leq \frac{p}{n\lambda_1(-\Delta_p)^p}\|u_n\|^{p-1} \to 0$$

as $n \to \infty$, where $\lambda_1(-\Delta_p)$ is the first eigenvalue of $-\Delta_p$. Since $I$ is bounded on a bounded set, we may assume that $\{u_n\}$ is a bounded Palais–Smale sequence of $I$. Because $I$ satisfies the bounded Palais–Smale condition (see Proposition 2), $u_n$ has a subsequence converging to some $v_0$ in $W_0^{1,p}(\Omega)$. It is clear that $I(v_0) \leq \sup_{n\geq n_0} I_{-n}(u_n) = I_{-n_0}(u_{n_0}) < 0$, and so $v_0$ is a nontrivial critical point of $I$. □

*Proof of Theorem 31*(ii). Using Lemma 30 and $\alpha_0 < \lambda_1(a_0)$, we can choose $q_0 \in (p, p^*]$ and $\rho > 0$ such that $\inf\{I(u) : \|u\|_{q_0} = \rho\} > 0$. Since $I_{+n}(u) \geq I(u) - \|u\|_{q_0}^p|\Omega|^{1-p/q_0}/n$ for every $u \in W_0^{1,p}(\Omega)$, we can take $n_0 \in \mathbb{N}$ such that $\alpha_0 + p/n_0 < \lambda_1(a_0)$ and $\delta_0 := \inf\{I_{+n_0}(u) : \|u\|_{q_0} = \rho\} > 0$. Hence, for every $n \geq n_0$, we have $\inf\{I_{+n}(u) : \|u\|_{q_0} = \rho\} \geq \delta_0$, because $I_{+n}(u) \geq I_{+n_0}(u)$ for every $n \geq n_0$ and $u \in W_0^{1,p}(\Omega)$. By noting that $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n > \alpha = \lambda_1(a_\infty)$ as $t \to +\infty$, and applying Lemma 28 to $f_{+n}$ instead of $f$, $I_{+n}$ satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every $n \geq n_0$, there exists a critical point $u_n \in W_0^{1,p}(\Omega)$ of $I_{+n}$ such that $I_{+n}(u_n) \geq \delta_0$. According to Lemma 32, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, because we have a bounded Palais–Smale sequence of $I$ due to a similar reason as in the case of (i), we can obtain a nontrivial critical point of $I$ (note that $\inf_{n\geq n_0} I(u_n) \geq \inf_{n\geq n_0} I_{+n}(u_n) \geq \delta_0 > 0$). □

We next turn to the case where $\alpha_0 = \lambda_1(a_0)$.

**Lemma 33.** *Assume* (H0−) *or* (HF0−) (*resp.* (H0+) *or* (HF0+)). *Let* $u_n \neq 0$ *be an element of* $W_0^{1,p}(\Omega)$ *satisfying* $I'_{\pm n}(u_n) = 0$ *for every* $n \in \mathbb{N}$ *and* $\inf_n I_{\pm n}(u_n) \geq 0$ (*resp.* $\sup_n I_{\pm n}(u_n) \leq 0$). *Then* $\liminf_{n\to\infty} \|u_n\|_p > 0$.

*Proof.* By way of contradiction, we assume that $\lim_{n\to\infty} \|u_n\|_p = 0$ by choosing a subsequence. Note that the boundedness of $\|u_n\|_p$ yields that $\|u_n\|$ and $\|u_n\|/\|u_n\|_p$ are bounded in view of

$$o(1)\|u_n\| = \langle I'_{\pm n}(u_n), u_n\rangle \geq \frac{C_0}{p-1}\|u_n\|^p - C(1 + \|(u_n)_+\|_p^p) \mp \frac{p}{n}\|(u_n)_+\|_p^p \tag{39}$$

