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We prove the factoriality of every nodal quartic threefold with 13 singular points that contains neither planes nor quadric surfaces. As a corollary, any nodal quartic threefold with 13 singular points that contains neither planes nor quadric surfaces is nonrational.

1. Introduction

All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$. A nodal variety is one that has, at most, isolated ordinary double points (nodes). A variety V is said to be factorial if each Weil divisor is Cartier, and $\mathbb{Q}$-factorial if a multiple of each Weil divisor of V is Cartier. This simple-looking definition is quite subtle when applied to projective varieties. It depends both on the kind of singularities and on their position. In the case of a Fano threefold X, $\mathbb{Q}$-factoriality is equivalent to the condition $\operatorname{rank}(H^2(X, \mathbb{Z})) = \operatorname{rank}(H_4(X, \mathbb{Z}))$. Thus a smooth Fano threefold is always $\mathbb{Q}$-factorial. The local class group at a node in a threefold has no torsion [Milnor 1968], so each Weil divisor that is $\mathbb{Q}$-Cartier must be a Cartier divisor on a nodal hypersurface in $\mathbb{P}^4$. Therefore, $\mathbb{Q}$-factoriality is equivalent to factoriality for a nodal hypersurface in $\mathbb{P}^4$. Moreover, the factoriality of a nodal quartic threefold implies its nonrationality; Mella [2004] proved that every factorial nodal quartic threefold is nonrational. This generalizes a classical result by Iskovskikh and Manin [1971] that every smooth quartic threefold is nonrational. On the other hand, there exist nonfactorial nodal quartic threefolds that are nonrational.

In view of Mella's result and the importance of rationality, studying the factoriality of nodal quartic threefolds is of interest. Here we consider this problem when the number of nodes is 13. This extends earlier results, which we now quote.

Throughout, X_4 will represent a nodal quartic threefold.

Theorem 1.1 [Cheltsov 2006]. *A quartic X_4 with at most 9 nodes is factorial if it contains no plane.*

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Theorem 1.2 [Shramov 2007, Theorem 1.3]. *A quartic X_4 with at most 11 nodes is factorial if it contains no planes. If X_4 has 12 nodes, then X_4 is factorial, with the exception of the case when X_4 contains a quadric surface.*

In Section 3 of this paper, we prove the following results.

Theorem 1.3. *A quartic X_4 with at most 13 nodes is factorial if it contains neither planes nor quadric surfaces.*

Corollary 1.4. *A quartic X_4 with at most 13 nodes is nonrational if it contains neither planes nor quadric surfaces.*

Theorem 1.3 improves on the degree-4 case of [Cheltsov 2006, Conjecture 13], which generalizes a well-known conjecture by Ciliberto [2004].

We present an example which motivates our study.

Example 1.5 [Cheltsov 2006, Example 10]. Let a_2, h_2, b_3 and g_1 be homogeneous polynomials of degrees 2, 2, 3 and 1, respectively. Consider the quartic threefold X_4 defined by the equation

$$a_2(x, y, z, w, t)h_2(x, y, z, w, t) = b_3(x, y, z, w, t)g_1(x, y, z, w, t);$$

it is the general quartic threefold passing through the quadric surface Q defined by $a_2 = g_1 = 0$. The quartic X_4 has 12 nodes, given by $h_2 = g_1 = a_2 = b_3 = 0$, and it is not factorial.

2. Preliminaries

If the nodes of a nodal quartic threefold X_4 impose independent linear conditions on hypersurfaces of degree 3 in $\mathbb{P}^4$, then X_4 is factorial [Cynk 2001]. Hence, if a nodal quartic threefold X_4 is factorial, it must have at most 35 simple double points because $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 35$. A nodal quartic threefold X_4 cannot have more than 45 nodes [Friedman 1986; Varchenko 1983]. Moreover, there is a unique nodal quartic threefold with 45 nodes [de Jong et al. 1990]. It is known as the Burkhardt quartic, which has too many nodes to be factorial.

The following result is one of the main tools.

Theorem 2.1 [Eisenbud and Koh 1989, Theorem 2]. *Let Σ be a set of points in $\mathbb{P}^N$ and let $d \geq 2$ be an integer. If no $dk+2$ of the points of Σ lie in a projective k-plane for all $k \geq 1$, then Σ imposes independent conditions on forms of degree d in $\mathbb{P}^N$.*

We see that the singular points of nodal threefolds are located in $\mathbb{P}^4$ with the following properties:

Lemma 2.2 [Cheltsov and Park 2006, Lemma 2.9]. *Let X_d be a nodal hypersurface of degree d in $\mathbb{P}^4$.*

(1) *A curve of degree k in $\mathbb{P}^4$ contains at most $k(d-1)$ nodes of X_d.*

(2) *If a 2-plane contains $d(d-1)/2+1$ nodes of X_d, then the 2-plane is contained in X_d.*

More generally, a nodal hypersurface X_d of degree d in $\mathbb{P}^4$ is factorial if and only if the singular points of X_d impose linearly independent conditions on hypersurfaces of degree $2d-5$ in $\mathbb{P}^4$ [Cynk 2001]. To prove the factoriality of X_d, we have to compute whether $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5)\otimes \mathcal{I}_{\mathrm{Sing}(X_d)})$ is 0 or not. Therefore, we need to study the dimension of the linear system $|\mathcal{O}_{\mathbb{P}^4}(2d-5)|$ having assigned base points.

