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For a pseudoconvex domain $D \subset \mathbb{C}^n$, we prove the equivalence of the local hypoellipticity of the system $(\bar\partial, \bar\partial^*)$ with the system $(\bar\partial_b, \bar\partial_b^*)$ induced at the boundary. This develops a former result of ours in which the theory of harmonic extension by Kohn was used. This technique is inadequate for the purpose of the present paper and must be replaced by that of the holomorphic extension.

Let D be a pseudoconvex domain of $\mathbb{C}^n$ defined by $r < 0$ with C^∞ boundary bD. We use the standard notation $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ for the complex Laplacian, $Q(u, u) = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ for the energy form, and some variants such as $Q_{\mathrm{Op}}(u, u) = \|\,\mathrm{Op}\,\bar\partial u\|^2 + \|\,\mathrm{Op}\,\bar\partial^* u\|^2$ for an operator Op. Here u is a $(0, k)$ form belonging to $D_{\bar\partial^*}$. We similarly define the tangential versions as $\Box_b$, $\bar\partial_b$, $\bar\partial_b^*$, and Q^b_{Op}. We take local coordinates (x, r) in $\mathbb{C}^n$, with $x \in \mathbb{R}^{2n-1}$ being the tangential coordinates and r, the equation of bD, serving as the last coordinate. We define the tangential s-Sobolev norm by $|||u|||_s := \|\Lambda^s u\|_0$, where Λ^s is the standard tangential pseudodifferential operator with symbol $\Lambda^s_\xi = (1 + |\xi|^2)^{s/2}$. We note that

$$\begin{cases} \|\bar\partial u\|_s^2 + \|\bar\partial^* u\|_s^2 = \sum_{j \le s} Q_{\Lambda^{s-j}\partial_r^j}(u, u), \\ |||\bar\partial u|||_s^2 + |||\bar\partial^* u|||_s^2 = Q_{\Lambda^s}(u, u), \\ \|\bar\partial_b u_b\|_s^2 + \|\bar\partial_b^* u_b\|_s^2 = Q^b_{\Lambda^s}(u_b, u_b). \end{cases} \tag{1-1}$$

We decompose u into a tangential and normal component; that is,

$$u = u^\tau + u^\nu,$$

and further decompose into microlocal components (see [Kohn 2002])

$$u^\tau = u^{\tau+} + u^{\tau-} + u^{\tau 0}.$$

We similarly decompose u_b as $u_b^+ + u_b^- + u_b^0$. We use the notation $\bar L_n$ for the normal $(0, 1)$-vector field and $\bar L_1, \dots, \bar L_{n-1}$ for the tangential ones. Therefore we have the

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description for the totally real tangential and normal vector fields, denoted by T and ∂_r respectively:

$$\begin{cases} T = i(L_n - \bar{L}_n), \\ \partial_r = L_n + \bar{L}_n. \end{cases}$$

From this, we get back $\bar{L}_n = \frac{1}{2}(\partial_r + iT)$. We denote the symbol of a (pseudo)-differential operator by σ and the partial tangential Fourier transform of u by $\tilde{u}$. We define a holomorphic extension (see [Khanh and Zampieri 2011]) $u^{\tau+(H)}$ of $u^{\tau+}|_{bD}$ by

$$u^{\tau+(H)} = (2\pi)^{-2n+1} \int_{\mathbb{R}^{2n-1}} e^{ix\xi} e^{r\sigma(\dot{T})} \psi^+(\xi)\tilde{u}(\xi, 0)\, d\xi, \tag{1-2}$$

where $\dot{T} := T(x, 0)$. Note that $\sigma(T) \gtrsim (1 + |\xi|^2)^{\frac{1}{2}}$ for ξ in supp ψ^+ and (x, r) in a local patch; thus in the integral the exponential is dominated by $e^{-|r|(1+|\xi|^2)^{1/2}}$ for $r < 0$. Differently from the harmonic extension by Kohn, the present one is well defined only in positive microlocalization. We can think of $u^{\tau+(H)}$ in two different ways: as a modification of $u^{\tau+}$, or as an extension of u_b^+. The property which motivates the terminology of holomorphic extension is

$$\|\bar{L}_n u^{\tau+(H)}\| = \|r \operatorname{Tan} u^{\tau+(H)}\| \le \|u_b^{\tau+}\|_{-\frac{1}{2}}. \tag{1-3}$$

