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ON THE STEINBERG CHARACTER OF A SEMISIMPLE p-ADIC GROUP

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Dedicated to Robert Steinberg on the occasion of his ninetieth birthday.

We show that the character of the Steinberg representation of a split semisimple p-adic group at a very regular element is given (up to sign) by a power of q, the number of elements in the residue field. We also show that (under an assumption on the characteristic) the character of an Iwahori-spherical representation at a split very regular element is given by a trace in the corresponding Hecke algebra module.

1. Introduction

1.1. Let K be a nonarchimedean local field and let $\underline{K}$ be a maximal unramified field extension of K. Let $\mathcal{O}$ be the ring of integers of K and let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$; the counterparts for $\underline{K}$ are denoted by $\underline{\mathcal{O}}$ and $\underline{\mathfrak{p}}$. Let $\underline{K}^* = \underline{K} - \{0\}$. We write $\mathcal{O}/\mathfrak{p} = F_q$, a finite field with q elements of characteristic p.

Let G be a semisimple almost simple algebraic group defined and split over K with a given $\mathcal{O}$-structure compatible with the K-structure.

If V is an admissible representation of $G(K)$ of finite length, we denote by ϕ_V the character of V in the sense of Harish-Chandra, viewed as a $\mathbb{C}$-valued function on the set $G(K)_{rs} := G_{rs} \cap G(K)$. (Here, G_{rs} is the set of regular semisimple elements of G, and $\mathbb{C}$ is the field of complex numbers.)

In this paper we study the restriction of the function ϕ_V to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, namely, the set of very regular elements in $G(K)$ (see 1.2) in the case where V is the Steinberg representation of $G(K)$, and

(b) a certain subset $G(K)_{svr}$ of $G(K)_{vr}$, namely, the set of split very regular elements in $G(K)$ (see 1.2) in the case where V is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

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In case (a), we show that $\phi_V(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \dots\}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{svr}$ (see Theorem 2.2) or when $g \in G(K)_{cvr}$ (see Theorem 3.2). In case (b) we show (with some restriction on characteristic) that $\phi_V(g)$ with $g \in G(K)_{svr}$ can be expressed as the trace of a certain element of an affine Hecke algebra on an irreducible module (see Theorem 4.3).

Note that the Steinberg representation **S** is an irreducible admissible representation of $G(K)$ with a one-dimensional subspace invariant under an Iwahori subgroup on which the corresponding affine Hecke algebra acts through the "sign" representation; see [Matsumoto 1969; Shalika 1970]. This is a p-adic analogue of the Steinberg representation [Steinberg 1951] of a reductive group over F_q. In [Rodier 1986], it is proven that $\phi_{\mathbf{S}}(g) \neq 0$ for any $g \in G(K)_{rs}$.

1.2. Let $g \in G_{rs} \cap G(\underline{K})$. Let $T' = T'_g$ be the maximal torus of G that contains g. We say that g is *very regular* if T' is split over $\underline{K}$ and for any root α with respect to T' viewed as a homomorphism $T'(\underline{K}) \to \underline{K}^*$ we have $\alpha(g) \notin (1 + \underline{\mathfrak{p}})$. If, in addition, $\alpha(g) \in \underline{\mathcal{O}}$, we say that g is *compact very regular*.

Let $G(\underline{K})_{vr}$ be the set of elements in $G(\underline{K})$ that are very regular, and $G(\underline{K})_{cvr}$ the set of compact very regular ones. We write $G(K)_{vr} = G(\underline{K})_{vr} \cap G(K)$ and $G(K)_{cvr} = G(\underline{K})_{cvr} \cap G(K)$. Let $G(K)_{svr}$ be the set of all $g \in G(K)_{vr}$ such that T'_g is split over K.

1.3. ***Notation.*** Let $K^* = K - \{0\}$, and let $v : K^* \to \mathbb{Z}$ be the unique (surjective) homomorphism such that $v(\mathfrak{p}^n - \mathfrak{p}^{n+1}) = n$ for any $n \in \mathbb{N}$. For $a \in K^*$ we set $|a| = q^{-v(a)}$.