Fixing a nodal hypersurface $X_d \subset \mathbb{P}^4$ and r singular points of X_d in $\mathbb{P}^4$, what is the dimension of the space of hypersurfaces of degree $2d-5$ in $\mathbb{P}^4$ passing through those points?

Let $\phi : \tilde{\mathbb{P}}^4 \to \mathbb{P}^4$ be the blowing up of $\mathbb{P}^4$ along $\mathrm{Sing}(X_d) = \{p_1, \dots, p_r\}$. Let $\tilde{X}_d$ be the strict transform of X_d, let H be the divisor class of the pullback of a hyperplane under ϕ, and let $E = \sum_{i=1}^r E_i$, where the E_i are the classes of exceptional divisors. Suppose that $\mathcal{F} \in \mathrm{Pic}\,\mathbb{P}^4$ and $\mathcal{F}' = \phi^*\mathcal{F}$. Then $\phi_*\mathcal{O}_{\tilde{\mathbb{P}}^4} \cong \mathcal{O}_{\mathbb{P}^4}$ and $R^i\phi_*\mathcal{F}' = 0$ for $i > 0$. Therefore, $H^j(\mathbb{P}^4, \mathcal{F}) \cong H^j(\tilde{\mathbb{P}}^4, \mathcal{F}')$ by the Leray spectral sequence. Moreover, we have the equalities

$$\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}(-E)) = \mathcal{I}_{\mathrm{Sing}(X_d)}, \quad R^i\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}(-E)) = 0 \quad \text{for } i > 0, \tag{2.3}$$

$$\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}(kE)) = \mathcal{O}_{\mathbb{P}^4}, \quad R^i\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}(kE)) = 0 \ \text{ for } i > 0,\ k = 0, 1, 2, 3. \tag{2.4}$$

By (2.3), we get $h^j(\tilde{\mathbb{P}}^4, \mathcal{O}_{\tilde{\mathbb{P}}^4}((2d-5)H - E)) = h^j(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5)\otimes \mathcal{I}_{\mathrm{Sing}(X_d)})$.

Let $L_4(2d-5;\, 1^r) = L_4(2d-5;\, 1_1, 1_2, \dots, 1_r)$ be the complete linear system $\left|(2d-5)H - \sum_{i=1}^r E_i\right|$ on $\tilde{\mathbb{P}}^4$. We will use the same notation to denote the corresponding line bundle on $\tilde{\mathbb{P}}^4$, as well as the push-forward of $\left|(2d-5)H - \sum_{i=1}^r E_i\right|$ to $\mathbb{P}^4$, i.e., the linear system of threefolds of degree $2d-5$ with multiplicity 1 at p_i.

Definition 2.5. A nonempty linear system $L_4(2d-5;\, 1^r)$ is special if

$$h^0(\tilde{\mathbb{P}}^4, L_4(2d-5;\, 1^r)) > (2d-1)(2d-2)(2d-3)(2d-4)/24 - r$$

or, which is the same, if $h^1(\tilde{\mathbb{P}}^4, L_4(2d-5;\, 1^r)) \neq 0$.

Note that $h^0(\tilde{\mathbb{P}}^4, L_4(2d-5;\, 1^r)) \geq h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5)) - r$. We call the system $L_4(2d-5;\, 1^r)$ nonspecial if $h^0(\tilde{\mathbb{P}}^4, L_4(2d-5;\, 1^r)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5)) - r$, or, which is the same, if $h^1(\tilde{\mathbb{P}}^4, L_4(2d-5;\, 1^r)) = 0$.

Lemma 2.6. $h^1(\tilde{X}_d, \mathcal{O}_{\tilde{X}_d}((2d-5)H - E)) = 0 \Leftrightarrow h^1(\tilde{\mathbb{P}}^4, \mathcal{O}_{\tilde{\mathbb{P}}^4}((2d-5)H - E)) = 0$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathbb{P}}^4}(-\tilde{X}_d + (2d-5)H - E) \to \mathcal{O}_{\tilde{\mathbb{P}}^4}((2d-5)H - E) \to \mathcal{O}_{\tilde{X}_d}((2d-5)H - E) \to 0.$$

We have $\tilde{X}_d \equiv dH - 2E$. By (2.4),

$$R^i\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}((d-5)H+E)) = \mathcal{O}_{\mathbb{P}^4}(d-5) \otimes R^i\phi_*(\mathcal{O}_{\tilde{\mathbb{P}}^4}(E)) = 0$$

for $i > 0$. Then we get $h^j(\tilde{\mathbb{P}}^4, \mathcal{O}_{\tilde{\mathbb{P}}^3}((d-5)H+E)) = h^j(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-5)) = 0$ for $0 < j < 4$. Thus

$$h^1(\tilde{X}_d, \mathcal{O}_{\tilde{X}_d}((2d-5)H-E)) = h^1(\tilde{\mathbb{P}}^4, \mathcal{O}_{\tilde{\mathbb{P}}^4}((2d-5)H-E)). \qquad \square$$

Therefore, studying linear systems of threefolds with assigned base points p_i is equivalent to studying complete linear systems on the fourfold $\tilde{\mathbb{P}}^4$ obtained by blowing up the points p_i. Also, a nodal hypersurface X_d of degree d in $\mathbb{P}^4$ is factorial if and only if a nonempty linear system $L_4(2d-5;\, 1^r)$ is nonspecial.

