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VARIATION OF COMPLEX STRUCTURES AND THE STABILITY OF KÄHLER–RICCI SOLITONS

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We investigate the linear stability of Kähler–Ricci solitons for perturbations induced by varying the complex structure within a fixed Kähler class. We calculate stability for the known examples of Kähler–Ricci solitons.

0. Introduction

We consider a stability problem for shrinking Kähler–Ricci solitons. These are critical points of the ν-functional, defined by Perelman on the space of Riemannian metrics on a closed manifold M. The main result is a formula for the second variation of this functional when restricted to perturbations obtained by varying the complex structure within a fixed Kähler class. Such perturbations were first studied by Tian and Zhu [2008] for Kähler–Einstein manifolds, and our paper attempts to extend their results to Kähler–Ricci solitons. Definitions and notation from the main theorem are explained below.

Theorem 0.1 (Main Theorem). *Let (M, g, f) be a normalised Kähler–Ricci soliton and let h be an f-essential variation. The second variation of the ν-functional at g, $\langle Nh, h\rangle_f$, is given as*

$$\langle Nh, h\rangle_f = 2\int_M f\|h\|^2 e^{-f}\, dV_g.$$

The main utility of this result is that if one had explicit knowledge of the metric and the function f then it is possible to calculate the quantity $\langle Nh, h\rangle_f$ quite easily. In Section 4, we do this for all the known examples of Kähler–Ricci solitons. Notice also that for Kähler–Einstein metrics $f = 0$ and so $N(h) = 0$, recovering a result of Tian and Zhu.

The structure of this paper is as follows: In Section 1, we begin with background on Ricci solitons and the stability problem. In Section 2, the space $\mathcal{W}(g)$ and the space of f-essential variations in the above theorem are studied. We obtain several

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useful characterizations of elements of these spaces. In Section 3, we give a proof of the main theorem. In Section 4, the stability of the known examples of Ricci solitons is investigated.

After a preliminary version of this work was posted on the arXiv, Yuanqi Wang kindly made us aware that he had independently obtained our Main Theorem as part of his Ph.D. thesis [Wang 2011]. His proof is similar to ours but proceeds by direct calculation rather than using the results of Dai, Wang and Wei. His thesis also contains interesting results about convergence of the Kähler–Ricci flow to a Kähler–Einstein metric when the complex structure is allowed to vary.

1. Ricci solitons and stability

Background on solitons. Throughout this paper, (M, g) is a smooth closed Riemannian manifold.

Definition 1.1 (Ricci soliton). Let $X \in \Gamma(TM)$ be a smooth vector field. The triple (M, g, X) is called a Ricci soliton if it satisfies the equation

$$\mathrm{Ric}(g) + L_X g = cg \tag{1-1}$$

for a constant $c \in \mathbb{R}$. If $c < 0, c = 0, c > 0$ then the soliton is referred to as expanding, steady and shrinking respectively. When $c \neq 0$, set $c = 1/2\tau$. If $X = \nabla f$ for a smooth function f then the soliton is called a gradient Ricci soliton and (1-1) becomes

$$\mathrm{Ric}(g) + \mathrm{Hess}(f) = \frac{1}{2\tau} g. \tag{1-2}$$

When the vector field X is Killing, an Einstein metric is recovered; Einstein metrics are therefore referred to as trivial Ricci solitons. We can set $c = 1$ to factor out homothety, and as one may change the soliton potential f by a constant, let us also require that

$$\int_M f e^{-f} \, dV_g = 0.$$

A soliton with these choices will be referred to as a *normalised gradient Ricci soliton*.

As well as being interesting as generalisations of Einstein metrics, Ricci solitons also occur as the fixed points of the Ricci flow

$$\frac{\partial g}{\partial t} = -2 \, \mathrm{Ric}(g) \tag{1-3}$$

up to diffeomorphism. In this paper we will be considering nontrivial Ricci solitons on compact manifolds. Foundational results due to Hamilton [1995] and Perelman [2002] imply that expanding and steady Ricci solitons on compact manifolds must be trivial. Hence our focus is on shrinking Ricci solitons. Perelman also showed that such solitons are necessarily gradient Ricci solitons. We will henceforth refer to these metrics as nontrivial shrinkers.