We fix a maximal torus T of G defined and split over K. Let Y (resp. X) be the group of cocharacters (resp. characters) of the algebraic group T. Let $\langle\ ,\ \rangle : Y \times X \to \mathbb{Z}$ be the obvious pairing. Let $R \subset X$ be the set of roots of G with respect to T, let R^+ be a set of positive roots for R, and let Π be the set of simple roots of R determined by R^+. We write $\Pi = \{\alpha_i : i \in I\}$. Let $R^- = R - R^+$. Let Y^+ (resp. Y^{++}) be the set of all $y \in Y$ such that $\langle y, \alpha\rangle \geq 0$ (resp. $\langle y, \alpha\rangle > 0$) for all $\alpha \in R^+$. We define $2\rho \in X$ by $2\rho = \sum_{a \in R^+} \alpha$.

We have canonically $T(K) = K^* \otimes Y$; we define a homomorphism $\chi : T(K) \to Y$ by $\chi(\lambda \otimes y) = v(\lambda) y$ for any $\lambda \in K^*$, $y \in Y$. For any $y \in Y$, we set $T(K)_y = \chi^{-1}(y)$. For $y \in Y$, let $T(K)^{\spadesuit}_y = T(K)_y \cap G(K)_{svr}$. Note that if $y \in Y^{++}$ then $T(K)^{\spadesuit}_y = T(K)_y$.

For each $\alpha \in R$ let U_α be the corresponding root subgroup of G.

2. Calculation of $\phi_{\mathbf{S}}$ on $G(K)_{svr}$

2.1. Let $\mathcal{W} \subset \mathrm{Aut}(T)$ be the Weyl group of G regarded as a Coxeter group; for $i \in I$, let s_i be the simple reflection in $\mathcal{W}$ determined by α_i. We can also view

$\mathcal{W}$ as a subgroup of Aut(Y) or Aut(X). Let $w = w_0$ be the longest element of $\mathcal{W}$. For any $J \subset I$, let $\mathcal{W}_J$ be the subgroup of $\mathcal{W}$ generated by $\{s_i : i \in J\}$ and let $R_J = R \cap \sum_{i\in J} \mathbb{Z}\alpha_i$. Let

$$R_J^+ = R_J \cap R^+ \quad \text{and} \quad R_J^- = R_J - R_J^+.$$

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T. For any $J \subset I$, let $\mathfrak{l}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{t}$ and the root spaces corresponding to the roots in R_J. Let $\mathfrak{n}_J$ be the Lie subalgebra of $\mathfrak{g}$ spanned by the root spaces corresponding to roots in $R^+ - R_J^+$.

According to [Casselman 1973], $\phi_{\mathbf{S}}$ is an alternating sum of characters of representations induced from one-dimensional representations of various parabolic subgroups of G defined over K. From this, one can deduce that if $t \in T(K) \cap G(K)_{rs}$ then

$$\phi_{\mathbf{S}}(t) = \sum_{J\subset I} (-1)^{\sharp J} \sum_{w\in {}^J\mathcal{W}} \delta_J(w(t))^{1/2} D_{I,J}(w(t))^{-1/2},$$

where for any $J \subset I$ and $t' \in T(K) \cap G(K)_{rs}$ we set

$$D_{I,J}(t') = \left|\det(1 - \mathrm{Ad}(t')|_{\mathfrak{g}/\mathfrak{l}_J})\right|,$$
$$\delta_J(t') = \left|\det(\mathrm{Ad}(t')|_{\mathfrak{n}_J})\right|,$$

and ${}^J\mathcal{W}$ is the set of representatives of minimal length for the cosets $\mathcal{W}_J \backslash \mathcal{W}$. Here for a real number $a \geq 0$ we denote by $a^{1/2}$ or $\sqrt{a}$ the nonnegative square root of a. Writing ϕ instead of $\phi_{\mathbf{S}}$, we have:

Theorem 2.2. *Let $y \in Y^+$ and let $t \in T(K)_y^\spadesuit$. Then $\phi(t) = q^{-\langle y, 2\rho\rangle}$.*

2.3. More generally, let $t \in T(K)_y^\spadesuit$, where $y \in Y$. By a standard property of Weyl chambers, there exists $w \in \mathcal{W}$ such that $w(y) \in Y^+$. Let $t_1 = w(t)$. Then the theorem is applicable to t_1, and we have $\phi(t) = \phi(t_1) = q^{-\langle w(y), 2\rho\rangle}$.

