Pacific Journal of Mathematics

ON VOLUME GROWTH OF GRADIENT STEADY RICCI SOLITONS

GUOFANG WEI AND PENG WU

Volume 265 No. 1 September 2013

PACIFIC JOURNAL OF MATHEMATICS
Vol. 265, No. 1, 2013
dx.doi.org/10.2140/pjm.2013.265.233

ON VOLUME GROWTH OF GRADIENT STEADY RICCI SOLITONS

GUOFANG WEI AND PENG WU

In this paper we study volume growth of gradient steady Ricci solitons. We show that if the potential function satisfies a uniform condition, then the soliton has at most euclidean volume growth.

1. Introduction

(M^n, g) is a gradient Ricci soliton if there is a smooth function $f : M \to \mathbb{R}$ and constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} + \mathrm{Hess} f = \lambda g. \tag{1-1}$$

We refer to f as the potential function. The soliton is called shrinking, steady, and expanding when $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ respectively.

Ricci solitons are self-similar solutions of the Ricci flow, and play an important role in the study of singularity formation. They are also natural extensions of Einstein manifolds, and special cases of smooth metric measure spaces.

Volume growth of gradient Ricci solitons is of particular interest to mathematicians. Estimates of the potential functions plays an important role in the study of volume growth. Hamilton [1995] proved the following identity for gradient Ricci solitons:

$$R + |\nabla f|^2 - 2\lambda f = \Lambda,$$

where Λ is a constant, and R is the scalar curvature.

For gradient shrinking Ricci solitons, the answer is complete. Perelman [2003] and Cao and Zhou [2010] proved that f always grows quadratically. Cao and Zhou [2010] further proved that any gradient Ricci shrinking soliton has at most euclidean volume growth. Recently, Munteanu and Wang [2012] proved that any noncompact gradient Ricci shrinking soliton has at least linear volume growth.

For gradient steady Ricci solitons, B. L. Chen [2009] proved that $R \geq 0$. Hence, $\Lambda \geq 0$, and equal to zero if and only if f is constant and (M, g) is Ricci flat. When

MSC2010: 53CXX.
Keywords: Ricci solitons, volume growth.

$\Lambda > 0$ we can assume $\Lambda = 1$ after scaling; that is,

$$R + |\nabla f|^2 = 1. \tag{1-2}$$

Combined with the trace of the steady Ricci soliton equation $R + \Delta f = 0$, we have

$$\Delta f - |\nabla f|^2 = -1. \tag{1-3}$$

Therefore, f has no local minimum. Equation (1-2) and $R \geq 0$ give $|\nabla f| \leq 1$. Namely, f decays at most linearly.

Cao and Chen [2012] proved that f decays linearly when Ricci curvature is positive, and R attains its maximum at some point. However, the simple example of $\mathbb{R}^2$ with the canonical metric g_0 and $f(x) = x_1$ shows that this is not the case; f is constant along the x_2 direction. Note that the Riemannian product of any two steady gradient Ricci solitons is still a steady gradient Ricci soliton. Hence, a steady gradient Ricci soliton multiplied with a trivial one (f is a constant) will have constant direction. Though one can take the product of two shrinking ones, all trivial shrinking ones are compact, so they will not give a constant direction by taking a product. Munteanu and Sesum [2013] and Wu [2013] independently showed that the infimum of f does decay linearly. In fact,

$$-r \leq \inf_{y \in \partial B(x,r)} f(y) - f(x) \leq -r + \sqrt{2n}(\sqrt{r} + 1), \quad r \gg 1. \tag{1-4}$$

In particular, $\liminf_{y \to \infty} R(y) = 0$. See also [Fernández-López and García-Río 2011; Chow et al. 2011].

We note that among all known examples of steady gradient Ricci solitons, the infimum of f is like $-r + O(\ln r)$. See the survey article [Cao 2010] for a list of examples. One naturally asks if one can improve the second order term in (1-4) from $\sqrt{r}$ to $\ln r$. We show this is indeed the case for a large class of steady gradient Ricci solitons. To study the second order term, write the potential function in polar coordinates:

$$f(r, \theta) = -r + \phi(r, \theta),$$

where $r(\cdot) = d(x, \cdot)$ for some $x \in M^n$, $\theta \in S^{n-1}$. Without loss of generality, we assume $\phi(0, \theta) = 0$ by adding a constant to f. Since $|\nabla f| \leq 1$, we have $|\partial f/\partial r| \leq 1$, so $f(r) \geq -r$, $\phi(r) \geq 0$ and $\phi(r, \theta)$ are nondecreasing in r for any fixed θ. We show that the estimate (1-4) can be improved to $\ln r$ if ϕ in one direction is comparable to the minimum of ϕ among all spherical directions for all large r.

