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SUMS OF SQUARES IN ALGEBRAIC FUNCTION FIELDS OVER A COMPLETE DISCRETELY VALUED FIELD

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A recently found local-global principle for quadratic forms over function fields of curves over a complete discretely valued field is applied to the study of quadratic forms, sums of squares, and related field invariants.

1. Introduction

Let K be a field of characteristic different from 2 and F/K an algebraic function field (i.e., a finitely generated extension of transcendence degree one). The study of quadratic forms over F is generally difficult, even in such cases where the quadratic form theory over all finite extensions of K is well understood. It can be considered complete in the cases where K is algebraically closed, real closed, or finite, but it is wide open for example when K is a number field.

A breakthrough was obtained recently in the situation where the base field K is a nondyadic local field. Parimala and Suresh [2010] proved that in this case any quadratic form of dimension greater than eight over F is isotropic. Harbater, Hartmann, and Krashen [Harbater et al. 2009] obtained the same result as a consequence of a new local–global principle for isotropy of quadratic forms over F. The local conditions are in geometric terms, relative to an arithmetic model for F. A less geometric version of the local–global principle, in terms of the discrete rank one valuations of F, was obtained by Colliot-Thélène, Parimala, and Suresh [Colliot-Thélène et al. 2012]; see Theorem 6.1 below. Both versions of the local–global principle hold more generally when K is complete with respect to a nondyadic discrete valuation.

In this article we apply the local–global principle to study sums of squares in F and to obtain further results on quadratic forms over F. This is of particular interest in the case where K is the field of Laurent series $k(\!(t)\!)$ over a (formally) real field k.

We outline the structure of this article and the main results. Section 2 provides some necessary basic results on valuations. In Section 3 we discuss discrete

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valuations on an algebraic function field over a complete discretely valued field and characterize their residue fields. In Section 4 we move on to the study of sums of squares in fields and corresponding field invariants, in the context of valuations. In Section 5 we do an analogous discussion of the u-invariant in the context of valuations. According to [Elman and Lam 1973], the u-invariant of a field is the supremum on the dimension of anisotropic torsion forms over that field. In Section 6 we finally apply the local-global principle to obtain new results for algebraic function fields and in particular the rational function field. Let us describe some of these results.

In Theorem 6.4 we show that the upper bound on the dimension of anisotropic torsion forms over algebraic function fields over K is the double of the corresponding upper bound for algebraic function fields over k, the residue field of the discrete valuation on K. We thus obtain an upper bound on the u-invariant of an algebraic function field over K. We obtain in Theorem 6.6 a refinement for the case of the rational function field, saying that the u-invariant of $K(X)$ is equal to the supremum of the u-invariant of all $\ell(X)$ where ℓ/k is a finite algebraic extension. In Corollary 6.9 we prove that the Pythagoras number of the rational function field $K(X)$ is equal to the supremum of the Pythagoras numbers of $\ell(X)$ for all finite field extensions ℓ/k. We conjecture in Conjecture 4.16 that this is equal to the Pythagoras number of $k(X)$. This is motivated by the observation — made previously in [Scheiderer 2001] — that both Pythagoras numbers are bounded by the same power of two. In the case where k is real closed we show in Theorem 6.12 that any sum of squares in F can be expressed as a sum of three squares and further prove the finiteness of $\sum F^2/D_F(2)$, the quotient of the group of nonzero sums of squares modulo the subgroup of sums of two squares in F.

Our methods are based on valuation theory and quadratic form theory, for which [Engler and Prestel 2005] and [Lam 2005] are our standard references. We also use some algebraic geometry, namely desingularization of arithmetic surfaces and the properties of blowing-ups in regular points. Our reference on this topic is [Liu 2002].

This article grew out of results obtained in the Ph.D. thesis of D. Grimm under the supervision of K. J. Becher at Universität Konstanz.

2. Valuations

For a ring R we denote by $R^\times$ its group of invertible elements.

Let K be a field. Given a valuation on K, we denote by $\mathcal{O}_v$ the valuation ring of v, by $\mathfrak{m}_v$ its maximal ideal, by κ_v the residue field, by K^v the completion of K with respect to v, and we further call v *dyadic* if κ_v has characteristic 2, *nondyadic* otherwise. Given a local ring R contained in K, we say that a valuation v of K *dominates* R if $\mathfrak{m}_v \cap R$ is the maximal ideal of R. Given a field extension L/K, we say that a valuation v of L is *unramified over* K if $v(L^\times) = v(K^\times)$.

A valuation with value group $\mathbb{Z}$ is called a *$\mathbb{Z}$-valuation*. Any discrete valuation of rank one can be identified (via a unique isomorphism of the value groups) with a $\mathbb{Z}$-valuation. A commutative ring is the valuation ring of a $\mathbb{Z}$-valuation if and only if it is a regular local ring of dimension one (see [Matsumura 1986, Theorem 11.2]); such rings are called *discrete valuation rings*.