2.4. Let $y' = w_0(y)$, $t' = w_0(t)$. We have $\phi_{\mathbf{S}}(t) = \phi_{\mathbf{S}}(t')$, $t' \in T(K)_{y'}^\spadesuit$, $-y' \in Y^+$. We show that

$$v(1 - \beta(t')) = \begin{cases} v(\beta(t')) & \text{if } \beta \in R^+, \\ 0 & \text{if } \beta \in R^-. \end{cases} \tag{1}$$

Assume first that $\beta \in R^+$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) < 0$ (since $\langle y', \beta\rangle \neq 0$ and $\langle y', \beta\rangle \leq 0$); hence, $v(1 - \beta(t')) = v(\beta(t'))$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1 - \beta(t')) = 0 = v(\beta(t'))$ as required.

Assume next that $\beta \in R^-$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) > 0$ (since $\langle y', \beta\rangle \neq 0$ and $\langle y', \beta\rangle \geq 0$); hence, $v(1 - \beta(t')) = 0$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1 - \beta(t')) = 0$ as required.

For any $w \in \mathcal{W}$, $J \subset I$ we have

$$D_{I,J}(w(t')) = \prod_{\alpha\in R-R_J} q^{-v(1-\alpha(w(t')))} = \prod_{\substack{\alpha\in R-R_J\\ w^{-1}\alpha\in R^+}} q^{-v(\alpha(w(t')))} = \prod_{\substack{\alpha\in R-R_J\\ w^{-1}\alpha\in R^+}} q^{-\langle y', w^{-1}\alpha\rangle}$$

and

$$\delta_J(w(t')) = \prod_{\alpha\in R^+-R_J^+} q^{-v(\alpha(w(t')))} = \prod_{\alpha\in R^+-R_J^+} q^{-\langle y', w^{-1}\alpha\rangle}.$$

(We have used (1) with $\beta = w^{-1}(\alpha)$.) We see that

$$\phi(t) = \phi(t') = \sum_{J\subset I}(-1)^{\sharp J}\sum_{w\in {}^J\mathcal{W}} \sqrt{q}^{\,-\langle y', x_{w,J}\rangle},$$

where for $w \in {}^J\mathcal{W}$ we have

$$\begin{aligned} x_{w,J} &= \sum_{\alpha\in R^+-R_J^+} w^{-1}\alpha - \sum_{\substack{\alpha\in R-R_J\\ w^{-1}\alpha\in R^+}} w^{-1}\alpha \\ &= \sum_{\substack{\alpha\in R^+-R_J^+\\ w^{-1}(\alpha)\in R^-}} w^{-1}\alpha - \sum_{\substack{\alpha\in R^--R_J^-\\ w^{-1}(\alpha)\in R^+}} w^{-1}\alpha \\ &= 2\sum_{\substack{\alpha\in R^+-R_J^+\\ w^{-1}\alpha\in R^-}} w^{-1}\alpha \in X. \end{aligned}$$

For $w \in {}^J\mathcal{W}$, we have $\alpha \in R_J^+ \implies w^{-1}\alpha \in R^+$; hence,

$$\sum_{\substack{\alpha\in R^+-R_J^+\\ w^{-1}\alpha\in R^-}} w^{-1}\alpha = \sum_{\substack{\alpha\in R^+\\ w^{-1}\alpha\in R^-}} w^{-1}\alpha,$$

so that $x_{w,J} = x_w$, where

$$x_w = 2\sum_{\substack{\alpha\in R^+\\ w^{-1}\alpha\in R^-}} w^{-1}\alpha \in X.$$

Thus, we have

$$\phi(t) = \sum_{J\subset I}(-1)^{\sharp J}\sum_{w\in {}^J\mathcal{W}} \sqrt{q}^{\,-\langle y', x_w\rangle} = \sum_{w\in\mathcal{W}} c_w \sqrt{q}^{\,-\langle y', x_w\rangle},$$

where for $w \in \mathcal{W}$ we set

$$c_w = \sum_{\substack{J\subset I\\ w\in {}^J\mathcal{W}}} (-1)^{\sharp J}.$$

For $w \in \mathcal{W}$, let $\mathcal{L}(w) = \{i \in I : s_i w > w\}$, where $>$ refers to the standard partial order on $\mathcal{W}$. For $J \subset I$, we have $w \in {}^J\mathcal{W}$ if and only if $J \subset \mathcal{L}(w)$; thus,

$$c_w = \sum_{J \subset \mathcal{L}(w)} (-1)^{\sharp J},$$

and this is 0 unless $\mathcal{L}(w) = \varnothing$ (that is $w = w_0$), in which case $c_w = 1$. Note also that $x_{w_0} = -4\rho$; thus, we have

$$\phi(t) = c_{w_0} \sqrt{q}^{\,-\langle y', x_{w_0} \rangle} = q^{\langle y', 2\rho \rangle} = q^{-\langle y, 2\rho \rangle}.$$