In the rest of this section we present tools to investigate the speciality of $L_4(3;\, 1^r)$ for $r \geq 12$. We need to consider the restriction on a quadric surface Q due to Example 1.5 and Lemma 2.2.

Before stating these results, let Q be a smooth quadric surface (when Q is a singular quadric, we don't have a proof yet). Let $\mathrm{Sing}(X_d) \cap Q = \{p_1, p_2, \ldots, p_\lambda\}$, where λ the maximal number of points of $\mathrm{Sing}(X_d)$ that can belong to the quadric Q. We consider a linear system $|\mathcal{O}(k_1, k_2) \otimes \mathcal{I}_{\mathrm{Sing}(X_d)\cap Q}| := L_Q((k_1, k_2);\, 1^\lambda)$ with $k_1 > 0$ and $k_2 > 0$ on the quadric Q (that is, a system of curves of type (k_1, k_2) through points p_i of multiplicity 1). Using a similar method to Definition 2.5, we define the speciality for $L_Q(k_1, k_2)$. Then we see that

$$h^0(Q, L_Q((k_1, k_2);\, 1^\lambda)) = k_1k_2 + k_1 + k_2 + 1 - \lambda$$

if and only if the system $L_Q((k_1, k_2);\, 1^\lambda)$ is nonspecial.

To prove the factoriality of a nodal quartic X_4, we have to investigate the speciality of the restriction system $L_4(3;\, 1^r)|_Q = L_Q((3, 3);\, 1^\lambda)$.

Lemma 2.7. *With the above notation, let* $\#|\mathrm{Sing}(X_4)| = r \geq 12$. *Suppose the smooth quadric surface* Q *is defined by* $\{f_2(x, y, z, w, t) = 0\} \cap A_1$, *where* f_2 *is a homogeneous polynomial of degree* 2 *and* A_1 *is a hyperplane in* $\mathbb{P}^4$ *such that* $\#|A_1 \cap \mathrm{Sing}(X_4)| \geq 12$. *Let* λ *be the maximal number of points of* $\mathrm{Sing}(X_4)$ *that can belong to the smooth quadric* Q.

(1) *Suppose that* $0 \leq r - \lambda \leq 2$. *Then a linear system* $L_Q((3, 3);\, 1^\lambda)$ *is special if and only if a linear system* $L_4(3;\, 1^r)$ *is special.*

(2) *Suppose that* $3 \leq r - \lambda$. *If a linear system* $L_Q((3, 3);\, 1^\lambda)$ *is special, then a linear system* $L_4(3;\, 1^r)$ *is special.*

Proof. Let $A_1 \equiv H - \widehat{E}$ and let $\{f_2 = 0\} \equiv 2H - \widehat{E}$, where $\widehat{E} = \sum_{i=1}^{\lambda} E_i$.

Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathbb{P}}^4}(-H + \widehat{E} + 3H - E) \to \mathcal{O}_{\tilde{\mathbb{P}}^4}(3H - E) \to \mathcal{O}_{\tilde{\mathbb{P}}^4}(3H - E)|_{A_1} \to 0.$$

We get the exact sequence

$$(2.8)\quad 0\to H^0\big(\mathcal{O}_{\tilde{\mathbb{P}}^4}\big(2H-\sum_{i=\lambda+1}^{r}E_i\big)\big)\to H^0(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E))$$
$$\to H^0(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E)|_{A_1})\to H^1\big(\mathcal{O}_{\tilde{\mathbb{P}}^4}\big(2H-\sum_{i=\lambda+1}^{r}E_i\big)\big)$$
$$\to H^1(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E))\to H^1(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E)|_{A_1})\to 0.$$

Notice that $R^j\phi_*\mathcal{O}_{\tilde{\mathbb{P}}^4}(2H-E)=\mathcal{O}_{\mathbb{P}^4}(2)\otimes R^j\phi_*\mathcal{O}_{\tilde{\mathbb{P}}^4}(-E)=0$ for all $j>0$. If $r=\lambda$, then $h^1\big(\mathcal{O}_{\tilde{\mathbb{P}}^4}\big(2H-\sum_{i=\lambda+1}^r E_i\big)\big)=h^1(\mathcal{O}_{\mathbb{P}^4}(2))=0$. Theorem 2.1 and Lemma 2.2(1) tell us that $h^1\big(\mathcal{O}_{\tilde{\mathbb{P}}^4}\big(2H-\sum_{i=\lambda+1}^r E_i\big)\big)=h^1(\mathbb{P}^4,\mathcal{I}_{\sum_{i=\lambda+1}^r p_i}(2))=0$ for $1\le r-\lambda\le 5$. We then have $h^1(L_4(3;1^r))=h^1(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E))=h^1(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E)|_{A_1})$ when $0\le r-\lambda\le 5$.

