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CHARACTERIZING ABELIAN VARIETIES BY THE REDUCTION OF THE MORDELL–WEIL GROUP

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Let A be an abelian variety defined over a number field K. Let $\mathfrak{p}$ be a prime of K of good reduction and $A_{\mathfrak{p}}$ the fiber of A over the residue field $k_{\mathfrak{p}}$. We call $A(K)_{\mathfrak{p}}$ the image of the Mordell–Weil group via reduction modulo $\mathfrak{p}$, which is a subgroup of $A_{\mathfrak{p}}(k_{\mathfrak{p}})$. We prove in particular that the size of $A(K)_{\mathfrak{p}}$, by varying $\mathfrak{p}$, encodes enough information to characterize the K-isogeny class of A, provided that the following necessary condition holds: the Mordell–Weil group $A(K)$ is Zariski dense in A. This is an analogue to a 1983 result of Faltings, considering instead the size of $A_{\mathfrak{p}}(k_{\mathfrak{p}})$.

1. Introduction

Statement of the theorems. Let K be a number field and A, A' be abelian varieties over K. Let $S(A, A')$ be the set of primes of K of good reduction for A and A', and let $A_{\mathfrak{p}}$, $A'_{\mathfrak{p}}$ be the respective fibers of A, A' over the residue field $k_{\mathfrak{p}}$ for $\mathfrak{p} \in S(A, A')$.

Faltings [1983] proved the following local-global principle for any $S \subseteq S(A, A')$ of Dirichlet density 1: A, A' are K-isogenous if and only if $A_{\mathfrak{p}}$, $A'_{\mathfrak{p}}$ are $k_{\mathfrak{p}}$-isogenous for every $\mathfrak{p} \in S$. The latter is equivalent, for a large class of abelian varieties, to the identities $\#A_{\mathfrak{p}}(k_{\mathfrak{p}}) = \#A'_{\mathfrak{p}}(k_{\mathfrak{p}})$ for $\mathfrak{p} \in S$. The motivation for this paper was to consider instead identities using the reductions of the Mordell–Weil groups $A(K)$, $A'(K)$, which we denote by $A(K)_{\mathfrak{p}}$, $A'(K)_{\mathfrak{p}}$ and which are subgroups of $A_{\mathfrak{p}}(k_{\mathfrak{p}})$, $A'_{\mathfrak{p}}(k_{\mathfrak{p}})$. We prove in particular the following result:

Theorem 1.1. *Suppose A, A' are abelian varieties over a number field K such that $A(K)$, $A'(K)$ are Zariski dense in A, A', respectively. Let $S \subseteq S(A, A')$ have Dirichlet density* 1. *If $\#A(K)_{\mathfrak{p}} = \#A'(K)_{\mathfrak{p}}$ holds for every $\mathfrak{p} \in S$, then A and A' are K-isogenous.*

In other words, if $A(K)$ is Zariski dense in A, then the function $\mathfrak{p} \in S \mapsto \#A(K)_{\mathfrak{p}}$ characterizes the K-isogeny class of A. Note, we define this function via a global object, namely the Mordell–Weil group $A(K)$, and it only "sees" the Zariski closure of $A(K)$, hence the reason we assume $A(K)$ is Zariski dense in A. This assumption

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is equivalent to the following: for every nontrivial abelian subvariety $B \subseteq A$, the Mordell–Weil group $B(K)$ is infinite.

For each prime number l and finite group G, we denote by $\mathrm{ord}_l(G)$, $\exp_l(G)$, and $\mathrm{rad}_l(G)$ the l-adic valuation of the order, exponent, and radical of the order of G, respectively. (Recall that the radical of a positive integer is the product of the primes dividing it, the empty product being 1; so $\mathrm{rad}_l(G)$ is 1 or 0 depending on whether or not l divides the order of G.)

We say an abelian variety A is *square-free* if and only if $B = 0$ is the only abelian variety for which there exists a K-homomorphism $B^2 \to A$ with finite kernel. Then we can prove the following stronger result:

Theorem 1.2. *Suppose A, A' are abelian varieties over a number field K such that $A(K)$, $A'(K)$ are Zariski dense in A, A' respectively. Let $S \subseteq S(A, A')$ have Dirichlet density* 1.

(i) *If for some prime number l and for some $m \geq 0$ the inequalities*

$$|\mathrm{ord}_l(A(K)_{\mathfrak{p}}) - \mathrm{ord}_l(A'(K)_{\mathfrak{p}})| \leq m \quad \textit{for all } \mathfrak{p} \in S$$

hold, then A and A' are K-isogenous.

(ii) *Suppose that A, A' are square-free.*
If for some prime number l and for some $m \geq 0$ the inequalities

$$|\exp_l(A(K)_{\mathfrak{p}}) - \exp_l(A'(K)_{\mathfrak{p}})| \leq m \quad \textit{for all } \mathfrak{p} \in S$$

hold, then A and A' are K-isogenous.

(iii) *Suppose that A, A' are square-free.*
There exists l_0 depending only on A, A', K such that if for some prime number $l \geq l_0$ the equalities

$$\mathrm{rad}_l(A(K)_{\mathfrak{p}}) = \mathrm{rad}_l(A'(K)_{\mathfrak{p}}) \quad \textit{for all } \mathfrak{p} \in S$$

hold, then A and A' are K-isogenous.

The assumption on A and A' that we required in the previous statement is in general necessary. Indeed, if C is a nonzero abelian variety having trivial Mordell–Weil group then A and $A' = A \times C$ can not be distinguished with the data as in the statement of the theorem, and indeed $A'(K) = A(K) \times \{0\}$ is not dense in A'. Moreover, we cannot distinguish between A and $A' = A^2$ just by looking at the exponent (or the radical of the size) of $A(K)_{\mathfrak{p}}$.