This follows from the relationships $\bar{L}_n = \frac{1}{2}(\partial_r + iT)$ and $T - \dot{T} = r \operatorname{Tan}$. We have our first relationship between a trace v_b and the general extension v ([Kohn 2002] p. 241); for any ϵ and suitable c_ϵ,

$$\|v_b\|_s \lesssim c_\epsilon |||v|||_{s+\frac{1}{2}} + \epsilon |||\partial_r v|||_{s-\frac{1}{2}}. \tag{1-4}$$

This is also seen in [Khanh and Zampieri 2011] as the small/large constant argument. As a specific property of our extension we have the reciprocal relation to (1-4):

$$\|r^k u^{\tau+(H)}\|_s \lesssim \|u_b^+\|_{s-k-\frac{1}{2}}. \tag{1-5}$$

This is readily checked; see [Khanh and Zampieri 2011, (1.12)].

A combination of (1-3) and (1-4) shows that $\bar{L}_n$ acts on $u^{\tau+(H)}$ as an operator of order 0. On the other hand, on the straightening of $b\Omega$ in which $r = x_n$, we have that $J\partial_r$ — i.e., T — coincides with ∂_{y_n}, and therefore $\bar{L}_n$ is the Cauchy–Riemann operator $\partial_{\bar{z}_n}$. A reference to the related literature is in order. The extension of generalized functions to half-spaces or wedges of $\mathbb{C}^n$ using the decomposition of the δ-function in plane waves as in (1-2) was introduced by Sato, Kashiwara, and Kawai in [Sato et al. 1973] as a general method for microlocal decomposition of the singularities. It has been used, among others, by Boutet de Monvel and Sjöstrand [1976] and by Hsiao [2010] in the study of the singularities of Szegő and Bergman kernels.

We denote by the symbol $\bar\partial^\tau$ the extension of the $\bar\partial_b$ from $b\Omega$ to Ω, which stays tangential to the level surfaces $r \equiv \text{const}$. It acts on tangential forms u^τ and its action is $\bar\partial^\tau u^\tau = (\bar\partial u^\tau)^\tau$. We denote its adjoint by $\bar\partial^{\tau\,*}$; thus $\bar\partial^{\tau\,*} u^\tau = \bar\partial^*(u^\tau)$. We use the notations $\Box^\tau$ and Q^τ for the corresponding Laplacian and energy forms. We notice that

$$Q(u^{\tau+(H)}, u^{\tau+(H)}) = Q^\tau(u^{\tau+(H)}, u^{\tau+(H)}) + \|\bar L_n u^{\tau+(H)}\|_0^2. \tag{1-6}$$

We have to describe how (1-4) and (1-5) are affected by $\bar\partial$ and $\bar\partial^*$.

Proposition 1.1. *We have for any extension v of v_b that*

$$Q^b(v_b, v_b) \lesssim Q^\tau_{\Lambda^{\frac12}}(v, v) + Q^\tau_{\partial_r \Lambda^{-\frac12}}(v, v), \tag{1-7}$$

and specifically for $u^{\tau+(H)}$,

$$Q^\tau(u^{\tau+(H)}, u^{\tau+(H)}) \lesssim Q^b_{\Lambda^{-\frac12}}(u_b^+, u_b^+) + \|u_b^+\|^2_{-\frac12}. \tag{1-8}$$

Proof. We have

$$\bar\partial^\tau v|_{bD} = \bar\partial_b v_b, \qquad \bar\partial^{\tau*} v|_{bD} = \bar\partial_b^* v_b.$$

Then, (1-7) follows from (1-4).

