Pacific Journal of Mathematics

GENERALIZED EIGENVALUE PROBLEMS OF NONHOMOGENEOUS ELLIPTIC OPERATORS AND THEIR APPLICATION

DUMITRU MOTREANU AND MIEKO TANAKA

Volume 265 No. 1 September 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 265, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.265.151

GENERALIZED EIGENVALUE PROBLEMS OF NONHOMOGENEOUS ELLIPTIC OPERATORS AND THEIR APPLICATION

DUMITRU MOTREANU AND MIEKO TANAKA

We consider the equation $-\operatorname{div}(a(x, |\nabla u|)\nabla u) = \lambda|u|^{p-2}u$ (whose special case $a(x, t) = t^{p-2}$ is the p-Laplace equation) on a bounded domain $\Omega \subset \mathbb{R}^N$ with C^2 boundary, with null boundary condition. We prove that there are $\lambda \in \mathbb{R}$ for which the equation has a nontrivial solution. As an application, by variational methods, we present the existence of a positive solution to $-\operatorname{div}(a(x, |\nabla u|)\nabla u) = f(x, u)$ in Ω, where f is asymptotically $(p-1)$-linear near zero and ∞, considering the nonresonant, resonant, and doubly resonant cases. We show that, generally, the spectrum of the operator $-\operatorname{div}(a(x, |\nabla u|)\nabla u)$ on $W_0^{1,p}(\Omega)$ is not discrete.

1. Introduction

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary $\partial\Omega$. We are interested in values of $\lambda \in \mathbb{R}$ such that a nontrivial solution exists to the equation

$$(\text{EV}; \lambda) \qquad \begin{cases} -\operatorname{div} A(x, \nabla u) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

such a λ is called an *eigenvalue* for A. Here $A \colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map that is strictly monotone in the second variable and satisfies the regularity conditions in Assumption A below.

The p-Laplace equation is the special case of (EV; λ) with $A(x, y) = |y|^{p-2}y$, and in this case the eigenvalues for A are the usual eigenvalues of the p-Laplacian. However, we do not suppose that A is $(p-1)$-homogeneous in the second variable. Instead, these are the assumptions we make on the map A:

Assumption A. $A(x, y) = a(x, |y|)y$, where $a(x, t) > 0$ for all $x \in \overline{\Omega}$ and all $t \in (0, +\infty)$; furthermore:

(i) $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$.

MSC2010: 35P30, 35J62, 49R05.
Keywords: quasilinear elliptic equations, nonhomogeneous operators, nonlinear eigenvalue problems, positive solutions, mountain pass theorem.

(ii) There exists $C_1 > 0$ such that

$$|D_y A(x, y)| \le C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.$$

(iii) There exists $C_0 > 0$ such that

$$D_y A(x, y)\xi \cdot \xi \ge C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x, y)| \le C_2 (1 + |y|^{p-1}) \quad \text{for every } x \in \overline{\Omega} \text{ and } y \in \mathbb{R}^N \setminus \{0\}.$$

(v) There exist $C_3 > 0$ and a positive $t_0 \le 1$ such that

$$|D_x A(x, y)| \le C_3 |y|^{p-1} (-\log |y|)$$

for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

From now on, we assume that $C_0 \le p - 1 \le C_1$ which leads to no loss of generality, as can be seen from Assumption A(ii)–(iii).

A similar hypothesis to Assumption A is considered in the study of quasilinear elliptic problems; see [Motreanu and Papageorgiou 2011, Example 2.2; Damascelli 1998; Motreanu et al. 2011; Miyajima et al. 2012; Tanaka 2012a]. We also refer to [García-Huidobro et al. 1995; Kim 2009; Kim and Kim 2010; Fukagai and Narukawa 2007; Prado and Ubilla 1998; Robinson 2004] for generalized p-Laplace operators. In particular, when $A(x, y) = |y|^{p-2} y$ — that is, when $\operatorname{div} A(x, \nabla u)$ is the usual p-Laplacian $\Delta_p u$ — we can take $C_0 = C_1 = p - 1$ in Assumption A. Conversely, if $C_0 = C_1 = p - 1$ in Assumption A, the inequalities in Remark 1(ii)–(iii) below show that $a(x, t) = |t|^{p-2}$, whence $A(x, y) = |y|^{p-2} y$. In the p-Laplace case, the first eigenvalue λ_1 is obtained by the Rayleigh quotient: $\lambda_1 = \inf\{\int_\Omega |\nabla u|^p\, dx / \|u\|_p^p : u \ne 0\}$. But since our operator is nonhomogeneous, $\inf\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of } A\}$ is in general not obtained by such a Rayleigh quotient corresponding to A. In Section 3, since the Rayleigh quotient plays an important role, we study its behavior as $\|u\|_p \to 0$ or $\|u\|_p \to \infty$ under an additional condition describing an asymptotic $(p-1)$-homogeneity. For example, we can consider

$$\operatorname{div} A(x, \nabla u) = \operatorname{div}\big(\big(a_0(x)|\nabla u|^{p-2} + a_\infty(x)|\nabla u|^{q-2}\big)(1 + |\nabla u|^q)^{(p-q)/q} \nabla u\big)$$

for $1 < p \le q < \infty$, $a_0, a_\infty \in C^1(\overline{\Omega})$ with $\min_{\overline{\Omega}} a_0 > 0$ and $\min_{\overline{\Omega}} a_\infty > 0$. This satisfies

$$A(x, y) - a_0(x)|y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to 0,$$
$$A(x, y) - a_\infty(x)|y|^{p-2} y = o(|y|^{p-1}) \quad \text{as } |y| \to \infty.$$

Under these these conditions (see (AH0) and (AH) in Section 3), we shall prove that

$$\min\left\{\int_\Omega\int_0^{|\nabla u(x)|}\frac{a(x,t)t}{r^p}\,dt\,dx : \|u\|_p = r\right\}$$

approaches $\lambda_1(a_0)/p$ as $r \to +0$ and $\lambda_1(a_\infty)/p$ as $r \to +\infty$; here

$$\lambda_1(a_0) = \min\left\{\int_\Omega a_0(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\},$$
$$\lambda_1(a_\infty) = \min\left\{\int_\Omega a_\infty(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\}.$$

Concerning the eigenvalue problem for a nonhomogeneous operator, we can refer to [Robinson 2004; Tanaka 2012b] under the Neumann boundary condition.

In Section 4, as an application of Section 3, we present the existence of a positive solution for the quasilinear elliptic equation

$$\text{(P)}\qquad \begin{cases} -\operatorname{div} A(x,\nabla u) = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where f satisfies the following assumption.

Assumption (f). f is a Carathéodory function on $\Omega \times \mathbb{R}$ with $f(x,0) = 0$ for a.e. $x \in \Omega$, f is bounded on bounded sets and f is asymptotically $(p-1)$-linear near $+0$ and $+\infty$ in the following sense:

$$\text{(i)}\qquad \lim_{u\to+0}\frac{f(x,u)}{u^{p-1}} = \alpha_0 \quad \text{uniformly in a.e. } x \in \Omega,$$

$$\text{(ii)}\qquad \lim_{u\to+\infty}\frac{f(x,u)}{u^{p-1}} = \alpha \quad \text{uniformly in a.e. } x \in \Omega,$$

for some constants α_0 and α.

Regarding the existence of a positive solution under the Dirichlet boundary condition, we can refer to [Fukagai and Narukawa 2007; Prado and Ubilla 1998] for nonhomogeneous operators. However, we can not apply these results to our nonlinear term which is only asymptotically $(p-1)$-linear near $+0$ and $+\infty$, and furthermore with possibly different weights. In [García-Huidobro et al. 1995], it is proved the existence of a positive radial solution for nonhomogeneous operators.

For the p-Laplace equation, it is well known that if $(\alpha-\lambda_1)(\alpha_0-\lambda_1) < 0$ (where λ_1 denotes the first eigenvalue of $-\Delta_p$ under a Dirichlet boundary condition),

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has a positive solution (see [Dancer and Perera 2001]). One of our main purposes is to extend this existence result from the p-Laplace equation to the corresponding problem involving our nonhomogeneous operator A. This is done in Theorem 25. We mention that in the special case of $A(x,y) = A(y)$, the result in [Kyritsi et al. 2010] provides the existence of a positive solution if $\alpha < \lambda_1 C_0/(p-1)$ and $\lambda_1 C_1/(p-1) < \alpha_0$ hold (note that we can apply this result only to the case where $\alpha < \alpha_0$). We emphasize that, for our general operator, the case $\lambda_1(a_0) \neq \lambda_1(a_1)$ can occur. Note that in such a situation, contrary to the p-Laplacian case, we can still apply our theorem when $\alpha_0 = \alpha$ provided this number is between $\lambda_1(a_0)$ and $\lambda_1(a_1)$. The known result for the p-Laplacian case is obtained from our theorem simply by setting $a_0 \equiv 1$ and $a_\infty \equiv 1$.

In particular, our theorem implies that if $\lambda_1(a_0) \neq \lambda_1(a_\infty)$, then every λ between $\lambda_1(a_0)$ and $\lambda_1(a_\infty)$ is an eigenvalue of A (see Corollary 26) and has a positive eigenfunction. This shows that, generally, the spectrum of the operator $-\mathrm{div}\, A(x, \nabla\cdot)$ on $W_0^{1,p}(\Omega)$ is not discrete.

In the final part of the paper, we treat the one side resonant and doubly resonant cases under additional conditions on f. For the p-Laplace equation, we refer to [Tanaka 2009] for the resonant and doubly resonant cases. Our Theorem 31 provides the existence of a positive solution in all cases of resonance for problem (P) with a nonhomogeneous operator in the principal part.

2. The properties of the map A

In what follows, the norm on $W_0^{1,p}(\Omega)$ is given by

$$\|u\|^p := \|\nabla u\|_p^p,$$

where $\|u\|_q$ denotes the usual norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \le q \le \infty)$. Setting

$$G(x, y) := \int_0^{|y|} a(x, t)t\, dt, \tag{1}$$

we can easily see that

$$\nabla_y G(x, y) = A(x, y) \quad \text{and} \quad G(x, 0) = 0$$

for every $x \in \overline{\Omega}$; see [Motreanu et al. 2011] for details.

Remark 1. The following assertions hold under Assumption A:

(i) For all $x \in \overline{\Omega}$, $A(x, y)$ is maximal monotone and strictly monotone in y.

(ii) $|A(x, y)| \le \dfrac{C_1}{p-1}|y|^{p-1}$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

(iii) $A(x, y)y \ge \dfrac{C_0}{p-1}|y|^p$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

(iv) $G(x, y)$ is strictly convex in y for all x and satisfies the inequalities

$$A(x, y)y \ge G(x, y) \ge \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x, y) \le \frac{C_1}{p(p-1)}|y|^p \tag{2}$$

for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$.

The following result is important for the proof of the Palais–Smale condition for the functionals related to our problem.

Proposition 2 [Motreanu et al. 2011, Proposition 1]. *Let $V : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ be the map defined by*

$$\langle V(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v \, dx$$

for $u, v \in W_0^{1,p}(\Omega)$. Then any sequence $\{u_m\}$ that converges weakly to u and satisfies $\limsup_{m\to\infty}\langle V(u_m), u_m - u\rangle \le 0$ also converges strongly to u.

Remark 3. (i) *If $u \in W_0^{1,p}(\Omega)$ is a solution of (P), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

(ii) *If $u \in W_0^{1,p}(\Omega)$ is a nontrivial solution of (P) such that $u \ge 0$, then $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$, where ν denotes the outward unit normal vector on $\partial\Omega$.*

Sketch of proof. (i) Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P). Then, because $u \in L^\infty(\Omega)$ as shown by using the Moser iteration process (cf. [Miyajima et al. 2012, Appendix]), we see that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ by the regularity result in [Lieberman 1988].
(ii) Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P) satisfying $u \ge 0$ and $u \not\equiv 0$. Then, by Assumption (f), we obtain a constant $\lambda > 0$ satisfying

$$-\mathrm{div}\, A(x, \nabla u) + \lambda u^{p-1} \ge 0 \quad \text{in } \Omega.$$

Noting that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ by (i), we have $u(x) > 0$ for every $x \in \Omega$ by [Miyajima et al. 2012, Appendix, Theorem B]. In addition, using the strong maximum principle [ibid., Appendix, Theorem A], we easily see that $\partial u(x)/\partial\nu < 0$ for every $x \in \partial\Omega$. □

Proposition 4. *Let $f_n : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$|f_n(x, t)| \le D(1 + |t|^{r-1}) \quad \text{for every } x \in \Omega,\ t \in \mathbb{R}$$

with some positive constant D independent of n and $r \in [p, p^*)$, where $p^* = \infty$ if $N \le p$ and $p^* = pN/(N - p)$ if $N > p$. Assume that $A_n : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a map satisfying parts (i)–(iv) *of Assumption A with positive constants C_1', C_0', and C_2' independent of n. If u_n is a solution for*

$$-\mathrm{div}\, A_n(x, \nabla u) = f_n(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

and $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, then there exist a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ and $u_0 \in C_0^1(\overline{\Omega})$ such that $u_{n_l} \to u_0$ in $C_0^1(\overline{\Omega})$ as $l \to \infty$.