Due to the work of many people [Cao 1996; Dancer and Wang 2011; Koiso 1990; Podestà and Spiro 2010; Wang and Zhu 2004] there are now many (infinitely many) examples of nontrivial shrinkers. One striking feature all known nonproduct examples share is that they are Kähler. This means that $\mathrm{Hess}(f)$ is J-invariant and so the real vector field ∇f is holomorphic (see [Besse 1987, 2.124]). In this case the underlying manifold M is in fact a smooth Fano variety.

Perelman [2002] showed that gradient Ricci solitons are the critical points of a functional, which is usually denoted by $\nu(g)$. Let $f \in C^\infty(M)$ and $\tau \in \mathbb{R}$. We say that (f, τ) is compatible if

$$\int_M e^{-f}(4\pi\tau)^{-n/2} = 1.$$

Definition 1.2. The ν-functional is given by

$$\nu(g) = \inf_{\text{compatible } (f,\tau)} \int_M [(R + |\nabla f|^2)\tau + f - n] e^{-f} (4\pi\tau)^{-n/2}\, dV_g,$$

where R is the scalar curvature of g.

As well as giving a variational characterization of Ricci solitons, Perelman showed that the functional is monotonically increasing under the Ricci flow. Hence, if one could perturb a soliton in a direction that increases ν and then continue the flow, one would not flow back to the soliton and the soliton would be regarded as unstable.

Linear stability. In order to determine the behaviour of the flow around a soliton one can investigate the second variation of $\nu(g)$ for an admissible perturbation.

Definition 1.3. Let $h \in s^2(T^*M)$. Then $g + th$, $t \in \mathbb{R}^+$ is said to be an admissible perturbation. We have $\partial g/\partial t|_{t=0} = h$.

If the second variation is strictly negative then the fixed point is stable and attracting. If the second variation has positive directions then one may perturb the soliton and then flow away. Natasha Sesum [2006] has obtained fundamental results on this topic.

Proposition 1.4 [Cao et al. 2004; Cao and Zhu 2012]. *Let $h \in s^2(TM^*)$ be an admissible variation of a Ricci soliton g. The second variation of ν is given by*

$$D_g^2\nu(h, h) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M \langle Nh, h\rangle e^{-f}\, dV_g,$$

where

$$Nh = \tfrac{1}{2}\Delta_f h + \mathrm{Rm}(h, \cdot\,) + \mathrm{div}^*\mathrm{div}_f h + \tfrac{1}{2}\mathrm{Hess}(v_h) - C(h, g)\mathrm{Ric}. \tag{1-4}$$

Here $\Delta_f(\,\cdot\,) = \Delta(\,\cdot\,) - \nabla_{\nabla f}(\,\cdot\,)$, $\mathrm{div}_f(\,\cdot\,) = \mathrm{div}(\,\cdot\,) - \iota_{\nabla f}$, v_h *is the solution of the equation*

$$\Delta_f v_h + \frac{v_h}{2\tau} = \mathrm{div}_f \mathrm{div}_f(h),$$

and

$$C(h, g) = \frac{\int_M \langle \mathrm{Ric}, h\rangle e^{-f}\, dV_g}{\int_M Re^{-f}\, dV_g}.$$

This operator allows us to define the concept of linear stability.

Definition 1.5. Let (M, g, f) be a Ricci soliton. The soliton is *linearly stable* if the operator N is nonpositive definite, and *unstable* otherwise.