Theorem 1.1. *Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying (1-2). Assume that there exist $\theta_0 \in S^{n-1}$, and constants $C_1 \geq 0$, $C_2 \geq 0$ such that*

$$\int_0^r \big(\phi(r, \theta_0) - \phi(t, \theta_0)\big)\, dt \leq C_1 \min_{\theta \in S^{n-1}} \int_0^r \big(\phi(r, \theta) - \phi(t, \theta)\big)\, dt + C_2 r \tag{1-5}$$

for sufficiently large r. Then for any $x \in M^n$, there exist constants $C \geq 0$ and $r_0 > 0$ such that for $r \geq r_0$,

$$-r \leq \inf_{y \in \partial B(x,r)} f(y) - f(x) \leq -r + \Big(\frac{n}{2} C_1 + C_2\Big) \ln r + C. \tag{1-6}$$

Remark 1.2. All known examples of gradient steady Ricci solitons satisfy the condition (1-5). We suspect that the estimate (1-6) holds for all gradient steady Ricci solitons.

In [Munteanu and Sesum 2013], it was proven that any gradient steady Ricci soliton has at least linear volume growth, and at most a growth rate of $e^{\sqrt{r}}$. We show that if the potential function satisfies a uniform condition in the spherical directions, then the gradient steady Ricci soliton has at most euclidean volume growth.

Theorem 1.3. *Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying (1-2). Assume that there exist constants $C_1, C_2 \geq 0$ such that*

$$\max_{\theta \in S^{n-1}} \int_0^r \big(\phi(r,\theta) - \phi(t,\theta)\big)\, dt \leq C_1 \min_{\theta \in S^{n-1}} \int_0^r \big(\phi(r,\theta) - \phi(t,\theta)\big)\, dt + C_2 r \tag{1-7}$$

for sufficiently large r. Then for any $x \in M^n$, there exist constants $C \geq 0$ and $r_0 > 0$ such that for any $r \geq r_0$,

$$-r \leq f(y) - f(x) \leq -r + C \ln r \tag{1-8}$$

for any $y \in \partial B(x,r)$. Moreover, the soliton has at most euclidean volume growth; that is, for any $x \in M^n$ there exists $r_0 > 0$, and for any $r \geq r_0$,

$$\operatorname{Vol} B(x,r) \leq C r^n.$$

If, in addition, $\phi(r) \geq \delta \ln r$ for large r, then

$$\operatorname{Vol} B(x,r) \leq C r^{n-\delta}.$$

Remark 1.4. (1) If ϕ increases uniformly along all spherical directions; that is, $\max_\theta \partial\phi/\partial r \leq C \min_\theta \partial\phi/\partial r$, where $\theta \in S^{n-1}$, then ϕ satisfies (1-7) with $C_1 = C$ and $C_2 = 0$.

(2) Theorem 1.3 can be considered an analogue of the volume growth theorem of [Cao and Zhou 2010], valid for gradient shrinking Ricci solitons. As the potential function for such solitons automatically satisfies a uniform condition, here too we need to impose a uniform condition for gradient steady Ricci solitons.

(3) If the soliton is rectifiable (see [Petersen and Wylie 2009]) — i.e., f is the distance function from a set — then ϕ satisfies (1-7) with $C_1 = 1$ if the set is bounded (this is the case with all nonproduct examples).

To prove the results, the following estimate for ϕ, which holds for all gradient steady Ricci solitons, is the key:

Proposition 1.5. *Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying* (1-2). *Then*

$$\min_{y\in\partial B(x,r)} \int_0^r \big(\phi(y)-\phi(t)\big)\,dt \le \frac{n}{2}\left(r+\sqrt{r}\right)+o\left(\frac{1}{r}\right). \tag{1-9}$$

This estimate improves the one in [Wu 2013]. In the next section we derive a volume comparison for the solitons by adapting the volume comparison for smooth metric measure spaces in [Wei and Wylie 2009]. Then we prove Proposition 1.5 by combining this with (1-3). In Section 3 we prove the main theorems using this estimate and an ODE.