We generalize the previous result by considering more general objects than the Mordell–Weil group. If A is an abelian variety, we call a subgroup $\Gamma \subseteq A(K)$ *dense* if and only if it is Zariski dense in A, and a *submodule* if and only if it is an $\mathrm{End}_K(A)$-submodule of $A(K)$. We denote $\Gamma_{\mathfrak{p}} \subseteq A(k_{\mathfrak{p}})$ the reduction of Γ modulo $\mathfrak{p}$.

Theorem 1.3. *Let A, A' be abelian varieties defined over a number field K, and $\Gamma \subseteq A(K)$, $\Gamma' \subseteq A'(K)$ be submodules. Suppose Γ is dense. Consider the following property*:

(1) $\exists \varphi \in \mathrm{Hom}_K(A, A')$ *such that* $\ker(\varphi)$ *and* $[\varphi(\Gamma) : \varphi(\Gamma) \cap \Gamma']$ *are finite.*

Let $S \subseteq S(A, A')$ have Dirichlet density 1.

(i) *If for some prime number l and for some $m \geq 0$*

(2) $$\mathrm{ord}_l(\Gamma_{\mathfrak{p}}) \leq \mathrm{ord}_l(\Gamma'_{\mathfrak{p}}) + m \quad \textit{for all } \mathfrak{p} \in S$$

holds, then (1) *holds.*

(ii) *Suppose that A is square-free. If for some prime number l and for some $m \geq 0$*

(3) $$\exp_l(\Gamma_{\mathfrak{p}}) \leq \exp_l(\Gamma'_{\mathfrak{p}}) + m \quad \textit{for all } \mathfrak{p} \in S$$

holds, then (1) *holds.*

(iii) *Suppose that A is square-free. There exists l_0 depending only on A, A', K, Γ, Γ' such that if for some prime number $l \geq l_0$*

(4) $$\mathrm{rad}_l(\Gamma_{\mathfrak{p}}) \leq \mathrm{rad}_l(\Gamma'_{\mathfrak{p}}) \quad \textit{for all } \mathfrak{p} \in S$$

holds, then (1) *holds.*

Conversely, property (1) implies that for all primes in $S(A, A')$ and for all but finitely many prime numbers l, we have $\mathrm{ord}_l(\Gamma_{\mathfrak{p}}) \leq \mathrm{ord}_l(\Gamma'_{\mathfrak{p}})$, and similarly for $\exp_l$ and rad_l (compare Lemma 3.4).

If A is a simple abelian variety and $\psi \in \mathrm{End}_K(A) \backslash \mathbb{Z}$, then for any nontorsion point $P \in A(K)$ and sufficiently large prime l, the subgroups $\Gamma = \mathbb{Z}P + \mathbb{Z}\psi(P)$ and $\Gamma' = \mathbb{Z}P$ satisfy $\exp_l(\Gamma_{\mathfrak{p}}) = \exp_l(\Gamma'_{\mathfrak{p}})$ for all $\mathfrak{p} \in S(A)$; however, Γ' does not contain a finite index subgroup of $\varphi(\Gamma)$ for $\varphi \neq 0 \in \mathrm{End}_K(A)$. This is the reason we suppose Γ, Γ' are submodules and not merely subgroups in Theorem 1.3.

Our results relate to the so-called support problem, and especially to the following result:

Theorem 1.4 [Demeyer and Perucca 2013, Theorem 1.2]. *Let A be an abelian variety defined over a number field K, and $P \in A(K)$ be a rational point. Suppose $\mathbb{Z}P$ is Zariski dense in A. If $S \subseteq S(A)$ has Dirichlet density* 1, *then the function $\mathfrak{p} \in S \mapsto \#(\mathbb{Z}P)_{\mathfrak{p}}$ characterizes the K-isomorphism class of A.*

It is not possible to characterize the K-isomorphism class of A by knowing the order and the exponent of $A(K)_{\mathfrak{p}}$ for every $\mathfrak{p} \in S(A)$. Indeed, there exist pairs of elliptic curves over a number field K which are not K-isomorphic, but such that for every prime number l there is a K-isogeny between them of degree coprime to l, as shown by Zarhin [2008, Section 12].

We will deduce Theorems 1.1 and 1.2 from Theorem 1.3. An overview of the proof of Theorem 1.3 is given at the end of this section (page 431). In Section 2 we develop the notion of *almost independent* points to compensate for the fact that the submodules we consider are in general not free. We also define what it means for points to dominate a submodule (page 434), and for an infinite submodule we show how to construct a finite dominating subset consisting of almost independent points. We bring these notions together in Section 3, with preparatory theorems (3.1 and 3.2) about the reduction of submodules and the proof of Theorem 1.3 (page 438).

Notation and conventions. We assume all abelian varieties, subvarieties, homomorphisms, etc. are defined over a fixed number field K. Given abelian varieties $A_1, \dots, A_r$ we write $S(A_1, \dots, A_r)$ for the primes of K of common good reduction.

If A is an abelian variety and $\mathfrak{p}$ is a prime in $S(A)$, we write $k_\mathfrak{p}$ for the residue field and $A_\mathfrak{p}$ for the fiber of A over $k_\mathfrak{p}$. Given a subgroup $\Gamma \subseteq A(K)$, we write $\Gamma_\mathfrak{p} \subseteq A_\mathfrak{p}(k_\mathfrak{p})$ for the reduction of Γ modulo $\mathfrak{p}$. The symbol l always denotes a prime number, and we define $\operatorname{ord}_l(\Gamma_\mathfrak{p})$, $\exp_l(\Gamma_\mathfrak{p})$, $\operatorname{rad}_l(\Gamma_\mathfrak{p})$ to be the l-adic valuation of the size, exponent, and radical of the size of $\Gamma_\mathfrak{p}$ respectively.

By the order of a point we mean the order of the subgroup that it generates.

Two main ingredients. The proof of Theorem 1.3 is based on two main ingredients: Theorem 1 of [Perucca 2011] and a basic structure theorem for abelian varieties, known as Poincaré's reducibility theorem. We recall these statements for the convenience of the reader and for future reference; aside from these two inputs, this paper will be self contained.