Theorem 2.2 is proved. □

2.5. Assume now that $\tau \in T(K)$ satisfies the following condition: for any $\alpha \in R$ we have $\alpha(\tau) - 1 \in \mathfrak{p} - \{0\}$ so that $\alpha(\tau) - 1 \in \mathfrak{p}^{n_\alpha} - \mathfrak{p}^{n_\alpha + 1}$ for a well defined integer $n_\alpha \geq 1$. Note that $n_{-\alpha} = n_\alpha$ and $v(1 - \alpha(\tau)) = n_\alpha \geq 1$ for all $\alpha \in R$; hence,

$$\phi(\tau) = \sum_{J \subset I} (-1)^{\sharp J} \sum_{w \in {}^J\mathcal{W}} q^{\sum_{\alpha \in R} n_\alpha/2 - \sum_{\alpha \in R_J} n_{w^{-1}(\alpha)}/2}.$$

Thus,

$$\phi(\tau) = \sharp(\mathcal{W}) q^{\sum_{\alpha \in R} n_\alpha/2} + \text{strictly smaller powers of } q. \tag{2}$$

In the case where K is the field of power series over F_q, the leading term in (2) is equal to $\sharp(\mathcal{W}) q^m$, where m is the dimension of the "variety" of Iwahori subgroups of $G(\underline{K})$ that contain the topologically unipotent element τ (see [Kazhdan and Lusztig 1988]).

3. Calculation of $\phi_{\mathbf{S}}$ on $G(K)_{vr}$

3.1. We will again write ϕ instead of $\phi_{\mathbf{S}}$. In this section we assume that we are given $\gamma \in G(K)_{vr}$. Let $T' = T'_\gamma$. Note that T' is defined over K; let A' be the largest K-split torus of T'. For any parabolic subgroup P of G defined over K such that $\gamma \in P$, we set $\delta_P(\gamma) = \left|\det(\mathrm{Ad}(\gamma)|_{\mathfrak{n}})\right|$, where $\mathfrak{n}$ is the Lie algebra of the unipotent radical of P.

Let $\mathcal{X}$ be the set of all pairs (P, A), where P is a parabolic subgroup of G defined over K and A is the unique maximal K-split torus in the center of some Levi subgroup of P defined over K. Then that Levi subgroup is uniquely determined by A and is denoted by M_A. Let $\mathcal{X}' = \{(P, A) \in \mathcal{X} : A \subset A'\}$. According to [Harish-Chandra 1973], we have

$$\phi(\gamma) = (-1)^{\dim T} \sum_{(P,A) \in \mathcal{X}'} (-1)^{\dim A} \delta_P(\gamma)^{1/2} D_{G/M_A}(\gamma)^{-1/2}, \tag{3}$$

where $D_{G/M_A}(\gamma) = \left|\det(1 - \mathrm{Ad}(\gamma)|_{\mathfrak{g}/\mathfrak{l}})\right|$ (we denote by $\mathfrak{l}$ the Lie algebra of M_A).

Theorem 3.2. *Assume in addition that $\gamma \in G(K)_{cvr}$. Then $\phi(\gamma) = (-1)^{\dim T - \dim A'}$.*

