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We use malleable deformations combined with spectral gap rigidity theory, in the framework of Popa's deformation/rigidity theory, to prove unique tensor product decomposition results for II_1 factors arising as tensor products of wreath product factors. We also obtain a similar result regarding measure equivalence decomposition of direct products of such groups.

Introduction

A major goal of the study of II_1 factors is the classification of these algebras based on the "input data" that goes into their construction. For example, given a countable discrete group Γ, one can construct the associated group von Neumann algebra $L(\Gamma)$. It is then natural to determine the properties/isomorphism class of the algebra based on those of the group. A significant landmark was the result, due to Connes [1976], that all group von Neumann algebras, $L(\Gamma)$, with Γ amenable i.c.c., are isomorphic. However, in the nonamenable realm there is a much greater variety, and a striking classification theory has developed.

One goal of this research is to determine if some algebra, which is constructed in one manner, can be obtained in some other manner. For example, if we have a II_1 factor that we know to be a free product of two II_1 factors, is it also possible for it to be the tensor product of two (possibly different) II_1 factors? Over the last decade many examples of such so-called "W*-rigidity" phenomena have been discovered and, in particular, wreath products, or their ergodic theory counterparts, Bernoulli shifts, have played a prominent role. In particular, they have led to the first examples [Ioana et al. 2013; Berbec and Vaes 2012] of W*-superrigid groups (i.e., groups Γ for which, for any Λ, isomorphism of $L(\Gamma)$ and $L(\Lambda)$ implies isomorphism of Γ and Λ). For a more detailed overview of the theory we refer the reader to [Popa 2007; Vaes 2011].

Here we study whether certain factors can be written as tensor products in distinct ways. Recall that a II_1 factor is prime if it is not the tensor product of two other II_1 factors. The first example of a prime II_1 factor was obtained by Popa [1983], who showed that the group von Neumann algebra of an uncountable free group is prime.

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Later, Ge [1998] used techniques from Voiculescu's free probability theory, in particular the tools of free entropy which were defined and developed in [Voiculescu 1993; 1994; 1996], to prove that all group factors coming from finitely generated free groups are prime. Note that Ge's result, unlike Popa's result mentioned above, gave the first example of a prime II_1 factor that is separable. Using C^*-algebraic techniques, this was greatly generalized in [Ozawa 2004] to show that all i.c.c. Gromov hyperbolic groups give rise to prime II_1 factors. Also, using his deformation/rigidity theory, Popa [2008] showed that all II_1 factors arising from the Bernoulli actions of nonamenable groups are prime. Further, Peterson [2009] used his derivation approach to deformation/rigidity to prove that any II_1 factor coming from a countable group with positive first l^2-Betti number is also prime. Finally we should also note that, using Popa's deformation/rigidity theory, Chifan and Houdayer [2010] gave many more examples of prime II_1 factors coming from amalgamated free products.

A natural question about prime factors is whether a tensor product of a finite number of such factors $P_1, P_2, \dots, P_n$ has a "unique prime factor decomposition"; i.e., if $P_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} P_n = Q_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q_m$, for some $m \geq n$ and some other prime factors Q_j, forces $n = m$ and P_i unitary conjugate to Q_i, modulo some permutation of indices and modulo some "rescaling" by appropriate amplifications of the prime factors involved. The first such result was obtained in [Ozawa and Popa 2004], where a combination of C^*-algebraic techniques from [Ozawa 2004] and intertwining techniques from [Popa 2006c] is used to show that any II_1 factor arising from a tensor product of II_1 factors of the form $L(\Gamma)$ with Γ hyperbolic, or more generally in Ozawa's class $\mathcal{S}$, has such a unique tensor product decomposition.

In this paper we prove an analogous unique prime factor decomposition result for tensor products of group von Neumann algebras coming from wreath product groups. More precisely, let us denote by $\mathcal{WR}_{\mathrm{NA}}$ the class of "amenable by nonamenable" wreath product groups, by which we means groups of the form $A \wr H$ where A is a nontrivial countable amenable group and H is a countable nonamenable group. Then we prove the following result:

Theorem 0.1. *Let $\Gamma_1 \dots, \Gamma_n \in \mathcal{WR}_{\mathrm{NA}}$ and $Q_1, \dots, Q_m$ be diffuse von Neumann algebras such that*

$$M = L(\Gamma_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} L(\Gamma_n) = Q_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q_k.$$

If $m \geq n$, then $n = m$, and after permutation of indices we have that $L(\Gamma_1) \simeq Q_i^{t_i}$ for some positive numbers $t_1, t_2, \dots t_n$ whose product is 1.