Lemma 2.1. *Let w_1 and w_2 be two valuations on K such that $\mathfrak{m}_{w_1} \subseteq \mathcal{O}_{w_2}$. Then $\mathcal{O}_{w_1} \subseteq \mathcal{O}_{w_2}$ or $\mathcal{O}_{w_2} \subseteq \mathcal{O}_{w_1}$.*

Proof. If $\mathfrak{m}_{w_1} \subseteq \mathfrak{m}_{w_2}$, then $\mathcal{O}_{w_1} \supseteq \mathcal{O}_{w_2}$, otherwise for any choice of $t \in \mathfrak{m}_{w_1} \setminus \mathfrak{m}_{w_2}$ we have $t^{-1} \in \mathcal{O}_{w_2}$ and $\mathcal{O}_{w_1} = t^{-1}(t\mathcal{O}_{w_1}) \subseteq t^{-1}\mathfrak{m}_{w_1} \subseteq \mathcal{O}_{w_2}$. □

The property for a valuation to be *henselian* is characterized by a list of equivalent conditions, including the statement of Hensel's Lemma, hence satisfied in particular by complete $\mathbb{Z}$-valuations; see [Engler and Prestel 2005, Section 4.1].

Proposition 2.2. *Let v be a henselian $\mathbb{Z}$-valuation on K. Then v is the unique $\mathbb{Z}$-valuation on K.*

Proof. By [Engler and Prestel 2005, Corollary 2.3.2] for distinct $\mathbb{Z}$-valuations w_1 and w_2 on K one has $\mathcal{O}_{w_1} \not\subseteq \mathcal{O}_{w_2}$ and $\mathcal{O}_{w_2} \not\subseteq \mathcal{O}_{w_1}$. Consider now a $\mathbb{Z}$-valuation w on K. Since v is henselian we have $1 + \mathfrak{m}_v \subseteq K^{\times n}$ for all $n \in \mathbb{N}$ prime to the characteristic of κ_v. As $w(K^\times) = \mathbb{Z}$, this implies that $1 + \mathfrak{m}_v \subseteq \mathcal{O}_w^\times$ and thus $\mathfrak{m}_v \subseteq \mathcal{O}_w$. Now Lemma 2.1 yields that $\mathcal{O}_w = \mathcal{O}_v$. □

Let X always denote a variable over a given ring or field.

Proposition 2.3. *Let R be a local domain with maximal ideal $\mathfrak{m}$ and residue field k. Let $p \in R[X]$ be monic and such that $\bar{p} \in k[X]$, the reduction of p modulo $\mathfrak{m}$, is irreducible. Then $R[X]/(p)$ is a local domain with maximal ideal $(\mathfrak{m}[X]+(p))/(p)$ and residue field $k[X]/(\bar{p})$. The ring $R[X]/(p)$ has the same dimension as R. Moreover, if R is regular, then $R[X]/(p)$ is regular.*

Proof. Note that $\mathfrak{m}[X] + (p)$ is a maximal ideal of $R[X]$. Consider a maximal ideal M of $R[X]$ containing p and set $\mathfrak{p} = M \cap R$. Since $R[X]/(p)$ is an integral extension of R, it follows using [Matsumura 1986, Theorems 9.3 and 9.4] that both rings have the same dimension. Moreover, the field $R[X]/M$ is an integral extension of $R/\mathfrak{p}$, whereby $R/\mathfrak{p}$ is a field. It follows that $\mathfrak{p} = \mathfrak{m}$ and thus $M = \mathfrak{m}[X] + (p)$. This shows that $\mathfrak{m}[X] + (p)$ is the unique maximal ideal of $R[X]$ containing p. Hence, $R[X]/(p)$ is a local domain with maximal ideal $(\mathfrak{m}[X] + (p))/(p)$ and residue field $k[X]/(\bar{p})$. Any set of generators of $\mathfrak{m}$ in R yields a set of generators of $(\mathfrak{m}[X] + (p))/(p)$ in $R[X]/(p)$. In particular, if R is regular, so is $R[X]/(p)$. □

Corollary 2.4. *Let T be a discrete valuation ring of K with residue field k. Let $p \in T[X]$ be monic with $\bar{p} \in k[X]$ irreducible. Then $T[X]/(p)$ is a discrete valuation ring with field of fractions $K[X]/(p)$ and residue field k-isomorphic to $k[X]/(\bar{p})$.*

Proof. Since a discrete valuation ring is the same as a regular local ring of dimension one, the statement follows from Proposition 2.3. □

We want to mention the following partial generalization of Corollary 2.4.

Proposition 2.5. *Let T be a valuation ring of K with residue field k and let ℓ/k be a finite field extension. There exists a finite field extension L/K with $[L:K]=[\ell:k]$ and a valuation v on L dominating T and unramified over K whose residue field is k-isomorphic to ℓ.*

Proof. It suffices to consider the case where $\ell = k[x]$ for some $x \in \ell$. Let $\mathfrak{m}$ denote the maximal ideal of T. Let $p \in T[X]$ be a monic polynomial whose residue $\bar{p}$ in $k[X]$ is the minimal polynomial of x over k. Then p is irreducible in $K[X]$, so $L = K[X]/(p)$ is a field. We obtain from Proposition 2.3 that $R = T[X]/(p)$ is a local domain with maximal ideal $M = (\mathfrak{m}[X] + (p))/(p)$ and residue field $k[X]/(\bar{p})$. Let v be a valuation on L dominating T. Then $T \subseteq R \subseteq \mathcal{O}_v$, and as M is generated by $\mathfrak{m}$, it follows that v dominates R. Hence, $k[X]/(\bar{p})$ embeds naturally into κ_v. In particular $[\kappa_v : k] \geq \deg(\bar{p}) = \deg(p) = [L : K]$. Using the Fundamental Inequality [Engler and Prestel 2005, Theorem 3.3.4] we conclude that v is unramified over K and $[\kappa_v : k] = \deg(\bar{p}) = [L : K]$, whereby κ_v is k-isomorphic to $k[X]/(\bar{p})$ and therefore to ℓ. □