for some $C > 0$ independent of $n$. Then, since $u_n$ is a positive solution of

$$-\mathrm{div}(A(x, \nabla u)) = f_{\pm n}(x, u_n) \quad \text{in } \Omega$$

(refer to Remarks 3 and 27), it follows from Proposition 4 that $u_n \to 0$ in $C^1(\overline{\Omega})$ (note that $|(f_{\pm n})_+(x, t)| \leq Ct_+^{p-1}$ (see Assumption $(f)$) and $u_n \to 0$ in $L^p(\Omega)$). Therefore, we may assume that $\|u_n\|_{C^1(\overline{\Omega})} \leq 1$ by considering a sufficiently large $n$. Since $|f_{\pm n}(x, \|u_n\|_p t)/\|u_n\|_p^{p-1}| \leq Ct^p$ for every $t \geq 0$, a.e. $x \in \Omega$ ($C > 0$ independent of $n$; see Assumption $(f)$ and (39)), by a similar argument to Theorem 13, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence converging to a positive solution $v_0$ in $C^1(\overline{\Omega})$ of

$$-\mathrm{div}(a_0(x)|\nabla u|^{p-2}\nabla u) = \alpha_0|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{40}$$

If $\alpha_0 \neq \lambda_1(a_0)$, we have a contradiction because (13) does not have a positive solution unless $\lambda = \lambda_1(a_0)$. So we may assume that $\alpha_0 = \lambda_1(a_0)$ and $v_0 = \varphi_{a_0}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$.

Assume (H0+) or (H0−). Then we can prove that

$$I := \int_\Omega \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^r}\, dx \to \pm\infty \tag{41}$$

(signs match), where $r \in [p, p^*)$ is a constant as in (H0+) or (H0−). Indeed, because $\|\nabla v_0\|_p > 0$, we can choose a constant $\varepsilon_0 > 0$ such that $|\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}| > 0$. With this $\varepsilon_0$, we have under assumption (H0+)

$$\begin{aligned} I &\geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r}|\nabla v_n|^r\, dx - \int_{|\nabla v_n| \leq \varepsilon_0} H_0|\nabla v_n|^r\, dx \\ &\geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r}|\nabla v_n|^r\, dx - \varepsilon_0^r H_0|\Omega|, \end{aligned}$$

where $H_0$ is a positive constant as in (H0+). Hence, applying Fatou's lemma, our claim is shown, because the Lebesgue measure of $\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}$ is positive. Similarly, by considering $\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2 - p\widetilde{G}_0(x, \nabla u_n)$, we can prove (41) under (H0−).

On the other hand, by using (H0+) or (H0−), we obtain

$$\pm II := \pm \int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^r}\,dx \ge -H_0 \int_\Omega (v_n)_+^r\,dx \ge -H_0\|v_n\|_r^r = -H_0\|v_0\|_r^r + o(1) \tag{42}$$

(note that $\|u_n\|_{C^1(\bar\Omega)} \le 1$ and $v_n \to v_0$ in $C^1(\bar\Omega)$). Now set $\Psi_n = I_{\pm n}$. Since

$$\pm(I + II) = \pm \frac{p\Psi_n(u_n) - \langle \Psi_n'(u_n), u_n\rangle}{\|u_n\|_p^r} = \pm \frac{p\Psi_n(u_n)}{\|u_n\|_p^r} \le 0 \tag{43}$$

if $\sup_n(\pm I_\pm(u_n)) \le 0$ (where the signs match throughout), we obtain a contradiction with (41) and (42) by taking the limit superior or inferior in (43).

Assume (HF0+) or (HF0−). As in the argument for $I$ in the case of (H0±), we can show that

$$\int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^r}\,dx = \int_{v_n>0} \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{(u_n)_+^r}(v_n)_+^r\,dx \to \pm\infty,$$

the sign matching that of (HF0±). Moreover, it is easily seen that

$$\pm \int_\Omega \frac{p\widetilde{G}_0(x,\nabla u_n) - \tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^r}\,dx \ge \mp H_0\|\nabla v_n\|_r^r = \mp H_0\|\nabla v_0\|_r^r + o(1).$$

(Note that $\|u_n\|_{C^1(\bar\Omega)} \le 1$ and $v_n \to v_0$ in $C^1(\bar\Omega)$.) Our conclusion follows from a similar argument as before. □