Also, note that $Q\equiv(2H-\widehat{E})|_{A_1}$. From the short exact sequence

$$0\to\mathcal{O}_{A_1}(-2H+\widehat{E}+3H-E)\to\mathcal{O}_{A_1}(3H-E)\to\mathcal{O}_{A_1}(3H-E)|_Q\to 0,$$

we obtain the sequence

$$(2.9)\quad 0\to H^0\big(\mathcal{O}_{A_1}\big(H-\sum_{i=\lambda+1}^{r}E_i\big)\big)\to H^0(\mathcal{O}_{A_1}(3H-E))$$
$$\to H^0(\mathcal{O}_{A_1}(3H-E)|_Q)\to H^1\big(\mathcal{O}_{A_1}\big(H-\sum_{i=\lambda+1}^{r}E_i\big)\big)$$
$$\to H^1(\mathcal{O}_{A_1}(3H-E))\to H^1(\mathcal{O}_{A_1}(3H-E)|_Q)\to 0.$$

Note that $h^1\big(\mathcal{O}_{A_1}\big(H-\sum_{i=\lambda+1}^{r}E_i\big)\big)=h^1(\mathbb{P}^3,\mathcal{I}_{\sum_{i=\lambda+1}^r p_i}(1))=0$ for $0\le r-\lambda\le 2$. We have $h^1(\mathcal{O}_{A_1}(3H-E))=h^1(\mathcal{O}_{A_1}(3H-E)|_Q)=h^1(L_Q((3,3);1^\lambda))$ when $0\le r-\lambda\le 2$.

The second statement follows from the last lines of (2.8) and (2.9). □

Corollary 2.10. *With the above notation, if $r=\lambda+j$ for $j=0,1,2$, then*

$$h^0(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3H-E))$$
$$=h^0\big(L_Q((3,3);1^\lambda)\big)+h^0\big(\mathcal{O}_{A_1}\big(H-\sum_{i=\lambda+1}^{r}E_i\big)\big)+h^0\big(\mathcal{O}_{\tilde{\mathbb{P}}^4}\big(2H-\sum_{i=\lambda+1}^{r}E_i\big)\big).$$

Proof. This follows immediately from Lemma 2.7(1). □

Lemma 2.11. *Suppose that a curve of type (a,b), with $0<a\le b\le 3$, and a curve of type $(3,3)$ meet in $3a+3b$ points, say $\Sigma_{(a,b)}=\{p_1,\dots,p_{3a+3b}\}$.*

(1) *Let $\Psi\subset\Sigma_{(2,2)}$ and let $\alpha=\#|\Psi|$. Then a linear system $L_Q((3,3);1^\alpha)$ is special if and only if $\alpha=12$, i.e., a curve of type $(2,2)$ contains $\Sigma_{(2,2)}$.*

(2) *Let* $\Omega \subset \Sigma_{(2,3)}$ (*or* $\Sigma_{(3,2)}$) *and let* $\beta = \#|\Omega|$. *Then a linear system* $L_Q((3,3); 1^\beta)$ *is special if and only if* $\beta = 14$.

(3) *Let* $\Upsilon \subset \Sigma_{(3,3)}$ *and let* $\gamma = \#|\Upsilon|$. *Then a linear system* $L_Q((3,3); 1^\gamma)$ *is special if and only if* $\gamma = 15$.

Proof. By Lemma 2.2(1), a system $L_Q((3,3); 1^\lambda)$ has no fixed curve. The number $h_{\Sigma_{(a,b)}}(3,3)$ of conditions imposed by $\Sigma_{(a,b)}$ on forms of bidegree $(3,3)$ satisfies

$$h_{\Sigma_{(a,b)}}(3,3) = h^0(L_Q(3,3)) - h^0(L_Q(3-a,3-b)) - 1.$$

There are four possible cases for the speciality of $L_Q((3,3); 1^\lambda)$.

When $(a,b) = (2,2)$, we get $h_{\Sigma_{(2,2)}}(3,3) = 11 < \#|\Sigma_{(2,2)}|$.

When $(a,b) = (2,3)$ (or $(3,2)$), we can write $h_{\Sigma_{(2,3)}}(3,3) = 13 < \#|\Sigma_{(2,3)}|$ (or $\#|\Sigma_{(3,2)}|$), so the statement (2) is true.

Finally, the inequality $h_{\Sigma_{(3,3)}}(3,3) = 14 < \#|\Sigma_{(3,3)}|$ implies statement (3). □

3. The proof of Theorem 1.3

Let X_4 be a nodal hypersurface in $\mathbb{P}^4$.

Definition 3.1. The set $\mathrm{Sing}(X_4)$ satisfies the property ∇ if the following conditions hold:

- There is a hyperplane A_1 in $\mathbb{P}^4$ which contains at least 11 points of $\mathrm{Sing}(X_4)$.
- Fix an arbitrary point p of $A_1 \cap \mathrm{Sing}(X_4)$. There is a reducible cubic surface in A_1 passing through $(A_1 \cap \mathrm{Sing}(X_4)) \setminus \{p\}$ but not passing through p.