3. Valuations on algebraic function fields

In this section we want to relate algebraic function fields over a valued field to algebraic function fields over the corresponding residue field. In particular we show in Proposition 3.4 that an algebraic function field over the residue field of a valuation on K can be realized as the residue field of an unramified extension to some algebraic function field over K, and we refine this statement in Theorem 3.5 for rational function fields.

In the sequel let T denote a valuation ring, K its field of fractions, and k the residue field of T. (That is, we have $T = \mathcal{O}_v$ for a valuation v on K and $k = \kappa_v$.) We consider the residue fields of valuations dominating T. (The reader may observe that we avoid to speak of extensions of valuations, as this can lead to confusion about the corresponding value groups.) For a field extension F/K and a valuation v on F dominating T, the field k is naturally embedded in the residue field κ_v. We often identify residue fields of valuations dominating T up to k-isomorphism, in order to simplify the language.

A finitely generated field extension F/K of transcendence degree one is called an *algebraic function field*. We say that F/K is *an algebrorational function field* if $F = L(x)$ for a finite extension L/K with $L \subseteq F$ and some element $x \in F$ that is transcendental over L; if this holds already with $L = K$, then F/K is called *a rational function field*.

Proposition 3.1. *Let F/K be an algebraic function field and v a valuation on F dominating T. The extension κ_v/k is either algebraic or an algebraic function field.*

Proof. This is a special case of the Dimension Inequality [Engler and Prestel 2005, Theorem 3.4.3]. □

The following is a refinement of Proposition 3.1 for rational function fields.

Theorem 3.2 (Ohm and Nagata). *Let F/K be a rational function field and v be a valuation on F dominating T. Then κ_v/k is either algebraic or algebrorational.*

Proof. This is shown in [Ohm 1983, Theorem], as a generalization of [Nagata 1967, Theorem 1]. □

We recall a construction to extend a valuation to a rational function field; in [Engler and Prestel 2005, Section 2.2] this is called the Gauss extension.

Proposition 3.3. *Let F/K be a rational function field. Let $x \in F$ be such that $F = K(x)$. Let T' be the localization of $T[x]$ with respect to the prime ideal $\mathfrak{m}[x]$ where $\mathfrak{m}$ is the maximal ideal of T. Then T' is a valuation ring with field of fractions F. The residue field of T' is $k(\bar{x})$ where $\bar{x}$ is the class of x modulo $\mathfrak{m}[x]$, which is transcendental over k. The corresponding valuation v on F with $\mathcal{O}_v = T'$, uniquely determined up to equivalence, is unramified over K.*

Proof. This follows from [Engler and Prestel 2005, Corollary 2.2.2]. □

Proposition 3.4. *Let E/k be an algebraic function field. There exists an algebraic function field F/K and a valuation v on F dominating T and unramified over K whose residue field is E.*

Proof. Let F'/K be a rational function field. Let $x \in F'$ be such that $F' = K(x)$ and let T' denote the valuation ring described in Proposition 3.3. We identify $\bar{x}$ with some element of E transcendental over k. Then $E/k(\bar{x})$ is a finite extension. By Proposition 2.5 there exists a finite field extension F/F' with $[F : F'] = [E : k(\bar{x})]$ and a valuation v on F dominating T' and unramified over F' with residue field E. Using Proposition 3.3 it follows that v is also unramified over K. □

Theorem 3.5. *Assume that $T \neq K$ and let F/K be a rational function field. Let ℓ/k be a finite separable field extension. There exists a valuation v on F dominating T and unramified over K for which κ_v/k is an algebrorational function field with field of constants ℓ.*

Proof. Let $y \in F$ and $\alpha \in \ell$ be such that $F = K(y)$ and $\ell = k(\alpha)$. Let $q \in T[Y]$ be monic and such that the residue $\bar{q}$ in $k[Y]$ is the minimal polynomial of α. Let $\mathfrak{m}$ be the maximal ideal of T. We choose $m \in \mathfrak{m} \setminus \{0\}$ and set $x = m^{-1}q(y) \in F$. Note that x is transcendental over K, and thus $F/K(x)$ is a finite extension. Let T' be the localization of $T[x]$ with respect to $\mathfrak{m}[x]$, the ideal consisting of the polynomials in x with coefficients in $\mathfrak{m}$. Let $\mathfrak{m}'$ be the maximal ideal of T'. By Proposition 3.3 T' is a valuation ring with field of fractions $K(x)$ and residue field $k(\bar{x})$, and $\bar{x}$ is transcendental over k. Note that $\bar{q}$ remains irreducible in $k(\bar{x})[Y]$.