Proposition 1.5 [Perucca 2011]. *Let $A_1, \dots, A_r$ be abelian varieties over K, and $P_i \in A_i(K)$ be a rational point for $1 \le i \le r$. If l is a prime number and $e_1, \dots, e_r$ are nonnegative integers, then the set of primes*

$$\{\mathfrak{p} \in S(A_1, \dots, A_r) : \operatorname{ord}_l(P_i \bmod p) = e_i \textit{ for all } i\}$$

admits a Dirichlet density.

If the rational point $P = (P_1, \dots, P_r)$ on $A = A_1 \times \cdots \times A_r$ generates a Zariski dense subgroup, then this Dirichlet density is positive.

Proof. This is the special case of [Perucca 2011, Theorem 1] where all semiabelian varieties are abelian and we consider only one prime number l. The existence of the density (in fact, it is a natural density) is proven there with a method from [Jones and Rouse 2010]. The fact that under the additional assumption the density is nonzero was first proven in [Pink 2004] and can also be proven with a method from [Khare and Prasad 2004]. The proof uses Kummer theory, results on the l-adic representation, and the Chebotarev density theorem. □

Theorem 1.6 (Poincaré's reducibility theorem [Mumford 1970, Theorem 1, p. 160]). *If A is an abelian variety over K and $B \subseteq A$ is an abelian subvariety, then there exists an abelian subvariety $C \subseteq A$ such that $B \cap C$ is finite, $A = B + C$, and A is isogenous to $B \times C$.*

By applying this result finitely many times, we have:

Corollary 1.7. *Let A be an abelian variety over K. There exist pairwise nonisogenous simple abelian varieties $B_1, \ldots, B_r$ uniquely determined up to isogeny and ordering, and positive integers $e_1, \ldots, e_r$ such that A is isogenous to $B_1^{e_1} \times \cdots \times B_r^{e_r}$.*

Overview of the proof of Theorem 1.3. Under the additional hypothesis that A and A' are simple and that Γ and Γ' are each generated by a rational point of infinite order, the proof of Theorem 1.3 becomes technically much easier. We present the proof of (i) in this special case.

Proposition 1.8. *Let A and A' be simple abelian varieties over K and let $\Gamma \subseteq A(K)$ and $\Gamma' \subseteq A'(K)$ be the submodules generated by points of infinite order $P \in A(K)$ and $P' \in A'(K)$ respectively. Let l be a prime number. Suppose that there exists a set $S \subseteq S(A, A')$ of Dirichlet density* 1 *and an integer $m \geq 0$ such that*

$$\operatorname{ord}_l(\Gamma_{\mathfrak{p}}) \leq \operatorname{ord}_l(\Gamma'_{\mathfrak{p}}) + m \quad \textit{for every } \mathfrak{p} \in S \tag{5}$$

holds. Then there exists an isogeny $\varphi : A \to A'$ such that the index $[\varphi(\Gamma) : \varphi(\Gamma) \cap \Gamma']$ is finite.

Proof. Consider the subgroup of $A \times A'$ generated by (P, P'), and denote by $B \subseteq A \times A'$ the connected component of the unity of its Zariski closure. A closed algebraic subgroup of an abelian variety has only finitely many connected components, hence there exists an integer $n \geq 1$ such that (nP, nP') is a rational point of B. Since A is simple and P is of infinite order, the Zariski closure of $\mathbb{Z}P$ in A is equal to A. The projection $\pi : B \to A$ is therefore surjective. For the same reason, the projection $\pi' : B \to A'$ is surjective. Again, because A and A' are simple, there are now two possibilities: either π and π' are isogenies, or else B is equal to $A \times A'$. In the first case, there exists an isogeny $\psi : A \to B$ such that $\psi \circ \pi$ is the multiplication-by-n' endomorphism of B for some nonzero integer n'. The composite isogeny $\varphi := \pi' \circ \psi : A \to A'$ has the required properties, since indeed

$$\varphi(nP) = \pi'(\psi(nP)) = \pi'(\psi(\pi(nP, nP'))) = \pi'(nn'P, nn'P') = nn'P'$$

holds. We are now left to show that (5) excludes the second possibility, that $B = A \times A'$. For this we use Proposition 1.5. Indeed, if $B = A \times A'$, then there exists by Proposition 1.5 a set $S' \subseteq S(A, A')$ of positive Dirichlet density, such that for all $\mathfrak{p} \in S'$,

$$\operatorname{ord}_l(P \bmod \mathfrak{p}) = m + 1 \quad \text{and} \quad \operatorname{ord}_l(P' \bmod \mathfrak{p}) = 0$$

holds. Hence, we have $\mathrm{ord}_l(\Gamma_\mathfrak{p}) > m$ and $\mathrm{ord}_l(\Gamma'_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in S'$, because on one hand $\Gamma_\mathfrak{p}$ contains $(P \bmod \mathfrak{p})$, and on the other hand $\Gamma'_\mathfrak{p}$ consists of images under endomorphisms of $(P' \bmod \mathfrak{p})$. The complement of S' has Dirichlet density < 1, contradicting (5). □

2. Preliminaries

Dense submodules. Let A be an abelian variety. We call a subgroup $\Gamma \subseteq A(K)$ a *submodule* if and only if Γ is an $\mathrm{End}(A)$-submodule of $A(K)$. We say that a subgroup of $A(K)$ is *dense* if and only if it satisfies the equivalent conditions of the following lemma:

Lemma 2.1. *If A is an abelian variety and Γ is a subgroup of $A(K)$, then the following are equivalent*:

(i) *Γ is Zariski dense in A.*

(ii) *$\varphi(\Gamma) \neq \{0\}$ for every abelian variety B and $\varphi \neq 0 \in \mathrm{Hom}(A, B)$.*

Proof. Let $C \subseteq A$ be the Zariski closure of Γ. If $\varphi(\Gamma) = 0$ for some nonzero $\varphi \in \mathrm{Hom}(A, B)$, then the kernel of φ is a proper subgroup containing Γ, so (i) implies (ii). Conversely, if $C \neq A$, then the projection $A \to A/C$ is a nonzero morphism between abelian varieties which kills Γ; therefore (ii) implies (i). □

If $\Gamma \subseteq A(K)$ is a finite subgroup, then either Γ is not dense or $A = 0$. If A is simple and if $\Gamma \subseteq A(K)$ is an infinite subgroup, then Γ is dense.