We now focus upon Kähler–Ricci solitons. The first result regarding stability is the following:

Theorem 1.6 [Cao et al. 2004; Hall and Murphy 2011; Tian and Zhu 2008]. *Let* (M, g, f) *be a Kähler–Ricci soliton. If* $dim H^{(1,1)}(M) > 1$ *then* (M, g, f) *is unstable.*

Kähler–Ricci solitons can be viewed as fixed points of a flow related to the Ricci flow (1-3) called the Kähler–Ricci flow, which in the Fano case can be written as

$$\frac{\partial g}{\partial t} = -\mathrm{Ric}(g) + g, \quad g(0) = g_0. \tag{1-5}$$

One important point about this flow is that it preserves the Kähler class. A foundational result about this flow, due to [Cao 1985], is that it exists for all time. The convergence of it is an extremely subtle issue because the complex structure can jump in the limit at infinity. Hence the type of convergence one expects is rather weak. This is illustrated by the following example:

Theorem 1.7 [Tian and Zhu 2007]. *Let* M *be a compact manifold which admits a Kähler–Ricci soliton* (g_{KRS}, f). *Then any solution of* (1-5) *will converge to* g_{KRS} *in the sense of Cheeger–Gromov if the initial metric* g_0 *is invariant under the maximal compact subset of the automorphism group of* M.

The unstable perturbations in Theorem 1.6 do not preserve the canonical class. Therefore, from the point of view of the Kähler–Ricci flow it is natural to consider perturbations which fix the Kähler class but allow the complex structure of the manifold to vary. This was initiated by Tian and Zhu [2008].

Definition 1.8. Let (M, g_{KRS}) be a Kähler–Ricci soliton with complex structure J_{KRS}. The space of perturbations is defined as

$$\mathcal{W}(g_{KRS}) = \Big\{ h \in s^2(TM^*) \;\Big|\; \text{there is a family of Kähler metrics } (g_t, J_t) \\ \text{with } \partial g_t/\partial t|_{t=0} = h,\ [g_t(J_t\,\cdot\,,\,\cdot\,)] = c_1(M, J_{KRS}), \\ (g_0, J_0) = (g_{KRS}, J_{KRS}) \Big\}.$$

The following result was our main motivation for considering this space of perturbations:

Theorem 1.9 [Tian and Zhu 2008]. *Let (M, g_{KE}) be a Kähler–Einstein metric and let $h \in \mathcal{W}(g_{KE})$. Then*

$$\langle N(h), h\rangle_f \le 0.$$

Tian and Zhu then conjectured that a similar result should be true for Ricci solitons. Our formula in Theorem 0.1 shows that this might not be true in general. The integral in the main theorem does not seem to have a sign in general. However, the examples we calculate in Section 4 do all have $\langle N(h), h\rangle_f = 0$; this seems be an artefact of their construction rather than a manifestation of some result in complex differential geometry.

We mention here the related study of stability by Dai, Wang, and Wei [Dai et al. 2007]. They prove that Kähler–Einstein metrics with negative scalar curvature are stable. There is also the recent work of Nefton Pali [2012] in this area. He considers a related functional known in the literature as the W-functional (here one is free to pick a volume form whereas in the definition of the ν-functional one is determined by the metric).

Notation and convention. We use the curvature convention that $\mathrm{Rm}(X, Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$. The convention for divergence that we adopt is $\mathrm{div}(h) = tr_{12}(\nabla h)$. The rough Laplacian

$$\Delta h = \mathrm{div}(\nabla h) = -\nabla^*\nabla h$$

is then negative definite. Set

$$\langle\,\cdot\,,\,\cdot\,\rangle_f = \int_M \langle\,\cdot\,,\,\cdot\,\rangle e^{-f}\, dV_g$$

to be the twisted inner product on tensors at a Ricci soliton (M, g, f). We will denote pointwise inner products induced on tensor bundles by g with round brackets $(\,\cdot\,,\,\cdot\,)$. The adjoint of a differential operator (such as ∇) with respect to this inner product will be denoted with a subscript f (for example, div_f) throughout.

2. Background on variations of complex structure

Variations of complex structure. We recall that an almost complex structure on a manifold M is a section J of the endomorphism bundle $\mathrm{End}(TM)$ satisfying $J^2 = -\,\mathrm{id}$. For M to be a complex manifold we require that the complex structure is *integrable*. By the Newlander–Nirenberg theorem we may take integrable to mean that the Nijenhuis tensor $\mathcal{N}(J) = 0$. We will be concerned with infinitesimal variations of complex structure that are modelled on those coming from a one-parameter family of complex structures J_t. As we are only working at an infinitesimal level, we don't actually mind if our variations are induced by such a family.