Consider $p = q - q(y) \in T'[Y]$. As $q(y) = mx$, taking residues modulo $\mathfrak{m}'[Y]$ we have $\bar{p} = \bar{q}$ in $k(\bar{x})[Y]$. It follows by Proposition 2.3 that $R = T'[Y]/(p)$ is a local ring with maximal ideal lying over $\mathfrak{m}'$, with field of fractions $K(x)[Y]/(p)$, and residue field $k(\bar{x})[Y]/(\bar{p})$. Note that $K(x)[Y]/(p)$ is $K(x)$-isomorphic to F. Using Chevalley's theorem [Engler and Prestel 2005, Theorem 3.1.1], we obtain a valuation v on F that dominates T'. Then v also dominates T. As $p(y) = 0$, we have that y is integral over T'. Since $\bar{p} = \bar{q}$ is irreducible in $k(\bar{x})[Y]$, we have that $\bar{p}(0) \neq 0$, whereby $p(0) \in T'^{\times}$. As v dominates T' and $p(y) = 0$, we obtain that $v(y) = 0$. Hence, $\bar{x}, \bar{y} \in \kappa_v$, and $\bar{y}$ is algebraic over k, because $\bar{q}(\bar{y}) = \bar{p}(\bar{y}) = 0$. As $\bar{q}$ is irreducible in $k(\bar{x})[Y]$ we obtain that

$$[\kappa_v : k(\bar{x})] \geq [k(\bar{x})[\bar{y}] : k(\bar{x})] = \deg(\bar{p}) = \deg(p) = [F : K(x)].$$

By the Fundamental Inequality [Engler and Prestel 2005, Theorem 3.3.4], it follows that v is unramified over $K(x)$ and $\kappa_v = k(\bar{x})[\bar{y}] = k[\bar{y}](\bar{x})$. Using Proposition 3.3 we obtain that v is unramified over K. Since $q(\bar{y}) = 0 = \bar{q}(\alpha)$ and since we consider residue fields up to k-isomorphism, we can identify $\ell = k[\alpha]$ with $k[\bar{y}]$. □

Remark 3.6. In Theorem 3.5, the hypothesis on the finite extension ℓ/k to be separable is not necessary. Given a finite extension ℓ/k we can obtain a regular model (see below for the definition) for F/T whose special fiber contains a component isomorphic to $\mathbb{P}^1_\ell$ in the following way: We choose $\alpha \in \ell$ and $\ell' = k(\alpha)$. Blowing up $\mathbb{P}^1_T$ in a point on the special fiber $\mathbb{P}^1_k$ with residue field ℓ', we obtain a new regular model whose special fiber has a component given by the exceptional fiber of this blowing-up and thus isomorphic to $\mathbb{P}^1_{\ell'}$. Iterating this process we eventually obtain a regular model for F/T whose special fiber has a component isomorphic to $\mathbb{P}^1_\ell$, and its generic point corresponds to a $\mathbb{Z}$-valuation whose residue field is a rational function field over ℓ.

Assume that the valuation ring T is discrete and consider an algebraic function field F/K. By a *regular model for* F/T we mean a 2-dimensional integral regular projective flat T-scheme $\mathscr{X}$ whose function field is K-isomorphic to F. Given a regular model $\mathscr{X}$ for F/K we denote by $\mathscr{X}_k$ its special fiber; by [Liu 2002, Chapter 8, Lemma 3.3] $\mathscr{X}_k$ is a curve.

Given an integral scheme $\mathscr{X}$, a point $P \in \mathscr{X}$, and a valuation v on the function field of $\mathscr{X}$, we say that v is *centered at* P if v dominates $\mathcal{O}_{\mathscr{X},P}$, the local ring at P.

Proposition 3.7. *Assume that T is a discrete valuation ring. Let F/K be an algebraic function field. Let $\mathscr{X}$ be a regular model for F/T. Let v be a $\mathbb{Z}$-valuation on F dominating T. Then v is centered at a point P of $\mathscr{X}$ lying in $\mathscr{X}_k$. Moreover, if the extension κ_v/k is neither algebraic nor algebrorational, then $\mathbb{O}_v = \mathbb{O}_{\mathscr{X},P}$ where P is the generic point of an irreducible component of $\mathscr{X}_k$.*

Proof. By [Liu 2002, Chapter 8, Definition 3.17] v is centered at a point P of the special fiber $\mathscr{X}_k$. Since $\mathscr{X}_k$ is a curve, P is either a closed point or the generic point of an irreducible component $\mathscr{X}_k$. In either case $\mathbb{O}_{\mathscr{X},P}$ is a regular local ring.

If P is a closed point of $\mathscr{X}_k$, then by [Abhyankar 1956, Proposition 3] the extension κ_v/k is either algebraic or algebrorational. Assume now that P is a generic point of $\mathscr{X}_k$. Then P has codimension one in $\mathscr{X}$, so $\mathbb{O}_{\mathscr{X},P}$ is a regular local ring of dimension one and thus a discrete valuation ring. As $\mathbb{O}_{\mathscr{X},P}$ is dominated by $\mathbb{O}_v$ and both are discrete valuation rings with the same field of fractions, it follows by [Engler and Prestel 2005, Corollary 2.3.2] that $\mathbb{O}_v = \mathbb{O}_{\mathscr{X},P}$. □

Proposition 3.8. *Assume that T is a complete discrete valuation ring. Let F/K be an algebraic function field. Then there exists a regular model for F/T.*

Proof. There exists a regular projective curve C over K whose function field is K-isomorphic to F. If the curve C is smooth, then by [Liu 2002, Chapter 10, Proposition 1.8)] there exists a regular model for F/T. Note that this applies in particular when $\text{char}(K) = 0$. Without assuming that C is smooth, we can follow the first steps in the proof of the proposition cited to obtain a 2-dimensional projective T-scheme $\mathscr{X}$ with function field F. Since the structure morphism $\mathscr{X} \to \text{Spec}(T)$ is surjective, it is flat (see [Liu 2002, Chapter 8, Definition 3.1]). Then T is an excellent ring (see [ibid., Corollary 2.40]), and $\mathscr{X}$, being locally of finite type over T, is excellent (see [ibid., Theorem 2.39]).