Almost independent points. In this subsection, we suppose $A_1, \dots, A_r$ are nonzero abelian varieties and $P_i \in A_i(K)$ is a rational point for $1 \le i \le r$. We let $A = A_1 \times \cdots \times A_r$ and $P = (P_1, \dots, P_r)$.

We say that $P_1, \dots, P_r$ are *independent* if and only if the Zariski closure of $\mathbb{Z}P$ satisfies $\overline{\mathbb{Z}P} = A$. Note that if $A_1 = \cdots = A_r$, the points $P_1, \dots, P_r$ are independent if and only if they form a basis for a free $\mathrm{End}(A_1)$-submodule of $A_1(K)$ (see Definition 3 and Remark 6 in [Perucca 2009]).

Lemma 2.2. *The following are equivalent*

(i) *$P_1, \dots, P_r$ are independent.*

(ii) *For every abelian variety B, the following implication holds*:

$$\sum_{i=1}^{r} \phi_i(P_i) = 0 \text{ for } (\phi_1, \dots, \phi_r) \in \mathrm{Hom}(A_1 \times \cdots \times A_r, B) \implies \phi_i = 0 \text{ for all } i. \tag{6}$$

Proof. If $A = A_1 \times \cdots \times A_r$ and $\Gamma = \mathbb{Z}P$, then conditions (i) and (ii) are equivalent to the respective conditions of Lemma 2.2. □

A weaker condition is the following:

Definition 2.3. We say $P_1, \dots, P_r$ are *almost independent* if and only if $\overline{\mathbb{Z}P}_1, \dots, \overline{\mathbb{Z}P}_r$ are nontrivial, connected, and satisfy

$$\overline{\mathbb{Z}P} = \overline{\mathbb{Z}P}_1 \times \overline{\mathbb{Z}P}_2 \times \cdots \times \overline{\mathbb{Z}P}_r.$$

The analogue of (6) for almost independent points is this:

$$\text{(7)} \quad \sum_{i=1}^{r} \phi_i(P_i) = 0 \text{ for } (\phi_1, \dots, \phi_r) \in \operatorname{Hom}(A_1 \times \cdots \times A_r, B) \implies \phi_i(P_i) = 0 \text{ for all } i.$$

Lemma 2.4. *Let $B_1, \dots, B_s$ be simple abelian varieties such that A is isogenous to $B_1 \times \cdots \times B_s$. If $\overline{\mathbb{Z}P}_1, \dots, \overline{\mathbb{Z}P}_r$ are connected and nontrivial, then the following are equivalent*:

(i) *$P_1, \dots, P_r$ are almost independent.*

(ii) *The implication (7) holds for every abelian variety B.*

(iii) *The implication (7) holds for $B = B_1, \dots, B_s$.*

Proof. If $P_1, \dots, P_r$ are almost independent and if B and $\phi \in \operatorname{Hom}(A, B)$ satisfy $\phi(P) = 0$, then $\phi_i(\overline{\mathbb{Z}P}_i) \subseteq \phi(\overline{\mathbb{Z}P}) = 0$, and so $\phi_i(P_i) = 0$ for each i; therefore (i) implies (ii). Conversely, if B is the quotient $A/\overline{\mathbb{Z}P}$ and $\phi : A \to B$ is the natural homomorphism, then (2) implies $\phi_i(\overline{\mathbb{Z}P}_i) = 0$, thus $\overline{\mathbb{Z}P}_1 \times \cdots \times \overline{\mathbb{Z}P}_r \subseteq \overline{\mathbb{Z}P}$. The reverse inclusion is trivial, thus (ii) implies (i).

It is clear (ii) implies (iii). We assume the latter holds and prove the converse. Suppose B is an abelian variety and $\phi \in \operatorname{Hom}(A, B)$ satisfies $\phi(P) = 0$. We must show $\phi_1(P_1) = \cdots = \phi_r(P_r) = 0$. In fact, the only finite quotients of $\overline{\mathbb{Z}P}_1, \dots, \overline{\mathbb{Z}P}_r$ are trivial since they are connected, hence it suffices to show $\phi_1(P_1), \dots, \phi_r(P_r)$ are torsion.

Up to rearranging $B_1, \dots, B_s$, there exists an isogeny $\psi : \phi(A) \to B_1 \times \cdots \times B_t$ for some $t \le s$. Let π_j be the projection onto the factor B_j. We have $0 = \pi_j \psi \phi(P) = \sum_i \pi_j \psi \phi_i(P_i)$, and by (iii) we deduce $\pi_j \psi \phi_i(P_i) = 0$ for every i, j. Then $\psi \phi_i(P_i) = 0$ for every i. The latter implies $\phi_1(P_1), \dots, \phi_r(P_r)$ are torsion as claimed since ψ is an isogeny. □

The following generalizes Proposition 1.5 to almost independent points:

Proposition 2.5. *Suppose $A_1, \dots, A_r$ are abelian varieties and $P_1 \in A_1(K), \dots, P_r \in A_r(K)$ are almost independent points. Let l be a prime number. If $e_1, \dots, e_r$ are nonnegative integers, then the following set has a positive Dirichlet density*:

$$\{\mathfrak{p} \in S(A_1, \dots, A_r) : \operatorname{ord}_l(P_i \bmod \mathfrak{p}) = e_i \textit{ for all } i\}.$$

Proof. Let $B_i \subseteq A_i$ be the abelian subvariety $\overline{\mathbb{Z}P}_i$. The point $P = (P_1, \dots, P_r)$ of $B := B_1 \times \cdots \times B_r$ satisfies $\overline{\mathbb{Z}P} = B$. The statement follows from Proposition 1.5. □

Domination of subgroups. Suppose $A, A_1, \dots, A_r$ are abelian varieties and let $\Gamma \subseteq A(K)$ be a subgroup.