Definition 2.1 (Infinitesimal variation of complex structure). Let (M, g, J) be a Kähler manifold. A tensor $\zeta \in \mathrm{End}(TM)$ is called an infinitesimal variation of complex structure if it satisfies the two equations

$$\zeta J + J\zeta = 0, \tag{2-1}$$

$$\dot{\mathcal{N}}(\zeta) = 0, \tag{2-2}$$

where $\dot{\mathcal{N}}(\zeta)$ is the infinitesimal variation in the Nijenhuis tensors $\mathcal{N}(J + t\zeta)$.

Equation (2-1) simply says that the J_t are almost complex structures, and (2-2) comes from requiring that they are integrable. In the above definition we are viewing ζ as a section of the bundle $\mathrm{End}(TM)$ which is defined for any manifold. Switching in the usual manner to the complex viewpoint, (2-1) can be thought of as saying that ζ is a section of the bundle $\Lambda^{(0,1)} \otimes TM^{(1,0)}$. We will variously view the variation as an element of the real bundle $\mathrm{End}(TM)$, a section of the bundle $\Lambda^{(0,1)} \otimes TM^{(1,0)}$, and, using the metric to lower indices, as a section of $TM^* \otimes TM^*$ and $\Lambda^{(0,1)} \otimes \Lambda^{(0,1)}$. We note that in complex coordinates Equations (2-1) and (2-2) become

$$\zeta_\alpha^\beta = 0 \quad \text{and} \quad \nabla_\alpha \zeta_{\beta\gamma} = \nabla_\beta \zeta_{\alpha\gamma}.$$

The bundle $\Lambda^{(0,1)} \otimes TM^{(0,1)}$ is an element of the Dolbeault complex

$$TM^{(1,0)} \xrightarrow{\bar\partial} \Lambda^{(0,1)} \otimes TM^{(1,0)} \xrightarrow{\bar\partial} \Lambda^{(0,2)} \otimes TM^{(1,0)} \xrightarrow{\bar\partial} \cdots,$$

where $\bar\partial$ is the usual d-bar operator associated to a holomorphic vector bundle over a complex manifold. Equation (2-2) is equivalent to requiring that $\bar\partial\zeta = 0$.

Analogous to [Tian and Zhu 2008] and following [Koiso 1983], we will decompose the space of infinitesimal variations into *trivial* variations and *f-essential* variations.

By analogy with the twisted inner product, set

$$\Delta_{\bar\partial, f} := \bar\partial\bar\partial_f^* + \bar\partial_f^*\bar\partial$$

to be the twisted $\bar\partial$-Laplacian.

Definition 2.2 (f-essential variation). Let ζ be an infinitesimal variation of the complex structure J. We say ζ is trivial if $\zeta = L_Z J$ for a smooth vector field $Z \in TM$. A variation ζ is said to be f-essential if

$$\int_M \langle \zeta, L_Z J\rangle e^{-f}\, dV_g = 0$$

for all $Z \in \Gamma(TM)$.

The following lemma gives a useful characterisation of f-essential variations:

Lemma 2.3 [Koiso 1983, Lemma 6.4]. *Let ζ be an f-essential variation and let $h(\,\cdot\,,\cdot\,) = \omega(\,\cdot\,, \zeta\,\cdot\,)$. If h is symmetric then*

(1) $\bar\partial_f^* \zeta = 0$, *and*

(2) $\operatorname{div}_f h = 0$.

In particular, an f-essential variation is $\Delta_{\bar\partial,f}$-harmonic.

Proof. (1) As ζ is f-essential,

$$\int_M \langle L_Z J, \zeta\rangle e^{-f} = 0$$

for all $Z \in \Gamma(TM)$. The Lie derivative of the complex stucture is related to the $\bar\partial$-operator by

$$\bar\partial.\, Z = -\tfrac{1}{2} J L_Z J(\,\cdot\,).$$

Hence, up to a constant, $\langle L_Z J, \zeta\rangle_f = \langle \bar\partial Z, \zeta\rangle_f$ and $\bar\partial_f^* \zeta = 0$, as claimed.