*Proof of Theorem 31*(iii). Let $n_0 \in \mathbb{N}$ such that $\alpha + p/n_0 < \lambda_1(a_\infty)$. The proof of Theorem 25(i) guarantees that, for every $n \ge n_0$, $I_{+n}$ has a global minimizer $u_n$ such that $I_{+n}(u_n) < 0$, because $f_{+n}(x,t)/t^{p-1} \to \alpha_0 + p/n > \alpha_0 = \lambda_1(a_0)$ as $t \to +0$ and $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n < \lambda_1(a_\infty)$ as $t \to +\infty$ if $n \ge n_0$. Noting that $I_{+n}(u) \ge I_{+n_0}(u)$ for every $u \in W_0^{1,p}(\Omega)$ and $n \ge n_0$, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ since $I_{+n_0}$ is coercive on $W_0^{1,p}(\Omega)$ by Lemma 29. Thus $\{u_n\}$ is a bounded Palais–Smale sequence of $I$ by the same argument as in (i). Therefore, $\{u_n\}$ has a convergent subsequence to some $u_0$ in $W_0^{1,p}(\Omega)$ because $I$ satisfies the bounded Palais–Smale condition. On the other hand, Lemma 33 guarantees that $u_0 \ne 0$ (note $\sup_{n\ge n_0} I_{+n}(u_n) \le 0$). Therefore $u_0$ is a nontrivial critical point of $I$. □

*Proof of Theorem 31*(iv). Let $n_0 \in \mathbb{N}$ be such that $\alpha - p/n_0 > \lambda_1(a_\infty)$. Applying Lemma 30 to $f_{-n}$ for $n \ge n_0$ (and since $\alpha_0 - p/n < \lambda_1(a_0)$), we can choose $q_0 \in (p, p^*]$ and $\rho_n > 0$ such that $\delta_n := \inf\{I_{-n}(u) : \|u\|_{q_0} = \rho_n\} > 0$. By noting that $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n > \lambda_1(a_\infty)$ as $t \to +\infty$ for every $n \ge n_0$, and applying Lemma 28 to $f_{-n}$ instead of $f$, we see that $I_{-n}$ satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every $n \ge n_0$, there exists a critical point $u_n \in W_0^{1,p}(\Omega)$ of $I_{-n}$ such that $I_{-n}(u_n) \geq \delta_n > 0$. By Lemma 32, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, by arguing as in case (i), we find a subsequence $\{u_n\}$ converging to some $u_0$ in $W_0^{1,p}(\Omega)$. Also, Lemma 33 yields $u_0 \neq 0$ (note that $\inf_{n \geq n_0} I_{-n}(u_n) \geq 0$). This shows that $u_0$ is a nontrivial critical point of $I$. □

*The doubly resonant case.* Choose smooth nonnegative functions $\varphi$ and $\psi$ on $[0, +\infty)$ satisfying $\varphi(t) = 1$ if $0 \leq t \leq 2$, $\varphi(t) = 0$ if $t \geq 4$, $\psi(t) = 0$ if $t \leq 5$, and $\psi(t) = 1$ if $t \geq 10$. Define approximate functionals on $W_0^{1,p}(\Omega)$ by

$$\tilde{I}_{\pm n}(u) := I(u) \mp \frac{1}{n}\psi(\|u\|_p^p)\|u_+\|_p^p \pm \frac{1}{n}\varphi(\|u\|_p^p)\|u_+\|_p^p.$$

Because $\tilde{I}_{\pm n}(u) = I_{\mp n}(u)$ provided $\|u\|_p \leq 2$, the following result can be proved by the same argument as in Lemma 33. We omit the proof.

**Lemma 34.** *Assume* (H0−) *or* (HF0−) (*resp.* (H0+) *or* (HF0+)). *Let* $u_n \neq 0$ *be an element of* $W_0^{1,p}(\Omega)$ *satisfying* $(\tilde{I}_{\pm n})'(u_n) = 0$ *for every* $n \in \mathbb{N}$ *and* $\inf_n \tilde{I}_{\pm n}(u_n) \geq 0$ (*resp.* $\sup_n \tilde{I}_{\pm n}(u_n) \leq 0$). *Then* $\liminf_{n\to\infty} \|u_n\|_p > 0$.