Let $\mathscr{X}' \to \mathscr{X}$ be the normalization of $\mathscr{X}$. Since $\mathscr{X}$ is excellent and projective over T, the normalization $\mathscr{X}' \to \mathscr{X}$ is a finite projective birational morphism (see [ibid., Theorem 8.2.39 and Lemma 3.47]). The singular locus of $\mathscr{X}'$ is closed in $\mathscr{X}'$ (see [ibid., Corollary 2.38]). We consider the blowing-up $\mathscr{X}'' \to \mathscr{X}'$ along the singular locus of $\mathscr{X}'$; this is a birational projective morphism (see [ibid., Propositions 1.12 and 1.22]).

We may alternate normalization and blowing-up until we reach a scheme that is regular. At each step we obtain a flat projective 2-dimensional T-scheme whose function field is F. By Lipman's desingularization theorem (see [ibid., Theorem 3.44]), after finitely many steps we come to a situation where the T-scheme is regular. □

Corollary 3.9. *Assume that T is a complete discrete valuation ring. Let F/K be an algebraic function field. Then there exist only finitely many $\mathbb{Z}$-valuations v on F dominating T for which the extension κ_v/k is neither algebraic nor algebrorational.*

Proof. By Proposition 3.8 there exists a regular model for F/T. The statement follows by applying Proposition 3.7 to any such model. □

The result Corollary 3.9 can be extended to the situation where T is an arbitrary discrete valuation ring. Moreover, one may ask to characterize the $\mathbb{Z}$-valuations on an algebraic function field that dominate a given discrete valuation ring of the base field and for which the residue field extension is neither algebraic nor algebrorational. We intend to develop these topics in a forthcoming article.

4. Sums of squares and valuations

From now on let K be a field of characteristic different from 2. We denote by $\sum K^2$ the subgroup of nonzero sums of squares in K and, for $n \in \mathbb{N}$, by $D_K(n)$ the set of nonzero elements that can be written as sums of n squares in K. One calls

$$s(K) = \inf\{n \in \mathbb{N} \mid -1 \in D_K(n)\} \in \mathbb{N} \cup \{\infty\}$$

the *level of* K. Recall that K is *real* if $s(K) = \infty$ and *nonreal* otherwise, and in the latter case $s(K)$ is a power of two (see [Lam 2005, Chapter XI, Section 2]).

The *Pythagoras number of* K is defined as

$$p(K) = \inf\left\{n \in \mathbb{N} \mid D_K(n) = \sum K^2\right\} \in \mathbb{N} \cup \{\infty\}.$$

We further define

$$p'(K) = \begin{cases} p(K) & \text{if } K \text{ is real,} \\ s(K)+1 & \text{if } K \text{ is nonreal.} \end{cases}$$

This field invariant has no independent interest, but it allows us to avoid case distinctions in statements about valuations and Pythagoras numbers, by formulating them for $p'(K)$ rather than for $p(K)$. As for nonreal field K we always have $s(K) \le p(K) \le s(K)+1 = p'(K)$. Hence, $p'(K)$ is always equal to $p(K)$ or to $p(K)+1$.

We now consider valuations in the context of sums of squares. We say that a valuation v on K is *real* or *nonreal*, respectively, if the residue field κ_v has the corresponding property.

Lemma 4.1. *Let v be a valuation on K and $n \in \mathbb{N}$. Then $s(\kappa_v) \ge n$ if and only if $v(a_1^2 + \cdots + a_n^2) = 2\min\{v(a_1), \ldots, v(a_n)\}$ holds for all $a_1, \ldots, a_n \in K$.*

Proof. Both conditions are easily seen to be equivalent to having that any sum of n squares of elements in $\mathcal{O}_v^\times$ lies in $\mathcal{O}_v^\times$. □

Let $\Omega(K)$ denote the set of nondyadic $\mathbb{Z}$-valuations on K.