We can view this as a "W*-rigidity" theorem in that it gives us large families of nonisomorphic II_1 factors. In particular, picking a specific amenable group, $\mathbb{Z}$, and a specific nonamenable group, $\mathbb{F}_n$, the free group on n generators, we get the new result that

$$L(\mathbb{Z}\wr\mathbb{F}_n)\mathbin{\overline{\otimes}}L(\mathbb{Z}\wr\mathbb{F}_n)\mathbin{\overline{\otimes}}L(\mathbb{Z}\wr\mathbb{F}_n)\not\simeq L(\mathbb{Z}\wr\mathbb{F}_n)\mathbin{\overline{\otimes}}L(\mathbb{Z}\wr\mathbb{F}_n).$$

Of course, the above theorem provides us with many such examples of rigidity phenomena.

Also we have a natural generalization of this theorem to unique measure-equivalence decomposition results of finite products of groups in the class $\mathscr{W}\mathscr{R}_{\mathrm{NA}}$. Such results were achieved for products of groups of the class $\mathscr{C}_{\mathrm{reg}}$ in [Monod and Shalom 2006, Theorem 1.16] and for products of biexact groups in [Sako 2009, Theorem 4] and, independently, in [Chifan and Sinclair 2010, Corollary C]. We refer the reader to the last section for the definition of measure equivalence for groups.

Before stating our second main result we would like to point out that Sako [2009, Theorem 7] has obtained measure equivalence rigidity results for certain classes of wreath products; however, his results were not of this type and used techniques different from the ones that we will employ.

Theorem 0.2. *Let $\Gamma_1,\dots,\Gamma_n,\Lambda_1,\dots,\Lambda_m\in\mathscr{W}\mathscr{R}_{\mathrm{NA}}$ be such that $\Gamma_1\times\cdots\times\Gamma_n$ is measure equivalent to $\Lambda_1\times\cdots\times\Lambda_m$. We denote this by*

$$\Gamma_1\times\cdots\times\Gamma_n\simeq_{\mathrm{ME}}\Lambda_1\times\cdots\times\Lambda_m.$$

If $m\geq n$, then $n=m$, and after permutation of indices we have that $\Gamma_i\simeq_{\mathrm{ME}}\Lambda_i$, Γ_i is measure equivalent to Λ_i.

We prove these results by using deformation/rigidity theory. More precisely, we use the malleable deformation for wreath product group factors in [Chifan et al. 2012], combined with Popa's spectral gap rigidity and intertwining by bimodules techniques.

1. Preliminaries

Intertwining by bimodules. Let us recall Popa's intertwining by bimodules technique. This is a crucial tool for locating subalgebras of II_1 factors, and is summed up in the following theorem:

Theorem 1.1 [Popa 2006c]. *Let (M,τ) be a finite von Neumann algebra with trace τ, and let $P,Q\subset M$ be von Neumann subalgebras. Then the following are equivalent*:

(1) *There exist nonzero projections $p\in P$, $q\in Q$, a nonzero partial isometry $v\in M$, and a *-homomorphism $\varphi: pPp\to qQq$ such that $vx=\varphi(x)v$ for all $x\in pPp$.*

(2) *There is a sub-P-Q-bimodule $\mathscr{H} \subset L^2(M)$ that is finitely generated as a right Q-module.*

(3) *There is no sequence $u_n \in \mathscr{U}(P)$ such that*

$$\lim_{n\to\infty} \|E_Q(xu_ny)\|_2 \to 0 \quad \textit{for all } x, y \in M.$$

If any of the above conditions hold, we say that *a corner of P embeds in Q inside M*, denoted by $P \prec_M Q$.

Following [Ozawa and Popa 2010] we have the following definition:

Definition 1.2. Let (M, τ) be a finite von Neumann algebra with trace τ, and let $P, Q \subset M$ be von Neumann subalgebras. We say that P *is amenable over Q inside M*, which we denote by $P \lessdot_M Q$, if there is a P-central state, φ, on $\langle M, e_Q\rangle$ such that $\varphi|_M = \tau$, where τ is the trace on M.