Lemma 3.2. *Let* $\#|\mathrm{Sing}(X_4)| = 11$. *Suppose that there is a hyperplane* A_1 *in* $\mathbb{P}^4$ *such that* $A_1 \cap \mathrm{Sing}(X_4) = \mathrm{Sing}(X_4)$, *and every quadric surface in* A_1 *does not contain all the points of* $\mathrm{Sing}(X_4)$. *Then* $\mathrm{Sing}(X_4)$ *satisfies the property* ∇.

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. Let $\mathrm{Sing}(X_4) = \{p_1, p_2, \dots, p_{10}, p\}$. Since every quadric surface does not contain all the points of $\mathrm{Sing}(X_4)$, we can find a quadric surface Q_1 in A_1 containing 9 points, say $\{p_1, p_2, \dots, p_8, p_9\}$, of $\mathrm{Sing}(X_4) \setminus \{p\}$ but not containing p. We shall take for the required cubic surface the union of Q_1 and a two-dimensional linear subspace in A_1 passing through p_{10} and not passing through p. □

Lemma 3.3. *Let* $\mathcal{M} \subseteq |\mathcal{O}_{\mathbb{P}^3}(2)|$ *be a linear subsystem that contains the set* $A_1 \cap \mathrm{Sing}(X_4)$, *where* A_1 *is a hyperplane in* $\mathbb{P}^4$. *Suppose that* $n = \#|A_1 \cap \mathrm{Sing}(X_4)| \geq 11$, X_4 *contains no* 2*-planes, and a space curve of degree* 4 *in* A_1 *contains at most* 10 *points of* $\mathrm{Sing}(X_4)$. *Then the base locus* $Bs(\mathcal{M})$ *is empty or two-dimensional.*

Proof. Suppose that $Bs(\mathcal{M})$ is zero-dimensional. Let M_1, M_2, and M_3 be the general surfaces of $\mathcal{M}$. Then the intersection number $M_1 \cdot M_2 \cdot M_3$ has at most 8, but $n \geqslant 11$ holds.

Now we suppose that the curve $B = Bs(\mathcal{M}) \subset A_1$. Then $\deg B \leq 4$. Since $n \geqslant 11$, by Lemma 2.2(1), $\deg B = 4$, and B must be reduced. Moreover, B is not contained in a two-dimensional linear subspace, because a two-dimensional linear subspace contains at most 6 points. This contradicts the assumption. □

Lemma 3.4. *Let* $\#|\mathrm{Sing}(X_4)| = 11$. *Suppose that* X_4 *contains no* 2*-planes, there is a hyperplane* A_1 *in* $\mathbb{P}^4$ *such that* $A_1 \cap \mathrm{Sing}(X_4) = \mathrm{Sing}(X_4)$, *every reducible quadric surface does not contain all the points of* $\mathrm{Sing}(X_4)$, *and a space curve of degree* 4 *in* A_1 *does not pass through all the points of* $\mathrm{Sing}(X_4)$. *Then* $\mathrm{Sing}(X_4)$ *satisfies the property* ∇.

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. Let $\mathrm{Sing}(X_4) = \{p_1, p_2, \ldots, p_{10}, p\}$. By Lemma 3.2, we assume that there is an irreducible quadric surface Q_2 in A_1 containing all the points of $\mathrm{Sing}(X_4)$. By Lemma 3.3, any quadric surface in A_1 passing through all the points of $\mathrm{Sing}(X_4)$ coincides with Q_2. Suppose that Q_2 is determined by 8 points, say $\{p_1, p_2, \ldots, p_8\}$, of $\mathrm{Sing}(X_4) \setminus \{p\}$ together with p. Then we can find a quadric Q_3 in A_1 containing $\{p_1, p_2, \ldots, p_8\}$ and not containing p. We can assume that $p_k \notin Q_3$ for $k = 9$ or 10; otherwise, take a two-dimensional linear subspace in A_1 containing the point $\mathrm{Sing}(X_4) \setminus Q_3$ but not containing p.

If $p \notin \overline{p_9, p_{10}}$, then we can easily construct a reducible cubic surface in A_1 that contains $\mathrm{Sing}(X_4) \setminus \{p\}$ and does not contain p.

Now we suppose that three points $\{p_9, p_{10}, p\}$ lie on a single line. By statement (1) of Lemma 2.2, the line determined by $\{p_i, p_9\}$ (or $\{p_i, p_{10}\}$), for $1 \leq i \leq 8$, does not pass through p. We consider the quadric surface Q_4 determined by $\{p_1, p_2, \ldots, p_8, p_9\}$. We can assume that the quadric surface Q_4 contains the points p; otherwise, take a two-dimensional linear subspace in A_1 containing the point p_{10} but not containing p.