Proof. From our assumptions we see that $\delta_P(\gamma) = 1 = D_{G/M_A}(\gamma)$ for all $(P, A) \in \mathcal{X}'$; hence, (3) becomes

$$\phi(\gamma) = (-1)^{\dim T} \sum_{(P,A)\in\mathcal{X}'} (-1)^{\dim A}.$$

Let $\mathcal{Y}$ be the group of cocharacters of A' and let $\mathfrak{H} = \mathcal{Y} \otimes \mathbb{R}$. The real vector space $\mathfrak{H}$ can be partitioned into facets $F_{P,A}$ indexed by $(P, A) \in \mathcal{X}'$ such that $F_{P,A}$ is homeomorphic to $\mathbb{R}^{\dim A}$. Note that the Euler characteristic with compact support of $F_{P,A}$ is $(-1)^{\dim A}$, and the Euler characteristic with compact support of $\mathfrak{H}$ is $(-1)^{\dim_{\mathbb{R}} \mathfrak{H}} = (-1)^{\dim A'}$. Using the additivity of the Euler characteristic with compact support we see that $\sum_{(P,A)\in\mathcal{X}'}(-1)^{\dim A} = (-1)^{\dim A'}$; thus, $\phi(\gamma) = (-1)^{\dim T - \dim A'}$, as required. □

3.3. In the setup of 3.1, let P_γ be the parabolic subgroup of G associated to γ as in [Casselman 1977]. Note that P_γ is defined over K. The following result can be deduced by combining Theorem 3.2 with the results in [Casselman 1977] and with Proposition 2 in [Rodier 1986].

Corollary 3.4. *We have* $\phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_\gamma}(\gamma)$.

The corollary provides another proof of Theorem 2.2.

4. Iwahori spherical representations: split elements

4.1. Let B be the subgroup of $G(K)$ generated by

$$\{U_\alpha(\mathcal{O}) : \alpha \in R^+\} \cup \{U_\alpha(\mathfrak{p}) : \alpha \in R^-\} \cup T(K)_0.$$

(The subgroups $U_\alpha(\mathcal{O})$, $U_\alpha(\mathfrak{p})$ of U_α are defined by the $\mathcal{O}$-structure of G.) Then B is an Iwahori subgroup of $G(K)$. For any $\alpha \in R$ we choose an isomorphism $x_\alpha : K \xrightarrow{\sim} U_\alpha(K)$ (the restriction of an isomorphism of algebraic groups from the additive group to U_α), which carries $\mathcal{O}$ onto $U_\alpha(\mathcal{O})$ and $\mathfrak{p}$ onto $U_\alpha(\mathfrak{p})$. We set $W := Y \cdot \mathcal{W}$ with Y normal in W (recall that $\mathcal{W}$ acts naturally on Y). Let Y' be the subgroup of Y generated by the coroots. Then $W' := Y' \cdot \mathcal{W}$ is naturally a subgroup of W. According to [Iwahori and Matsumoto 1965], W is an extended Coxeter group (the semidirect product of the Coxeter group W' with the finite abelian group Y/Y') with length function

$$l(yw) = \sum_{\substack{\alpha\in R^+ \\ w^{-1}(\alpha)\in R^+}} \|\langle y, \alpha\rangle\| + \sum_{\substack{\alpha\in R^+ \\ w^{-1}(\alpha)\in R^-}} \|\langle y, \alpha\rangle - 1\|,$$

where $\|a\| = a$ if $a \geq 0$ and $\|a\| = -a$ if $a < 0$. From the same reference we know that the set of double cosets $B\backslash G(K)/B$ is in bijection with W; to yw (where $y \in Y$, $w \in \mathcal{W}$) corresponds the double coset Ω_{yw} containing $T(K)_y \dot{w}$ (here $\dot{w}$ is an element in $G(\mathcal{O})$ which normalizes $T(K)_0$ and acts on it in the same way as w); moreover, $\sharp(\Omega_{yw}/B) = \sharp(B\backslash\Omega_{yw}) = q^{l(yw)}$ for any $y \in Y$, $w \in \mathcal{W}$. For example, if $y \in Y^{++}$ then $l(y) = \langle y, 2\rho\rangle$.

Let H be the algebra of B-biinvariant functions $G(K) \to \mathbb{C}$ with compact support with respect to convolution (we use the Haar measure dg on $G(K)$ for which $\mathrm{vol}(B) = 1$). For y, w as above, let $\mathfrak{T}_{yw} \in H$ be the characteristic function of Ω_{yw}. Then the functions $\mathfrak{T}_{\underline{w}}$, $\underline{w} \in W$ form a $\mathbb{C}$-basis of H, and according to [Iwahori and Matsumoto 1965], we have

$$\mathfrak{T}_{\underline{w}}\mathfrak{T}_{\underline{w}'} = \mathfrak{T}_{\underline{w}\underline{w}'} \quad \text{for } \underline{w}, \underline{w}' \in W \text{ with } l(\underline{w}\underline{w}') = l(\underline{w}) + l(\underline{w}'),$$
$$(\mathfrak{T}_{\underline{w}} + 1)(\mathfrak{T}_{\underline{w}} - q) = 0 \quad \text{for } \underline{w} \in W' \text{ with } l(\underline{w}) = 1.$$

In other words, H is what one now calls the Iwahori–Hecke algebra of the (extended) Coxeter group W with parameter q.