Let us note that, by [Ozawa and Popa 2010, Theorem 2.1], $P \lessdot_M Q$ is equivalent to $L^2(P)$ being weakly contained in $\bigoplus L^2(\langle M, e_Q\rangle)$ as P-bimodules. Further, if $P \prec_M Q$ then $L^2(M)$ contains a sub-P-Q-module, $\mathscr{H}$, that is finitely generated as a right Q-module. Therefore, the projection onto this module will commute with the right action of Q and will have finite trace. Therefore, it will be a vector in $L^2(\langle M, e_Q\rangle)$. Further, it will also commute with P, so, if we look at $L^2(\langle M, e_Q\rangle)$ as a P-bimodule, it will contain a central vector. Since strong containment implies weak containment we get the following observation.

Proposition 1.3. *Let (M, τ) be a finite von Neumann algebra with trace τ, and let $P, Q \subset M$ be von Neumann subalgebras. If $P \prec_M Q$ then $P \lessdot_M Q$.*

Deformation of wreath products. Let A and H be countable discrete groups. Now let us consider the infinite direct sum, $\bigoplus_H A$, indexed by H. Now notice that H acts on $\bigoplus_H A$ by acting on the index set on the left. The resulting semidirect product group $\bigoplus_H A \rtimes H = A \wr H$ is known as the wreath product. Throughout this paper we will consider trace preserving actions of $A \wr H$ on a finite von Neumann algebra N with trace τ, and we consider the resulting crossed product algebra $M = N \rtimes A \wr H$.

Now let us describe the construction of a deformation for von Neumann algebras coming from wreath products as above. This is the same deformation that the first author used in [Chifan et al. 2012], and is inspired by similar free malleable deformations in [Popa 2006b; Ioana et al. 2008; Ioana 2007]. We refer to this previous work for additional discussion.

Let $\tilde{A} = A * \mathbb{Z}$. If we let $u \in L(\tilde{A})$ denote the Haar unitary that generates $L(\mathbb{Z})$ then we can find a selfadjoint element $h \in L(\mathbb{Z})$ such that $u = \exp(ih)$. Thus, for every $t \in \mathbb{R}$, we define $u^t \doteq \exp(ith) \in L\mathbb{Z}$. This allows us to define $\theta_t \in \mathrm{Aut}(L(\tilde{A}))$ by $\theta_t(x) = u^t x (u^*)^t$. By applying this automorphism in each coordinate we can get an automorphism of $L(\tilde{A}^H)$. Since the action of H is by permuting the coordinates, it commutes with θ_t and so we can extend it to $L(\tilde{A} \wr H)$. If we now declare that the Haar unitaries in each coordinate do not act on the algebra N, then we can extend to an automorphism, which we still denote by θ_t, of $\tilde{M} = N \rtimes \tilde{A} \wr H$.

It is easy to see that $\lim_{t\to 0} \|u^t - 1\|_2 = 0$, and hence $\lim_{t\to 0} \|\theta_t(x) - x\|_2 = 0$ for all $x \in \tilde{M}$. Therefore, the path $(\theta_t)_{t\in\mathbb{R}}$ is a deformation by automorphisms of $\tilde{M}$.

Next we show that θ_t admits a "symmetry"; i.e., there exists an automorphism β of $\tilde{M}$ satisfying the following relations:

$$\beta^2 = \mathrm{id}, \quad \beta_{|_M} = \mathrm{id}_{|_M}, \quad \beta\theta_t\beta = \theta_{-t} \text{ for all } t \in \mathbb{R}.$$

To see this, we first define $\beta_{|_{LA^H}} = \mathrm{id}_{|_{LA^H}}$ and then for every $h \in H$ we let $(u)_h$ be the element in $L\tilde{A}^H$ whose h-th entry is u and whose other entries are 1. On elements of this form we define $\beta((u)_h) = (u^*)_h$, and, since β commutes with the actions of H on A^H, it extends to an automorphism of $L(\tilde{A} \wr H)$ by acting identically on $L(H)$. Finally, the automorphism β extends to an automorphism of $\tilde{M}$, still denoted by β, which acts trivially on A.

Let us note that, with this choice of β, θ_t is an *s-malleable deformation* of $\tilde{M}$ in the sense of [Popa 2006c].

2. Intertwining techniques for wreath products

In this section we prove the necessary intertwining results for II_1 factors arising from wreath product groups that we will need in order to prove our desired uniqueness of tensor product decomposition.

The following proposition is a relative version of [Chifan et al. 2012, Lemma 4.2], and will follow a similar proof.