2. The preliminary estimate

In this section we prove Proposition 1.5 by applying a weighted volume comparison argument for smooth metric measure spaces as in [Wei and Wylie 2009; Wu 2013].

Recall that a smooth metric measure space is a triple $(M^n, g, e^{-f}\,d\mathrm{vol}_g)$, where (M^n, g) is a smooth Riemannian manifold, and $f : M^n \to \mathbb{R}$ is a smooth function. Write the volume element in polar coordinates $d\mathrm{vol} = J(r,\theta)\,dr\,d\theta$. Define the weighted volume element as $J_f(r,\theta) = e^{-f} J(r,\theta)$ and the weighted volume as $\mathrm{Vol}_f B(x,r) = \int_{B(x,r)} e^{-f}\,d\mathrm{vol}$.

Wei and Wylie [2009] obtained the following f-volume comparison theorem for smooth metric measure spaces:

Theorem 2.1 (f-volume comparison). *Suppose $(M^n, g, e^{-f}\,d\mathrm{vol})$ is a smooth metric measure space with $\mathrm{Ric}_f \ge (n-1)H$. Fix $x\in M$. If $|f|\le\Lambda$, then for $R\ge r>0$ (and $R\le \pi/4\sqrt{H}$ if $H>0$),*

$$\frac{\mathrm{Vol}_f B(x,R)}{\mathrm{Vol}_f B(x,r)} \le \frac{V_H^{n+4\Lambda}(B_R)}{V_H^{n+4\Lambda}(B_r)}$$

where $V_H^n(B_r)$ is the volume of the ball of radius r in M_H^n (the simply connected model space of dimension n with constant sectional curvature H).

One observes that the dimension of the model space in the volume comparison depends on the potential function f. Further investigation of the dimension will lead to Proposition 1.5.

Denote the f-mean curvature by $m_f = (\ln J_f)'$. For $0<r_1\le r_2$, let $A(x,r_1,r_2) = \{y \mid r_1 \le d(x,y)\le r_2\}$ be the annulus, and

$$a = \min_{y\in A(x,r_1,r_2)} \frac{2}{r(y)^2}\int_0^{r(y)} \big(\phi(y)-\phi(t)\big)\,dt.$$

Clearly $a \geq 0$. By (1-4), we have $a \leq C/\sqrt{r_1}$ for $r_1 \gg 1$. For the rest we assume $r_1 \gg 1$ and therefore we can assume $a < 1$.

Proposition 2.2. *For a gradient steady Ricci soliton, we have*

$$m_f(r,\theta) \leq \frac{n-1}{r} + 1 - \frac{2}{r^2}\int_0^r \big(\phi(r,\theta)-\phi(t,\theta)\big)\,dt \leq \frac{n-1}{r}+1, \tag{2-1}$$

$$\frac{\mathrm{Vol}_f\,\partial B(x,r_2)}{\mathrm{Vol}_f A(x,r_1,r_2)} \leq \frac{n/r_2+1-a}{1-(r_1/r_2)^{n+(1-a)r_2}}. \tag{2-2}$$

Proof. For a smooth metric space (M^n, g, f) with $\mathrm{Ric}_f \geq 0$, recall the following estimate for m_f from [Wei and Wylie 2009, (3.19)]:

$$m_f(r,\theta) \leq \frac{n-1}{r} + \frac{2}{r^2}\int_0^r \big(f(t)-f(r)\big)\,dt.$$

Plugging in $f = -r+\phi$ gives (2-1).