Definition 2.6. Given subsets $M_i \subseteq A_i(K)$ for $1 \le i \le r$, we say $M_1, \dots, M_r$ *dominate* Γ if and only if the submodule $\Gamma' \subseteq A(K)$, which is generated by $\operatorname{Hom}(A_1, A)M_1, \dots, \operatorname{Hom}(A_r, A)M_r$, is such that $\Gamma \cap \Gamma'$ has finite index in Γ.

We understand that an empty set of points dominates any finite subgroup.

Lemma 2.7. *If A is an abelian variety and $\Gamma, \Gamma' \subseteq A(K)$ are submodules, then the following are equivalent:*

(i) $\Gamma \cap \Gamma'$ *has finite index in* Γ.

(ii) $\Gamma \cap \Gamma' \cap B(K)$ *has finite index in* $\Gamma \cap B(K)$ *for every simple abelian subvariety* $B \subseteq A$.

Proof. The implication (i) $\Rightarrow$ (ii) is an easy remark about abelian groups, so we only have to prove the converse. If A is simple, then (i) and (ii) are trivially equivalent, so suppose $A_1, A_2 \subseteq A$ are nontrivial complementary abelian subvarieties.

If C is an abelian variety and $\varphi : A \to C$ is an isogeny, then $\varphi(\Gamma), \varphi(\Gamma')$ have finite index in the respective submodules $\Gamma_0, \Gamma'_0 \subseteq C(K)$ they generate, thus $[\Gamma : \Gamma \cap \Gamma']$ is finite if and only if $[\Gamma_0 : \Gamma_0 \cap \Gamma'_0]$ is finite. Moreover, if $B \subseteq A$ is a simple abelian subvariety, then $[\Gamma \cap B(K) : \Gamma \cap \Gamma' \cap B(K)]$ is finite if and only if $[\Gamma_0 \cap \varphi(B)(K) : \Gamma_0 \cap \Gamma'_0 \cap \varphi(B)(K)]$ is finite. We may then suppose without loss of generality that $A = A_1 \times A_2$, so that $\Gamma = \Gamma_1 \times \Gamma_2$ and $\Gamma' = \Gamma'_1 \times \Gamma'_2$, where Γ_i, Γ'_i are submodules of $A_i(K)$ for $i = 1, 2$. By induction on $\dim(A)$, we suppose that (i) and (ii) are equivalent for $A = A_1, A_2$. If (ii) holds for Γ, Γ', A, it also holds for Γ_i, Γ'_i, A_i, where $i = 1, 2$. We deduce that $[\Gamma_i : \Gamma_i \cap \Gamma'_i]$ is finite for $i = 1, 2$, hence $[\Gamma : \Gamma \cap \Gamma']$ is finite. Thus, (ii) implies (i) as claimed. ☐

Proposition 2.8. *Suppose $A, A_1, \dots, A_r$ are abelian varieties and $\Gamma \subseteq A(K)$ is a submodule. If $P_1 \in A_1(K), \dots, P_r \in A_r(K)$ are almost independent, then either they dominate Γ, or there exists a simple abelian subvariety $A_{r+1} \subseteq A$ and a point $P_{r+1} \in \Gamma \cap A_{r+1}(K)$ such that $P_1, \dots, P_{r+1}$ are almost independent.*

Proof. Let $\Gamma' \subseteq A(K)$ be the submodule generated by $\operatorname{Hom}(A_i, A)P_i$ for $1 \le i \le r$. Suppose $P_1, \dots, P_r$ do not dominate Γ and thus $\Gamma \cap \Gamma'$ has infinite index in Γ. Then Lemma 2.7 implies $\Gamma \cap \Gamma' \cap A_{r+1}(K)$ has infinite index in $\Gamma \cap A_{r+1}(K)$ for some simple abelian subvariety $A_{r+1} \subseteq A$. Let $\Gamma_0 = \Gamma \cap A_{r+1}(K)$ and $\Gamma'_0 = \Gamma' \cap A_{r+1}(K)$. The index of $\Gamma_0 \cap \Gamma'_0$ in Γ_0 is infinite. Then since Γ_0 is a finitely generated abelian group, there exists a point $P_{r+1} \in \Gamma_0$ of infinite order such that $\mathbb{Z}P_{r+1} \cap \Gamma'_0 = \{0\}$. We will show $P_1, \dots, P_{r+1}$ are almost independent.

Let $\Gamma''_0 \subseteq \Gamma_0$ be the $\operatorname{End}(A_{r+1})$-submodule generated by P_{r+1}. If $\varphi \in \operatorname{End}(A_{r+1})$ is nonzero, then there exist $\psi \in \operatorname{End}(A_{r+1})$ and $m \ge 1$ such that $\psi\varphi$ is multiplication by m. In particular, the identity $\mathbb{Z}P_{r+1} \cap \Gamma'_0 = \{0\}$ implies $\Gamma''_0 \cap \Gamma'_0 = \{0\}$. Since P_{r+1} has infinite order and A_{r+1} is simple, we have that $\overline{\mathbb{Z}P}_{r+1} = A_{r+1}$ is nontrivial and connected.