Proposition 2.1. *Let N be a finite von Neumann algebra. Let A, H be groups with A nontrivial amenable and H nonamenable. Let $Q \subset N \rtimes A \wr H = M$ be an inclusion of von Neumann algebras. Assume Q is not amenable over N inside M; then $Q' \cap \tilde{M}^\omega \subseteq M^\omega$.*

Proof. As mentioned above this proof follows closely the proof of [Chifan et al. 2012, Lemma 4.2] as well as [Popa 2008, Lemma 5.1] and other similar results in the literature.

We will prove the contrapositive so let us assume that $Q' \cap \tilde{M}^\omega \nsubseteq M^\omega$. Then, proceeding as in [Popa 2008, Lemma 5.1], we see that

$$L^2(Q) \prec L^2(\tilde{M}) \ominus L^2(M)$$

as Q-bimodules. Now we decompose $L^2(\tilde{M}) \ominus L^2(M)$ as an M-bimodule.

One can see that the above M-bimodule can be written as a direct sum of M-bimodules $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$, where the cyclic vectors $\tilde{\eta}_s$ correspond to an enumeration of all elements of $\tilde{A}^H$ whose nontrivial coordinates start and end with nonzero powers of u.

Next, for every s, we denote by η_s the element of A^H that remains from $\tilde{\eta}_s$ after deleting all nontrivial powers of u. Also for every s let $\Delta_s \subset H$ be the support of $\tilde{\eta}_s$ and observe that if $\mathrm{Stab}_H(\tilde{\eta}_s)$ denotes the stabilizing group of $\tilde{\eta}_s$ inside H then we have $\mathrm{Stab}_H(\tilde{\eta}_s)(H \setminus \Delta_s) \subset H \setminus \Delta_s$.

Hence we can consider the von Neumann algebra

$$K_s = N \rtimes (A \wr_{H \setminus \Delta_s} \mathrm{Stab}_H(\tilde{\eta}_s))$$

and, using similar computations to those in [Popa 2008, Lemma 5.1], one can easily check that the map $x\tilde{\eta}_s y \to x\eta_s e_{K_s} y$ implements an M-bimodule isomorphism between $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$ and $L^2(\langle M, e_{K_s}\rangle)$.

Therefore, as M-bimodules, we have the isomorphism

$$L^2(\tilde{M}) \ominus L^2(M) = \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Thus we can get the weak containment of Q-bimodules

$$L^2(Q) \prec \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Notice that, since Δ_s is finite, and the action of H on itself is free, then $\mathrm{Stab}_H(\tilde{\eta}_s)$ is finite for all s. Also, since A is an amenable group we have that $K_s \lessdot_N N$ for all s. Thus for all s we have the weak containment of K_s-bimodules

$$L^2(K_s) \prec \bigoplus L^2(\langle K_s, e_N\rangle) \simeq \bigoplus L^2(K_s) \otimes_N L^2(K_s).$$

Now if we induce to M-bimodules and restrict to Q-bimodules and use continuity of weak containment under induction and restriction we get the inclusions of Q-bimodules

$$\begin{aligned} L^2(Q) &\prec \bigoplus L^2(\langle M, e_{K_s}\rangle) \simeq \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_{K_s} L^2(M) \\ &\prec \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_N L^2(K_s) \otimes_{K_s} L^2(M) \\ &\simeq \bigoplus L^2(M) \otimes_N L^2(M) \simeq \bigoplus L^2(\langle M, e_N\rangle). \end{aligned}$$

Thus $Q \lessdot_M N$. □

To state the next result let us recall the following standard definition.

Definition 2.2. Given an inclusion of von Neumann algebras $P \subset M$ the *normalizer of P inside M* is the set

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : uPu^* = P\}.$$

We say that, for such an inclusion, P is a *regular subalgebra* if $\mathcal{N}_M(P)'' = M$.

We finish this section with a theorem that allows us to locate regular subfactors with large commutant.

Theorem 2.3. *Let N be a finite von Neumann algebra. Let A and H be groups with A nontrivial amenable and H nonamenable. Let $Q \subset N \rtimes A \wr H = M$ be a von Neumann subalgebra that is not amenable over N. Let $P = Q' \cap M$. If P is a regular subfactor of M then $P \prec_M N$.*

Proof. Applying Proposition 2.1 and following the proof of [Chifan et al. 2012, Theorem 4.1] we see that the deformation θ_t converges uniformly on the unit ball of P, and thus by [Chifan et al. 2012, Theorem 3.1] we have that $P \prec_M N \rtimes A^H$ or $P \prec_M N \rtimes H$.