Proof. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we may assume that u_n converges weakly to some u_0 in $W_0^{1,p}(\Omega)$ by choosing a subsequence. We can show that there exists a $C > 0$ depending only on $|\Omega|$, p, N, D, C_0', C_1', and the embedding constant of

$W_0^{1,p}(\Omega)$ into $L^{\bar{p}^*}(\Omega)$ such that $\|u_n\|_\infty \le C \max\{1, \|u_n\|^{(\bar{p}^*-p)/(\bar{p}^*-r)}\}$ by the Moser iteration process to [Miyajima et al. 2012, Theorem C], where $\bar{p}^* = p^*$ if $N > p$ and $\bar{p}^* > r$ is any constant if $N \le p$. Since D, C_1', and C_0' are independent of n, $\|u_n\|_\infty$ is bounded. Therefore, the regularity result in [Lieberman 1988] guarantees that there exist $\gamma \in (0, 1)$ and $M > 0$ independent of n such that $u_n \in C_0^{1,\gamma}(\overline{\Omega})$ and $\|u_n\|_{C_0^{1,\gamma}(\overline{\Omega})} \le M$ (where we use the fact that C_2' is independent of n). Since the inclusion of $C_0^{1,\gamma}(\overline{\Omega})$ to $C_0^1(\overline{\Omega})$ is compact, u_n converges to u_0 in $C_0^1(\overline{\Omega})$ (note that $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$). □

3. Eigenvalue problems

We introduce a function $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega G(x, \nabla u)\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3}$$

It is clear that J is of class C^1. We also note that

$$rS := \{u \in W_0^{1,p}(\Omega) : \|u\|_p = r\} \quad \text{for } r > 0 \tag{4}$$

is a C^1 Finsler manifold (cf. [Deimling 1985, Sections 27.4 and 27.5]) because r is a regular value of the function $u \mapsto \|u\|_p$ on $W_0^{1,p}(\Omega)$. Hence the norm of the derivative at $u \in (rS)$ of the restriction $\tilde{J}$ of J to rS is defined by

$$\begin{aligned} \|\tilde{J}'(u)\|_* &:= \min\{\|J'(u) - t\Phi'(u)\|_{W_0^{1,p}(\Omega)^*} : t \in \mathbb{R}\} \\ &= \sup\{\langle J'(u), v\rangle : v \in T_u(rS),\ \|v\| = 1\}, \end{aligned}$$

where $\Phi(u) := (1/p)\|u\|_p^p$ and $T_u(rS)$ denotes the tangent space of rS at u, that is, $T_u(rS) = \{v \in W_0^{1,p}(\Omega) : \int_\Omega |u|^{p-2}uv\, dx = 0\}$. It follows that the restriction $\tilde{J} = J|_{(rS)}$ is a C^1-function on rS in the sense of manifolds.

Proposition 5. *For $r > 0$, the infimum*

$$\mu_1(A, r) = \inf_{u \in (rS)} \int_\Omega G(x, \nabla u)\, dx \tag{5}$$

is attained at points $\pm\hat{u}_r \in (rS)$ with $\hat{u}_r \in C^{1,\alpha}(\overline{\Omega})$ and $\hat{u}_r > 0$ in Ω. Moreover, $\pm\hat{u}_r$ are solutions of (EV; λ) with $\lambda = \lambda_1(A, \hat{u}_r)/r^p$, where

$$\lambda_1(A, \hat{u}_r) = \int_\Omega A(x, \nabla\hat{u}_r)\nabla\hat{u}_r\, dx \ge \frac{C_0}{p-1}\lambda_1 r^p. \tag{6}$$

Proof. Let $\{u_n\} \subset (rS)$ be a minimizing sequence for (5). Using (2), it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, so along a relabeled subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$, thus $u \in (rS)$. Since

$G(x,\cdot)$ is convex and continuous for all $x \in \Omega$, J is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ [Mawhin and Willem 1989, Theorem 1.2]. Therefore, we derive that

$$\mu_1(A,r) \le \int_\Omega G(x,\nabla u)\,dx \le \liminf_{n\to\infty} \int_\Omega G(x,\nabla u_n)\,dx,$$

which yields

$$\mu_1(A,r) = \int_\Omega G(x,\nabla u)\,dx.$$

The fact that the functional J is even implies that $|u|$ is also a global minimizer of $\tilde{J}_r$. Consequently, we may assume that $u \ge 0$. On the other hand, the Lagrange multiplier rule leads to the existence of $t \in \mathbb{R}$ such that

$$\int_\Omega A(x,\nabla u)\nabla v\,dx = t\int_\Omega u^{p-1} v\,dx \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{7}$$

Inserting $v = u$ in (7) entails

$$\mathrm{t}r^p = \int_\Omega A(x,\nabla u)\nabla u\,dx \ge \frac{C_0}{p-1}\|\nabla u\|_p^p \ge \frac{C_0\lambda_1}{p-1}\|u\|_p^p = \frac{C_0\lambda_1}{p-1} r^p. \tag{8}$$

Therefore, we have

$$t = \frac{\lambda_1(A,u)}{r^p} \ge \frac{C_0\lambda_1}{p-1}.$$

From (7), it follows that u is a solution of (EV; λ) with $\lambda = t = \lambda_1(A,u)/r^p$. According to Remark 3 with $f(x,u) = t|u|^{p-2}u$, it follows that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0<\alpha<1)$ and $u > 0$ in Ω. Since J is even and $\lambda_1(A,u) = \lambda_1(A,-u)$, we have that $J(-u) = J(u) = \mu_1(A,r)$ and $-u$ is a negative solution of (EV; λ) with $\lambda = t = \lambda_1(A,u)/r^p$. The result is thus established with $\hat{u}_r = u$. □

We define

$$K_1(A,r) := \{u \in (rS) : J(u) = \mu_1(A,r)\}.$$

Then it follows from Proposition 5 that $K_1(A,r)$ is not empty for each $r > 0$.

Because we do not know whether the minimizers of $\tilde{J}_r$ are only $\pm\hat{u}_r$, we introduce the following:

$$\underline{\lambda}_1(A,r) := \inf\left\{\int_\Omega A(x,\nabla u)\nabla u\,dx : u \in K_1(A,r)\right\},$$
$$\bar{\lambda}_1(A,r) := \sup\left\{\int_\Omega A(x,\nabla u)\nabla u\,dx : u \in K_1(A,r)\right\}.$$

Lemma 6. *For every $r > 0$, $\underline{\lambda}_1(A,r)$ and $\bar{\lambda}_1(A,r)$ are attained.*

Proof. We only deal with $\underline{\lambda}_1(A,r)$ because $\bar{\lambda}_1(A,r)$ can be treated similarly. Fix any $r > 0$. Let $u_n \in K_1(A,r)$ satisfy $\lambda_1(A,u_n) \to \underline{\lambda}_1(A,r)$ as $n\to\infty$. Then we see that $\|\nabla u_n\|_p$ is bounded from the inequality

$$\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \le \int_\Omega G(x,\nabla u_n)\,dx = \mu_1(A,r) \le \int_\Omega G(x,\nabla w)\,dx$$

for $w \in rS$, where we use the definition of $\mu_1(A,r)$ and (2). Recall that each u_n is a solution of (EV; λ) with $\lambda = \lambda_1(A,u_n)/r^p$. Moreover, we have

$$\frac{C_0}{p-1}\lambda_1 r^p \le \lambda_1(A,u_n) \le \frac{C_1}{p-1}\|\nabla u_n\|_p^p$$

by Remark 1(ii) (see (6) for the first inequality), whence $\lambda_1(A,u_n)/r^p$ is bounded. As a result, due to Proposition 4, we may assume that there exists $u_0 \in W_0^{1,p}(\Omega)$ such that $u_n \to u_0$ in $C_0^1(\overline{\Omega})$ by choosing a subsequence if necessary. Since J and $\lambda_1(A,\cdot)$ are continuous in $W_0^{1,p}(\Omega)$, we see that $J(u_0) = \lim_{n\to\infty} J(u_n) = \mu_1(A,r)$, $u_0 \in K_1(A,r)$, and $\lambda_1(A,u_0) = \lim_{n\to\infty}\lambda_1(A,u_n) = \underline{\lambda}_1(A,r)$. Thus, our conclusion holds. □

Define

$$\lambda_1(A) := \inf_{u\neq 0}\int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx \quad\text{and}\quad \mu_1(A) := \inf_{u\neq 0}\int_\Omega \frac{G(x,\nabla u)}{\|u\|_p^p}\,dx.$$

Lemma 7.

$$\frac{C_0}{p-1}\lambda_1 \le \lambda_1(A) \le \min\left\{\inf_{r>0}\frac{\underline{\lambda}_1(A,r)}{r^p},\ \frac{C_1}{p-1}\lambda_1\right\} \quad\text{and}\quad \mu_1(A) = \inf_{r>0}\frac{\mu_1(A,r)}{r^p}.$$

Proof. First, we consider $\lambda_1(A)$. For every $0 \neq u \in W_0^{1,p}(\Omega)$, we have

$$\frac{C_0}{p-1}\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \le \int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx \le \frac{C_1}{p-1}\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \tag{9}$$

by Remark 1(ii)–(iii). Thus $(C_0/(p-1))\lambda_1 \le \lambda_1(A) \le (C_1/(p-1))\lambda_1$ by taking the infimum with respect to u.

Here we fix any $\varepsilon > 0$. Then there exists an $r_\varepsilon > 0$ such that $\underline{\lambda}_1(A,r_\varepsilon)/r_\varepsilon^p \le \inf_{r>0}(\underline{\lambda}_1(A,r)/r^p) + \varepsilon$. By Lemma 6, we can choose $u_\varepsilon \in (r_\varepsilon S)$ such that $\lambda_1(A,u_\varepsilon) = \underline{\lambda}_1(A,r_\varepsilon)$, that is, $\int_\Omega A(x,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx = \underline{\lambda}_1(A,r_\varepsilon)$. By the definition of $\lambda_1(A)$, we obtain

$$\lambda_1(A) \le \int_\Omega \frac{A(x,\nabla u_\varepsilon)\nabla u_\varepsilon}{\|u_\varepsilon\|_p^p}\,dx = \frac{\underline{\lambda}_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \inf_{r>0}\frac{\underline{\lambda}_1(A,r)}{r^p} + \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we have $\lambda_1(A) \le \inf_{r>0}(\underline{\lambda}_1(A,r)/r^p)$.

Next we treat $\mu_1(A)$. Fix any $\varepsilon > 0$. Then there exists an $r_\varepsilon > 0$ such that $\mu_1(A,r_\varepsilon)/r_\varepsilon^p \le \inf_{r>0}(\mu_1(A,r)/r^p) + \varepsilon$. On the other hand, because $\mu_1(A,r_\varepsilon)$ is attained at some $u_\varepsilon \in (r_\varepsilon S)$, we have

$$\inf_{u\neq 0}\int_\Omega \frac{G(x,\nabla u)}{\|u\|_p^p}\,dx \le \int_\Omega \frac{G(x,\nabla u_\varepsilon)}{\|u_\varepsilon\|_p^p}\,dx = \frac{\mu_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \inf_{r>0}\frac{\mu_1(A,r)}{r^p}+\varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, this yields that $\mu_1(A) \le \inf_{r>0}(\mu_1(A,r)/r^p)$.

For any $\varepsilon > 0$, we take $v_\varepsilon \neq 0$ such that $\int_\Omega (G(x,\nabla v_\varepsilon)/\|v_\varepsilon\|_p^p)\,dx \le \mu_1(A)+\varepsilon$. Then $r_\varepsilon := \|v_\varepsilon\|_p > 0$ and so

$$\frac{\mu_1(A,r_\varepsilon)}{r_\varepsilon^p} \le \int_\Omega \frac{G(x,\nabla v_\varepsilon)}{\|v_\varepsilon\|_p^p}\,dx \le \mu_1(A)+\varepsilon.$$

This leads to $\mu_1(A) \ge \inf_{r>0}(\mu_1(A,r)/r^p)$. □

Proposition 8. *If $\lambda < \lambda_1(A)$, (EV; λ) has no nontrivial solutions.*

Proof. Let u be a nontrivial solution of (EV; λ) with $\lambda < \lambda_1(A)$. Then we have

$$\lambda_1(A) \le \int_\Omega \frac{A(x,\nabla u)\nabla u}{\|u\|_p^p}\,dx = \lambda$$

by the definition of $\lambda_1(A)$. This is a contradiction. □

Set

$$A_p := \frac{C_1}{p-1}\left(\frac{C_1}{C_0}\right)^{p-1} \ge 1, \tag{10}$$

which is equal to 1 exactly in the case of $A(x,y)=|y|^{p-2}y$ (that is, the special case of the p-Laplacian) because we can choose $C_0 = C_1 = p-1$.

Lemma 9 [Tanaka 2012a, Lemma 16]. *Let $\varepsilon > 0$. For every*

$$u,\varphi \in W^{1,p}(\Omega)\cap C^1(\Omega)\cap L^\infty(\Omega)$$

with $u \ge 0$ and $\varphi \ge 0$ in Ω, we have

$$\int_\Omega A(x,\nabla u)\nabla\left(\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\right)dx \le A_p\|\nabla\varphi\|_p^p.$$

Proposition 10. *Any nontrivial solution of (EV; λ) with $\lambda > A_p\lambda_1$ changes sign.*

Proof. By way of contradiction, assume there is a solution u that does not change sign. Then we may suppose that $u \ge 0$ because A is odd. Due to the strong maximum principle and the regularity theorem (see Remark 3), it follows that $u \in C_0^1(\overline{\Omega})$ and $u > 0$ in Ω. Let φ_1 be the positive eigenfunction of $-\Delta_p$ corresponding to λ_1 such that $\|\varphi_1\|_p = 1$. According to Lemma 9, we obtain

$$A_p\lambda_1 = A_p\|\nabla\varphi_1\|_p^p \ge \int_\Omega A(x,\nabla u)\nabla\left(\frac{\varphi_1^p}{(u+\varepsilon)^{p-1}}\right)dx = \lambda\int_\Omega\left(\frac{u}{u+\varepsilon}\right)^{p-1}\varphi_1^p\,dx$$

for every $\varepsilon > 0$. By taking $\varepsilon \downarrow 0$, we have $\lambda \le A_p\lambda_1$. This is a contradiction. □

Proposition 11. *Assume* $A_p\lambda_1 < C_0\lambda_2/(p-1)$, *where* $\lambda_2 > \lambda_1$ *is the second eigenvalue of* $-\Delta_p$. *If* $A_p\lambda_1 < \lambda < C_0\lambda_2/(p-1)$, (EV; λ) *has no nontrivial solutions.*

Proof. By way of contradiction, we assume that (EV; λ) has a nontrivial solution u. Then it follows from Proposition 10 that u changes sign. Moreover, by taking $u_\pm$ as a test function in (EV; λ), we have

$$\frac{C_0}{p-1}\|\nabla u_\pm\|_p^p \le \int_\Omega A(x,\nabla u)(\pm\nabla u_\pm)\,dx = \lambda\|u_\pm\|_p^p,$$

whence

$$\|\nabla u_\pm\|_p^p < \lambda_2\|u_\pm\|_p^p. \tag{11}$$

This inequality guarantees the existence of a continuous path γ_0 on S such that $\gamma_0(0)=\varphi_1$, $\gamma_0(1)=-\varphi_1$ and $\max_{t\in[0,1]}\|\nabla\gamma_0(t)\|_p^p < \lambda_2$; refer to [Cuesta et al. 1999, Lemma 5.3]. This contradicts the equality

$$\lambda_2 = \inf_{\gamma\in\Sigma}\max_{t\in[0,1]}\Phi(\gamma(t)),$$

where $\Phi(u) := \|\nabla u\|_p^p$ and $\Sigma := \{\gamma\in C([0,1],S) : \gamma(0)=\varphi_1,\ \gamma(1)=-\varphi_1\}$; see [Anane 1987; Cuesta et al. 1999]. This contradiction proves our result.