4.2. Let $\mathcal{C}_0^\infty(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbb{C}$. Let (V, σ) be an irreducible admissible representation of $G(K)$ such that the space V^B of B-invariant vectors in V is nonzero. If $f \in \mathcal{C}_0^\infty(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(g)(x)\,dg$. This linear map has finite rank; hence, it has a well defined trace $\mathrm{tr}(\sigma_f) \in \mathbb{C}$. From the definitions we see that for $f, f' \in \mathcal{C}_0^\infty(G(K))$ we have $\sigma_{f*f'} = \sigma_f\sigma_{f'} : V \to V$ where $*$ denotes convolution. If $f \in H$ then σ_f maps V into V^B and $\mathrm{tr}(\sigma_f) = \mathrm{tr}(\sigma_f|_{V^B})$. (Recall that $\dim V^B < \infty$.) We see that the maps $\sigma_f|_{V^B}$ define a (unital) H-module structure on V^B. It is known that the H-module V^B is irreducible [Borel 1976]. Moreover, for $\underline{w} \in W$ we have $\mathrm{tr}(\sigma_{\mathfrak{T}_{\underline{w}}}) = \mathrm{tr}(\mathfrak{T}_{\underline{w}})$, where the trace in the right side is taken in the H-module V^B.

Theorem 4.3. *Assume that K has characteristic zero and that p is sufficiently large. Let $y \in Y^+$ and $t \in T(K)_y^\spadesuit$. We have*

$$\phi_V(t) = q^{-\langle y, 2\rho\rangle}\mathrm{tr}(\mathfrak{T}_y),$$

where the trace in the right side is taken in the irreducible H-module V^B.

An equivalent statement is that

$$\phi_V(t) = \mathrm{tr}(\sigma_{\mathfrak{T}_y})/\,\mathrm{vol}(\Omega_y).$$

(Recall that $\mathfrak{T}_y$ on the right side is the characteristic function of $\Omega_y = BT(K)_yB$.)

The assumption on characteristic in the theorem is needed only to be able to use a result from [Adler and Korman 2007]; see (5) below. We expect that the theorem holds without that assumption.

In the case where $y = 0$, the theorem becomes

$$t \in T(K) \cap G_{cvr} \implies \phi_V(t) = \dim(V^B). \tag{4}$$

As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where $y \in Y^{++}$, Theorem 4.3 can be deduced from results in [Casselman 1977].

4.4. In the case where $V = \mathbf{S}$ (see 1.1), for any $y \in Y^+$, $\mathfrak{T}_y$ acts on the one-dimensional vector space V^B as the identity map, so that $\phi_V(t) = q^{-\langle y, 2\rho\rangle}$ for all $t \in T(K)^{\spadesuit}_y$. We thus recover Theorem 2.2 (which holds in any characteristic).

5. Proof of Theorem 4.3

5.1. Let $B = B_0, B_1, B_2, \ldots$ be the strictly decreasing Moy–Prasad [1994] filtration of B. This is a sequence associated to a point x in the building such that $B = G_{x,0}$. Each B_i/B_{i+1} is abelian. Let $T_n := T(K) \cap B_n$. Applying [Adler and Korman 2007, Corollary 12.11] to ϕ_V, we conclude that

(5) ϕ_V is constant on the Ad(G)-orbit ${}^G(tT_1)$ of tT_1.

Lemma 5.2. *Let $n \geq 1$. For any $t' \in T(K)^{\spadesuit}_y$ and $z \in B_n$, there exist $g \in B_n$, $t'' \in T_n$, and $z' \in B_{n+1}$ such that* $\mathrm{Ad}(g)(t'z) = t't''z'$.