Following the same argument as [Chifan et al. 2012, Theorem 4.1], if we assume that $P \prec_M N \rtimes A^H$ and $P \not\prec_M N$, then we get $Q \prec_M N \rtimes A \wr H_0$ for some finite subgroup $H_0 \subset H$. Since A is amenable and H_0 is finite then $N \rtimes A \wr H_0 \lessdot_M N$. So, since $Q \prec_M N \rtimes A \wr H_0$, then, by Proposition 1.3, we have $Q \lessdot_M N \rtimes A \wr H_0$. Then by [Ozawa and Popa 2010, part 3 of Proposition 2.4] we have that $Q \lessdot_M N$, contradicting our assumption.

Thus $P \prec_M N \rtimes H$. Therefore, by Theorem 1.1, there exist nonzero projections $p \in P$, $q \in N \rtimes H$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \to q(N \rtimes H)q$ such that $vx = \varphi(x)v$ for all $x \in pPp$. Furthermore we have that $v^*v = p$ and $vv^* = \hat{q} \in \varphi(pPp)' \cap qMq$. Also, by [Popa 2006c, Lemma 3.5] we know that pPp is a regular subalgebra of pMp.

Then for all $u \in \mathcal{N}_{pMp}(pPp)$ let us calculate

$$\begin{aligned}\varphi(x)vuv^* &= vxuv^* = vu(u^*xu)v^* = vuv^*v(u^*xu)v^* \\ &= vuv^*\varphi(u^*xu)vv^* = vuv^*\varphi(u^*xu).\end{aligned}$$

Now assume that $P \not\prec_M N$; then by [Chifan et al. 2012, part (3) of Lemma 2.4] we have that $vuv^* \in N \rtimes H$. Since pPp is regular in pMp we would then get that $M \prec_M N \rtimes H$. However, this is impossible since the fact that A is nontrivial implies that $[M : N \rtimes H] = \infty$. □

3. Proof of main theorems

In this section we prove our main theorems. Our main technical tool is the following, which is [Popa and Vaes 2011, Proposition 2.7]. Before we state the result let us recall that two von Neumann subalgebras $M_1, M_2 \subset M$ of a finite von Neumann algebra M are said to form a commuting square if $E_{M_1}E_{M_2} = E_{M_2}E_{M_1}$, where E_{M_i} denotes the unique trace-preserving conditional expectation from M onto M_i.

Theorem 3.1 [Popa and Vaes 2011]. *Let (M, τ) be a tracial von Neumann algebra with von Neumann subalgebras $M_1, M_2 \subset M$. Assume that M_1 and M_2 form a commuting square and that M_1 is regular in M. If a von Neumann subalgebra $Q \subset pMp$ is amenable relative to both M_1 and M_2, then Q is amenable relative to $M_1 \cap M_2$.*

Notice that this theorem allows us to eliminate the case where Q is amenable over M_1. More specifically we have the following observation.

Proposition 3.2. *Let G_1 and G_2 be groups. Let A be a finite amenable von Neumann algebra with an action of $G_1 \times G_2$, and let $Q \subset A \rtimes G_1 \times G_2$ be a nonamenable subalgebra. Then there exists an i such that Q is not amenable over $A \rtimes G_i$.*

Proof. If we let $A \rtimes G_i = M_i$ then it is easy to see that M_1, $M_2 \subset M$ form a commuting square. So if Q is amenable over both M_i we would have that it would be amenable over the intersection, which is A. Since A is amenable this would imply that Q is amenable. □

Finally combining the above results we can prove Theorem 0.1.

Proof. First let us mention that, for the case $n = 1$, this is equivalent to the primeness of II_1 factors arising from Bernoulli shifts, which was proven in [Popa 2008].

Now, since we have that $\Gamma_i \in \mathcal{WR}_{\mathrm{NA}}$, there is a nontrivial amenable group A_i and a nonamenable group H_i such that $\Gamma_i = A_i \wr H_i$. Let us note, since the A_i are nontrivial and H_i are infinite, that $L(A_i \wr H_i)$ and $L(A_1 \wr H_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} L(A_{i-1} \wr H_{i-1})$ are II_1 factors. Thus we must also have that $Q_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q_m$ is as well and thus each Q_i is a II_1 factor.