For the reader's convenience, we give the sketch of the construction of a path γ_0 as required above. Define paths as follows:

$$\gamma_1(t) := \frac{tu+(1-t)u_+}{\|tu+(1-t)u_+\|_p} = \frac{u_+-tu_-}{\|u_+-tu_-\|_p}, \qquad \gamma_2(t) := \frac{tu_++(1-t)u_-}{\|tu_++(1-t)u_-\|_p},$$
$$\gamma_3(t) := \frac{(1-t)u-tu_-}{\|(1-t)u-tu_-\|_p} = \frac{(1-t)u_+-u_-}{\|(1-t)u_+-u_-\|_p}$$

for $t\in[0,1]$. Then, setting $\widetilde{\Phi} := \Phi|_S$, we obtain by (11)

$$\max_{t\in[0,1]}\widetilde{\Phi}(\gamma_i(t)) < \lambda_2, \quad \text{for } i=1,2,3.$$

We recall that any component of $\mathcal{O}(r) := \{u\in S : \widetilde{\Phi}(u) < r\}$ contains at least one critical point of $\widetilde{\Phi}$, where $r>0$ [Cuesta et al. 1999, Lemma 3.6]. Note that $\mathcal{O}(\lambda_2)$ contains just two critical points φ_1 and $-\varphi_1$ because a critical value c of $\widetilde{\Phi}$ corresponds to the eigenvalue c of the negative p-Laplacian. Since any component of $\mathcal{O}(\lambda_2)$ is path connected [ibid., Lemma 3.5], there exists a path γ_4 joining from $u_-/\|u_-\|_p$ to φ_1 or $-\varphi_1$ in $\mathcal{O}(\lambda_2)$. Thus, by noting that Φ is even, we can construct a path $\gamma_0\in\Sigma$ such that $\max_t\widetilde{\Phi}(\gamma_0(t)) < \lambda_2$ by considering $\gamma_4^{-1}\cdot\gamma_2\cdot\gamma_1\cdot\gamma_3\cdot(-\gamma_4)$ or its inverse, where $\gamma_i^{-1}(t) := \gamma_i(1-t)$ and $\gamma_i\cdot\gamma_j$ denotes the path defined by $\gamma_i(2t)$ if $0\le t\le\frac{1}{2}$ and $\gamma_j(2t-1)$ if $\frac{1}{2} < t \le 1$. □

3.1. ***Asymptotically homogeneous case near zero.*** We now consider the case where A is asymptotically $(p-1)$-homogeneous near zero in the following sense.

(AH0) *There exist a positive function* $a_0 \in C^1(\overline{\Omega}, \mathbb{R})$ *and a continuous function* $\tilde{a}_0(x,t)$ *on* $\overline{\Omega} \times [0, +\infty)$ *such that*

$$A(x, y) = a_0(x)|y|^{p-2}y + \tilde{a}_0(x, |y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$

where

$$\lim_{t \to +0} \frac{\tilde{a}_0(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For this weight function a_0, we define

$$\lambda_1(a_0) := \inf\left\{ \int_\Omega a_0(x)|\nabla u|^p\, dx : \|u\|_p = 1 \right\}. \tag{12}$$

Because $0 < \min_{x \in \overline{\Omega}} a_0(x) \le \max_{x \in \overline{\Omega}} a_0(x) < \infty$, by the same argument as the one for the first eigenvalue of the negative p-Laplacian, we can prove that $\lambda_1(a_0)$ is the first eigenvalue of

$$-\operatorname{div}\bigl(a_0(x)|\nabla u|^{p-2}\nabla u\bigr) = \lambda |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{13}$$

Moreover, $\lambda_1(a_0)$ has a positive eigenfunction $\varphi_{a_0} \in C^1(\overline{\Omega})$ and it is simple. It is proved that (13) has no constant sign solutions other than 0 provided $\lambda \neq \lambda_1(a_0)$.

Theorem 12. *Assume* (AH0)*. For every* $\varepsilon > 0$ *there exists* $r_0 > 0$ *such that equation* (EV; λ) *has no nontrivial solutions in* $B_p(r_0) := \{v \in W_0^{1,p}(\Omega) : \|v\|_p < r_0\}$ *provided* $\lambda < \lambda_1(a_0) - \varepsilon$.

Proof. We argue by contradiction. Thus we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n < \lambda_1(a_0) - \varepsilon_0$, $u_n \in B_p(1/n)$ and u_n is a nontrivial solution of (EV; λ_n). By taking u_n as a test function in (EV; λ_n), we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\nabla u_n\, dx = \lambda_n \|u_n\|_p^p \le (\lambda_1(a_0) - \varepsilon_0)/n^p \to 0 \tag{14}$$

as $n \to \infty$. Therefore, $u_n \to 0$ in $W_0^{1,p}(\Omega)$. In addition, by noting that u_n is a nontrivial solution of (EV; λ_n) and $0 \le \lambda_n < \lambda_1(a_0) - \varepsilon_0$, Proposition 4 yields that u_n converges to 0 in $C^1(\overline{\Omega})$.

Set $v_n := u_n/\|u_n\|_p$. Then it follows from (14) and the boundedness of $\{\lambda_n\}$ that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may assume that v_n converges to some v_0 weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Again by taking $u_n/\|u_n\|_p^p$ as a test function in (EV; λ_n), we obtain

$$\begin{aligned}\lambda_1(a_0)-\varepsilon_0>\lambda_n&=\int_\Omega\frac{a_0(x)|\nabla u_n|^p}{\|u_n\|_p^p}\,dx+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx\\&=\int_\Omega a_0(x)|\nabla v_n|^p\,dx+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\\&\ge\lambda_1(a_0)+\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}=:\lambda_1(a_0)+I\end{aligned}$$

because of the characterization of $\lambda_1(a_0)$. Hypothesis (AH0) guarantees that for every $\delta>0$ there exists $\rho_0>0$ such that $|\tilde a_0(x,t)|\le\delta|t|^{p-2}$ if $|t|\le\rho_0$. Since $\|u_n\|_{C^1(\bar\Omega)}\to 0$ and in view of (14), we can get

$$|I|\le\delta\int_\Omega|\nabla v_n|^p\,dx\le\frac{\delta(p-1)}{C_0}\lambda_n\le\frac{\delta(p-1)}{C_0}(\lambda_1(a_0)-\varepsilon_0)$$

for sufficiently large n. As a result, by taking a sufficiently small $\delta>0$, we have a contradiction for sufficiently large n. □

Theorem 13. *Assume* (AH0). *For every $\varepsilon>0$ there exists $r_1>0$ such that* (EV; λ) *has no constant sign solutions in $B_p(r_1)\setminus\{0\}$ provided $\lambda>\lambda_1(a_0)+\varepsilon$.*

Proof. By way of contradiction, we assume that there exist $\varepsilon_0>0$, $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n>\lambda_1(a_0)+\varepsilon_0$, $0\ne u_n\in B_p(1/n)$ and u_n is a constant sign solution of (EV; λ_n). Because A is odd, we may suppose that $u_n\ge 0$ by considering $-u_n$ if necessary. Thus, by Remark 3(i)–(ii), $u_n\in C^1(\bar\Omega)$ and $u_n>0$ in Ω. We note that $\lambda_n\le A_p\lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of A_p), and so $\{\lambda_n\}$ is bounded. Therefore, we may assume that λ_n converges to some λ_0 by choosing a subsequence. In addition, by the same argument as in Theorem 12, we can show that $u_n\to 0$ in $C^1(\bar\Omega)$.

Set $A_n(x,y):=A(x,\|u_n\|_p y)/\|u_n\|_p^{p-1}$ and $f_n(x,t):=\lambda_n|t|^{p-2}t$. Then A_n satisfies Assumption A(i)–(iv) with the same constants C_0, C_1, and C_2. Moreover, $|f_n(x,t)|\le\lambda_n|t|^{p-1}\le A_p\lambda_1(-\Delta_p)|t|^{p-1}$ for every $t\in\mathbb{R}$, a.e. $x\in\Omega$. Note also that we have the boundedness of $\|v_n\|$ due to the inequality $C_0\|\nabla u_n\|_p^p/(p-1)\le\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx=\lambda_n\|u_n\|_p^p$. Since $v_n:=u_n/\|u_n\|_p$ is a positive solution of

$$-\mathrm{div}(A_n(x,\nabla u))=f_n(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega,$$

Proposition 4 guarantees that $\{v_n\}$ has a convergent subsequence in $C^1(\bar\Omega)$. By choosing a subsequence, we may suppose that $v_n\to v_0\ne 0$ in $C^1(\bar\Omega)$ (note that $\|v_0\|_p=1$). Using that we obtain, for every $w\in W_0^{1,p}(\Omega)$, that

$$\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla w\,dx=\int_\Omega\frac{\tilde a_0(x,|\nabla u_n|)\nabla u_n}{|\nabla u_n|^{p-1}}\nabla w|\nabla v_n|^{p-1}\,dx\to 0$$

as $n\to\infty$ in view of (AH0) and the convergence $u_n\to 0$. As a result, letting

$n \to \infty$ in the equality

$$\int_\Omega a_0(x)|\nabla v_n|^{p-2}\nabla v_n \nabla w\,dx + \int_\Omega \frac{\tilde{a}_0(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla w\,dx = \lambda_n \int_\Omega |v_n|^{p-2} v_n w\,dx$$

for each $w \in W_0^{1,p}(\Omega)$, we see that $v_0 \neq 0$ is a positive solution of (13) with $\lambda = \lambda_0$ (see Remark 3(ii) for $v_0 > 0$). This yields that $\lambda_0 = \lambda_1(a_0)$, because (13) has no positive solutions other that $\lambda = \lambda_1(a_0)$. Therefore we have a contradiction, because $\lambda_0 = \lim_{n\to\infty} \lambda_n \ge \lambda_1(a_0) + \varepsilon_0$. □

Proposition 14. *Assume (AH0). Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that*

$$\frac{\underline{\lambda}_1(A,r)}{r^p} \ge \lambda_1(a_0) - \varepsilon \quad \text{for every } 0 < r < r_0.$$

Proof. Assume that there exist $\varepsilon > 0$ and $r_n > 0$ such that $r_n \to 0$ as $n \to \infty$ and $\underline{\lambda}_1(A, r_n)/r_n^p < \lambda_1(a_0) - \varepsilon$ for every $n \in \mathbb{N}$. Because of Proposition 5 and Lemma 6 (note that A is odd in the second variable), we can choose a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n), \qquad \min_{v\in r_n S}\int_\Omega G(x,\nabla v)\,dx = \int_\Omega G(x,\nabla u_n)\,dx.$$

Note that

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A,r_n) < (\lambda_1(a_0)-\varepsilon) r_n^p \to 0, \tag{15}$$

and so $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Because u_n is a solution of (EV; λ) with $\lambda = \underline{\lambda}_1(A, r_n)/r_n^p$ (see Proposition 5), by combining the inequality

$$\lambda_1(a_0) - \varepsilon > \frac{\underline{\lambda}_1(A,r_n)}{r_n^p} = \int_\Omega a_0(x)|\nabla v_n|^p\,dx + \int_\Omega \frac{\tilde{a}_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx$$

and an argument as in Theorem 12 with $\lambda_n = \underline{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

Proposition 15. *Assume (AH0). Then, for every $\varepsilon > 0$, there exists $r_1 > 0$ such that*

$$\frac{\bar{\lambda}_1(A,r)}{r^p} \le \lambda_1(a_0) + \varepsilon \quad \text{for every } 0 < r < r_1.$$

Proof. Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to 0$ as $n \to \infty$ and $\bar{\lambda}_1(A, r_n)/r_n^p > \lambda_1(a_0) + \varepsilon_0$ for every $n \in \mathbb{N}$. According to Lemma 6 and Proposition 5, we can take a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \bar{\lambda}_1(A,r_n), \qquad \min_{v\in r_n S}\int_\Omega G(x,\nabla v)\,dx = \int_\Omega G(x,\nabla u_n)\,dx.$$

Noting that, with φ_{a_0} the positive eigenfunction corresponding to $\lambda_1(a_0)$ satisfying

$\|\varphi_{a_0}\|_p = 1$, we have

$$\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \le \int_\Omega G(x,\nabla u_n)\,dx \le \int_\Omega G(x, r_n\nabla\varphi_{a_0})\,dx \le \frac{C_1 r_n^p}{p(p-1)}\|\nabla\varphi_{a_0}\|_p^p,$$

we see that $u_n \to 0$ in $C^1(\overline{\Omega})$ due to Proposition 4, because u_n is a positive solution of (EV; λ) with $\lambda = \bar{\lambda}_1(A, r_n)/r_n^p$ and $(\lambda_1(a_0)+\varepsilon_0 <)\bar{\lambda}_1(A, r_n)/r_n^p \le A_p\lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of A_p). Therefore, by the same argument as in Theorem 13 with $\lambda_n = \bar{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

The following result follows from Propositions 14 and 15, (note $\underline{\lambda}_1(A, r) \le \bar{\lambda}_1(A, r)$ for every $r > 0$).