Suppose $B_1, \dots, B_s$ are simple abelian varieties such that $A_1 \times \cdots \times A_{r+1}$ is isogenous to $B_1 \times \cdots \times B_s$, and either $B_i = B_j$ or B_i, B_j are nonisogenous if $i \neq j$. We may suppose $B_s = A_{r+1}$. Let $B \in \{B_1, \dots, B_s\}$ and $(\phi_1, \dots, \phi_{r+1}) \in \operatorname{Hom}(A_1 \times \cdots \times A_{r+1}, B)$ satisfy $\sum_{i=1}^{r+1} \phi_i(P_i) = 0$. Let $Q = \sum_{i=1}^{r} \phi_i(P_i)$. If $B \neq A_{r+1}$, then $\operatorname{Hom}(A_{r+1}, B) = \{0\}$, hence $\phi_{r+1}(P_{r+1}) = 0$. If $B = A_{r+1} \subseteq A$, we have $\phi_{r+1}(P_{r+1}) = -Q \in \Gamma'_0$, hence $\phi_{r+1}(P_{r+1})$ lies in $\Gamma'_0 \cap \Gamma''_0 = \{0\}$.

Either way, $\phi_{r+1}(P_{r+1}) = 0$, hence Lemma 2.4 implies $\phi_i(P_i) = 0$ for $1 \leq i \leq r$ since $P_1, \dots, P_r$ are almost independent. In particular, Lemma 2.4 also implies $P_1, \dots, P_{r+1}$ are almost independent as claimed. □

Any infinite submodule $\Gamma \subseteq A(K)$ contains an almost independent point (it suffices to take any point P in Γ of infinite order and should $\overline{\mathbb{Z}P}$ not be connected, replacing P by a suitable multiple). One can then use the following corollary to find finitely many points of Γ which are almost independent and dominate Γ:

Corollary 2.9. *Suppose $A, A_1, \dots, A_r$ are abelian varieties and let $\Gamma \subseteq A(K)$ be a submodule. If $P_1 \in A_1(K), \dots, P_r \in A_r(K)$ are almost independent, then either they dominate Γ, or there exist $s > r$ and points $P_{r+1}, \dots, P_s \in \Gamma$ such that $P_1, \dots, P_s$ are almost independent and dominate Γ.*

Proof. Repeated application of Proposition 2.8 yields a sequence $P_1, \dots, P_r, P_{r+1}, \dots$ of almost independent points and a strictly increasing sequence of subgroups of Γ which are dominated by those points. This process must terminate after finitely many iterations because Γ is a finitely generated abelian group, and when it does, by Proposition 2.8, the given points dominate Γ. □

3. Proof of the theorems

Order of reductions of submodules.

Theorem 3.1. *Let A, B be abelian varieties, and suppose that no element of* $\operatorname{Hom}(A, B)$ *has finite kernel. Let $\Gamma \subseteq A(K)$ be a dense submodule and l be a prime number.*

(i) *For every $e \geq 0$ the following set has positive Dirichlet density*:

$$O(A, B, \Gamma)_e := \{\mathfrak{p} \in S(A, B) : \operatorname{ord}_l(\Gamma_{\mathfrak{p}}) \geq \operatorname{ord}_l(B(K)_{\mathfrak{p}}) + e\}.$$

(ii) *If A is square-free, then for every $e \geq 0$ the following set has positive Dirichlet density*:

$$E(A, B, \Gamma)_e := \{\mathfrak{p} \in S(A, B) : \exp_l(\Gamma_{\mathfrak{p}}) \geq \exp_l(B(K)_{\mathfrak{p}}) + e\}.$$

(iii) *If A is square-free and if l is larger than a constant depending only on A, B, Γ, K, then the following set has positive Dirichlet density*:

$$R(A, B, \Gamma) := \{\mathfrak{p} \in S(A, B) : \mathrm{rad}_l(\Gamma_\mathfrak{p}) = 1, \mathrm{rad}_l(B(K)_\mathfrak{p}) = 0\}.$$

Proof. By our hypothesis on the elements of $\mathrm{Hom}(A, B)$, there exists a simple abelian variety C which occurs with multiplicity a as an isogeny factor of A, and with strictly smaller multiplicity b as an isogeny factor of B. Let $A' \subseteq A$ and $B' \subseteq B$ be abelian subvarieties for which there exist isogenies $\varphi : A \to A' \times C^a$ and $\psi : C^b \times B' \to B$. There is $d_1 > 0$ satisfying

$$\left|\mathrm{ord}_l(B(K)_\mathfrak{p}) - \mathrm{ord}_l((B' \times C^b)(K)_\mathfrak{p})\right| \leq v_l(d_1) \quad \text{for every } \mathfrak{p} \in S(A, B).$$

Moreover, if $\varphi_*(\Gamma) \subseteq (A' \times C^a)(K)$ is the submodule generated by $\varphi(\Gamma)$, then there exists $d_2 > 0$ satisfying

$$|\mathrm{ord}_l(\Gamma_\mathfrak{p}) - \mathrm{ord}_l(\varphi_*(\Gamma)_\mathfrak{p})| \leq v_l(d_2)$$

because $\varphi(\Gamma)$ has finite index in $\varphi_*(\Gamma)$; therefore,

$$O(A' \times C^a, B' \times C^b, \varphi_*(\Gamma))_{v_l(d_1 d_2)+e} \subseteq O(A, B, \Gamma)_e.$$

Similarly we have $d_3 > 0$ such that

$$E(A' \times C^a, B' \times C^b, \varphi_*(\Gamma))_{v_l(d_3)+e} \subseteq E(A, B, \Gamma)_e,$$

and such that for $l \nmid d_3$ we have:

$$R(A' \times C^a, B' \times C^b, \varphi_*(\Gamma)) \subseteq R(A, B, \Gamma).$$

Up to replacing A, B, Γ by $A' \times C^a$, $B' \times C^b$, $\varphi_*(\Gamma)$, we may suppose without loss of generality that φ, ψ are the respective identity maps.