Now notice that we can write M as $M = N_i \rtimes_\sigma A_i \wr H_i$, where $N_i = L(A_1 \wr H_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} L(A_{i-1} \wr H_{i-1}) \mathbin{\overline{\otimes}} L(A_{i+1} \wr H_{i+1}) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} L(A_n \wr H_n)$ and σ is the trivial action. Therefore, since we can view M as a crossed product by a wreath product group, we can use the above intertwining statements to determine the location of algebras which are not amenable over N_i for some i.

In order to proceed in this manner, let us define

$$\widehat{Q_i} = (Q_i)' \cap M = Q_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q_{i-1} \mathbin{\overline{\otimes}} Q_{i+1} \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} Q_k.$$

Since each H_i is nonamenable this implies, in particular, that there is a j such that Q_j is nonamenable. Moreover, by Proposition 3.2, where we let $A = \mathbb{C}$, we know that there is an i such that Q_j is not amenable over N_i. With this information we can then apply our results above to finish the proof.

Specifically, since $\widehat{Q_j}$ is a regular subalgebra of M, then by Theorem 2.3 we get that $\widehat{Q_j} \prec_M N$.

We complete the argument by following Proposition 12 and the induction argument of the proof of Theorem 1 in [Ozawa and Popa 2004]. □

Before we prove our final theorem let us recall the following definition:

Definition 3.3. We say that two group Γ and Λ are *measure equivalent*, $\Gamma \simeq_{\mathrm{ME}} \Lambda$, if there is a diffuse abelian von Neumann algebra, A, and free ergodic trace preserving actions, σ, ρ, of Γ and Λ, respectively, such that $A \rtimes_\sigma \Gamma \simeq (A \rtimes_\rho \Lambda)^t$, and the isomorphism takes A onto A^t.

With this definition we can now prove our final result (Theorem 0.2).

Proof. Our argument here follows closely a similar argument in the proof of [Chifan and Sinclair 2010, Corollary C]. For this reason we sketch the proof here but refer the reader to the cited work for any remaining details. Let $\Gamma_1, \dots, \Gamma_n, \Lambda_1, \dots, \Lambda_m \in \mathcal{WR}_{\text{NA}}$. Then there are nontrivial amenable groups A_i and B_j as well as nonamenable groups H_i and G_j such that $\Gamma_i = A_i \wr H_i$ and $\Lambda_j = B_j \wr G_j$. Note, for all i and j, Γ_i and Λ_j are nonamenable.

Now we know that there are actions on $\Gamma = A_1 \wr H_1 \times \cdots \times A_n \wr H_n \curvearrowright L^\infty(X)$ and $\Lambda = K_1 \times \cdots \times K_m \curvearrowright L^\infty(Y)$ such that $M_1 = L^\infty(X) \rtimes \Gamma$ is isomorphic to $M_2 = (L^\infty(Y) \rtimes \Lambda)^t$ via an isomorphism $\phi : M_1 \to M_2$ such that $\phi(L^\infty(X)) = (L^\infty(Y))^t$. Note that the intertwining statements which we will use remain true under amplifications; thus we may assume that $t = 1$.

Following [ibid.] we fix the following notation. Given a subset $F \subset \{1, \dots, n\}$, we denote by $\hat{\Gamma}_F$ the subgroup of $\Gamma = A_1 \wr H_1 \times \cdots \times A_n \wr H_n$ which consists of all elements with trivial i-th coordinate, for all $i \in F$, and similarly for Λ. Also for any subset $F \subset \{1, \dots, n\}$ and $K \subset \{1, \dots, m\}$ we define $\hat{M}_{1,F} = L^\infty(X) \rtimes \hat{\Gamma}_F$ and $\hat{M}_{2,K} = L^\infty(Y) \rtimes \hat{\Lambda}_K$.

As in [ibid.] we will show that for any subset $F \subset \{1, \dots, n\}$ there is a subset $K \subset \{1, \dots, m\}$ with $|F| = |K|$ such that

$$\phi(L(\hat{\Gamma}_F)) \prec \hat{M}_{2,k}. \tag{1}$$

We will prove this via induction on $|F|$. For $|F| = 1$ we are considering $L(\hat{\Gamma}_i)$. As in the proof of the previous theorem, since the $\phi(L(\Gamma_i))$ are nonamenable, there is a j such that $\phi(L(\Gamma_i))$ is nonamenable over $\hat{M}_{2,\{j\}}$. Now by the proof of Theorem 2.3 this implies that $\phi(L(\Gamma_i))' \cap M_2 \prec \hat{M}_{2,\{j\}} \rtimes G_j$, and, since we have that $\phi(L(\hat{\Gamma}_i)) \subset \phi(L(\Gamma_i))' \cap M_2$, we get that $\phi(L(\hat{\Gamma}_i)) \prec \hat{M}_{2,\{j\}} \rtimes G_j$.