(2) We begin by noting that ζ being f-essential means that

$$\langle L_Z J, \zeta\rangle_f = \langle \omega(\,\cdot\,, L_Z J(\,\cdot\,)), h\rangle_f = 0.$$

Rewriting and using the Cartan formula we have

$$\omega(\,\cdot\,, L_Z J(\,\cdot\,)) = L_Z g(\,\cdot\,,\cdot\,) - L_Z \omega(\,\cdot\,,\cdot\,) = 2\operatorname{div}^* Z^\flat(\,\cdot\,,\cdot\,) - (d\circ\iota_Z\omega)(\,\cdot\,, J\cdot\,).$$

The result follows by noting that

$$\langle (d\circ\iota_Z\omega)(\,\cdot\,, J\cdot\,), h\rangle_f = -\langle (d\circ\iota_Z\omega)(\,\cdot\,,\cdot\,), h(\,\cdot\,, J\cdot\,)\rangle_f,$$

and that $h(\,\cdot\,, J\cdot\,)$ is symmetric. □

In the previous lemma we have assumed that h is symmetric. This is not strictly necessary by the following argument: If there existed an antisymmetric, $\Delta_{\bar\partial,f}$-harmonic section of $\Lambda^{(0,1)} \otimes TM^{(1,0)}$ then there would have to exist an antisymmetric $\Delta_{\bar\partial}$-harmonic section of $\Lambda^{(0,1)} \otimes TM^{(1,0)}$ as

$$\mathcal{H}^{p,q}(E) \equiv H^q(M, E\otimes\Lambda^{(p,0)}) \equiv \mathcal{H}_f^{p,q}(E)$$

for any holomorphic vector bundle E. The Dai–Wei–Wang Weitzenböck formula (Lemma 3.3) and Lemma 3.4 then imply that the associated (0, 2)-form is parallel. This would imply that $h^{0,2}(M) > 0$. One can then appeal to a classical result of Bochner to show that on a Fano manifold such a holomorphic form cannot exist (see [Besse 1987, 11.24]). Tian and Zhu [2008] give a straightforward proof of this fact in the case one is at a Kähler–Einstein metric.

Tian and Zhu decompose the space $\mathcal{W}(g)$ modulo the action of the diffeomorphism group. They show that

$$\mathcal{W}(g)/\mathcal{D}(M) = \mathcal{A}^{(1,1)} \oplus H^1(M, TM),$$

where $\mathcal{A}^{(1,1)}$ is the space of $\partial\bar{\partial}$-exact (1,1)-forms and $H^1(M, TM)$ is the usual cohomology for the holomorphic vector bundle TM. Tian and Zhu then show that for a general Kähler–Ricci soliton, $N|_{\mathcal{A}^{(1,1)}} \leq 0$ so that potentially destabilising elements of $\mathcal{W}$ actually lie in $H^1(M, TM)$ (they then show that N vanishes on this space when g is an Einstein metric). Hence we will only consider perturbations in $H^1(M, TM)$ and we will use the special representatives given by f-essential perturbations. Formally:

Proposition 2.4 [Tian and Zhu 2008]. *Let (M, g_{KRS}, J) be a Kähler–Ricci soliton. Then we have the following decomposition*:

$$\mathcal{W}(g_{KRS})/\mathcal{D}(M) \cong \mathcal{A}^{(1,1)}(M, J) \oplus H^1(M, TM),$$

where $\mathcal{D}(M)$ is the diffeomorphism group of M. The operator N is nonpositive when restricted to $\mathcal{A}^{(1,1)}(M, J)$.

3. Proof of Main Theorem

Consider an f-essential variation of the complex structure $h \in H^1(M, TM)$. Firstly, as h is J-anti-invariant it is apparent that $C(h, g) = 0$. Thus

$$\langle N(h), h\rangle_f = \langle \tfrac{1}{2}\Delta_f h + \mathrm{Rm}(h, \cdot), h\rangle_f.$$

In order to evaluate the above we will use a Weitzenböck formula. In order to explain the formula we will digress briefly into the spinorial construction used in [Dai et al. 2007]. This is a powerful generalisation of the techniques used by Koiso [1983].