Lemma 2.1 implies $\mathrm{Hom}(A, C)\Gamma$ is infinite since Γ is dense. It follows that $\Gamma \cap \mathrm{Hom}(C, A)(C(K))$ is infinite; thus it contains a point P which is almost independent. Let $\Gamma_0 \subseteq \Gamma$ and $\Gamma'_0 \subseteq B(K)$ be the respective submodules generated by P and $\mathrm{Hom}(A, B)P$. They are respectively isomorphic to a and b copies of the submodule of $C(K)$ generated by P; thus

$$\mathrm{ord}_l(\Gamma_\mathfrak{p}) \geq \mathrm{ord}_l(\Gamma_{0,\mathfrak{p}}) \geq \mathrm{ord}_l(\Gamma'_{0,\mathfrak{p}}) + (a - b)\,\mathrm{ord}_l(P \bmod \mathfrak{p}) \quad \text{for } \mathfrak{p} \in S(A, B).$$

Moreover, if A is square-free, then $a = 1$ and $b = 0$, so in particular, $\mathrm{Hom}(C, B) = \{0\}$ and $\Gamma'_0 = \{0\}$.

Corollary 2.9 implies there exist $t \geq 0$ points $Q_i \in B(K)$ such that together with P they are almost independent and dominate $B(K)$. Moreover, for every $m \geq 0$, the set

$$S_m := \{\mathfrak{p} \in S(A, B) : \mathrm{ord}_l(P \bmod \mathfrak{p}) = m, \mathrm{ord}_l(Q_i \bmod \mathfrak{p}) = 0 \text{ for all } i\}$$

has a positive density by Proposition 2.5. Thus it suffices to show that each of $O(A, B, \Gamma)_e$, $E(A, B, \Gamma)_e$, $R(A, B, \Gamma)$ contains S_m for some $m \geq 1$.

Let $\Gamma'_1 \subseteq B(K)$ be the submodule generated by the points Q_i. If $\Gamma' = \Gamma'_0 + \Gamma'_1$, then the index d of Γ' in $B(K)$ is finite; therefore, if $\mathfrak{p} \in S_m$, then

$$\operatorname{ord}_l(B(K)_{\mathfrak{p}}) \leq v_l(d) + \operatorname{ord}_l(\Gamma'_{0,\mathfrak{p}}) + \operatorname{ord}_l(\Gamma'_{1,\mathfrak{p}})$$

and

$$\operatorname{ord}_l(\Gamma_{\mathfrak{p}}) - \operatorname{ord}_l(B(K)_{\mathfrak{p}}) \geq (a-b)\cdot m - v_l(d) \geq m - v_l(d).$$

In particular, if $m \geq e + v_l(d)$, then $S_m \subseteq O(A, B, \Gamma)_e$, hence (i) holds. If $\mathfrak{p} \in S_m$ and if $\Gamma'_0 = \{0\}$ (for example, if A is square-free), then

$$\exp_l(B(K)_{\mathfrak{p}}) \leq v_l(d) + \exp_l(\Gamma'_{1,\mathfrak{p}}) = v_l(d),$$

while

$$\exp_l(\Gamma_{\mathfrak{p}}) \geq \operatorname{ord}_l(P \bmod \mathfrak{p}) = m \geq \exp_l(B(K)_{\mathfrak{p}}) + (m - v_l(d)).$$

Thus, if A is square-free and $m \geq e + v_l(d)$, then $S_m \subseteq E(A, B, \Gamma)_e$, hence (ii) holds. If $l \nmid d$, then $S_1 \subseteq R(A, B, \Gamma)$, hence (iii) holds. $\square$

Exponents of reductions of submodules.

Theorem 3.2. *Let $A_1, \ldots, A_r$ be abelian varieties, and $\Gamma_i \subseteq A_i(K)$ for $1 \leq i \leq r$ be submodules. Suppose that $e_r \geq \cdots \geq e_1 \geq 0$, and that for $1 \leq i < r$ we have $e_{i+1} = 0$ whenever $\Gamma_1, \ldots, \Gamma_i$ dominate Γ_{i+1}. Then there exists $d \geq 1$ (depending only on $A_1, \ldots, A_r, K, \Gamma_1, \ldots, \Gamma_r$) such that the following set has positive Dirichlet density for every prime number l:*

$$E_{l,d} = \{\mathfrak{p} \in S(A_1, \ldots, A_r) : e_i \leq \exp_l(\Gamma_i \bmod \mathfrak{p}) \leq e_i + v_l(d) \textit{ for all } i\}.$$

Proof. For $i = 1, \ldots, r$ we apply Corollary 2.9 and choose $M_i \subseteq \Gamma_i$ such that the elements of $\mathcal{B}_i = M_1 \cup \cdots \cup M_i$ are almost independent and dominate Γ_i. Let $\Gamma'_i \subseteq A_i(K)$ be the submodule generated by $\operatorname{Hom}(A_1, A_i)M_1, \ldots, \operatorname{Hom}(A_i, A_i)M_i$ so that $d_i = [\Gamma_i : \Gamma_i \cap \Gamma'_i]$ is finite.

If $\mathcal{B}_i = \mathcal{B}_{i-1}$, then $\Gamma_1, \ldots, \Gamma_{i-1}$ dominate Γ_i, hence $e_i = 0$ by hypothesis. In particular, if we define $\exp_l(M \bmod \mathfrak{p}) = \max_{P \in M \cup \{0\}} \exp_l(P \bmod \mathfrak{p})$ for a finite set M, then Proposition 2.5 implies the following set has positive density for every l:

$$S_l = \{\mathfrak{p} \in S(A_1, \ldots, A_r) : \exp_l(M_i \bmod \mathfrak{p}) = e_i \text{ for all } i\}.$$

We claim S_l is contained in $E_{l,d}$ for $d = d_1 \ldots d_r$, and thus the latter has positive density.