Proof. Let $Z = \{\alpha \in R : U_\alpha \cap B_n \supsetneqq U_\alpha \cap B_{n+1}\}$. If $Z = \varnothing$ then $B_n = T_n B_{n+1}$; hence, $z = t''z'$ for some $t'' \in T_n$ and $z' \in B_{n+1}$, and one can take $g = 1$. If $Z \neq \varnothing$ then we can find $a_\alpha \in K$ for each $\alpha \in Z$ such that $x_\alpha(a_\alpha) \in B_n$ and $z \equiv \prod_{\alpha\in Z} x_\alpha(a_\alpha) \pmod{T_n B_{n+1}}$. Such a_α can be chosen independent of the order of the product since B_n/T_nB_{n+1} is abelian. Take $g = \prod_{\alpha\in Z} x_\alpha((1-\alpha(t'^{-1}))^{-1}a_\alpha)$. Then $g \in B_n$ since $|1-\alpha(t'^{-1})| \geq 1$ for $y \in Y^+$. (To show $|1-\alpha(t'^{-1})| \geq 1$ for $y \in Y^+$, we argue as for (1). Assume first that $\alpha \in R^+$. If $v(\alpha(t'^{-1})) \neq 0$ then $v(\alpha(t'^{-1})) < 0$ (since $\langle y, \alpha\rangle \neq 0$, $\langle y, \alpha\rangle \geq 0$); therefore, $v(1-\alpha(t'^{-1})) = v(\alpha(t'^{-1})) < 0$ and $|1-\alpha(t'^{-1})| > 1$. If $v(\alpha(t'^{-1})) = 0$ then $\alpha(t'^{-1}) - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1-\alpha(t'^{-1})) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required. Assume next that $\alpha \in R^-$. If $v(\alpha(t'^{-1})) \neq 0$ then $v(\alpha(t'^{-1})) > 0$ (since $\langle y, \alpha\rangle \neq 0$, $\langle y, \alpha\rangle \leq 0$); hence, $v(1-\alpha(t'^{-1})) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required. If $v(\alpha(t'^{-1})) = 0$ then $\alpha(t'^{-1}) - 1 \in \mathcal{O} - \mathfrak{p}$; hence, $v(1-\alpha(t'^{-1})) = 0$ and $|1-\alpha(t'^{-1})| = 1$ as required.) Now, we have $t'^{-1}gt'g^{-1} \equiv z^{-1} \pmod{T_nB_{n+1}}$.

Writing $\mathrm{Ad}(g)(t'z) = t' \cdot (t'^{-1}gt'g^{-1}) \cdot (gzg^{-1})$, we observe that $gzg^{-1} \equiv z \pmod{B_{n+1}}$ and $t'^{-1}gt'g^{-1}z \in T_nB_{n+1}$; hence, $\mathrm{Ad}(g)(t'z)$ can be written as $t't''z'$ with $t'' \in T_n$ and $z' \in B_{n+1}$. □

Lemma 5.3. $B_1tB_1 \subset {}^{B_1}(tT_1)$.

Proof. It is enough to show that $tB_1 \subset {}^{B_1}(tT_1)$. Let $t_0z_1 \in tB_1$ with $t_0 = t$ and $z_1 \in B_1$. We will construct inductively sequences $g_1, g_2, \ldots, t_1, t_2, \ldots$, and $z_1, z_2, \ldots$ such that $\mathrm{Ad}(g_k \cdots g_2g_1)(t_0z_1) = \mathrm{Ad}(g_k)(t_0t_1\cdots t_{k-1}z_k) = (t_0t_1\cdots t_k)z_{k+1}$ with $g_i \in B_i$, $t_i \in T_i$, and $z_i \in B_i$.

Applying Lemma 5.2 to $n = 1$, $t' = t_0$, and $z = z_1$, we find $t_1 \in T_1$ and $z_2 \in B_2$ such that $g_1 t_0 z_1 g_1^{-1} = t_0 t_1 z_2$ with $t_1 \in T_1$ and $z_2 \in B_2$. Suppose we found $g_i \in B_i$, $z_{i+1} \in B_{i+1}$, and $t_i \in T_i$ for $i = 1, \dots, k$ where $k \geq 1$. Applying Lemma 5.2 to $n = k+1$, $t' = t_0 t_1 \cdots t_k$, and $z = z_{k+1}$, we find $g_{k+1} \in B_{k+1}$, $t_{k+1} \in T_{k+1}$, and $z_{k+2} \in B_{k+2}$ so that $g_{k+1} t_0 t_1 \cdots t_k z_{k+1} g_{k+1}^{-1} = \mathrm{Ad}(g_{k+1} \cdots g_2 g_1)(t_0 z_1) = t_0 t_1 t_2 \cdots t_{k+1} z_{k+2}$. (To apply Lemma 5.2 we note that $t' \in T(K)_y^{\spadesuit}$ since $t_0 \in T(K)_y^{\spadesuit}$ and $t_1 \cdots t_k \in T_1$, so that for any $\alpha \in R$ we have $\alpha(t_1 \cdots t_k) \in 1 + \mathfrak{p}$.) Taking $g \in B_1$ to be the limit of $g_k \cdots g_2 g_1$ as $k \to \infty$, we have $\mathrm{Ad}(g)(t_0 z_1) \in t T_1$. □