Corollary 16. *Under (AH0), we have*

$$\lim_{r\to+0}\frac{\bar{\lambda}_1(A,r)}{r^p} = \lim_{r\to+0}\frac{\underline{\lambda}_1(A,r)}{r^p} = \lambda_1(a_0).$$

Proposition 17. *Under (AH0), we have*

$$\lim_{r\to+0}\frac{\mu_1(A,r)}{r^p} = \frac{\lambda_1(a_0)}{p}.$$

Proof. Due to Proposition 5, for every $r > 0$, there exists a positive solution $u_r \in (rS)\cap C^1(\overline{\Omega})$ of (EV; λ) with $\lambda = \lambda_1(A, u_r)/r^p$ and $\mu_1(A, r) = J(u_r)$. Then we can prove that $u_r \to 0$ in $C^1(\overline{\Omega})$ as $r \to +0$ and $u_r/\|u_r\|_p$ is bounded in $W_0^{1,p}(\Omega)$ as $r \to +0$ by a similar reason to the one in Proposition 15 (note that $\lambda_1(A, u_r)/r^p$ is bounded as $r \to +0$ by the inequality below and Corollary 16).

Set $\widetilde{G}_0(x, y) := \int_0^{|y|}\tilde{a}_0(x,t)t\,dt$ for $y \in \mathbb{R}^N$. We point out that

$$\underline{\lambda}_1(A, r) \le \lambda_1(A, u_r) \le \bar{\lambda}_1(A, r)$$

and

$$\begin{aligned}\mu_1(A, r) &= \int_\Omega G(x,\nabla u_r)\,dx = \frac{1}{p}\int_\Omega a_0(x)|\nabla u_r|^p\,dx + \int_\Omega \widetilde{G}_0(x,\nabla u_r)\,dx\\ &= \frac{\lambda_1(A,u_r)}{p} - \frac{1}{p}\int_\Omega \tilde{a}_0(x,|\nabla u|)|\nabla u_r|^2\,dx + \int_\Omega \widetilde{G}_0(x,\nabla u_r)\,dx.\end{aligned}$$

Thus, by Corollary 16 and $r = \|u_r\|_p$, it suffices to prove

$$\lim_{r\to+0}\int_\Omega \frac{\tilde{a}_0(x,|\nabla u|)|\nabla u_r|^2}{\|u_r\|_p^p}\,dx = 0 \quad\text{and}\quad \lim_{r\to+0}\int_\Omega \frac{\widetilde{G}_0(x,\nabla u_r)}{\|u_r\|_p^p}\,dx = 0.$$

Now we fix any $\varepsilon > 0$. Then, by (AH0), there exists $\delta > 0$ such that

$$|\tilde{a}_0(x,t)| \le \varepsilon t^{p-2} \quad\text{and}\quad |\widetilde{G}_0(x,y)| \le \varepsilon|y|^p/p \quad \text{for every } 0 < t \le \delta,\ |y| \le \delta.$$

Because $u_r \to 0$ in $C^1(\overline{\Omega})$ as $r \to +0$, we may assume that $\|u_r\|_{C^1(\overline{\Omega})} \le \delta$ for sufficiently small $r > 0$. Therefore, we obtain

$$\left|\int_\Omega \frac{\tilde{a}_0(x, |\nabla u|)|\nabla u_r|^2}{\|u_r\|_p^p}\,dx\right| \le \varepsilon \frac{\|\nabla u_r\|_p^p}{\|u_r\|_p^p}, \qquad \left|\int_\Omega \frac{\widetilde{G}_0(x, \nabla u_r)}{\|u_r\|_p^p}\,dx\right| \le \varepsilon \frac{\|\nabla u_r\|_p^p}{p\|u_r\|_p^p}.$$

Since $\|\nabla u_r\|_p / \|u_r\|_p$ is bounded as $r \to +0$ and $\varepsilon > 0$ is arbitrary, our conclusion holds. □

3.2. ***Asymptotically homogeneous case near*** ∞**.** In this subsection, we consider the case where A is asymptotically $(p-1)$-homogeneous near ∞ in the following sense.

(AH) *There exist a positive function* $a_\infty \in C^1(\overline{\Omega}, \mathbb{R})$ *and a continuous function* $\tilde{a}(x, t)$ *on* $\overline{\Omega} \times \mathbb{R}$ *such that*

$$A(x, y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x, |y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$

where

$$\lim_{t \to +\infty} \frac{\tilde{a}(x, t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For the weight function a_∞, we define

$$\lambda_1(a_\infty) := \inf\left\{\int_\Omega a_\infty(x)|\nabla u|^p\,dx : \|u\|_p = 1\right\}. \tag{16}$$

Because $0 < \min_{x\in\overline{\Omega}} a_\infty(x) \le \max_{x\in\overline{\Omega}} a_\infty(x) < \infty$, by the same argument as for the first eigenvalue of $-\Delta_p$, we can prove the following elementary results:

(i) $\lambda_1(a_\infty)$ is the first eigenvalue of

$$-\mathrm{div}(a_\infty(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{17}$$

(ii) $\lambda_1(a_\infty)$ has a positive eigenfunction $\varphi_{a_\infty} \in C^1(\overline{\Omega})$ with $\|\varphi_{a_\infty}\|_p = 1$ and it is simple.

(iii) If $\lambda \ne \lambda_1(a_\infty)$, then (17) has no constant sign solutions other than 0.

Theorem 18. *Assume (AH). For every* $\varepsilon > 0$ *there exists* $R_0 > 0$ *such that equation (EV; λ) has no solutions in* $W_0^{1,p}(\Omega) \setminus B_p(R_0)$ *provided* $\lambda < \lambda_1(a_\infty) - \varepsilon$.

To prove the theorem, we need the following result.

Lemma 19. *Assume (AH) and let* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *be a sequence satisfying* $\|u_n\|_p \to \infty$ *as* $n \to \infty$*. If* $v_n := u_n/\|u_n\|_p$ *is bounded in* $W_0^{1,p}(\Omega)$*, the following assertions hold*:

(i) $\displaystyle\lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\,dx = 0.$

(ii) *For every* $w \in W_0^{1,p}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\, \nabla u_n \nabla w}{\|u_n\|_p^{p-1}}\, dx = 0.$$

(iii) $\displaystyle\lim_{n\to\infty} \int_\Omega \frac{\widetilde{G}(x, \nabla u_n)}{\|u_n\|_p^p}\, dx = 0,$ *where* $\widetilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t)t\, dt$ *for* $y \in \mathbb{R}^N$.

Proof. (i) Fix any $\varepsilon > 0$. By the property of the function $\tilde{a}$, there exist $R > 0$ and $C > 0$ such that

(18) $$|\tilde{a}(x, t)| \le \varepsilon |t|^{p-2} \text{ if } t \ge R \quad \text{and} \quad |\tilde{a}(x, t)| \le C \text{ if } 0 \le t \le R.$$

Therefore, we obtain

$$\begin{aligned}\left|\int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx\right| &\le \int_{|\nabla u_n|>R} \varepsilon |\nabla v_n|^p\, dx + \int_{|\nabla u_n|\le R} \frac{C|\nabla u_n|^2}{\|u_n\|_p^p}\, dx \\ &\le \varepsilon \|\nabla v_n\|_p^p + \frac{CR^2|\Omega|}{\|u_n\|_p^p} \le \varepsilon D^p + \frac{CR^2|\Omega|}{\|u_n\|_p^p}\end{aligned}$$

by (18), where $D := \sup_n \|\nabla v_n\|_p$. Letting $n \to \infty$, we have

$$\limsup_{n\to\infty} \left|\int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx\right| \le \varepsilon D^p,$$

because $\|u_n\|_p \to \infty$ as $n \to \infty$. Thus, since $\varepsilon > 0$ is arbitrary, our conclusion holds.

(ii) For any $\varepsilon > 0$ and $w \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}&\left|\int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\, \nabla u_n \nabla w}{\|u_n\|_p^{p-1}}\, dx\right| \\ &\qquad \le \int_{|\nabla u_n|>R} \varepsilon |\nabla v_n|^{p-1} |\nabla w|\, dx + \int_{|\nabla u_n|\le R} \frac{C|\nabla u_n||\nabla w|}{\|u_n\|_p^{p-1}}\, dx \\ &\qquad \le \varepsilon \|\nabla v_n\|_p^{p-1} \|\nabla w\|_p + \frac{CR\|\nabla w\|_p |\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}\end{aligned}$$

by Hölder's inequality and (18). By combining this inequality and a similar argument to that used in (i), our conclusion is shown.

(iii) It is easily shown that, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$|\widetilde{G}(x, y)| \le \varepsilon |y|^p + C \quad \text{for every } y \in \mathbb{R}^N.$$

Therefore, $\left|\int_\Omega \widetilde{G}(x, \nabla u_n)\, dx\right| \le \varepsilon \|\nabla u_n\|_p^p + C|\Omega|$. This implies our conclusion. □

Proof of Theorem 18. By way of contradiction, we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$, and $\{u_n\}$ such that $\lambda_n < \lambda_1(a_\infty) - \varepsilon_0$, $\lim_{n\to\infty} \|u_n\|_p = \infty$, and u_n is a solution of (EV; λ_n). By taking u_n as a test function in (EV; λ_n), we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\nabla u_n \, dx = \lambda_n \|u_n\|_p^p \le (\lambda_1(a_\infty) - \varepsilon_0)\|u_n\|_p^p;$$

refer to Remark 1(iii). Therefore, $v_n := u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$.

Again by taking $u_n/\|u_n\|_p^p$ as a test function in (EV; λ_n), we obtain

$$\begin{aligned}\lambda_1(a_\infty) - \varepsilon_0 > \lambda_n &= \int_\Omega \frac{a_\infty(x)|\nabla u_n|^p}{\|u_n\|_p^p}\, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx \\ &= \int_\Omega a_\infty(x)|\nabla v_n|^p \, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^p}\, dx \\ &\ge \lambda_1(a_\infty) + o(1),\end{aligned}$$

using the definition of $\lambda_1(a_\infty)$ and Lemma 19(i). This is a contradiction. □

Theorem 20. *Assume* (AH). *For every $\varepsilon > 0$ there exists $R_1 > 0$ such that* (EV; λ) *has no constant sign solutions in $W_0^{1,p}(\Omega) \setminus B_p(R_1)$ provided $\lambda > \lambda_1(a_\infty) + \varepsilon$.*

Proof. By way of contradiction, we assume that there exist $\varepsilon_0 > 0$, $\{\lambda_n\}$, and $\{u_n\}$ such that $\lambda_n > \lambda_1(a_\infty) + \varepsilon_0$, $\lim_{n\to\infty} \|u_n\|_p = \infty$, and u_n is a constant sign solution of (EV; λ_n). Because A is odd, we may suppose that $u_n \ge 0$ by considering $-u_n$ if necessary. Thus, by Remark 3, $u_n \in C^1(\overline{\Omega})$ and $u_n > 0$ in Ω. Here we note that $\lambda_n \le A_p \lambda_1(-\Delta_p)$ by Proposition 10, where $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p$ (see (10) for the definition of A_p), and so $\{\lambda_n\}$ is bounded. Hence we may assume, by taking a subsequence, that λ_n converges to some

$$\lambda_0 \in \left[\lambda_1(a_\infty) + \varepsilon_0,\ A_p\lambda_1(-\Delta_p)\right].$$

In addition, we know that $v_n := u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\, dx = \lambda_n \|u_n\|_p^p,$$

where we take u_n as a test function in (EV; λ_n). Thus, by choosing a subsequence, we may suppose that v_n converges to some v weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$.

We claim that v is a positive solution of

$$(19) \qquad -\mathrm{div}(a_\infty(x)|\nabla v|^{p-2}\nabla v) = \lambda_0 |v|^{p-2} v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

that is, v is a positive eigenfunction corresponding to λ_0. If our claim holds, then $\lambda_0 = \lambda_1(a_\infty)$ occurs because (17) has no positive solutions in the case of $\lambda \ne \lambda_1(a_\infty)$. Hence this contradicts $\lambda_1(a_\infty) + \varepsilon_0 \le \lim_{n\to\infty} \lambda_n = \lambda_0$.