Proposition 4.2. *Let $v \in \Omega(K)$. If v is real, then $v\left(\sum K^2\right) = 2\mathbb{Z}$. If v is nonreal, then for $s = s(\kappa_v)$ we have $v(D_K(s)) = 2\mathbb{Z}$ and $v(D_K(s+1)) = v\left(\sum K^2\right) = \mathbb{Z}$.*

Proof. If v is real, then it follows from Lemma 4.1 that $v\big(\sum K^2\big) = 2\mathbb{Z}$. Assume now that v is nonreal and let $s = s(\kappa_v)$. Then it follows from Lemma 4.1 that $v(D_K(s)) = 2\mathbb{Z}$ and that there exist $x_0, \dots, x_s \in K$ such that $v(x_0^2 + \cdots + x_s^2) \neq 2\min\{v(x_0), \dots, v(x_s)\}$. Dividing by one of the elements $x_0, \dots, x_s$ with minimal value, we can assume that $\min\{v(x_0), \dots, v(x_s)\} = 0$. Hence $v(x_0^2 + \cdots + x_s^2) \geq 1$. If $v(x_0^2 + \cdots + x_s^2) > 1$, then we choose $t \in K$ with $v(t) = 1$, and we conclude that $v((x_0 + t)^2 + x_1^2 \cdots + x_s^2) = 1$, as v is nondyadic. We may therefore assume that $v(x_0^2 + \cdots + x_s^2) = 1$. Since $K^{\times 2} \cdot D_K(s+1) = D_K(s+1)$, we conclude that $v(D_K(s+1)) = \mathbb{Z}$, and thus in particular that $v\big(\sum K^2\big) = \mathbb{Z}$. □

Proposition 4.3. *Let $v \in \Omega(K)$. Then $p'(K) \geq p(K) \geq p'(\kappa_v)$. Moreover, if v is henselian, then $p'(K) = p(K) = p'(\kappa_v)$.*

Proof. Note that $p(K) \geq p(\kappa_v)$. If v is real, then κ_v and K are real, and we obtain that $p'(K) = p(K) \geq p(\kappa_v) = p'(\kappa_v)$. If v is nonreal, then for $s = s(\kappa_v)$ we conclude that $D_K(s) \subsetneq D_K(s+1)$ by Proposition 4.2, and therefore $p'(K) \geq p(K) \geq s+1 = p'(\kappa_v)$.

Assume finally that v is henselian. Then $s(K) = s(\kappa_v)$, and further $p(K) = p(\kappa_v)$ in case v is real. This yields that $p'(K) = p'(\kappa_v)$. □

Recall that the completion of K with respect to a valuation v is denoted by K^v.

Corollary 4.4. *For $v \in \Omega(K)$ we have $p(K) \geq p(K^v) = p'(\kappa_v)$.*

Proof. Since v extends to a $\mathbb{Z}$-valuation on K^v with the same residue field κ_v, we obtain using both statements in Proposition 4.3 that $p(K) \geq p'(\kappa_v) = p(K^v)$. □

Corollary 4.5. *We have $p'(K(t)) = p(K(t)) \geq p'(K(\!(t)\!)) = p(K(\!(t)\!)) = p'(K)$.*

Proof. We have $p(K(t)) \geq p(K(\!(t)\!))$ by Corollary 4.4 and $p'(K(\!(t)\!)) = p(K(\!(t)\!)) = p'(K)$ by Proposition 4.3. If K is real, then $K(t)$ is real, thus $p'(K(t)) = p(K(t))$ by the definition. If K is nonreal, then $p(K(t)) = s(K) + 1 = s(K(t)) + 1 = p'(K(t))$. □

Corollary 4.6. *Let F/K be an algebrorational function field. Then $p'(F) = p(F)$.*

Proof. Replacing K by its relative algebraic closure in F, we have $F = K(t)$ for some $t \in F$ transcendental over K. We conclude using Corollary 4.5. □

This does not generalize to arbitrary algebraic function fields:

Example 4.7. Consider the function field F of the curve $Y^2 = -(X^2+1)(X^2+1+t)$ over $\mathbb{R}(\!(t)\!)$. By [Becher and Van Geel 2009, Example 5.13] we have $p(F) = s(F) = 2$, and therefore $p'(F) = 3 > p(F)$. In particular $-1 \notin F^{\times 2}$ whereas -1 is a square in F^v for any $v \in \Omega(F)$ by Corollary 4.4.

We apply Proposition 4.3 to give a short argument for a well-known fact:

Corollary 4.8. *Assume that K is a finitely generated nonalgebraic extension of a subfield. Then $p(K) \geq 2$.*

Proof. It follows from the hypotheses that there exists $v \in \Omega(K)$ such that κ_v is nonreal. From Proposition 4.3 we obtain that $p(K) \geq p'(\kappa_v) = s(\kappa_v) + 1 \geq 2$. □

Remark 4.9. If $K = k(t)$ for a subfield k and $t \in K$ transcendental over k, then $1 + t^2 \notin K^{\times 2}$ and thus $p(K) \geq 2$. An alternative proof of Corollary 4.8 is therefore obtained by reduction to the case of a rational function field via the Diller–Dress Theorem [Lam 2005, Chapter VIII, Theorem 5.7], which says that if $p(K) \geq 2$ then $p(L) \geq 2$ for every finite field extension L/K.

For $S \subseteq \Omega(K)$ we define a homomorphism

$$\Phi_S : K^\times \to \mathbb{Z}^S, \quad x \mapsto (v(x))_{v \in S}.$$

If $S \subseteq \Omega(K)$ is a finite subset, then it follows from the Approximation Theorem (see [Engler and Prestel 2005, Theorem 2.4.1] or [Liu 2002, Chapter 9, Lemma 1.9]) that Φ_S is surjective.

The following statement extends Proposition 4.2 from a single $\mathbb{Z}$-valuation to finitely many $\mathbb{Z}$-valuations on K.