**Lemma 35.** *If* $\alpha \pm p/n \neq \lambda_1(a_\infty)$, *then* $\tilde{I}_{\pm n}$ (*with the matching sign*) *satisfies the Palais–Smale condition.*

*Proof.* Let $\{u_m\}$ be a Palais–Smale sequence of $\tilde{I}_{+n}$ or $\tilde{I}_{-n}$. If $\|u_m\|_p \to \infty$ occurs, then $\tilde{I}_{\pm n}(u_m) = I_{\pm n}(u_m)$ for sufficiently large $m$. So, by applying Lemma 28 to $f_{\pm n}$ (note that $\alpha \pm p/n \neq \lambda_1(a_\infty)$), we have a contradiction if $\|u_m\|_p \to \infty$. Consequently, we see that $\|u_m\|_p$ is bounded. Then, by the same reason as in Lemma 28, $\{u_m\}$ has a convergent subsequence in $W_0^{1,p}(\Omega)$. □

Because $\tilde{I}_{\pm n}(u) = I_{\pm n}(u)$ provided $\|u\|_p \geq 10$, the following result can be proved by the same argument as in Lemma 32. We omit the proof.

**Lemma 36.** *If either* (H+) *or* (HF+) (*resp. either* (H−) *or* (HF−)) *and* $\{u_n\}$ *satisfies*

$$\sup_{n\in\mathbb{N}} \tilde{I}_{\pm n}(u_n) < +\infty \quad \text{and} \quad \lim_{n\to\infty} \|(\tilde{I}_{\pm n})'(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0$$

$$\Big(\textit{resp.}\ \inf_{n\in\mathbb{N}} \tilde{I}_{\pm n}(u_n) > -\infty \quad \text{and} \quad \lim_{n\to\infty} \|(\tilde{I}_{\pm n})'(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0\Big),$$

$\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

*Proof of Theorem 31*(v). Note that $\tilde{I}_{-n}(u) = I_{-n}(u)$ provided $\|u\|_p \geq 10$ and $\tilde{I}_{-n}(u) = I_{+n}(u)$ if $\|u\|_p \leq 2$. So, by a similar argument to that in (i), $\tilde{I}_{-n}$ has a global minimizer $u_n$. Moreover, by a similar argument to that in (iii) (note that $f_{+n}(x,t)/t^{p-1} \to \alpha_0 + p/n > \lambda_1(a_0)$ as $t \to +0$ and $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n < \lambda_1(a_\infty)$ as $t \to +\infty$), we have $\tilde{I}_{-n}(u_n) < 0$, whence $u_n \neq 0$. Because Lemma 36 implies the boundedness of $\|u_n\|$, by the same argument as in (i), we see that $\{u_n\}$

is a bounded Palais–Smale sequence of $I$. Therefore, we may assume that $u_n$ converges to some $u_0$ in $W_0^{1,p}(\Omega)$ by choosing a subsequence. On the other hand, Lemma 33 yields $\liminf_{n\to\infty} \|u_n\|_p > 0$. Hence $u_0 \neq 0$. This means that $u_0$ is a nontrivial critical point of $I$. □

*Proof if Theorem 31*(vi). Note that $\tilde{I}_{+n}(u) = I_{+n}(u)$ provided $\|u\|_p \geq 10$ and $\tilde{I}_{+n}(u) = I_{-n}(u)$ if $\|u\|_p \leq 2$. So, because $f_{-n}(x,t)/t^{p-1} \to \alpha_0 - p/n < \lambda_1(a_0)$ as $t \to +0$ and $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n > \lambda_1(a_\infty)$ as $t \to +\infty$, by a similar argument to those in (ii) and (iv), for each $n$, we have a nontrivial critical point $u_n$ of $\tilde{I}_{+n}$ with $\tilde{I}_{+n}(u_n) > 0$. As a result, by a similar reasoning as in (v), we can obtain a nontrivial critical point of $I$. □

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