We proceed to prove (1-8). We have $\bar\partial^\tau = \bar\partial_b + r\,\mathrm{Tan}$, and $\bar\partial^{\tau*} = \bar\partial_b^* + r\,\mathrm{Tan}$, which yields

$$\begin{aligned} \bar\partial^\tau u^{\tau+(H)} &= (\bar\partial_b u_b)^{\tau+(H)} + r\,\mathrm{Tan}\; u^{\tau+(H)}, \\ \bar\partial^{\tau*} u^{\tau+(H)} &= (\bar\partial_b^* u_b)^{\tau+(H)} + r\,\mathrm{Tan}\; u^{\tau+(H)}. \end{aligned} \tag{1-9}$$

Application of (1-5) yields

$$\begin{aligned} &\|\bar\partial^\tau u^{\tau+(H)}\|^2 + \|\bar\partial^{\tau*} u^{\tau+(H)}\|^2 \\ &\qquad = \|(\bar\partial_b u_b)^{\tau+(H)}\|^2 + \|(\bar\partial_b^* u_b)^{\tau+(H)}\|^2 + \|r\,\mathrm{Tan}\; u^{\tau+(H)}\|^2 \\ &\qquad \lesssim \|\bar\partial_b u_b^+\|^2_{-\frac12} + \|\bar\partial_b^* u_b^+\|^2_{-\frac12} + \|u_b^+\|^2_{-\frac12}. \end{aligned}$$ □

We decompose $u^{\tau+}$ as $u^{\tau+(H)} + u^{\tau+(0)}$, which also serves as a definition of $u^{\tau+(0)}$. Let ζ and ζ' be cut-offs with $\zeta \prec \zeta'$ in the sense that $\zeta'|_{\mathrm{supp}\,\zeta} \equiv 1$.

Proposition 1.2. *Each of the forms $u^\# = u^\nu, u^{\tau\,-}, u^{\tau\,0}, u^{\tau+(0)}, u_b^-$, and u_b^0 enjoy elliptic estimates; that is,*

$$\|\zeta u^\#\|_s \lesssim \|\zeta' \bar\partial u^\#\|_{s-1} + \|\zeta' \bar\partial^* u^\#\|_{s-1} + \|u^\#\|_0, \quad s \geq 2. \tag{1-10}$$

Proof. Estimate (1-10) follows by iteration from

$$\|\zeta u^\#\|_s \lesssim \|\zeta \bar\partial u^\#\|_{s-1} + \|\zeta \bar\partial^* u^\#\|_{s-1} + \|\zeta' u^\#\|_{s-1}. \tag{1-11}$$

As for u^ν and $u^{\tau+(0)}$, this latter follows from $u^\nu|_{bD}\equiv 0$ and $u^{\tau+(0)}|_{bD}\equiv 0$. For the terms with $-$ and 0, this follows from the fact that $|\sigma(T)|\lesssim|\sigma(\bar\partial)|$ in the region of 0-microlocalization, and from $\sigma[\bar\partial,\bar\partial^*]\le 0$ and $\sigma(T)<0$ in the negative microlocalization. We refer to (1) in the Main Theorem of [Folland and Kohn 1972] as a general reference, but also give an outline of the proof. We start from

$$\tag{1-12} |||\zeta u^\#|||_1^2\lesssim Q\big(\zeta u^\#,\zeta u^\#\big)+\|\zeta' u^\#\|_0^2;$$

this is the basic estimate in the case of u^ν and $u^{\tau+(0)}$ (which vanish at bD), and it is Lemma 8.6 of [Kohn 2002] for $u^{\tau\,-}$, $u^{\tau\,0}$ and u_b^-, u_b^0. Applying (1-12) to $\zeta\Lambda^{s-1}\zeta u^\#$ one gets the estimate of tangential norms for any s; that is, (1-11) with the usual norm replaced by the triplet norm. Finally, by noncharacteristicity of $(\bar\partial,\bar\partial^*)$, one passes from tangential to full norms along the guidelines of [Zampieri 2008, Theorem 1.9.7]. The version of this argument for $\Box$ can be found in [Kohn 2002, second part of p. 245]. $\Box$

Let s and l be indices.