We now prove our claim. First, we show that v_n converges to v strongly in $W_0^{1,p}(\Omega)$. Indeed, by taking $(v_n - v)/\|u_n\|_p^{p-1}$ as a test function in (EV; λ_n), we have

$$\begin{aligned}\lambda_n &\int_\Omega v_n^{p-1}(v_n - v)\,dx \\ &= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}}\nabla(v_n - v)\,dx \\ &= \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx + o(1)\end{aligned}$$

as $n \to \infty$ due to Lemma 19(i)–(ii). Since $v_n \to v$ in $L^p(\Omega)$, this implies that $\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\,dx$ converges to 0 as $n \to \infty$. Noting that

$$\begin{aligned}o(1) &= \int_\Omega a_\infty(x)(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla(v_n - v)\,dx \\ &\geq \min_{\overline{\Omega}} a_\infty \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla(v_n - v)\,dx \\ &\geq \min_{\overline{\Omega}} a_\infty (\|\nabla v_n\|_p^{p-1} - \|\nabla v\|_p^{p-1})(\|\nabla v_n\|_p - \|\nabla v\|_p) \geq 0,\end{aligned}$$

we have $v_n \to v$ in $W_0^{1,p}(\Omega)$ (note $0 < \min_{\overline{\Omega}} a_\infty \leq \max_{\overline{\Omega}} a_\infty < \infty$). As a result, v is a solution of (19) by letting $n \to \infty$ in the equality

$$\int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla w\,dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n \nabla w}{\|u_n\|_p^{p-1}}\,dx = \lambda_n \int_\Omega v_n^{p-1} w\,dx$$

for every $w \in W_0^{1,p}(\Omega)$; note that, by Lemma 19(ii), the second term converges to zero. Since $v_n = u_n/\|u_n\|_p > 0$ in Ω, v is nonnegative, and so v is positive by Remark 3(i) and $\|v\|_p = 1$. Thus our claim is shown. □

Proposition 21. *Assume (AH). Then, for every $\varepsilon > 0$, there exists $R_0 > 0$ such that*

$$\frac{\underline{\lambda}_1(A, r)}{r^p} \geq \lambda_1(a_\infty) - \varepsilon \quad \text{for every } r > R_0.$$

Proof. Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ and $\underline{\lambda}_1(A, r_n)/r_n^p < \lambda_1(a_\infty) - \varepsilon_0$ for every $n \in \mathbb{N}$. Because of Proposition 5 and Lemma 6, we can choose a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x, \nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A, r_n), \qquad \min_{v \in r_n S} \int_\Omega G(x, \nabla v)\,dx = \int_\Omega G(x, \nabla u_n)\,dx.$$

Note that

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \leq \int_\Omega A(x, \nabla u_n)\nabla u_n\,dx = \underline{\lambda}_1(A, r_n) < (\lambda_1(a_\infty) - \varepsilon_0) r_n^p,$$

and so $u_n/r_n = u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$. Because u_n is a solution of (EV; λ) with $\lambda = \underline{\lambda}_1(A, r_n)/r_n^p$ (see Proposition 5), by the same argument as in Theorem 18 with $\lambda_n = \underline{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

Proposition 22. *Assume* (AH). *Then, for every* $\varepsilon > 0$, *there exists* $R_1 > 0$ *such that*

$$\frac{\bar{\lambda}_1(A, r)}{r^p} \le \lambda_1(a_\infty) + \varepsilon \quad \text{for every } r > R_1.$$

Proof. Assume that there exist $\varepsilon_0 > 0$ and $r_n > 0$ such that $r_n \to \infty$ as $n \to \infty$ and $\bar{\lambda}_1(A, r_n)/r_n^p > \lambda_1(a_\infty) + \varepsilon_0$ for every $n \in \mathbb{N}$. According to Lemma 6 and Proposition 5, we can take a positive function $u_n \in (r_n S) \cap C^1(\overline{\Omega})$ satisfying

$$\int_\Omega A(x, \nabla u_n)\nabla u_n \, dx = \bar{\lambda}_1(A, r_n), \qquad \min_{v \in r_n S} \int_\Omega G(x, \nabla v)\, dx = \int_\Omega G(x, \nabla u_n)\, dx.$$

Note that, with φ_{a_∞} as in item (ii) of page 165, we have

$$\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \le \int_\Omega G(x, \nabla u_n)\, dx \le \int_\Omega G(x, r_n \nabla \varphi_{a_\infty})\, dx \le \frac{C_1 r_n^p}{p(p-1)}\|\nabla \varphi_{a_\infty}\|_p^p.$$

Hence $u_n/r_n = u_n/\|u_n\|_p$ is bounded in $W_0^{1,p}(\Omega)$. Since u_n is a positive solution of (EV; λ) with $\lambda = \bar{\lambda}_1(A, r_n)/r_n^p$, by the same argument as in Theorem 20 with $\lambda_n = \bar{\lambda}_1(A, r_n)/r_n^p$, we have a contradiction. □

By Propositions 21 and 22, we have the following result.

Corollary 23. *Under* (AH), *we have*

$$\lim_{r\to+\infty} \frac{\bar{\lambda}_1(A, r)}{r^p} = \lim_{r\to+\infty} \frac{\underline{\lambda}_1(A, r)}{r^p} = \lambda_1(a_\infty).$$

Proposition 24. *Under* (AH), *we have*

$$\lim_{r\to+\infty} \frac{\mu_1(A, r)}{r^p} = \frac{\lambda_1(a_\infty)}{p}.$$

Proof. Due to Proposition 5, for every $r > 0$, there exists a positive solution $u_r \in (rS) \cap C^1(\overline{\Omega})$ of (EV; λ) with $\lambda = \lambda_1(A, u_r)/r^p$ and $\mu_1(A, r) = J(u_r)$. Then $u_r/\|u_r\|_p = u_r/r$ is bounded in $W_0^{1,p}(\Omega)$, as seen from

$$\frac{C_0}{p(p-1)}\|\nabla u_r\|_p^p \le \int_\Omega G(x, \nabla u_r)\, dx \le \int_\Omega G(x, r\nabla w)\, dx \le \frac{r^p C_1}{p(p-1)}\|\nabla w\|_p^p$$

for any $w \in W_0^{1,p}(\Omega)$ with $\|w\|_p = 1$.

Set

$$\widetilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t)t \, dx \quad \text{for } y \in \mathbb{R}^N.$$

Note that

$$\underline{\lambda}_1(A, r) \le \lambda_1(A, u_r) \le \bar{\lambda}_1(A, r)$$

and

$$\mu_1(A,r)=\int_\Omega G(x,\nabla u_r)\,dx=\frac{1}{p}\int_\Omega a_\infty(x)|\nabla u_r|^p\,dx+\int_\Omega \widetilde{G}(x,\nabla u_r)\,dx$$
$$=\frac{\lambda_1(A,u_r)}{p}-\frac{1}{p}\int_\Omega \tilde{a}(x,|\nabla u|)|\nabla u_r|^2\,dx+\int_\Omega \widetilde{G}(x,\nabla u_r)\,dx.$$

According to Corollary 23 and Lemma 19(i) and (iii) (note $\|u_r\|_p=r\to+\infty$), our conclusion is achieved. □

4. Existence of a positive solution

In this section, we provide the existence of a positive solution to the equation

$$\text{(P)}\qquad \begin{cases} -\operatorname{div} A(x,\nabla u)=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where the nonlinear term f satisfies Assumption (f).

Theorem 25. *Assume (AH0), (AH), and (f). Let $\lambda_1(a_0)$ and $\lambda_1(a_\infty)$ be the first eigenvalues of, respectively, (13) and (17) (see the discussion there). If one of the following conditions holds, (P) has at least one positive solution.*

(i) $\alpha_0>\lambda_1(a_0)$ *and* $\alpha<\lambda_1(a_\infty)$.

(ii) $\alpha_0<\lambda_1(a_0)$ *and* $\alpha>\lambda_1(a_\infty)$.

This addresses the existence of an eigenvalue for our operator because we can apply Theorem 25 to $f(x,u)=\lambda|u|^{p-2}u$.

Corollary 26. *Assume (AH0), (AH), and $\lambda_1(a_0)\neq\lambda_1(a_\infty)$. Then, for every λ between $\lambda_1(a_0)$ and $\lambda_1(a_\infty)$, (EV; λ) has a nontrivial (positive) solution. Therefore λ is an eigenvalue of A*

To show the existence of a positive solution, we define a C^1 functional I on $W_0^{1,p}(\Omega)$ by

$$I(u):=\int_\Omega G(x,\nabla u)\,dx-\int_\Omega F_+(x,u)\,dx\quad\text{for } u\in W_0^{1,p}(\Omega),$$

where $F_+(x,u):=\int_0^u f_+(x,u)\,dx$, with $f_+(x,t)$ given by $f(x,t)$ if $t\geq 0$ and 0 if $t\leq 0$.

Remark 27. If $u\in W_0^{1,p}(\Omega)$ is a nontrivial critical point of I, then u is a positive solution of (P).

Indeed, by taking $-u_-$ as a test function, we obtain

$$0=\langle I'(u),-u_-\rangle=\int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx-\int_\Omega f_+(x,u)(-u_-)\,dx$$
$$=\int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx\geq\frac{C_0}{p-1}\|\nabla u_-\|_p^p.$$

Thus $u \geq 0$. By Remark 3(ii) (note that $u \not\equiv 0$), we see that u is a positive solution of (P) (note that $f_+(x, u) = f(x, u)$).

Convention. From now on, let Assumption (f) be satisfied.

Lemma 28. *If $\alpha \neq \lambda_1(a_\infty)$, then I satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of I, which means that

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{W_0^{1,p}(\Omega)^*} \to 0 \quad \text{as } n \to \infty$$

for some $c \in \mathbb{R}$. In view of Proposition 2 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, it is sufficient to prove the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Then, in view of the inequality

$$\begin{aligned}(20)\qquad \frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p \leq \int_\Omega G(x, \nabla u_n)\,dx &= I(u_n) + \int_\Omega F_+(x, u_n)\,dx \\ &\leq I(u_n) + C\|u_n\|_p^p,\end{aligned}$$

it is sufficient to prove the boundedness of $\{u_n\}$ in $L^p(\Omega)$. By way of contradiction we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Set $v_n := u_n/\|u_n\|_p$. The inequality (20) ensures that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, by choosing a subsequence, we may suppose that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ for some v_0.

First, we see that $v_0 \geq 0$ for a.e. $x \in \Omega$. Indeed, by taking $-(u_n)_-$ as a test function, we have

$$\begin{aligned}o(1)\|\nabla(u_n)_-\|_p &= \langle I'(u_n), -(u_n)_-\rangle \\ &= \int_\Omega A(x, \nabla u_n)(-\nabla(u_n)_-)\,dx \geq \frac{C_0}{p-1}\|\nabla(u_n)_-\|_p^p.\end{aligned}$$

Because $p > 1$, we have $\|\nabla(u_n)_-\|_p \to 0$ as $n \to \infty$. Thus $(v_n)_- \to 0$ in $W_0^{1,p}(\Omega)$, and hence $(v_0)_- = 0$ for a.e. $x \in \Omega$.

Now we prove that

$$(21)\qquad \lim_{n\to\infty} \frac{\|f_+(\cdot, u_n) - \alpha(u_n)_-^{p-1}\|_{p'}}{\|u_n\|_p^{p-1}} = 0,$$

where $p' = p/(p-1)$. Fix an arbitrary $\varepsilon > 0$. It follows from condition (ii) of Assumption (f) that there exists a $C_\varepsilon > 0$ such that

$$|f(x, u) - \alpha u^{p-1}| \leq \varepsilon|u|^{p-1} + C_\varepsilon \quad \text{for every } u \geq 0, \text{ a.e. } x \in \Omega.$$

Then we obtain

$$\int_\Omega |f_+(x, u_n) - \alpha(u_n)_+^{p-1}|^{p'}\,dx \leq 2^{p'-1}(\varepsilon^{p'-1}\|(u_n)_+\|_p^p + C_\varepsilon^{p'-1}|\Omega|).$$

Since we are assuming that $\|u_n\|_p \to \infty$ as $n \to \infty$, this shows that

$$\lim_{n\to\infty} \left\| f_+(\cdot, u_n) - \alpha(u_n)_+^{p-1} \right\|_{p'} / \|u_n\|_p^{p-1} = 0,$$

because $\varepsilon > 0$ is arbitrary.

Here we recall the following result proved in Lemma 19:

$$\text{(22)}\quad \lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla(v_n - v_0)\, dx = \lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla\varphi\, dx = 0$$

for every $\varphi \in W_0^{1,p}(\Omega)$. Thus, by considering

$$o(1) = \frac{\langle I'(u_n), v_n - v_0\rangle}{\|u_n\|_p^{p-1}} = \int_\Omega a_\infty(x)|\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v_0)\, dx + o(1),$$

and using Proposition 2, we see that v_n converges strongly to v_0 in $W_0^{1,p}(\Omega)$. Hence, by passing to the limit in $o(1) = \langle I'(u_n), \varphi\rangle / \|u_n\|_p^{p-1}$ for any $\varphi \in W_0^{1,p}(\Omega)$ and by noting (21) and (22), we infer that v_0 is a nontrivial solution of

$$-\operatorname{div}(a_\infty |\nabla u|^{p-2}\nabla u) = \alpha |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

(note that $\|v_0\|_p = 1$ and $v_0 \geq 0$ for a.e. $x \in \Omega$). Since $v_0 \geq 0$ for a.e. $x \in \Omega$, v is a positive solution of (17) with $\lambda = \alpha$ (see Remark 3). This implies that $\alpha = \lambda_1(a_\infty)$, because (17) has no positive solutions if $\lambda \neq \lambda_1(a_\infty)$. It contradicts the hypothesis $\alpha \neq \lambda_1(a_\infty)$. Hence $\|u_n\|_p$ is bounded, which completes the proof. □

Lemma 29. *Assume (AH) and $\alpha < \lambda_1(a_\infty)$. Then I is coercive, bounded from below and weakly lower semicontinuous (wlsc) on $W_0^{1,p}(\Omega)$.*

Proof. Because $\alpha < \lambda_1(a_\infty)$, we can take sufficiently small constants $\varepsilon > 0$ and $0 < \delta < 1$ satisfying

$$\text{(23)}\qquad (1-\delta)(\lambda_1(a_\infty) - \varepsilon) > \alpha + \varepsilon.$$

By condition (ii) of Assumption (f), there exists a $C > 0$ such that

$$|F_+(x, u)| \leq (\alpha + \varepsilon)\frac{u^p}{p} + C$$

for every $u \geq 0$ and a.e. $x \in \Omega$. Due to Proposition 24 and the definition of $\mu_1(A, r)$, there exists an $R > 0$ such that, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_p \geq R$,

$$\text{(24)}\qquad \int_\Omega G(x, \nabla u)\, dx \geq \mu_1(A, \|u\|_p) \geq \frac{\lambda_1(a_\infty) - \varepsilon}{p}\|u\|_p^p.$$

Hence, for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_p \geq R$, we obtain

$$I(u) \geq \frac{(1-\delta)(\lambda_1(a_\infty)-\varepsilon)}{p}\|u\|_p^p + \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\alpha+\varepsilon}{p}\|u_+\|_p^p - C|\Omega|$$
$$\geq \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - C|\Omega|$$

by (2), (23), and (24), where $u_+ := \max\{0, u\}$. This yields that I is coercive. Moreover, because I is bounded from below on $B_p(R)$, we see that I is bounded from below on $W_0^{1,p}(\Omega)$. Since J is wlsc (see the proof of Proposition 5) and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, I is wlsc on $W_0^{1,p}(\Omega)$. □