As M is Fano it has a canonical spin^c structure and parallel spinor $\sigma_0 \in \Gamma(\mathcal{S}^c)$, where $\mathcal{S}^c \to M$ is the spin^c spinor bundle. This induces a map

$$\Phi : s^2(TM^*) \to \mathcal{S}^c \otimes TM^*,$$
$$\Phi(h) = h_{ij} e_i \cdot \sigma_0 \otimes e^j,$$

where $\{e_i\}$ is an orthonormal basis of TM and $e_i \cdot \sigma_0$ denotes Clifford multiplication in $\mathcal{S}^c$.

For $1 \le i \le m$, following [Dai et al. 2007], choose

$$X_i = \frac{e_i - \sqrt{-1}Je_i}{\sqrt{2}} \quad \text{and} \quad \bar{X}_i = \frac{e_i + \sqrt{-1}Je_i}{\sqrt{2}}.$$

Then $\{X_1, \dots, X_m\}$ is a local unitary frame for $T^{1,0}M$. Set $\{\theta^1, \dots, \theta^m\}$ to be its dual frame. Then

$$\Phi(h) = h(\bar{X}_i, \bar{X}_j)\bar{\theta^i} \otimes \bar{\theta^j}.$$

This can be identified with

$$\Psi(h) = h(\bar{X}_i, \bar{X}_j)\bar{\theta^i} \otimes X_j \in \textstyle\bigwedge^{0,1}(TM^{1,0}),$$

where $TM^{1,0}$ is the holomorphic tangent bundle.

Lemma 3.1 [Dai et al. 2005, Lemma 2.3]. *For $h, \tilde{h} \in s^2(TM^*)$,*

$$\mathrm{Re}(\Phi(h), \Phi(\tilde{h})) = (h, \tilde{h}).$$

We will also need the following, which is a result of the calculations on page 680 of [Dai et al. 2007]:

Lemma 3.2. *Let (M, g) be a Fano manifold with canonical spin^c spinor bundle $\mathcal{S}^c$ and Dirac operator D. Let Φ and Ψ be defined as above. Then*

$$D\Phi(h) = \sqrt{2}(\bar{\partial} - \bar{\partial}^*)\Psi(h).$$

The main result we need is the following Weitzenböck formula:

Lemma 3.3 [Dai et al. 2007, Lemma 2.3]. *Let $h \in s^2(TM^*)$ and let D be the Dirac operator. Then*

$$D^*D(\Phi(h)) = \Phi(\nabla^*\nabla h - 2\mathrm{Rm}(h, \cdot) + \mathrm{Ric} \circ h - h \circ i\rho), \tag{3-1}$$

where ρ is the Ricci form.

In order to deal with the Ricci curvature terms we use the following lemma, which is implicit in the proof of Theorem 2.5 in [Dai et al. 2007]:

Lemma 3.4. *Let h be a skew-hermitian section of $s^2(TM^*)$. Then*

$$(\mathrm{Ric} \circ h - h \circ i\rho, h) = 0.$$

Proof. This is a pointwise calculation. Choose normal coordinates at $p \in M$, $\{e_1, \dots, e_{2m}\}$, where $e_{m+i} = Je_i$ for $1 \le i \le m$. We can also choose this basis so that the Ricci tensor is diagonalised; that is, $\mathrm{Ric}(e_i, e_j) = c_i\delta_{ij}$, where $c_{m+i} = c_i$. We have

$$\mathrm{Re}(\Phi(\mathrm{Ric}\circ h), \Phi(h)) = \sum_{i,j=1}^{2m} c_i h_{ij}^2,$$

$$-\mathrm{Re}(\Phi(h\circ i\rho), \Phi(h)) = -2\sum_{j=1}^{m}\sum_{i=1}^{m} c_j(h_{(i+m)j}h_{i(j+m)} - h_{ij}h_{(i+m)(j+m)}).$$

If h is skew-Hermitian then

$$h_{ij} = -h_{(i+m)(j+m)} \quad \text{and} \quad h_{i(j+m)} = h_{(i+m)j}.$$

Hence

$$-\mathrm{Re}(\Phi(h\circ i\rho), \Phi(h)) = -2\sum_{i=1}^{m}\sum_{j=1}^{2m} c_j(h_{ij}^2) = -\sum_{i,j=1}^{2m} c_i h_{ij}^2,$$

and the result follows. □

The final lemma we need to prove the main result in this section is a technical lemma to deal with the extra term one obtains by using the rescaled volume form $e^{-f}dV_g$.