Theorem 1.3. *Consider the estimates*

$$\tag{1-13} \|\zeta u_b\|_s\lesssim\|\zeta'\bar\partial_b u_b\|_{s+l}+\|\zeta'\bar\partial_b^* u_b\|_{s+l}+\|u_b\|_0\quad\text{for any } u_b\in C^\infty(b\Omega),$$

$$\tag{1-14} \|\zeta u\|_s\lesssim\|\zeta'\bar\partial u\|_{s+l}+\|\zeta'\bar\partial^* u\|_{s+l}+\|u\|_0\quad\text{for any } u\in D_{\bar\partial^*}\cap C^\infty(\bar\Omega),$$

$$\tag{1-15} \|\zeta u\|_s\le\epsilon(\|\zeta\bar\partial u\|_s+\|\zeta\bar\partial^* u\|_s)+c_\epsilon\|u\|$$
$$\text{for any } \epsilon, \text{for suitable } c_\epsilon, \text{and for any } u\in D_{\bar\partial^*}\cap C^\infty(\bar\Omega).$$

Then (1-13) *implies* (1-14) *and* (1-15) *implies* (1-13) *for* $l=0$.

Remark 1.4. (i) The above estimates (1-13) and (1-14) for any s, ζ, ζ' and for suitable l, characterize the local hypoellipticity of the system $(\bar\partial_b,\bar\partial_b^*)$ and $(\bar\partial,\bar\partial^*)$ respectively (see [Kohn 2005]). When $l>0$, one says that the system has a loss of l derivatives; when $l<0$, one says that it has a gain of $-l$ derivatives.

(ii) The point in (1-15), as opposed to (1-13) and (1-14), is that we have the same cut-off ζ in both sides, and also that there is a factor ϵ of compactness. Though (1-15) is stronger than (1-14), there are wide classes of domains Ω for which it holds, including all domains of infraexponential type, for which a superlogarithmic estimate holds (see [Baracco et al. 2014]). Indeed, let R^s be the pseudodifferential operator defined by $\widetilde{R^s u}=\Lambda_\xi^{s\sigma(x)}\tilde u$ (see [Kohn 2002, p. 234]). On one hand, we have $R^s\sim\Lambda^s$ modulo operators of order $-\infty$ over u such that $\sigma|_{\mathrm{supp}\,u}\equiv 1$. On the other, we have that $[R^s,\zeta']$ has order $-\infty$ if $\zeta'|_{\mathrm{supp}\,\sigma}\equiv 1$ and hence the supports of σ and $\dot\zeta'$ are disjoint. Finally, we have

$$|\zeta''[\bar\partial,R^s]\zeta'|\lesssim\log\Lambda R^s\zeta'$$

in the sense of operators when $\sigma \prec \zeta' \prec \zeta''$. Using R^s as a substitute for Λ^s, we can prove (1-15) whenever a superlogarithmic estimate holds (see [Kohn 2002, § 7]).

Proof. First, it is clearly not restrictive that u and u_b have compact support. Because of Proposition 1.2, it suffices to prove (1-13) for u_b^+ and (1-14) for $u^{\tau\,+}$. It is also obvious that we can consider cut-off functions ζ and ζ' only in tangential coordinates, not in r. We start by proving that (1-13) implies (1-14). We recall the decomposition

$$u^{\tau+} = u^{\tau+(H)} + u^{\tau+(0)}$$

and begin by estimating $u^{\tau+(H)}$. We then have

$$\begin{aligned}
\text{(1-16)}\qquad |||\zeta u^{\tau+(H)}|||_s^2 &\underset{\text{(1-5)}}{\lesssim} \|\zeta u_b^+\|_{s-\frac{1}{2}}^2 \\
&\underset{\text{(1-13)}}{\lesssim} Q^b_{\Lambda^{s+l-\frac{1}{2}}\zeta'}(u_b^+, u_b^+) + \|u_b^+\|_{-\frac{1}{2}}^2 \\
&\underset{\text{(1-7)}}{\lesssim} Q^\tau_{\Lambda^{s+l}\zeta'}(u^{\tau+}, u^{\tau+}) + Q^\tau_{\partial_r\Lambda^{s+l-1}\zeta'}(u^{\tau+}, u^{\tau+}) + \|u^{\tau+}\|_0^2.
\end{aligned}$$