5.4. Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and (5), for the characteristic function f_t of $B_1 t B_1$, we have

$$\mathrm{tr}(\sigma_{f_t}) = \int_G f_t(g)\phi_V(g)\,dg = \int_{B_1 t B_1} \phi_V(t)\,dg = \mathrm{vol}(B_1 t B_1)\phi_V(t).$$

Thus it remains to show

$$\frac{\mathrm{tr}(\sigma_{f_t})}{\mathrm{vol}(B_1 t B_1)} = \frac{\mathrm{tr}(\sigma_{\mathfrak{T}_y})}{\mathrm{vol}(BtB)}.$$

Since B_1 is normalized by B, B acts on V^{B_1}; moreover, since V is irreducible and $V^B \neq 0$, B acts trivially on V^{B_1}. (Otherwise, there would exist a nonzero subspace of V on which B acts through a nontrivial character of B/B_1; since $V^B \neq 0$, we see that (V, σ) would have two distinct cuspidal supports, a contradiction.) Thus we have $V^{B_1} = V^B$. Since σ_{f_t} and $\sigma_{\mathfrak{T}_y}$ have images contained in $V^{B_1} = V^B$, it is enough to show

$$\frac{\mathrm{tr}(\sigma_{f_t}|_{V^B})}{\mathrm{vol}(B_1 t B_1)} = \frac{\mathrm{tr}(\sigma_{\mathfrak{T}_y}|_{V^B})}{\mathrm{vol}(BtB)}. \tag{6}$$

We can find a finite subset L of $T(K)_0$ such that $BtB = \bigsqcup_{\tau \in L} B_1 t B_1 \tau$. It follows that

$$\mathrm{vol}(BtB) = \mathrm{vol}(B_1 t B_1)\sharp(L) \tag{7}$$

and $\sigma_{\mathfrak{T}_y} = \sum_{\tau \in L} \sigma_{f_t}\sigma(\tau)$ as linear maps $V \to V$. Restricting this equality to V^B and using the fact that $\sigma(\tau)$ acts as identity on V^B, we obtain

$$\sigma_{\mathfrak{T}_y}|_{V^B} = \sharp(L)\sigma_{f_t}|_{V^B} \tag{8}$$

as linear maps $V^B \to V^B$. Clearly, (6) follows from (7) and This completes the proof of Theorem 4.3. □

The following result will not be used in the rest of the paper:

Proposition 5.5. *If $y \in Y^{++}$ and $t \in T(K)_y$ then $BtB \subset {}^{B_1}T(K)_y$.*

Proof. It is enough to show that $tz \subset {}^{B_1}T(K)_y$ for any $z \in B$. We can write $z = t_0 z'$, where $t_0 \in T(K)_0$, $z' \in B_1$. We have $tz = t t_0 z'$, where $t t_0 \in T(K)_y = T(K)_y^{\spadesuit}$ (here we use that $y \in Y^{++}$). Using Lemma 5.3, we have $t t_0 z' \in {}^{B_1}(t t_0 T_1) \subset {}^{B_1}T(K)_y$. □

5.6. In the remainder of this section we assume that G is adjoint. In this case, the irreducible representations (V, σ) as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite-dimensional representations of the Hecke algebra H (see [Borel 1976]) by $(V, \sigma) \mapsto V^B$. The irreducible finite-dimensional representations of H have been classified in [Kazhdan and Lusztig 1987] in terms of geometric data; moreover, in [Lusztig 2010], an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\mathfrak{T}_y : y \in Y^+\}$ on any tempered H module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that theorem) is computable when V is tempered.

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