Then Q_4 must be Q_2, that is, the quadric surface Q_2 is determined by the point p and $\{p_1, p_2, \ldots, p_8, p_9\} \setminus \{p_j\}$ for $1 \leq j \leq 9$. Therefore, we can find a quadric surface Q_5 passing through $\{p_1, p_2, \ldots, p_8, p_9\} \setminus \{p_j\}$, for $1 \leq j \leq 8$ and not passing through p. Let l be the line determined by two points $\mathrm{Sing}(X_4) \setminus Q_5 \setminus \{p\}$. Then p cannot lie on the line l. Let $\bar{A}_1$ be a two-dimensional linear subspace in A_1 containing the line l but not containing p. Then the union of Q_5 and $\bar{A}_1$ is the desired form of degree 3. □

Lemma 3.5. *Let* $\#|\mathrm{Sing}(X_4)| = 11$. *Suppose that* X_4 *contains no* 2*-planes, there is a hyperplane* A_1 *in* $\mathbb{P}^4$ *such that* $A_1 \cap \mathrm{Sing}(X_4) = \mathrm{Sing}(X_4)$, *every reducible quadric surface does not contain all the points of* $\mathrm{Sing}(X_4)$, *and there is a space* *curve D of degree* 4 *in* A_1 *that passes through all the points of* $\mathrm{Sing}(X_4)$. *Then* $\mathrm{Sing}(X_4)$ *satisfies the property* ∇.

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. Let $\mathrm{Sing}(X_4) = \{p_1, p_2, \ldots, p_{10}, p\}$. By Lemma 3.2, we assume that there is an irreducible quadric surface containing all the points of $\mathrm{Sing}(X_4)$. By Lemma 2.2(1), a twisted cubic contains at most 9 singular points of X_4. Lemma 3.3 tells us that D can be written as an intersection of two different quadric surfaces in A_1. Then there is a quadric surface in A_1 containing 7 points, say $\{p_1, \ldots, p_7\}$, of $\mathrm{Sing}(X_4) \setminus \{p\}$ but not containing p. We consider the two-dimensional linear subspace $\hat{A}_1$ in A_1 determined by $\{p_8, p_9, p_{10}\}$. We can assume that $p \in \hat{A}_1$; otherwise, one can easily construct the required cubic surface in A_1.

By Lemma 2.2(1), renumbering p_8, p_9, and p_{10} if necessary, we can assume that $\{p_9, p_{10}, p\}$ span $\hat{A}_1$. By Lemma 2.2(2), $\hat{A}_1$ contains at most 2 points of $\{p_1, \ldots, p_7\}$. We can assume that $\hat{A}_1$ contains 2 points, say $\{p_6, p_7\}$, of $\{p_1, \ldots, p_7\}$ (a similar method applies to the case when $\hat{A}_1$ passes through one or none of $\{p_1, \ldots, p_7\}$). Assume that all the quadric surfaces in A_1 containing 7 points, $\{p_1, \ldots, p_7, p_8\} \setminus \{p_i\}$ for $1 \leq i \leq 5$, also pass through p. Then each quadric surface containing the points p_6, p_7, and p_8 also contains p, and hence p_6, p_7, p_8, and p lie on a single line. This is a contradiction to Lemma 2.2(1). Thus, we can find a quadric surface Q_6 in A_1 containing 7 points, $\{p_1, \ldots, p_7, p_8\} \setminus \{p_i\}$ for $1 \leq i \leq 5$, and not containing p. Take the two-dimensional linear subspace A_1' in A_1 determined by three points $\mathrm{Sing}(X_4) \setminus Q_6 \setminus \{p\}$. Then A_1' does not contain p, and hence $Q_6 + A_1'$ is the required cubic surface. □

Proposition 3.6. *Let* $\#|\mathrm{Sing}(X_4)| = 11$. *Suppose that* X_4 *contains no* 2*-planes*, *a hyperplane in* $\mathbb{P}^4$ *contains all the points of* $\mathrm{Sing}(X_4)$, *and every reducible quadric surface does not contain all the points of* $\mathrm{Sing}(X_4)$. *Then* $\mathrm{Sing}(X_4)$ *satisfies the property* ∇.

Proof. By Lemma 3.2, we can assume that there is an irreducible quadric surface containing all the points of $\mathrm{Sing}(X_4)$. It immediately follows from Lemmas 3.4 and 3.5. □

The following result is proved in [Shramov 2007].

Corollary 3.7 [Shramov 2007, Corollary 3.4]. *Let* $\mathrm{Sing}(X_4) = \{p_1, \ldots, p_s\}$. *Assume that* $s \geq 11$, *and no* 2*-plane contains* 7 *points of* $\mathrm{Sing}(X_4)$. *Let* $p_1, \ldots, p_s$ *be points lying in a reducible quadric surface* (*that is*, *in a pair of planes spanning a three-dimensional surface*). *Then either* $p_1, \ldots, p_s$ *impose independent conditions on the forms of degree* 3 *or* $p_1, \ldots, p_s$ *also lie in an irreducible quadric surface.*

Proposition 3.8. *Let* $\#|\mathrm{Sing}(X_4)| = 13$. *Suppose that* X_4 *contains no* 2*-planes*, *and a hyperplane in* $\mathbb{P}^4$ *contains at most* 11 *points of* $\mathrm{Sing}(X_4)$. *Then* X_4 *is factorial.*

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. It is enough to construct a cubic threefold T that contains $\mathrm{Sing}(X_4) \setminus \{p\}$ and does not contain p.

Suppose that there is a hyperplane A_1 in $\mathbb{P}^4$ containing 11 points of $\mathrm{Sing}(X_4)$; otherwise, X_4 is factorial due to Theorem 2.1 and Lemma 2.2.