If $\mathfrak{p} \in S(A_1, \ldots, A_r)$, then

$$\exp_l(\Gamma_i \cap \Gamma'_i \bmod \mathfrak{p}) \leq \exp_l(\Gamma_i \bmod \mathfrak{p}) \leq \exp_l(\Gamma_i \cap \Gamma'_i \bmod \mathfrak{p}) + v_l(d).$$

If, moreover, $\mathfrak{p} \in S_l$, then

$$e_i = \exp_l(M_i \bmod \mathfrak{p}) \le \exp_l(\Gamma_i \cap \Gamma_i' \bmod \mathfrak{p}) \le \max\{e_1, \dots, e_i\} = e_i$$

because either $M_i = \varnothing$ and $e_i = 0$, or $\varnothing \ne M_i \subseteq \Gamma_i \cap \Gamma_i'$; therefore, $S_l \subseteq E_{l,d}$ as claimed. □

In particular, we may apply this theorem as soon as no Γ_i is dominated by the other submodules. We deduce that this is a necessary and sufficient condition for the set

$$\{\mathfrak{p} \in S(A_1, \dots, A_r) : \exp_l(\Gamma_i \bmod \mathfrak{p}) = e_i \text{ for all } i\}$$

to have a positive Dirichlet density for all but finitely many prime numbers l, and for every $e_1, \dots, e_r \ge 0$.

Corollary 3.3. *Suppose $A_1, \dots, A_r$ are abelian varieties, and $\Gamma_i \subseteq A_i(K)$ for $1 \le i \le r$ are submodules. If $\Gamma_1, \dots, \Gamma_i$ do not dominate Γ_{i+1} for $1 \le i < r$, then for every $m \ge 0$ and for every prime number l, the following set has positive Dirichlet density*:

$$O_{l,m} = \{\mathfrak{p} \in S(A_1, \dots, A_r) : \mathrm{ord}_l(\Gamma_{i+1} \bmod \mathfrak{p}) > \mathrm{ord}_l(\Gamma_i \bmod \mathfrak{p}) + m \text{ for } 1 \le i < r\}.$$

Proof. Let $d \ge 1$ and $E_{l,d}$ be as in Theorem 3.2. Choose $e_1 \ge 0$ and $e_{i+1} > \dim(A_i)(e_i + v_l(d)) + m$ for $1 \le i < r$. Then $E_{l,d}$ has positive density by Theorem 3.2, and it lies in $O_{l,m}$ since

$$\exp_l(\Gamma_i \bmod \mathfrak{p}) \le \mathrm{ord}_l(\Gamma_i \bmod \mathfrak{p}) \le \dim(A_i(K)) \cdot \exp_l(\Gamma_i \bmod \mathfrak{p})$$

holds for $\mathfrak{p} \in S(A_i)$. □

Proof of the main theorems.

Proof of Theorem 1.3. Suppose that property (1) fails. We show that (2), (3), and (4) fail accordingly.

If there is no homomorphism $A \to A'$ with finite kernel, then Theorem 3.1 (i) shows that (2) fails for every l and m, that (3) fails for every l and m if A is square-free, and that (4) fails if A is square-free and l is greater than a constant depending only on A, A', Γ, K.

Suppose now that there is $\varphi \in \mathrm{Hom}(A, A')$ with finite kernel. Since (1) fails, then $\varphi(\Gamma) \cap \Gamma'$ has infinite index in $\varphi(\Gamma)$, which means that Γ' does not dominate $\varphi(\Gamma)$. Consequently, Γ' does not dominate Γ. Let $A_1 = A'$, $A_2 = A$, $\Gamma_1 = \Gamma'$, and $\Gamma_2 = \Gamma$. Corollary 3.3 implies (2) fails for every l and m. Theorem 3.2 (applied with $e_1 = 0$ and $e_2 > v_l(d) + m$) implies (3) fails for every l and m, moreover (applied with $e_1 = 0$ and $e_2 = 1$), it implies (4) fails for l greater than a constant depending only on A, A', K, Γ, Γ'. □

Proof of Theorem 1.2. Applying Theorem 1.3 by taking $\Gamma = A(K)$, $\Gamma' = A'(K)$, we find in particular that A is isogenous to an abelian subvariety of A'. Moreover, by reversing the roles of Γ, Γ' we analogously find that A' is isogenous to an abelian subvariety of A, so we deduce that A, A' are isogenous. □

Proof of Theorem 1.1. This is an immediate consequence of Theorem 1.2. □

We conclude with a converse to Theorem 1.3:

Lemma 3.4. *With the notations of Theorem 1.3, property (1) implies that, for some integer $d > 0$, we have*

$$\mathrm{ord}_l(\Gamma_{\mathfrak{p}}) \le \mathrm{ord}_l(\Gamma'_{\mathfrak{p}}), \quad \exp_l(\Gamma_{\mathfrak{p}}) \le \exp_l(\Gamma'_{\mathfrak{p}}), \quad \mathrm{rad}_l(\Gamma_{\mathfrak{p}}) \le \mathrm{rad}_l(\Gamma'_{\mathfrak{p}})$$

for every $\mathfrak{p} \in S(A, A')$ and for every prime number $l \nmid d$.

Proof. Let $\varphi \in \mathrm{Hom}(A, A')$ be as in (1). Let k be the size of the kernel of φ, and c the index of $\varphi(\Gamma) \cap \Gamma'$ in $\varphi(\Gamma)$. If $\mathfrak{p} \in S(A, A')$ and letting $d = kc$, we have

$$\mathrm{ord}_l(\Gamma_{\mathfrak{p}}) \le v_l(k) + v_l(c) + \mathrm{ord}_l((\varphi(\Gamma) \cap \Gamma')_{\mathfrak{p}}) \le v_l(d) + \mathrm{ord}_l(\Gamma'_{\mathfrak{p}}).$$

Similarly, we have

$$\exp_l(\Gamma_{\mathfrak{p}}) \le v_l(d) + \exp_l(\Gamma'_{\mathfrak{p}}), \quad \mathrm{rad}_l(\Gamma_{\mathfrak{p}}) \le v_l(d) + \mathrm{rad}_l(\Gamma'_{\mathfrak{p}}).$$ □

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