Lemma 30. *Assume (AH0) and $\alpha_0 < \lambda_1(a_0)$. Let $p < q \leq p^*$, where $p^* = Np/(N-p)$ if $N > p$ and $p^* = +\infty$ if $N \leq p$. Then there exists $\rho_0 > 0$ such that*

$$\inf\{I(u) : \|u\|_q = \rho\} > 0 \quad \text{for every } 0 < \rho < \rho_0.$$

Proof. Because $\alpha_0 < \lambda_1(a_0)$, we can take some sufficiently small $\varepsilon > 0$ and $0 < \delta < 1$ satisfying

$$(1-\delta)(\lambda_1(a_0)-\varepsilon) > \alpha_0 + \varepsilon. \tag{25}$$

According to Proposition 17, there exists an $r_0 > 0$ such that

$$\frac{\mu_1(A,r)}{r^p} \geq \frac{\lambda_1(a_0)-\varepsilon}{p} \quad \text{for every } 0 < r < r_0. \tag{26}$$

In addition, Assumption (f) guarantees the existence of $D_q > 0$ satisfying

$$F_+(x,u) \leq \frac{\alpha_0+\varepsilon}{p}u^p + D_q u^q \quad \text{for every } u \geq 0, \text{ a.e. } x \in \Omega. \tag{27}$$

Because $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, we can take a positive constant C_q such that $\|u\|_q \leq C_q\|\nabla u\|_p$ for every $W_0^{1,p}(\Omega)$. We choose a positive constant ρ satisfying

$$\rho < \min\left\{ r_0|\Omega|^{1/q-1/p},\ \left(\frac{\delta C_0}{2p(p-1)D_q C_q^p}\right)^{1/(q-p)} \right\} =: \rho_0. \tag{28}$$

Note that $\|u\|_p < r_0$ if $\|u\|_q = \rho$, by Hölder's inequality and (28). Therefore, for every $\|u\|_q = \rho$, we have

$$I(u) = (1-\delta)\int_\Omega G(x,\nabla u)\,dx + \delta\int_\Omega G(x,\nabla u)\,dx - \int_\Omega F_+(x,u)\,dx$$
$$\geq (1-\delta)\frac{\mu_1(A,\|u\|_p)}{\|u\|_p^p}\|u\|_p^p + \frac{\delta C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\alpha_0+\varepsilon}{p}\|u_+\|_p^p - D_q\|u_+\|_q^q$$
$$\geq \frac{1}{p}\{(1-\delta)(\lambda_1(a_0)-\varepsilon)-\alpha_0-\varepsilon\}\|u\|_p^p + \left(\frac{\delta C_0}{p(p-1)C_q^p} - D_q\|u\|_q^{q-p}\right)\|u\|_q^p$$

$$\geq \frac{\delta C_0}{2p(p-1)C_q^p}\rho^p,$$

by the definition of $\mu_1(A,r)$, (2), (27), (26), (25), and (28). This ensures our conclusion. □

Proof of Theorem 25. (i) Lemma 29 guarantees the existence of a global minimizer of I. Thus it suffices to prove that $\min_{W_0^{1,p}(\Omega)} I < 0$ to show the existence of a nontrivial critical point of I. Choose a positive constant $\varepsilon > 0$ such that $\alpha_0 > \lambda_1(a_0) + 2\varepsilon$. Let $\varphi_{a_0} \in C^1(\overline{\Omega})$ be a positive eigenfunction corresponding to $\lambda_1(a_0)$ with $\|\varphi_{a_0}\|_p = 1$ (refer to the text below (13) and note that (13) is a homogeneous equation). It is easily seen that $\int_\Omega \widetilde{G}_0(x, r\nabla\varphi_{a_0})\,dx/r^p \to 0$ as $r \to +0$ (refer to the proof of Proposition 17 with $\|r\varphi_{a_0}\|_p = r$). Hence there exists $r_0 > 0$ such that

$$\begin{aligned}(29)\quad \int_\Omega G(x, r\nabla\varphi_{a_0})\,dx &= \frac{r^p}{p}\int_\Omega a_0(x)|\nabla\varphi_{a_0}|^p\,dx + r^p\int_\Omega \frac{\widetilde{G}_0(x, r\nabla\varphi_{a_0})}{r^p}\,dx \\ &\leq \frac{\lambda_1(a_0)+\varepsilon}{p}r^p = \frac{\lambda_1(a_0)+\varepsilon}{p}\|r\varphi_{a_0}\|_p^p\end{aligned}$$

for every $0 < r < r_0$. On the other hand, it follows from part (i) of Assumption (f) that there exists a $\delta > 0$ such that

$$(30)\qquad F_+(x,u) \geq \frac{\alpha_0 - \varepsilon}{p}u^p \quad \text{for every } u \in [0,\delta],\ \text{a.e. } x \in \Omega.$$

Therefore, for every $0 < r < \min\{r_0, \delta/\|\varphi_{a_0}\|_\infty\}$, we have

$$I(ru_0) \leq \frac{r^p}{p}(\lambda_1(a_0) + 2\varepsilon - \alpha_0)\|\varphi_{a_0}\|_p^p < 0,$$

by (29) and (30) (note $\lambda_1(a_0) + 2\varepsilon - \alpha_0 < 0$), whence $\min_{W_0^{1,p}(\Omega)} I < 0$.

(ii) Let $p < q \leq p^*$. Then, by Lemma 30, we obtain $\rho > 0$ satisfying

$$\delta_0 := \inf\{I(u) : \|u\|_q = \rho\} > 0.$$

Now we claim the existence of $w \in W_0^{1,p}(\Omega)$ such that

$$(31)\qquad \|w\|_q > \rho \quad \text{and} \quad I(w) < \delta_0.$$

Admitting this claim, we define

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \qquad \Gamma := \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = w\}.$$

It is obvious that $\Gamma \neq \varnothing$ and $\gamma([0,1]) \cap \{u \in W_0^{1,p}(\Omega) : \|u\|_q = \rho\} \neq \varnothing$ for every $\gamma \in \Gamma$, since $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Thus the mountain pass theorem guarantees that $c(\geq \delta_0)$ is a nontrivial critical value of I because I satisfies the Palais–Smale condition by Lemma 28.

Finally, we prove the existence of w satisfying (31). Because $\alpha > \lambda_1(a_\infty)$, we can choose a positive constant $\varepsilon_0 > 0$ such that

$$\alpha > \lambda_1(a_\infty) + 2\varepsilon_0. \tag{32}$$

Using item (ii) on page 165, we can take $\varphi_{a_\infty} \in C^1(\overline{\Omega})$ be a positive eigenfunction corresponding to $\lambda_1(a_\infty)$ with $\|\varphi_{a_\infty}\|_p = 1$. It follows from Lemma 19(iii) that

$$\int_\Omega \widetilde{G}(x, r\nabla\varphi_{a_\infty})\,dx/r^p \to 0$$

as $r \to +\infty$ (note that $\|r\varphi_{a_\infty}\|_p = r$). Hence there exists $R_0 > 0$ such that

$$\begin{aligned} \int_\Omega G(x, r\nabla\varphi_{a_\infty})\,dx &= \frac{r^p}{p}\int_\Omega a_\infty(x)|\nabla\varphi_{a_\infty}|^p\,dx + r^p\int_\Omega \frac{\widetilde{G}_0(x, r\nabla\varphi_{a_\infty})}{r^p}\,dx \\ &\le \frac{\lambda_1(a_\infty)+\varepsilon_0}{p} r^p = \frac{\lambda_1(a_\infty)+\varepsilon_0}{p}\|r\varphi_{a_\infty}\|_p^p \end{aligned} \tag{33}$$

for every $r \ge R_0$. In addition, it follows from condition (ii) of Assumption (f) that there exists $D > 0$ such that

$$F_+(x, u) \ge \frac{\alpha - \varepsilon_0}{p} u^p - D \quad \text{for every } u \ge 0, \text{ a.e. } x \in \Omega. \tag{34}$$

Consequently, by (32), (33), and (34), we obtain

$$I(r\varphi_{a_0}) \le \frac{r^p}{p}(\lambda_1(a_\infty) + 2\varepsilon_0 - \alpha)\|\varphi_{a_0}\|_p^p + D|\Omega| \to -\infty$$

as $t \to +\infty$. This implies the existence of w satisfying (31). □

4.1. *Resonant cases.* To consider the resonant cases, we introduce the following hypotheses for

$$\widetilde{G}(x, y) := \int_0^{|y|} \tilde{a}(x, t)t\,dt \quad \text{and} \quad \widetilde{G}_0(x, y) := \int_0^{|y|} \tilde{a}_0(x, t)t\,dt,$$

where $\tilde{a}$ and $\tilde{a}_0$ are as in (AH) and (AH0).

(H+) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$\begin{aligned} &\lim_{|y|\to\infty} \frac{p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2}{|y|^q} = +\infty && \text{for a.e. } x \in \Omega, \\ &p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \ge -H_0(1 + |y|^q) && \text{for a.e. } x \in \Omega, \text{ every } y \in \mathbb{R}^N, \\ &f(x, t)t - pF(x, t) \ge -H_0(1 + t^q) && \text{for a.e. } x \in \Omega, \text{ every } t \ge 0. \end{aligned}$$

(H−) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$\lim_{|y|\to\infty} \frac{p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2}{|y|^q} = -\infty \quad \text{for a.e. } x \in \Omega,$$
$$p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \le H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega,\ \text{every } y \in \mathbb{R}^N,$$
$$f(x, t)t - pF(x, t) \le H_0(t^q + 1) \quad \text{for a.e. } x \in \Omega,\ \text{every } t \ge 0.$$

(HF+) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \ge -H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega,\ \text{every } y \in \mathbb{R}^N,$$
$$f(x, t)t - pF(x, t) \ge -H_0(1 + t^q) \quad \text{for every } t \ge 0,\ \text{a.e. } x \in \Omega,$$
$$\lim_{t\to+\infty} \frac{f(x, t)t - pF(x, t)}{t^q} = +\infty \quad \text{for a.e. } x \in \Omega.$$

(HF−) There exist $1 \le q < p$ and $H_0 > 0$ such that

$$p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \le H_0(1 + |y|^q) \quad \text{for a.e. } x \in \Omega,\ \text{every } y \in \mathbb{R}^N,$$
$$f(x, t)t - pF(x, t) \le H_0(1 + t^q) \quad \text{for every } t \ge 0,\ \text{a.e. } x \in \Omega,$$
$$\lim_{t\to+\infty} \frac{f(x, t)t - pF(x, t)}{t^q} = -\infty \quad \text{for a.e. } x \in \Omega.$$

(H0+) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y|\to 0} \frac{p\widetilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2}{|y|^r} = +\infty \quad \text{for a.e. } x \in \Omega,$$
$$p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \ge -H_0|y|^r \quad \text{for a.e. } x \in \Omega,\ \text{every } |y| \le 1,$$
$$f(x, t)t - pF(x, t) \ge -H_0 t^r \quad \text{for a.e. } x \in \Omega,\ \text{every } t \in [0, 1].$$

(H0−) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\lim_{|y|\to 0} \frac{p\widetilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2}{|y|^r} = -\infty \quad \text{for a.e. } x \in \Omega,$$
$$p\widetilde{G}(x, y) - \tilde{a}(x, |y|)|y|^2 \le H_0|y|^r \quad \text{for a.e. } x \in \Omega,\ \text{every } |y| \le 1,$$
$$f(x, t)t - pF(x, t) \le H_0 t^r \quad \text{for a.e. } x \in \Omega,\ \text{every } t \in [0, 1].$$

(HF0+) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$p\widetilde{G}_0(x, y) - \tilde{a}_0(x, |y|)|y|^2 \ge -H_0|y|^r \quad \text{for a.e. } x \in \Omega,\ \text{every } |y| \le 1,$$
$$f(x, t)t - pF(x, t) \ge -H_0 t^r \quad \text{for every } t \in [0, 1],\ \text{a.e. } x \in \Omega,$$
$$\lim_{t\to+0} \frac{f(x, t)t - pF(x, t)}{t^r} = +\infty \quad \text{for a.e. } x \in \Omega.$$

(HF0−) There exist $p \le r < p^*$ and $H_0 > 0$ such that

$$\begin{aligned} p\widetilde{G}_0(x,y) - \tilde{a}_0(x,|y|)|y|^2 &\le H_0|y|^r && \text{for a.e. } x \in \Omega,\ \text{every } |y| \le 1, \\ f(x,t)t - pF(x,t) &\le H_0 t^r && \text{for every } t \in [0,1],\ \text{a.e. } x \in \Omega, \\ \lim_{t\to+0} \frac{f(x,t)t - pF(x,t)}{t^r} &= -\infty && \text{for a.e. } x \in \Omega. \end{aligned}$$

Theorem 31. *Let Assumption (f), (AH0), and (AH) hold. If any of the following conditions is satisfied, (P) has at least one positive solution.*

(i) $\alpha_0 > \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *and (HF+) or (H+).*

(ii) $\alpha_0 < \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *and (HF−) or (H−).*

(iii) $\alpha_0 = \lambda_1(a_0)$, $\alpha < \lambda_1(a_\infty)$, *and (HF0+) or (H0+).*

(iv) $\alpha_0 = \lambda_1(a_0)$, $\alpha > \lambda_1(a_\infty)$, *and (HF0−) or (H0−).*

(v) $\alpha_0 = \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *(HF0+) or (H0+), and (HF+) or (H+).*

(vi) $\alpha_0 = \lambda_1(a_0)$, $\alpha = \lambda_1(a_\infty)$, *(HF0−) or (H0−), and (HF−) or (H−).*

The rest of this section is devoted to the proof of this theorem, which involves some preparatory steps.

The singly resonant case. Set $f_{\pm n}(x,t) := f(x,t) \pm \dfrac{p}{n}|t|^{p-2}t$ and define approximate functionals on $W_0^{1,p}(\Omega)$ by

$$I_{\pm n}(u) := \int_\Omega G(x,\nabla u)\,dx - \int_\Omega (F_{\pm n})_+(x,u)\,dx = I(u) \mp \frac{1}{n}\|u_+\|_p^p.$$

From now on, assume f satisfies Assumption (f). Take first the case $\alpha = \lambda_1(a_\infty)$.