Suppose that $p \notin A_1$. We can find a quadric threefold G passing through two points $\mathrm{Sing}(X_4) \setminus A_1 \setminus \{p\}$ but not passing through p. Then $T = A_1 + G$.

Now assume that $p \in A_1$. We divide the case into two subcases. Let $\{q_1, q_2\} = \mathrm{Sing}(X_4) \setminus A_1$.

First, assume that a reducible quadric surface in A_1 does not contain all the points of $\mathrm{Sing}(X_4)$. By Proposition 3.6, we obtain a cubic surface $W = Q' + \Lambda$, where Q' is a quadric surface and Λ is a two-dimensional linear subspace, in A_1 containing $\mathrm{Sing}(X_4) \cap (A_1 \setminus \{p\})$ but not containing p. Take the cone Q'' over Q' with vertex q_1 and a hyperplane Λ' in $\mathbb{P}^4$ containing Λ together with q_2. Then we can get the required cubic threefold T as the union of Q'' and Λ'.

Second, assume that a reducible quadric surface in A_1 contains all the points of $\mathrm{Sing}(X_4)$. Applying the proof of Corollary 3.7, we can construct a reducible cubic surface Y in A_1 passing through $(\mathrm{Sing}(X_4) \cap A_1) \setminus \{p\}$ and not passing through p such that Y consists of three two-dimensional linear subspaces, say L_1, L_2, and L_3. Note that $q_1, q_2 \notin A_1$. Then we obtain the required cubic threefold T as the union of a hyperplane in $\mathbb{P}^4$ containing $\{L_1, q_1\}$, a hyperplane in $\mathbb{P}^4$ containing $\{L_2, q_2\}$, and a hyperplane in $\mathbb{P}^4$ containing L_3 but not containing p. □

The following three results are proved in [Shramov 2007]. They are very useful for the proof of Theorem 1.3. We let $\mathrm{Sing}(X_4) = \{p_1, \dots, p_s\}$.

Lemma 3.9 [Shramov 2007, Lemma 3.5]. *Assume that $s \leq 12$, and no 2-plane contains 7 points of $\mathrm{Sing}(X_4)$. Let $p_1, \dots, p_s$ be points in a 3-dimensional subspace $\mathbb{P}^3 \subset \mathbb{P}^4$ not lying in some quadric surface. Then $p_1, \dots, p_s$ impose independent conditions on the forms of degree 3 in $\mathbb{P}^3$ (and therefore also in $\mathbb{P}^4$).*

Lemma 3.10 [Shramov 2007, Lemma 3.8]. *Assume that no 2-plane contains 7 points of $\mathrm{Sing}(X_4)$. Let $p_1, \dots, p_{12}$ be points in a quadric surface. Then either $p_1, \dots, p_{12}$ impose independent conditions on the forms of degree 3 in $\mathbb{P}^4$ or $p_1, \dots, p_{12}$ lie in a pencil of quadric surfaces in some 3-dimensional subspace.*

Lemma 3.11 [Shramov 2007, Lemma 3.9]. *Assume that no 2-plane contains 7 points of $\mathrm{Sing}(X_4)$. Let $p_1, \dots, p_{12}$ be points lying in a pencil of quadric surfaces in a 3-dimensional subspace. Then X_4 contains a quadric surface.*

Proposition 3.12. *Let $\#|\mathrm{Sing}(X_4)| = 13$. Suppose that X_4 contains no 2-planes, and a hyperplane in $\mathbb{P}^4$ contains at most 12 points of $\mathrm{Sing}(X_4)$. Then X_4 is either factorial or contains a quadric surface.*

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. We can assume that there is a hyperplane A_1 in $\mathbb{P}^4$ containing 12 points of $\mathrm{Sing}(X_4)$. Let $\{q\} = \mathrm{Sing}(X_4) \setminus A_1$.

First, suppose that a quadric surface contains at most 11 points of $\mathrm{Sing}(X_4)$. We can assume that $p \in A_1$; otherwise, one can easily check that X_4 is factorial. By Lemma 3.9, we can find a cubic surface U in A_1 passing through $(\mathrm{Sing}(X_4) \cap A_1) \setminus \{p\}$ and not passing through p. Taking a cone over U with vertex q, we obtain a cubic threefold passing through $\mathrm{Sing}(X_4) \setminus \{p\}$ and not passing through p. In this case, X_4 is factorial.

Second, suppose that there is a quadric surface $\widehat{Q}$ containing 12 points, say Ξ, of $\mathrm{Sing}(X_4)$. We can assume that Ξ cannot lie on a pencil of quadric surface in A_1; otherwise, by Lemma 3.11, X_4 contains a quadric surface.

Now we have to prove that X_4 is factorial. We can assume that $p \in \widehat{Q}$. Let $\Xi = \{p_1, \dots, p_{11}, p\}$. Applying the proof of Lemma 3.10, we obtain a reducible cubic surface K in A_1 containing $\Xi \setminus \{p\}$ but not containing p. Note that $q \notin A_1$. Let $K = S + L$, where S is a quadric surface and L is a two-dimensional linear subspace. Then we can construct a cubic threefold as the union of the cone over S with vertex q and a hyperplane in $\mathbb{P}^4$ containing L but not containing p. Thus, X_4 is factorial. □

Proposition 3.13. *Let* $\#|\mathrm{Sing}(X_4)| = 13$. *Suppose that* X_4 *contains no* 2*-planes, and there is a hyperplane* A_1 *in* $\mathbb{P}^4$ *containing all the points of* $\mathrm{Sing}(X_4)$. *Then* X_4 *is either factorial or contains a quadric surface.*

Proof. Fix an arbitrary point p of $\mathrm{Sing}(X_4)$. Let $\mathrm{Sing}(X_4) = \{p_1, \dots, p_{12}, p\}$.