Proposition 4.10. *Let S be a finite subset of $\Omega(K)$ and $n \in \mathbb{N}$. Then*

$$\Phi_S(D_K(n)) = \{(e_v)_{v \in S} \in \mathbb{Z}^S \mid e_v \in 2\mathbb{Z} \textit{ for } v \in S \textit{ with } s(\kappa_v) \geq n\}.$$

Proof. For $v \in \Omega(K)$ with $s(\kappa_v) \geq n$ we have $v(D_K(n)) \subseteq 2\mathbb{Z}$ by Lemma 4.1. This shows that

$$\Phi_S(D_K(n)) \subseteq \{(e_v)_{v \in S} \in \mathbb{Z}^S \mid e_v \in 2\mathbb{Z} \text{ for } v \in S \text{ with } s(\kappa_v) \geq n\}.$$

It remains to show the other inclusion. Consider a tuple $(e_v)_{v \in S} \in \mathbb{Z}^S$ such that $e_v \in 2\mathbb{Z}$ for all $v \in S$ with $s(\kappa_v) \geq n$. The aim is to find an element $x \in D_K(n)$ with $\Phi_S(x) = (e_v)_{v \in S}$. We explain how to obtain such an element, using the Approximation Theorem (see above) several times.

For $v \in S$ with $e_v \notin 2\mathbb{Z}$, as $s(\kappa_v) < n$ we may choose $x_{v,2}, \ldots, x_{v,n} \in \mathbb{O}_v$ such that $v(1 + x_{v,2}^2 + \cdots + x_{v,n}^2) > 0$. For $v \in S$ with $e_v \in 2\mathbb{Z}$ we set $x_{v,2} = \cdots = x_{v,n} = 0$. For $i = 2, \ldots, n$ we choose $x_i \in K^\times$ such that $v(x_i - x_{v,i}) > 0$ for all $v \in S$. We set $y = x_2^2 + \cdots + x_n^2$. For $v \in S$ we have $v(1 + y) = 0$ if $e_v \in 2\mathbb{Z}$ and $v(1 + y) > 0$ otherwise. We choose $t \in K^\times$ such that, for all $v \in S$, we have $v(t) = 1$ if $v(1 + y) > 1$, and $v(t) > 1$ otherwise. Note that $(1 + t)^2 + y \in D_K(n)$. For any $v \in S$ the value $v((1 + t)^2 + y)$ is either 0 or 1 and such that $v((1 + t)^2 + y) \equiv e_v \bmod 2\mathbb{Z}$. Now choose $z \in K^\times$ such that $2v(z) = e_v - v((1 + t)^2 + y)$ for all $v \in S$ and set $x = z^2((1 + t)^2 + y)$. Then $x \in D_K(n)$ and $\Phi_S(x) = (v(x))_{v \in S} = (e_v)_{v \in S}$. □

Corollary 4.11. *Let $n \in \mathbb{N}$ and S a finite subset of $\Omega(K)$ such that $s(\kappa_v) = 2^n$ for all $v \in S$. Then Φ_S induces a surjective homomorphism*

$$D_K(2^{n+1})/D_K(2^n) \to (\mathbb{Z}/2\mathbb{Z})^S.$$

In particular, $|D_K(2^{n+1})/D_K(2^n)| \geq 2^{|S|}$.

Proof. By the hypotheses on S and by Proposition 4.10, we have $\Phi_S(D_K(2^{n+1})) = \mathbb{Z}^S$ and $\Phi_S(D_K(2^n)) = (2\mathbb{Z})^S$. From this the statement follows. □

Theorem 4.12. *Let K be a real field. For $n \in \mathbb{N}$ the following are equivalent*:

(i) $p(K(X)) \leq 2^n$.

(ii) $p(L) < 2^n$ *for all finite real extensions L/K.*

(iii) $s(L) \leq 2^{n-1}$ *for all finite nonreal extensions L/K.*

(iv) $p'(L) < 2^n$ *for all finite extensions L/K with $-1 \notin L^{\times 2}$.*

Proof. See [Becher and Van Geel 2009, Theorem 3.3] for the equivalence of (i)–(iii); the equivalence of these conditions with (iv) is obvious. □

Corollary 4.13. *Let $n \in \mathbb{N}$ be such that $p(K(X)) \leq 2^n$. Then $p(L(X)) \leq 2^n$ for any finite field extension L/K.*

Proof. If K is nonreal, then $p(L(X)) = s(L)+1 \leq s(K)+1 = p(K(X)) \leq 2^n$. If K is real and L is nonreal, then $s(L) \leq 2^{n-1}$ by Theorem 4.12 and thus $p(L(X)) \leq 2^n$. If L is real, then since any finite real extension of L is a finite real extension of K, the equivalence of (i) and (ii) in Theorem 4.12 allows us to conclude that $p(L(X)) \leq 2^n$. □

Theorem 4.14. *Let K be endowed with a $\mathbb{Z}$-valuation with residue field k. Then $p(K(X)) \geq p(k(X))$. Moreover*, *if the valuation is henselian and $n \in \mathbb{N}$ is such that $p(k(X)) \leq 2^n$*, *then $p(K(X)) \leq 2^n$.*

Proof. Using Proposition 3.3 the given $\mathbb{Z}$-valuation on K extends to a $\mathbb{Z}$-valuation on $K(X)$ with residue field $k(X)$. Hence, $p(K(X)) \geq p'(k(X)) = p(k(X))$ by Proposition 4.3 and Corollary 4.5.