Lemma 32. *If either (H+) or (HF+) (resp. either (H−) or (HF−)) hold and $\{u_n\}$ satisfies*

$$\sup_{n\in\mathbb{N}} I_{\pm n}(u_n) < +\infty \quad \text{and} \quad \lim_{n\to\infty} \|I'_{\pm n}(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0$$

$$\Big(\textit{resp.}\ \inf_{n\in\mathbb{N}} I_{\pm n}(u_n) > -\infty \quad \text{and} \quad \lim_{n\to\infty} \|I'_{\pm n}(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0\Big),$$

then $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. The boundedness of $\|u_n\|_p$ guarantees that $\|u_n\|$ is bounded, since

$$o(1)\|u_n\| = \langle I'_{\pm n}(u_n), u_n\rangle \ge \frac{C_0}{p-1}\|u_n\|^p - C(1+\|u_n\|_p^p) \mp \frac{1}{n}\|(u_n)_+\|_p^p$$

for some $C > 0$ independent of n. So, by way of contradiction, we assume that $\|u_n\|_p \to \infty$ as $n \to \infty$. Then, by the same argument as in Lemma 28, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence strongly converging to a positive solution v_0 of

$$(35) \qquad -\mathrm{div}(a_\infty|\nabla u|^{p-2}\nabla u) = \alpha|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

If $\alpha \neq \lambda_1(a_\infty)$, we have a contradiction, because (35) does not have a positive solution except when $\lambda = \lambda_1(a_\infty)$. So we may assume that $\alpha = \lambda_1(a_\infty)$ and $v_0 = \varphi_{a_\infty}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$. Thus $u_n(x) \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega$ (note $v_0 = \varphi_{a_\infty} > 0$ in Ω).

Assume (HF+) or (HF−). We show that

$$I := \int_\Omega \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{\|u_n\|_p^q}\, dx \to \pm\infty, \tag{36}$$

where the sign on ∞ matches (HF±) and q is a constant as in (HF±). Indeed, it follows from (HF+) that $(f_+(x,t)t - pF_+(x,t))/t^q$ is bounded from below on $\Omega \times [1, +\infty)$. Therefore, since $u_n(x) \to \infty$ for a.e. $x \in \Omega$, we have (36) if (HF+) holds, by applying Fatou's lemma to the inequality

$$I \geq \int_{u_n(x)\geq 1} \frac{f_+(x, u_n)u_n - pF_+(x, u_n)}{u_n^q} v_n^q\, dx - \frac{2H_0}{\|u_n\|_p^p}|\Omega|,$$

where $H_0 > 0$ is a constant as in (HF+). The case of (HF−) is handled by the same argument, with $-f$ instead of f. This shows (36).

Furthermore, by Hölder's inequality, we have

$$\begin{aligned} II &:= \int_\Omega \frac{p\widetilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q}\, dx \\ &\leq H_0 \int_\Omega (|\nabla v_n|^q + \frac{1}{\|u_n\|_p^q})\, dx \leq H_0 \|\nabla v_n\|_p^q |\Omega|^{(p-q)/p} + o(1) \\ &\leq H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1) \end{aligned} \tag{37}$$

in the case of (HF−), because $v_n \to v_0$ in $W_0^{1,p}(\Omega)$, where $q \in [1, p)$ and $H_0 > 0$ are constants as in (HF−). Similarly, we obtain

$$II \geq -H_0 \|\nabla v_0\|_p^q |\Omega|^{(p-q)/p} + o(1) \tag{38}$$

in the case of (HF+).

Hence we have a contradiction because of (36), (37) or (38) by taking the limit inferior or superior in the equality

$$\frac{pI_{\pm n}(u_n) - \langle I'_{\pm n}(u_n), u_n\rangle}{\|u_n\|_p^q} = II + I.$$

Assume (H+) or (H−). Because v_0 is a positive solution of (35), we have $|\nabla u_n(x)| \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega_0 := \{x' \in \Omega : |\nabla v_0(x')| \neq 0\}$. Because $|\Omega_0| > 0$, we can show, by an argument similar to the one used for f, that

$$\int_\Omega \frac{p\widetilde{G}(x, \nabla u_n) - \tilde{a}(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^q}\, dx \to \pm\infty,$$

where again the sign matches that of (H±). In addition, we easily obtain that

$$\pm\int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^q}\,dx \geq -H_0\|v_n\|_q^q + o(1) = -H_0\|v_0\|_q^q + o(1)$$

(again, the sign matches). Hence we have a contradiction by considering the limit of $(pI_{\pm n}(u_n) - \langle I'_{\pm n}(u_n), u_n\rangle)/\|u_n\|_p^q$. □

Proof of Theorem 31(i). Because $\alpha_0 > \lambda_1(a_0)$, there exists an $n_0 \in \mathbb{N}$ such that $\alpha_0 - p/n_0 > \lambda_1(a_0)$. Note that $f_{-n}(x,t)/t^{p-1} \to \alpha_0 - p/n > \lambda_1(a_0)$ as $t \to +0$ for $n \geq n_0$ and $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n = \lambda_1(a_\infty) - p/n < \lambda_1(a_\infty)$ as $t \to +\infty$. Hence, by using the proof of Theorem 25(i) to f_{-n}, we can find a global minimizer u_n of I_{-n} with $I_{-n}(u_n) < 0$ for each $n \geq n_0$. Here we remark that $\sup_{n\geq n_0} I_{-n}(u_n) < 0$. In fact, for every $n \geq n_0$, we have

$$I_{-n}(u_n) \leq I_{-n}(u_{n_0}) = I(u_{n_0}) + \frac{1}{n}\|u_{n_0}\|_p^p \leq I(u_{n_0}) + \frac{1}{n_0}\|u_{n_0}\|_p^p = I_{-n_0}(u_{n_0}) < 0,$$

where, in the first inequality, we use the fact that u_n is a global minimizer of I_{-n}. Now, due to Lemma 32, we see that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore,

$$\|I'(u_n)\|_{W_0^{1,p}(\Omega)^*} = \|I'(u_n) - I'_{-n}(u_n)\|_{W_0^{1,p}(\Omega)^*} \leq \frac{p}{n\lambda_1(-\Delta_p)^p}\|u_n\|^{p-1} \to 0$$

as $n \to \infty$, where $\lambda_1(-\Delta_p)$ is the first eigenvalue of $-\Delta_p$. Since I is bounded on a bounded set, we may assume that $\{u_n\}$ is a bounded Palais–Smale sequence of I. Because I satisfies the bounded Palais–Smale condition (see Proposition 2), u_n has a subsequence converging to some v_0 in $W_0^{1,p}(\Omega)$. It is clear that $I(v_0) \leq \sup_{n\geq n_0} I_{-n}(u_n) = I_{-n_0}(u_{n_0}) < 0$, and so v_0 is a nontrivial critical point of I. □

Proof of Theorem 31(ii). Using Lemma 30 and $\alpha_0 < \lambda_1(a_0)$, we can choose $q_0 \in (p, p^*]$ and $\rho > 0$ such that $\inf\{I(u) : \|u\|_{q_0} = \rho\} > 0$. Since $I_{+n}(u) \geq I(u) - \|u\|_{q_0}^p|\Omega|^{1-p/q_0}/n$ for every $u \in W_0^{1,p}(\Omega)$, we can take $n_0 \in \mathbb{N}$ such that $\alpha_0 + p/n_0 < \lambda_1(a_0)$ and $\delta_0 := \inf\{I_{+n_0}(u) : \|u\|_{q_0} = \rho\} > 0$. Hence, for every $n \geq n_0$, we have $\inf\{I_{+n}(u) : \|u\|_{q_0} = \rho\} \geq \delta_0$, because $I_{+n}(u) \geq I_{+n_0}(u)$ for every $n \geq n_0$ and $u \in W_0^{1,p}(\Omega)$. By noting that $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n > \alpha = \lambda_1(a_\infty)$ as $t \to +\infty$, and applying Lemma 28 to f_{+n} instead of f, I_{+n} satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every $n \geq n_0$, there exists a critical point $u_n \in W_0^{1,p}(\Omega)$ of I_{+n} such that $I_{+n}(u_n) \geq \delta_0$. According to Lemma 32, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, because we have a bounded Palais–Smale sequence of I due to a similar reason as in the case of (i), we can obtain a nontrivial critical point of I (note that $\inf_{n\geq n_0} I(u_n) \geq \inf_{n\geq n_0} I_{+n}(u_n) \geq \delta_0 > 0$). □

We next turn to the case where $\alpha_0 = \lambda_1(a_0)$.

Lemma 33. *Assume* (H0−) *or* (HF0−) (*resp.* (H0+) *or* (HF0+)). *Let* $u_n \neq 0$ *be an element of* $W_0^{1,p}(\Omega)$ *satisfying* $I'_{\pm n}(u_n) = 0$ *for every* $n \in \mathbb{N}$ *and* $\inf_n I_{\pm n}(u_n) \geq 0$ (*resp.* $\sup_n I_{\pm n}(u_n) \leq 0$). *Then* $\liminf_{n\to\infty} \|u_n\|_p > 0$.

Proof. By way of contradiction, we assume that $\lim_{n\to\infty} \|u_n\|_p = 0$ by choosing a subsequence. Note that the boundedness of $\|u_n\|_p$ yields that $\|u_n\|$ and $\|u_n\|/\|u_n\|_p$ are bounded in view of

$$o(1)\|u_n\| = \langle I'_{\pm n}(u_n), u_n\rangle \geq \frac{C_0}{p-1}\|u_n\|^p - C(1 + \|(u_n)_+\|_p^p) \mp \frac{p}{n}\|(u_n)_+\|_p^p \tag{39}$$

for some $C > 0$ independent of n. Then, since u_n is a positive solution of

$$-\mathrm{div}(A(x, \nabla u)) = f_{\pm n}(x, u_n) \quad \text{in } \Omega$$

(refer to Remarks 3 and 27), it follows from Proposition 4 that $u_n \to 0$ in $C^1(\overline{\Omega})$ (note that $|(f_{\pm n})_+(x, t)| \leq Ct_+^{p-1}$ (see Assumption (f)) and $u_n \to 0$ in $L^p(\Omega)$). Therefore, we may assume that $\|u_n\|_{C^1(\overline{\Omega})} \leq 1$ by considering a sufficiently large n. Since $|f_{\pm n}(x, \|u_n\|_p t)/\|u_n\|_p^{p-1}| \leq Ct^p$ for every $t \geq 0$, a.e. $x \in \Omega$ ($C > 0$ independent of n; see Assumption (f) and (39)), by a similar argument to Theorem 13, we see that $v_n := u_n/\|u_n\|_p$ has a subsequence converging to a positive solution v_0 in $C^1(\overline{\Omega})$ of

$$-\mathrm{div}(a_0(x)|\nabla u|^{p-2}\nabla u) = \alpha_0|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{40}$$

If $\alpha_0 \neq \lambda_1(a_0)$, we have a contradiction because (13) does not have a positive solution unless $\lambda = \lambda_1(a_0)$. So we may assume that $\alpha_0 = \lambda_1(a_0)$ and $v_0 = \varphi_{a_0}$ (note $\|v_0\|_p = 1$). For simplicity, we still denote the subsequence under discussion by $\{v_n\}$.

Assume (H0+) or (H0−). Then we can prove that

$$I := \int_\Omega \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^r}\, dx \to \pm\infty \tag{41}$$

(signs match), where $r \in [p, p^*)$ is a constant as in (H0+) or (H0−). Indeed, because $\|\nabla v_0\|_p > 0$, we can choose a constant $\varepsilon_0 > 0$ such that $|\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}| > 0$. With this ε_0, we have under assumption (H0+)

$$\begin{aligned} I &\geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r}|\nabla v_n|^r\, dx - \int_{|\nabla v_n| \leq \varepsilon_0} H_0|\nabla v_n|^r\, dx \\ &\geq \int_{|\nabla v_n| > \varepsilon_0} \frac{p\widetilde{G}_0(x, \nabla u_n) - \tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2}{|\nabla u_n|^r}|\nabla v_n|^r\, dx - \varepsilon_0^r H_0|\Omega|, \end{aligned}$$

where H_0 is a positive constant as in (H0+). Hence, applying Fatou's lemma, our claim is shown, because the Lebesgue measure of $\{x \in \Omega : |\nabla v_0| > 2\varepsilon_0\}$ is positive. Similarly, by considering $\tilde{a}_0(x, |\nabla u_n|)|\nabla u_n|^2 - p\widetilde{G}_0(x, \nabla u_n)$, we can prove (41) under (H0−).

On the other hand, by using (H0+) or (H0−), we obtain

$$\begin{aligned}(42)\qquad \pm II := \pm\int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^r}\,dx &\geq -H_0\int_\Omega (v_n)_+^r\,dx \\ &\geq -H_0\|v_n\|_r^r = -H_0\|v_0\|_r^r + o(1)\end{aligned}$$

(note that $\|u_n\|_{C^1(\overline{\Omega})} \leq 1$ and $v_n \to v_0$ in $C^1(\overline{\Omega})$). Now set $\Psi_n = I_{\pm n}$. Since

$$(43)\qquad \pm(I+II) = \pm\frac{p\Psi_n(u_n) - \langle \Psi_n'(u_n), u_n\rangle}{\|u_n\|_p^r} = \pm\frac{p\Psi_n(u_n)}{\|u_n\|_p^r} \leq 0$$

if $\sup_n(\pm I_\pm(u_n)) \leq 0$ (where the signs match throughout), we obtain a contradiction with (41) and (42) by taking the limit superior or inferior in (43).