It remains to estimate $u^{\tau+(0)}$. Since $u^{\tau+(0)}|_{bD} \equiv 0$, then by 1-elliptic estimates

$$\begin{aligned}
\text{(1-17)}\quad & |||\zeta u^{\tau+(0)}|||_s^2 \\
&\underset{\text{(1-11)}}{\lesssim} Q_{\Lambda^{s-1}\zeta}(u^{\tau+(0)}, u^{\tau+(0)}) + |||\zeta' u^{\tau+(0)}|||_{s-1}^2 \\
&\lesssim Q_{\Lambda^{s-1}\zeta}(u^{\tau+}, u^{\tau+}) + Q^\tau_{\Lambda^{s-1}\zeta}(u^{\tau+(H)}, u^{\tau+(H)}) + |||r\zeta u^{\tau+(H)}|||_s^2 + |||\zeta' u^{\tau+(0)}|||_{s-1}^2 \\
&\lesssim Q_{\Lambda^{s-1}\zeta}(u^{\tau+}, u^{\tau+}) + |||\zeta u^{\tau+(H)}|||_s^2 + |||\zeta' u^{\tau+(H)}|||_{s-1}^2 + |||\zeta' u^{\tau+(0)}|||_{s-1}^2,
\end{aligned}$$

where we have used $Q = Q^\tau + O(r)\Lambda$ over $h^{\tau+(H)}$; that is, (1-6) in addition to (1-3) in the second inequality, together with the estimate

$$Q^\tau_{\Lambda^{s-1}} \lesssim \Lambda^s$$

in the third. We estimate terms in the last line. First, the term $|||\zeta u^{\tau+(H)}|||_s^2$ is estimated by means of (1-16). Next, the terms in $(s-1)$-norm can be brought to 0-norm by combined inductive use of (1-16) and (1-17), and eventually their sum is controlled by $\|u^{\tau+}\|_0^2$. We put together (1-16) and (1-17) (with the above further reductions), recall the first part of (1-1) in order to estimate $Q^\tau_{\Lambda^{s+l}\zeta'} + Q^\tau_{\partial_r\Lambda^{s+l-1}\zeta'}$ in the right side of (1-16), and end up with

$$\text{(1-18)}\qquad |||\zeta u^{\tau+}|||_s \lesssim \|\zeta'\bar\partial u^{\tau+}\|_{s+l} + \|\zeta'\bar\partial^* u^{\tau+}\|_{s+l} + \|u^{\tau+}\|_0.$$

Finally, by noncharacteristicity of $(\bar\partial, \bar\partial^*)$, one passes from tangential to full norms in the left side of (1-18) along the guidelines of [Zampieri 2008, Theorem 1.9.7]. The version of this argument for $\Box$ can be found in [Kohn 2002] in the second part of p. 245. Thus we get (1-14).

We prove that (1-15) implies (1-13) for $l=0$. Thanks to $\partial_r=\bar L_n+\mathrm{Tan}$ and to (1-3), we have

$$\partial_r u^{\tau+(H)}=\mathrm{Tan}\,u^{\tau+(H)}\quad\text{and}\quad \bar L_n u^{\tau+(H)}=r\,\mathrm{Tan}\,u^{\tau+(H)}.$$