Now let

$$\overline{m}(r) = \begin{cases} \dfrac{n-1}{r}+1 & \text{if } r \leq r_1,\\[2mm] \dfrac{n-1+(1-a)r_2}{r} & \text{if } r_1 < r \leq r_2. \end{cases}$$

Then

$$m_f(r) \leq \overline{m}(r) \quad \text{for } 0 < r \leq r_2. \tag{2-3}$$

Let $\bar{A}(r) = e^{\int_0^r \overline{m}(t)\,dt}$ and $\overline{V}(r_0, r) = \int_{r_0}^r \bar{A}(t)\,dt$. From the mean curvature relation (2-3), we have $(A_f/\bar{A})' \leq 0$; therefore

$$\frac{\mathrm{Vol}_f\;\partial B(x,r_2)}{\mathrm{Vol}_f A(x,r_1,r_2)} \leq \frac{\bar{A}(r_2)}{\overline{V}(r_1,r_2)}.$$

We compute

$$\begin{aligned}\frac{\bar{A}(r_2)}{\overline{V}(r_1,r_2)} &= \frac{e^{\int_0^{r_2}\overline{m}(t)\,dt}}{\int_{r_1}^{r_2} e^{\int_0^s \overline{m}(t)\,dt}\,ds} = \frac{e^{\int_{r_1}^{r_2}\overline{m}(t)\,dt}}{\int_{r_1}^{r_2} e^{\int_{r_1}^s \overline{m}(t)\,dt}\,ds}\\ &= \frac{(r_2/r_1)^{n-1+(1-a)r_2}}{\int_{r_1}^{r_2}(s/r_1)^{n-1+(1-a)r_2}\,ds} = \frac{n/r_2+1-a}{1-(r_1/r_2)^{n+(1-a)r_2}}.\end{aligned}$$

This gives (2-2). □

Proof of Proposition 1.5. Integrating (1-3) and using $|\nabla f| \leq 1$ we have, for any $x \in M$,

$$\begin{aligned}\int_{B(x,r)} 1\cdot e^{-f}\,d\mathrm{vol} &= -\int_{B(x,r)} (\Delta f - |\nabla f|^2)\cdot e^{-f}\,d\mathrm{vol}\\ &= -\int_{\partial B(x,r)} \frac{\partial f}{\partial n} e^{-f}\,d\mathrm{vol} \leq \int_{\partial B(x,r)} e^{-f}\,d\mathrm{vol}.\end{aligned}$$

Therefore,

$$\frac{\mathrm{Vol}_f \partial B(x,r)}{\mathrm{Vol}_f B(x,r)} \geq 1. \tag{2-4}$$

Combining (2-2) and (2-4) we have

$$a \leq \frac{n}{r_2} + \left(\frac{r_1}{r_2}\right)^{n+(1-a)r_2}.$$

Let $r_1 = r$ and $r_2 = r + \sqrt{r}$. Then $r_1/r_2 = (1 + 1/\sqrt{r})^{-1}$. When r is large,

$$\left(1 + \frac{1}{\sqrt{r}}\right)^{-(n+(1-a)(r+\sqrt{r}))} = O\big(e^{-(1-a)\sqrt{r}}\big).$$

Therefore, for all r large enough,

$$a = \min_{y \in A(x,r,r+\sqrt{r})} \frac{2}{r(y)^2} \int_0^{r(y)} \big(\phi(y) - \phi(t)\big)\, dt \leq \frac{n}{r+\sqrt{r}} + O\big(e^{-(1-a)\sqrt{r}}\big).$$

Suppose the minimum above is attained at $y_0 = (r_0, \theta_1)$ with $r \leq r_0 \leq r + \sqrt{r}$. Then

$$\begin{aligned}
\min_{y \in \partial B(x,r)} \int_0^r \big(\phi(y) - \phi(t)\big)\, dt &\leq \int_0^{r_0} \big(\phi(y_0) - \phi(t)\big)\, dt \\
&\leq \frac{r_0^2}{2} \left(\frac{n}{r+\sqrt{r}} + O\big(e^{-(1-a)\sqrt{r}}\big)\right) \\
&\leq \frac{n}{2}\left(r + \sqrt{r}\right) + o\left(\frac{1}{r}\right). \qquad \square
\end{aligned}$$

3. Proof of main results

Proof of Theorem 1.1. From (1-9) and (1-5), we have

$$\int_0^r \big(\phi(r,\theta_0) - \phi(t,\theta_0)\big)\, dt \leq \frac{nC_1}{2}\big(r + \sqrt{r}\big) + C_2 r + o\left(\frac{1}{r}\right). \tag{3-1}$$