Assume (HF0+) or (HF0−). As in the argument for I in the case of (H0±), we can show that

$$\int_\Omega \frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{\|u_n\|_p^r}\,dx = \int_{v_n>0}\frac{f_+(x,u_n)u_n - pF_+(x,u_n)}{(u_n)_+^r}(v_n)_+^r\,dx \to \pm\infty,$$

the sign matching that of (HF0±). Moreover, it is easily seen that

$$\pm\int_\Omega \frac{p\widetilde{G}_0(x,\nabla u_n) - \tilde{a}_0(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^r}\,dx \geq \mp H_0\|\nabla v_n\|_r^r = \mp H_0\|\nabla v_0\|_r^r + o(1).$$

(Note that $\|u_n\|_{C^1(\overline{\Omega})} \leq 1$ and $v_n \to v_0$ in $C^1(\overline{\Omega})$.) Our conclusion follows from a similar argument as before. □

Proof of Theorem 31(iii). Let $n_0 \in \mathbb{N}$ such that $\alpha + p/n_0 < \lambda_1(a_\infty)$. The proof of Theorem 25(i) guarantees that, for every $n \geq n_0$, I_{+n} has a global minimizer u_n such that $I_{+n}(u_n) < 0$, because $f_{+n}(x,t)/t^{p-1} \to \alpha_0 + p/n > \alpha_0 = \lambda_1(a_0)$ as $t \to +0$ and $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n < \lambda_1(a_\infty)$ as $t \to +\infty$ if $n \geq n_0$. Noting that $I_{+n}(u) \geq I_{+n_0}(u)$ for every $u \in W_0^{1,p}(\Omega)$ and $n \geq n_0$, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ since I_{+n_0} is coercive on $W_0^{1,p}(\Omega)$ by Lemma 29. Thus $\{u_n\}$ is a bounded Palais–Smale sequence of I by the same argument as in (i). Therefore, $\{u_n\}$ has a convergent subsequence to some u_0 in $W_0^{1,p}(\Omega)$ because I satisfies the bounded Palais–Smale condition. On the other hand, Lemma 33 guarantees that $u_0 \neq 0$ (note $\sup_{n\geq n_0} I_{+n}(u_n) \leq 0$). Therefore u_0 is a nontrivial critical point of I. □

Proof of Theorem 31(iv). Let $n_0 \in \mathbb{N}$ be such that $\alpha - p/n_0 > \lambda_1(a_\infty)$. Applying Lemma 30 to f_{-n} for $n \geq n_0$ (and since $\alpha_0 - p/n < \lambda_1(a_0)$), we can choose $q_0 \in (p, p^*]$ and $\rho_n > 0$ such that $\delta_n := \inf\{I_{-n}(u) : \|u\|_{q_0} = \rho_n\} > 0$. By noting that $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n > \lambda_1(a_\infty)$ as $t \to +\infty$ for every $n \geq n_0$, and applying Lemma 28 to f_{-n} instead of f, we see that I_{-n} satisfies the Palais–Smale condition. Therefore, the proof of Theorem 25(ii) implies that, for every $n \geq n_0$, there exists a critical point $u_n \in W_0^{1,p}(\Omega)$ of I_{-n} such that $I_{-n}(u_n) \geq \delta_n > 0$. By Lemma 32, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, by arguing as in case (i), we find a subsequence $\{u_n\}$ converging to some u_0 in $W_0^{1,p}(\Omega)$. Also, Lemma 33 yields $u_0 \neq 0$ (note that $\inf_{n\geq n_0} I_{-n}(u_n) \geq 0$). This shows that u_0 is a nontrivial critical point of I. □

The doubly resonant case. Choose smooth nonnegative functions φ and ψ on $[0, +\infty)$ satisfying $\varphi(t) = 1$ if $0 \leq t \leq 2$, $\varphi(t) = 0$ if $t \geq 4$, $\psi(t) = 0$ if $t \leq 5$, and $\psi(t) = 1$ if $t \geq 10$. Define approximate functionals on $W_0^{1,p}(\Omega)$ by

$$\tilde{I}_{\pm n}(u) := I(u) \mp \frac{1}{n}\psi(\|u\|_p^p)\|u_+\|_p^p \pm \frac{1}{n}\varphi(\|u\|_p^p)\|u_+\|_p^p.$$

Because $\tilde{I}_{\pm n}(u) = I_{\mp n}(u)$ provided $\|u\|_p \leq 2$, the following result can be proved by the same argument as in Lemma 33. We omit the proof.

Lemma 34. *Assume* (H0−) *or* (HF0−) (*resp.* (H0+) *or* (HF0+)). *Let* $u_n \neq 0$ *be an element of* $W_0^{1,p}(\Omega)$ *satisfying* $(\tilde{I}_{\pm n})'(u_n) = 0$ *for every* $n \in \mathbb{N}$ *and* $\inf_n \tilde{I}_{\pm n}(u_n) \geq 0$ (*resp.* $\sup_n \tilde{I}_{\pm n}(u_n) \leq 0$). *Then* $\liminf_{n\to\infty} \|u_n\|_p > 0$.

Lemma 35. *If* $\alpha \pm p/n \neq \lambda_1(a_\infty)$, *then* $\tilde{I}_{\pm n}$ (*with the matching sign*) *satisfies the Palais–Smale condition.*

Proof. Let $\{u_m\}$ be a Palais–Smale sequence of $\tilde{I}_{+n}$ or $\tilde{I}_{-n}$. If $\|u_m\|_p \to \infty$ occurs, then $\tilde{I}_{\pm n}(u_m) = I_{\pm n}(u_m)$ for sufficiently large m. So, by applying Lemma 28 to $f_{\pm n}$ (note that $\alpha \pm p/n \neq \lambda_1(a_\infty)$), we have a contradiction if $\|u_m\|_p \to \infty$. Consequently, we see that $\|u_m\|_p$ is bounded. Then, by the same reason as in Lemma 28, $\{u_m\}$ has a convergent subsequence in $W_0^{1,p}(\Omega)$. □

Because $\tilde{I}_{\pm n}(u) = I_{\pm n}(u)$ provided $\|u\|_p \geq 10$, the following result can be proved by the same argument as in Lemma 32. We omit the proof.

Lemma 36. *If either* (H+) *or* (HF+) (*resp. either* (H−) *or* (HF−)) *and* $\{u_n\}$ *satisfies*

$$\sup_{n\in\mathbb{N}} \tilde{I}_{\pm n}(u_n) < +\infty \quad \text{and} \quad \lim_{n\to\infty} \|(\tilde{I}_{\pm n})'(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0$$

$$\Big(\textit{resp.} \ \inf_{n\in\mathbb{N}} \tilde{I}_{\pm n}(u_n) > -\infty \quad \text{and} \quad \lim_{n\to\infty} \|(\tilde{I}_{\pm n})'(u_n)\|_{W_0^{1,p}(\Omega)^*} = 0\Big),$$

$\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof of Theorem 31(v). Note that $\tilde{I}_{-n}(u) = I_{-n}(u)$ provided $\|u\|_p \geq 10$ and $\tilde{I}_{-n}(u) = I_{+n}(u)$ if $\|u\|_p \leq 2$. So, by a similar argument to that in (i), $\tilde{I}_{-n}$ has a global minimizer u_n. Moreover, by a similar argument to that in (iii) (note that $f_{+n}(x,t)/t^{p-1} \to \alpha_0 + p/n > \lambda_1(a_0)$ as $t \to +0$ and $f_{-n}(x,t)/t^{p-1} \to \alpha - p/n < \lambda_1(a_\infty)$ as $t \to +\infty$), we have $\tilde{I}_{-n}(u_n) < 0$, whence $u_n \neq 0$. Because Lemma 36 implies the boundedness of $\|u_n\|$, by the same argument as in (i), we see that $\{u_n\}$

is a bounded Palais–Smale sequence of I. Therefore, we may assume that u_n converges to some u_0 in $W_0^{1,p}(\Omega)$ by choosing a subsequence. On the other hand, Lemma 33 yields $\liminf_{n\to\infty} \|u_n\|_p > 0$. Hence $u_0 \neq 0$. This means that u_0 is a nontrivial critical point of I. □

Proof if Theorem 31(vi). Note that $\tilde{I}_{+n}(u) = I_{+n}(u)$ provided $\|u\|_p \geq 10$ and $\tilde{I}_{+n}(u) = I_{-n}(u)$ if $\|u\|_p \leq 2$. So, because $f_{-n}(x,t)/t^{p-1} \to \alpha_0 - p/n < \lambda_1(a_0)$ as $t \to +0$ and $f_{+n}(x,t)/t^{p-1} \to \alpha + p/n > \lambda_1(a_\infty)$ as $t \to +\infty$, by a similar argument to those in (ii) and (iv), for each n, we have a nontrivial critical point u_n of $\tilde{I}_{+n}$ with $\tilde{I}_{+n}(u_n) > 0$. As a result, by a similar reasoning as in (v), we can obtain a nontrivial critical point of I. □

Acknowledgements

The second author would like to express her sincere thanks to Professor Shizuo Miyajima for helpful comments and encouragement.

References

[Anane 1987] A. Anane, *Etude des valeurs propres et de la résonnance pour l'opérateur p-Laplacien*, Ph.D. thesis, Université Libre de Bruxelles, 1987.

[Cuesta et al. 1999] M. Cuesta, D. de Figueiredo, and J.-P. Gossez, "The beginning of the Fučik spectrum for the p-Laplacian", *J. Differential Equations* **159**:1 (1999), 212–238. MR 2001f:35308 Zbl 0947.35068

[Damascelli 1998] L. Damascelli, "Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15**:4 (1998), 493–516. MR 99e:35081 Zbl 0911.35009

[Dancer and Perera 2001] N. Dancer and K. Perera, "Some remarks on the Fučik spectrum of the p-Laplacian and critical groups", *J. Math. Anal. Appl.* **254**:1 (2001), 164–177. MR 2001k:35238 Zbl 0970.35056

[Deimling 1985] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985. MR 86j:47001 Zbl 0559.47040

[Fukagai and Narukawa 2007] N. Fukagai and K. Narukawa, "On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems", *Ann. Mat. Pura Appl.* (4) **186**:3 (2007), 539–564. MR 2008d:35052 Zbl 1223.35132

[García-Huidobro et al. 1995] M. García-Huidobro, R. Manásevich, and P. Ubilla, "Existence of positive solutions for some Dirichlet problems with an asymptotically homogeneous operator", *Electron. J. Differential Equations* **1995**:10 (1995), 1–22. MR 96f:35053 Zbl 0823.35057

[Kim 2009] Y.-H. Kim, "A global bifurcation for nonlinear equations with nonhomogeneous part", *Nonlinear Anal.* **71**:12 (2009), 738–743. MR 2011d:35048 Zbl 1238.35009

[Kim and Kim 2010] I.-S. Kim and Y.-H. Kim, "Global bifurcation for equations involving nonhomogeneous operators in an unbounded domain", *Nonlinear Anal.* **73**:4 (2010), 1057–1064. MR 2011e:35023 Zbl 1194.35492

[Kyritsi et al. 2010] S. T. Kyritsi, D. O'Regan, and N. S. Papageorgiou, "Existence of multiple solutions for nonlinear Dirichlet problems with a nonhomogeneous differential operator", *Adv. Nonlinear Stud.* **10**:3 (2010), 631–657. MR 2011e:35116 Zbl 1216.35037

[Lieberman 1988] G. M. Lieberman, "Boundary regularity for solutions of degenerate elliptic equations", *Nonlinear Anal.* **12**:11 (1988), 1203–1219. MR 90a:35098 Zbl 0675.35042

[Mawhin and Willem 1989] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences **74**, Springer, New York, 1989. MR 90e:58016 Zbl 0676.58017

[Miyajima et al. 2012] S. Miyajima, D. Motreanu, and M. Tanaka, "Multiple existence results of solutions for the Neumann problems via super- and sub-solutions", *J. Funct. Anal.* **262**:4 (2012), 1921–1953. MR 2873865 Zbl pre06012152

[Motreanu and Papageorgiou 2011] D. Motreanu and N. S. Papageorgiou, "Multiple solutions for nonlinear Neumann problems driven by a nonhomogeneous differential operator", *Proc. Amer. Math. Soc.* **139**:10 (2011), 3527–3535. MR 2012c:35166 Zbl 1226.35021

[Motreanu et al. 2011] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, "Multiple constant sign and nodal solutions for nonlinear Neumann eigenvalue problems", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **10**:3 (2011), 729–755. MR 2012m:35087 Zbl 1234.35169

[Prado and Ubilla 1998] H. Prado and P. Ubilla, "Existence of nonnegative solutions for generalized p-Laplacians", pp. 289–298 in *Reaction diffusion systems* (Trieste, 1995), edited by G. Caristi and E. Mitidieri, Lecture Notes in Pure and Appl. Math. **194**, Dekker, New York, 1998. MR 98j:35069 Zbl 0913.35044

[Robinson 2004] S. B. Robinson, "On the second eigenvalue for nonhomogeneous quasi-linear operators", *SIAM J. Math. Anal.* **35**:5 (2004), 1241–1249. MR 2005a:35219 Zbl 1061.35071

[Tanaka 2009] M. Tanaka, "Existence of constant sign solutions for the p-Laplacian problems in the resonant case with respect to Fučik spectrum", *SUT J. Math.* **45**:2 (2009), 149–166. MR 2011d:35207 Zbl 1196.35115

[Tanaka 2012a] M. Tanaka, "The antimaximum principle and the existence of a solution for the generalized p-Laplace equations with indefinite weight", *Differ. Equ. Appl.* **4**:4 (2012), 581–613.

[Tanaka 2012b] M. Tanaka, "Existence of the Fučik type spectrums for the generalized p-Laplace operators", *Nonlinear Anal.* **75**:7 (2012), 3407–3435. MR 2891177 Zbl 1241.35091

Received June 19, 2012.

DUMITRU MOTREANU
DÉPARTMENT DE MATHÉMATIQUES
UNIVERSITÉ DE PERPIGNAN
52 AVENUE PAUL ALDUY
66860 PERPIGNAN
FRANCE
motreanu@univ-perp.fr

MIEKO TANAKA
DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF SCIENCE
KAGURAZAKA 1-3
SHINJYUKU-KU
TOKYO 162-8601
JAPAN
tanaka@ma.kagu.tus.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US $400/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 265 No. 1 September 2013