Lemma 3.5. *Let $A \in \Omega^1(M)$ be a one-form and $B \in \bigotimes^k TM^*$. Then*

(1) $\mathrm{div}(A\otimes B) = \mathrm{div}(A)\otimes B + \nabla_{A^\sharp}B$,

(2) $\mathrm{div}(df\otimes h) = (\Delta f)h + \nabla_{\nabla f}h$, *and*

(3) $-\langle\nabla_{\nabla f}h, h\rangle_f = \frac{1}{2}\int_M \Delta_f f\|h\|^2 e^{-f}\, dV_g$.

Proof. (1) We calculate using a normal, orthonormal basis $\{e_i\}$,

$$\mathrm{div}(A\otimes B) = \nabla_{e_i}(A\otimes B)(e_i,\cdot\,) = \mathrm{div}(A)\otimes B + \nabla_{A^\sharp}B.$$

(2) We use $A = df$, $B = h$ in (1).

(3) We note that

$$\langle\nabla_{\nabla f}h, h\rangle_f = \langle\iota_{\nabla f}\nabla h, h\rangle_f = \langle\nabla h, df\otimes h\rangle_f = -\langle h, \mathrm{div}_f(df\otimes h)\rangle_f.$$

Now using (2) we have

$$\begin{aligned}\langle\nabla_{\nabla f}h, h\rangle_f &= \int_M |\nabla f|^2\|h\|^2 e^{-f}\, dV_g - \langle h, \mathrm{div}(df\otimes h)\rangle_f\\ &= -\int_M (\Delta_f f)\|h\|^2 e^{-f}\, dV_g - \langle\nabla_{\nabla f}h, h\rangle_f.\end{aligned}$$ □

As is well known, the soliton potential function of a normalised gradient Ricci soliton solves the equation

$$\Delta_f f = -2f.$$

Proof of Main Theorem. Lemmas 3.1, 3.2, and 3.3 yield that pointwise

$$\begin{aligned}\big(\tfrac{1}{2}\Delta h+\mathrm{Rm}(h,\cdot\,),h\big)&=\mathrm{Re}\big(\Phi(\tfrac{1}{2}\Delta h+\mathrm{Rm}(h,\cdot\,)),\Phi(h)\big)\\&=\mathrm{Re}\big((D^*D\Phi(h),\Phi(h))\big)\\&=\mathrm{Re}\big(-2\Delta_{\bar\partial}\Psi(h),\Psi(h)\big).\end{aligned}$$

However, as h is f-essential then $\Psi(h)$ is orthogonal to the image of $\Delta_{\bar\partial}$ with respect to the global inner product. Hence

$$\int_M\big(\tfrac{1}{2}\Delta h+\mathrm{Rm}(h,\cdot\,),h\big)e^{-f}\,dV_g=0. \qquad \square$$

4. Examples and applications

Setup. As mentioned in the introduction, there are three main sources for concrete examples of Kähler–Ricci solitons: the Dancer–Wang, Podestà–Spiro, and the Wang–Zhu examples. The Wang–Zhu solitons exist on toric-Kähler manifolds and are nontrivial precisely when the Futaki invariant is nonzero. Unfortunately, this class of manifold does not admit any nontrivial deformations of complex structure:

Theorem 4.1 [Bien and Brion 1996, Theorem 3.2]. *Every Fano toric-Kähler manifold M has $H^1(M,TM)=0$.*

Similarly, one can see the Podestà–Spiro examples are rigid. The next class of examples to investigate are provided by the Dancer–Wang solitons. These solitons are generalisations of the soliton on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ constructed by Koiso [1990] and Cao [1996]. We begin by reviewing their construction.