Suppose that every quadric surface does not contain all the points of $\mathrm{Sing}(X_4)$. Then we find a quadric surface containing 9 points, say $\{p_1, \dots, p_9\}$, of $\mathrm{Sing}(X_4)$ but not containing p. We consider the two-dimensional linear subspace $\hat{A}_1$ in A_1 determined by $\{p_{10}, p_{11}, p_{12}\}$. We can assume that $p \in \hat{A}_1$; otherwise, one can easily check that X_4 is factorial.

By Lemma 2.2(1), renumbering p_{10}, p_{11}, and p_{12} if necessary, we can assume that $\{p_{11}, p_{12}, p\}$ span $\hat{A}_1$. By Lemma 2.2(2), $\hat{A}_1$ contains at most 2 points of $\{p_1, \dots, p_9\}$. We can assume that $\hat{A}_1$ contains 2 points, say $\{p_8, p_9\}$, of $\{p_1, \dots, p_9\}$ (a similar method applies to the case when $\hat{A}_1$ passes through one or none of $\{p_1, \dots, p_9\}$). Assume that all the quadric surfaces in A_1 containing 9 points, $\{p_1, \dots, p_9, p_{10}\} \setminus \{p_i\}$ for $1 \le i \le 7$, also pass through p. Then each quadric surface containing the points p_8, p_9, and p_{10} also contains p, and hence p_8, p_9, p_{10}, and p lie on a single line. This is a contradiction to Lemma 2.2(1). Thus, we can find a quadric surface Q_7 in A_1 containing 9 points, $\{p_1, \dots, p_9, p_{10}\} \setminus \{p_i\}$ for $1 \le i \le 7$, and not containing p. The union of Q_7 and the two-dimensional linear subspace A_1'' in A_1 determined by three points $\mathrm{Sing}(X_4) \setminus Q_7 \setminus \{p\}$ is a cubic surface containing $\mathrm{Sing}(X_4) \setminus \{p\}$ but not containing p. It implies that X_4 is factorial.

Now suppose that there is a quadric surface Q_8 containing all the points of $\mathrm{Sing}(X_4)$. Then, by Lemma 2.2(2), Q_8 is irreducible. We may assume that Q_8 is a quadric cone; otherwise, by Lemma 2.7(1) and 2.11(1), X_4 is either factorial or contains a quadric surface. For instance, since a curve of type (1, 1) contains at most 6 points of $\mathrm{Sing}(X_4)$, X_4 cannot contain a 2-plane.

If there is a quadric surface different from Q_8 containing 12 points of $\mathrm{Sing}(X_4)$, then X_4 contains a quadric surface due to Lemma 3.11.

We can assume Q_8 is unique; that is, any quadric surface different from Q_8 passes through at most 11 points of $\mathrm{Sing}(X_4)$. Consider a nodal quartic threefold $\widehat{X}_4$ defined by

$$a_1(x, y, z, t, w)h_3(x, y, z, t, w) + b_2(x, y, z, t, w)g_2(x, y, z, t, w) = 0,$$

where a_1, h_3, b_2 and g_2 are homogeneous polynomials of degree 1, 3, 2, and 2, respectively. Suppose that the quadric Q_8 is the quadric cone given by $\{a_1 = b_2 = 0\}$, and V_2 is a quadric surface given by $\{a_1 = g_2 = 0\}$. Then the nodes of $\widehat{X}_4$ are $\{a_1 = h_3 = b_2 = g_2 = 0\}$ and the vertex of Q_8. The quartic $\widehat{X}_4$ has 13 nodes with $\mathrm{Sing}(\widehat{X}_4) = \mathrm{Sing}(X_4)$, and all the points of $\mathrm{Sing}(\widehat{X}_4)$ lie on a hyperplane $\{a_1 = 0\}$. By the uniqueness of Q_8, X_4 must be $\widehat{X}_4$. Since V_2 contains 12 points of $\mathrm{Sing}(X_4)$, this contradicts the assumption. □

Proof of Theorem 1.3. Suppose that every two dimensional linear subspace contains at most 6 singular points of a nodal quartic X_4, i.e, by Lemma 2.2(2), X_4 contains no 2-planes; otherwise, X_4 is defined by an equation of the form

$$y_1(x, y, z, t, w) f_3(x, y, z, t, w) + \hat{y}_1(x, y, z, t, w)g_3(x, y, z, t, w) = 0,$$

where y_1, f_3, $\hat{y}_1$, and g_3 are homogeneous polynomials of degree 1, 3, 1, and 3, respectively. Then X_4 is not factorial.

Theorem 1.3 immediately follows from Propositions 3.8, 3.12, and 3.13. □

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