For simplicity, when there is no confusion we omit θ_0 in the function. Let $\Phi(r) = \int_0^r \phi(t)\, dt$; then (3-1) can be written as

$$\Phi'(r) - \frac{1}{r}\Phi(r) \leq \frac{nC_1}{2} + C_2 + O\left(\frac{1}{\sqrt{r}}\right). \tag{3-2}$$

Multiplying by the integrating factor $1/r$ and integrating from some fixed $t_0 \gg 1$ to r, we get

$$\frac{\Phi(r)}{r} \leq \left(\frac{nC_1}{2} + C_2\right) \ln r + C_3.$$

So, we have

$$\begin{aligned}\phi(r,\theta_0)=\Phi'(r,\theta_0) &\le \frac{\Phi(r,\theta_0)}{r}+\frac{nC_1}{2}+C_2+O\left(\frac{1}{\sqrt{r}}\right)\\ &\le \left(\frac{nC_1}{2}+C_2\right)\ln r+C_4\\ f(r,\theta_0)=-r+\phi(r,\theta_0) &\le -r+\left(\frac{nC_1}{2}+C_2\right)\ln r+C_4.\end{aligned}$$

This gives (1-6). □

Proof of Theorem 1.3. From (1-9) and (1-7), we have

$$\int_0^r \big(\phi(r,\theta)-\phi(t,\theta)\big)\,dt \le \frac{nC_1}{2}\big(r+\sqrt{r}\big)+C_2r+o\left(\frac{1}{r}\right)$$

for all $\theta\in S^{n-1}$. Therefore, (1-6) holds for all y. Namely, for all $y\in\partial B(x,r)$,

$$-r\le f(y)-f(x)\le -r+\left(\frac{nC_1}{2}+C_2\right)\ln r+C_4.$$

By (2-1), for all $r>0$,

$$\begin{aligned}\frac{\partial}{\partial r}\ln J &= m_f(r)+\langle\nabla f,\nabla r\rangle\\ &\le \frac{n-1}{r}+1-\frac{2}{r}\phi(r)+\frac{2}{r^2}\int_0^r\phi(t)\,dt+\langle\nabla f,\nabla r\rangle.\end{aligned}$$

Integrating from 1 to r and performing integration by parts for the double integral, we get

$$\begin{aligned}\text{(3-3)}\quad &\ln J(r)-\ln J(1)\\ &\le (n-1)\ln r+(r-1)-\int_1^r\frac{2}{s}\phi(s)\,ds+\left(-\frac{2}{s}\int_0^s\phi(t)\,dt\right)\bigg|_1^r\\ &\qquad\qquad +\int_1^r\frac{2}{s}\phi(s)\,ds+f(r)-f(1)\\ &=(n-1)\ln r+\phi(r)-\frac{2}{r}\int_0^r\phi(t)\,dt+2\int_0^1\phi(t)\,dt-f(1)-1\\ &=(n-1)\ln r-\phi(r)+2\left(\phi(r)-\frac{1}{r}\int_0^r\phi(t)\,dt\right)+2\int_0^1\phi(t)\,dt-f(1)-1.\end{aligned}$$

Using (3-2) we have, for large r,

$$\ln J(r)\le (n-1)\ln r-\phi(r)+C\le (n-1)\ln r+C.$$

Hence,

$$J(r)\le e^Ce^{(n-1)\ln r}=e^Cr^{n-1},$$

and the volume of a geodesic ball centered at x satisfies

$$\operatorname{Vol} B(x,r) \le C' r^n.$$

If further $\phi(s) \ge \delta \ln s$, then we have

$$J(r) \le C r^{n-1} \exp\bigl(-\phi(r)\bigr) \le C r^{n-\delta-1},$$

therefore the volume growth is strictly less than euclidean volume growth:

$$\operatorname{Vol} B(x,r) \le C r^{n-\delta}. \qquad \square$$

For general gradient steady Ricci solitons, the estimate of a potential function can be reduced to the following:

Question 3.1. Suppose ϕ is nondecreasing along any minimal geodesic starting from x. Assume that for sufficiently large r, $\inf_{y\in\partial B(x,r)} \phi(y) \le C\sqrt{r}$, and

$$\inf_{y\in\partial B(x,r)} \int_1^r \bigl(\phi(y) - \phi(\gamma_y(t))\bigr)\, dt \le \frac{nr}{2}.$$

Does the following hold?

$$\inf_{y\in\partial B(x,r)} \phi(y) \le C \ln r$$

Remark 3.2. From (3-3), we see that if

$$-r \le f(y) - f(x) \le -r + C\ln r$$

for $y \in \partial B(x,r)$, then for any $x \in M^n$, there exists $r_0 > 0$ such that for any $r \ge r_0$,

$$\operatorname{Vol} B(x,r) \le C' r^{n+C}.$$

Acknowledgements

The second author would like to thank Professor Thomas Sideris for helpful discussions.