Let (V_i,r_i,J_i), $1\le i\le r$ be Fano Kähler–Einstein manifolds with first Chern class $c_1(V_i,J_i)=p_ia_i$, where p_i are positive integers and $a_i\in H^2(V_i;\mathbb{Z})$ are indivisible classes. The Kähler–Einstein metrics r_i are normalised so that $\mathrm{Ric}(r_i)=p_ir_i$. For $q=(q_1,\dots,q_r)$ with $q_i\in\mathbb{Z}-\{0\}$, let P_q be the total space of the principal $U(1)$-bundle over $B:=V_1\times V_2\times\cdots\times V_r$ with Euler class $\sum_1^r q_i\pi_i^*a_i$, where

$$\pi_i:V_1\times\cdots\times V_r\to V_i$$

is the projection onto the i-th factor. Denote by M_0 the product $I\times P_q$ for the unit interval I. We denote by θ the principal $U(1)$-connection on P_q with curvature

$$\Omega:=\sum_{i=1}^r q_i\pi_i^*\eta_i,$$

where η_i is the Kähler form of r_i. There is a one-parameter family of metrics on P_q given by

$$g_t:=f^2(t)\theta\otimes\theta+\sum_{i=1}^r l_i^2(t)\pi_i^*r_i,$$

where f and l_i are smooth functions on I with prescribed boundary behaviour. Finally, consider the metric on M_0 given by

$$g = dt^2 + g_t,$$

with the correct boundary behaviour of f and the l_i. This metric then extends to a metric on a compactification of M_0, which we denote M.

The complex structure on this manifold can be described explicitly by lifting the complex structure on the base and requiring that $J(N) = -f(t)^{-1}Z$, where $N = \partial_t$ is normal to the hypersurfaces, and Z is the Killing vector that generates the isometric $U(1)$ action on P_q.

Deformations of Dancer–Wang solitons. The Ricci soliton equations in this setting reduce to a system of ODEs. We have the following existence theorem:

Theorem 4.2 [Dancer and Wang 2011, Theorem 4.30]. *Let M denote the compactification of M_0 as above. Then M admits a Kähler–Ricci soliton (M, g, u) which is Einstein if and only if the associated Futaki invariant vanishes.*

We refer to [Dancer and Wang 2011] for details of the constructions. If one chooses the components V_i to be homogeneous Kähler–Einstein manifolds then the resulting M is toric. However, by choosing the components V_i to be nonhomogeneous, Fano and Kähler–Einstein, and calculating the Futaki invariant, they give examples of nontoric Kähler–Ricci solitons. It is these that may admit complex deformations.

Suppose that V_i is a Fano, Kähler–Einstein manifold admitting deformations of its complex structure J_i. We consider an essential variation h_i in the Kähler metric r_i such that the Kähler form $\eta_i = r_i(J_i\,\cdot\,,\,\cdot\,)$ remains in the class $c_1(V_i, J_0)$. This induces a variation in the metric on the whole space given by

$$h = l_i^2(t)\pi^* h_i.$$

Clearly the same procedure works for any product of Kähler–Einstein manifolds with some (or all) of the factors admitting complex deformations. Here it is simply stated for one factor for simplicity. Let us state our final result:

Theorem 4.3. *For this perturbation h, one has $N(h) = 0$.*

Proof. It follows from the construction of h that the pointwise norm $\|h\|$ is independent of t. It also follows that if h_i is essential then h is u-essential. We see now that

$$\langle Nh, h\rangle = \int_M u\|h\|^2 e^{-u}\, dV_g = \|h\|^2_{L^2}(V_i) \int_I u e^{-u}\, dt = 0. \qquad \square$$

Remark 4.4. The significance of this result is that it verifies Tian–Zhu's conjecture for every obvious example of a complex deformation of the known Kähler–Ricci solitons. We do not know of any explicit deformations beyond these.

It is notable that for all f-essential perturbations h known to us, one has $N(h) = 0$. Understanding if this is always the case would involve calculating $H^1(M, TM)$, which is not easy to calculate in general.

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