It follows that

$$\begin{aligned}
&(1\text{-}19)\quad \|\zeta u_b^+\|_s^2\\
&\underset{(1\text{-}4)}{\lesssim} |||\zeta u^{\tau+(H)}|||^2_{s+\frac12}+|||\partial_r\zeta u^{\tau+(H)}|||^2_{s-\frac12}\\
&\lesssim |||\zeta u^{\tau+(H)}|||^2_{s+\frac12}+\|\bar L_n\zeta u^{\tau+(H)}\|^2_{s-\frac12}\\
&\underset{(1\text{-}15)}{\lesssim} \epsilon\big(Q^\tau_{\Lambda^{s+\frac12}\zeta}(u^{\tau+(H)},u^{\tau+(H)})+|||\zeta\bar L_n u^{\tau+(H)}|||^2_{s+\frac12}\big)\\
&\qquad\qquad\qquad\qquad+c_\epsilon\big(|||\zeta' u^{\tau+(H)}|||^2_{s-\frac12}+|||u^{\tau+(H)}|||^2_0\big)\\
&\underset{(1\text{-}8)}{\lesssim} \epsilon\big(Q^b_{\Lambda^s\zeta}(u_b^+,u_b^+)+\|\zeta u_b^+\|^2_s\big)+c_\epsilon\big(Q^b_{\Lambda^{s-1}\zeta'}(u_b^+,u_b^+)+\|\zeta' u_b^+\|^2_{s-1}+\|u_b^{\tau+}\|^2_{-\frac12}\big)\\
&\lesssim Q^b_{\Lambda^s\zeta'}(u_b^+,u_b^+)+\epsilon\|\zeta u_b^+\|^2_s+c_\epsilon\big(\|\zeta' u_b^+\|^2_{s-1}+\|u_b^{\tau+}\|^2_{-\frac12}\big),
\end{aligned}$$

where in the second-to-last line we have calculated $[\zeta,\#^{(H)}]$, which yields

$$|||\zeta u^{\tau+(H)}|||_{s+\frac12}\lesssim\|\zeta u_b^+\|_s+\|\zeta' u_b^+\|_{s-1}$$

(and similarly for $[\zeta,Q^{(H)}]$). We absorb the term with ϵ and get (1-13). $\square$

Since on a pseudoconvex domain the H^0-ranges of $\Box$ and $\Box_b$ are closed by basic estimates and by [Kohn 1986] respectively, then there are well defined H^0-inverses denoted by N and G, and named the Neumann and Green operators.

Remark 1.5. Equations (1-13) and (1-14) imply local regularity in degree ≥ 2 of G and N respectively. We first prove regularity for N. We start by remarking that

$$(1\text{-}20)\qquad \begin{aligned}\bar\partial^*N_q &\text{ is regular over } \mathrm{Ker}\,\bar\partial &&\text{if } q\ge 2,\\ \bar\partial N_q &\text{ is regular over } \mathrm{Ker}\,\bar\partial^* &&\text{if } q\ge 0.\end{aligned}$$

In the first case, we set $u=\bar\partial^*Nf$ for $f\in\mathrm{Ker}\,\bar\partial$. We have ($\bar\partial u=f$, $\bar\partial^*u=0$), and hence by (1-14)

$$\|\zeta u\|_s\lesssim\|\zeta' f\|_{s+l}+\|u\|_0.$$

To prove the second case, we simply set $u=\bar\partial Nf$ for $f\in\mathrm{Ker}\,\bar\partial^*$ and reason likewise. It follows from (1-20) that the Bergman projection B_q is regular in any degree $q\ge 0$. (Notice that even if one started from exact regularity by assuming (1-15), this is perhaps lost by taking the additional $\bar\partial$ in $B:=\mathrm{Id}-\bar\partial^*N\bar\partial$.) Finally, we exploit formula (5.36) in [Straube 2010] in unweighted norms; that is, for $t = 0$:

$$N_q = B_q(N_q\bar\partial)(\mathrm{Id} - B_{q-1})(\bar\partial^* N_q)B_q + (\mathrm{Id} - B_q)(\bar\partial^* N_{q+1})B_{q+1}(N_{q+1}\bar\partial)(\mathrm{Id} - B_q). \tag{1-21}$$

Now, in the right side, the $\bar\partial N$'s and $\bar\partial^* N$'s are evaluated over $\operatorname{Ker}\bar\partial^*$ and $\operatorname{Ker}\bar\partial$ respectively; thus they are regular for $q \geq 2$. The B's are also regular and therefore such is N. This concludes the proof of the regularity of N. The proof of the regularity of G is similar, apart from replacing (1-21) by its version for the Green operator G stated in Section 5 of [Khanh 2010].

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