References

[Cao 2010] H.-D. Cao, “Recent progress on Ricci solitons”, pp. 1–38 in *Recent advances in geometric analysis* (Taipei, 2007), edited by Y.-I. Lee et al., Adv. Lect. Math. **11**, International Press, Somerville, MA, 2010. MR 2011d:53061 Zbl 1201.53046 arXiv 0908.2006

[Cao and Chen 2012] H.-D. Cao and Q. Chen, “On locally conformally flat gradient steady Ricci solitons”, *Trans. Amer. Math. Soc.* **364**:5 (2012), 2377–2391. MR 2888210 Zbl 1245.53038

[Cao and Zhou 2010] H.-D. Cao and D. Zhou, “On complete gradient shrinking Ricci solitons”, *J. Differential Geom.* **85**:2 (2010), 175–185. MR 2011k:53040 Zbl 1246.53051

[Chen 2009] B.-L. Chen, “Strong uniqueness of the Ricci flow”, *J. Differential Geom.* **82**:2 (2009), 363–382. MR 2010h:53095 Zbl 1177.53036

[Chow et al. 2011] B. Chow, P. Lu, and B. Yang, "Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons", *C. R. Math. Acad. Sci. Paris* **349**:23-24 (2011), 1265–1267. MR 2861997 Zbl 1230.53036

[Fernández-López and García-Río 2011] M. Fernández-López and E. García-Río, "Maximum principles and gradient Ricci solitons", *J. Differential Equations* **251**:1 (2011), 73–81. MR 2012d:53136 Zbl 1217.53042

[Hamilton 1995] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Proceedings of the conference on geometry and topology* (Cambridge, MA, 1993), edited by C. C. Hsiung and S.-T. Yau, Surveys in Differential Geometry **2**, International Press, Somerville, MA, 1995. MR 97e:53075 Zbl 0867.53030

[Munteanu and Sesum 2013] O. Munteanu and N. Sesum, "On gradient Ricci solitons", *J. Geom. Anal.* **23**:2 (2013), 539–561. MR 3023848

[Munteanu and Wang 2012] O. Munteanu and J. Wang, "Analysis of weighted Laplacian and applications to Ricci solitons", *Comm. Anal. Geom.* **20**:1 (2012), 55–94. MR 2903101 Zbl 1245.53039 arXiv 1112.3027

[Perelman 2003] G. Y. Perelman, "Ricci flow with surgery on three manifolds", preprint, 2003. Zbl 1130.53002 arXiv math.DG/0303109

[Petersen and Wylie 2009] P. Petersen and W. Wylie, "On gradient Ricci solitons with symmetry", *Proc. Amer. Math. Soc.* **137**:6 (2009), 2085–2092. MR 2010a:53073 Zbl 1168.53021

[Wei and Wylie 2009] G. Wei and W. Wylie, "Comparison geometry for the Bakry–Emery Ricci tensor", *J. Differential Geom.* **83**:2 (2009), 377–405. MR 2011a:53064 Zbl 1189.53036

[Wu 2013] P. Wu, "On the potential function of gradient steady Ricci solitons", *J. Geom. Anal.* **23**:1 (2013), 221–228. MR 3010278 Zbl 06136841

Received July 12, 2012. Revised November 9, 2012.

GUOFANG WEI
MATH DEPARTMENT
UC SANTA BARBARA
SANTA BARBARA, CA 93106
UNITED STATES
wei@math.ucsb.edu

PENG WU
MATH DEPARTMENT
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
wupenguin@math.cornell.edu

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2013 is US \$400/year for the electronic version, and \$485/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 265 No. 1 September 2013

0030-8730(201309)265:1;1-1