Thus by [Chifan et al. 2012, Lemma 2.2] we have that $\phi(L^\infty(X) \rtimes \hat{\Gamma}_i) \prec \hat{M}_{2,\{j\}} \rtimes G_j$. Now since $\phi(L^\infty(X) \rtimes \hat{\Gamma}_i)$ is a regular subalgebra we have by Theorem 2.3 that $\phi(L^\infty(X) \rtimes \hat{\Gamma}_i) \prec \hat{M}_{2,\{j\}}$. This proves the base case and the inductive case follows exactly as in [Chifan and Sinclair 2010].

Again following [ibid.] we can apply the same logic to ϕ^{-1} to get that for each $i \in F$ there is a $\rho(i) \in K$ such that $\phi(L^\infty(X) \rtimes \hat{\Gamma}_i) \prec \hat{M}_{2,\{j\}}$ and for each $\rho(i) \in K$ there is a $\pi(\rho(i)) \in F$ with

$$\phi(L^\infty(X) \rtimes \Gamma_i) \prec L^\infty(Y) \rtimes \Lambda_{\rho(i)}$$

and

$$\phi^{-1}(L^\infty(y) \rtimes \Lambda_{\rho(i)}) \prec L^\infty(X) \rtimes \Gamma_{\pi(\rho(i))}.$$

Thus we have

$$\phi(L^\infty(X) \rtimes \Gamma_i) \prec L^\infty(X) \rtimes \Gamma_{\pi(\rho(i))},$$

and so we have that π and ρ are permutations. Thus using [Ioana et al. 2008, Proposition 8.4] we get unitaries u_i such that

$$u_i\phi(L^\infty(X) \rtimes \Gamma_i)u_i^* = L^\infty(Y) \rtimes \Lambda_{\rho(i)}. \tag{2}$$

This further gives that $\phi_{u_i} = \mathrm{Ad}(u_i) \circ \phi$ is an isomorphism from $L^\infty(X) \rtimes \Gamma_i$ onto $L^\infty(Y) \rtimes \Lambda_{\rho(i)}$ which satisfies

$$\phi_{u_i}(a)u_i = u_i\phi(a)$$

for all $a \in L^\infty(X)$.

Now we would like to finish the proof by showing that we can map the Cartan subalgebras onto each other. Toward this goal let us consider $L^\infty(Y) \rtimes (\Lambda_{\rho(i)} \times \hat{\Lambda}_{\rho(i)}) = (L^\infty(Y) \rtimes \Lambda_{\rho(i)}) \rtimes \hat{\Lambda}_{\rho(i)}$. Then we can consider the Fourier decomposition $u = \sum_{\lambda \in \hat{\Lambda}_{\rho(i)}} x_\lambda v_\lambda$ with $x_\lambda \in L^\infty(Y) \rtimes \Lambda_{\rho(i)}$ and, using the above equation, there exists a nonzero element $x_\lambda \in L^\infty(Y) \rtimes \Lambda_{\rho(i)}$ such that for all $a \in L^\infty(X)$ we have

$$\phi_{u_i}(a)x_\lambda = x_\lambda\sigma_\lambda(\phi(a)),$$

where σ_λ represents the actions of v_λ on $L^\infty(Y) \rtimes \Lambda_{\rho(i)}$.

Now we can take the polar decomposition of x_λ to get a partial isometry w_λ such that

$$\phi_{u_i}(a)w_\lambda = w_\lambda\sigma_\lambda(\phi(a)). \tag{3}$$

Notice that the left side of the above equation is $\phi_{u_i}(L^\infty(X))$ while the right side is $\phi(L^\infty(X)) = L^\infty(Y)$. Thus (3) implies that we know $\phi_{u_i}(A) \prec_{L^\infty(Y) \rtimes \Lambda_{\rho(i)}} L^\infty(Y)$. Since they are both Cartan subalgebras then by [Popa 2006a, Theorem A2] we can extend this to unitary conjugacy and thus get our result. □

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