Assume now that the $\mathbb{Z}$-valuation on K is henselian. If K is nonreal, then $p(K(X)) = s(K) + 1 = s(k) + 1 = p(k(X))$. Assume that K is real. Then k and $k(X)$ are real. Let $n \in \mathbb{N}$ be such that $p(k(X)) \leq 2^n$. By Theorem 4.12, to prove that $p(K(X)) \leq 2^n$ it suffices to show that $p'(L) < 2^n$ for all finite extensions L/K with $-1 \notin L^{\times 2}$. Consider such an extension L/K. Then L is endowed with a henselian $\mathbb{Z}$-valuation whose residue field ℓ is a finite extension of k. Then $-1 \notin \ell^{\times 2}$ and thus $p'(L) = p'(\ell) < 2^n$ by Proposition 4.3 and Theorem 4.12. □

The last two statements motivate us to formulate the following two conjectures.

Conjecture 4.15. *For any finite field extension L/K, one has $p(L(X)) \leq p(K(X))$.*

Conjecture 4.16. *If K is complete with respect to a nondyadic $\mathbb{Z}$-valuation with residue field k, then $p(K(X)) = p(k(X))$.*

Note that both conjectures hold trivially if K is a nonreal field. In the case where K is real, Conjecture 4.16 was raised originally by C. Scheiderer [2001, Remark 5.18.2] as a question. We shall prove in Corollary 6.9 that the two conjectures are equivalent.

5. The u-invariant for algebraic function fields

We refer to [Lam 2005] for basic facts and terminology from the theory of quadratic forms over fields of characteristic different from two. The u-invariant of K was defined in [Elman and Lam 1973] as

$$u(K) = \sup\{\dim\varphi \mid \varphi \text{ anisotropic torsion form over } K\} \in \mathbb{N}\cup\{\infty\},$$

where a *torsion form* is a regular quadratic form that corresponds to a torsion element in the Witt ring.

Proposition 5.1. *Let $v \in \Omega(K)$. Let ψ be a torsion form over κ_v. There exist $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{O}_v^\times$, and $t \in K^\times$ with $v(t) = 1$ such that $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_n\rangle$ is a torsion form over K and such that ψ is Witt equivalent to $\langle \bar{a}_1, \dots, \bar{a}_n\rangle$.*

Proof. Assume first that v is nonreal. Then by Proposition 4.2 there exists $t \in \sum K^2$ with $v(t) = 1$. For $n = \dim\psi$ and $a_1, \dots, a_n \in \mathbb{O}_v^\times$ such that ψ is isometric to $\langle \bar{a}_1, \dots, \bar{a}_n\rangle$, we obtain that $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_n\rangle$ is a torsion form over K.

Assume now that v is real. Then ψ is Witt equivalent to a sum of binary torsion forms over κ_v (see [Pfister 1966, Satz 22]). Every binary torsion form over κ_v is of the shape $\langle \bar{a}_1, \bar{a}_2\rangle$ with $a_1, a_2 \in \mathbb{O}_v^\times$ such that $-a_1a_2 \in \sum K^2$. Hence, there exist $r \in \mathbb{N}$ and $a_1, \dots, a_{2r} \in \mathbb{O}_v^\times$ such that ψ is Witt equivalent to $\langle \bar{a}_1, \dots, \bar{a}_{2r}\rangle$ and $-a_{2i-1}a_{2i} \in \sum K^2$ for $i = 1, \dots, r$. Then $\langle a_1, \dots, a_{2r}\rangle$ is torsion form over K. We choose any $t \in K^\times$ with $v(t) = 1$. Then also $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_{2r}\rangle$ is a torsion form over K. □

The following statement was independently obtained in [Scheiderer 2009, Proposition 5] using different arguments, based on the theory of spaces of orderings.

Proposition 5.2. *For $v \in \Omega(K)$ we have $u(K) \geq u(K^v) = 2u(\kappa_v)$.*

Proof. Let $v \in \Omega(K)$. Let ψ be an anisotropic torsion form over κ_v. Using Proposition 5.1 we choose $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{O}_v^\times$, and $t \in K^\times$ with $v(t) = 1$ such that ψ is Witt equivalent to $\langle \bar{a}_1, \dots, \bar{a}_n\rangle$ and such that $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_n\rangle$ is a torsion form over K. Let φ denote its anisotropic part. Then φ is a torsion form and isometric to $\langle b_1, \dots, b_s\rangle \perp -t\langle c_1, \dots, c_r\rangle$ for certain $r, s \in \mathbb{N}$ and $c_1, \dots, c_r, b_1, \dots, b_s \in \mathbb{O}_v^\times$. Applying residue homomorphisms (see [Lam 2005, Chapter VI, §1]), it follows that the forms $\langle \bar{b}_1, \dots, \bar{b}_s\rangle$ and $\langle \bar{c}_1, \dots, \bar{c}_r\rangle$ over κ_v are

Witt equivalent to ψ. As ψ is anisotropic we conclude that $\dim\varphi = r+s \geq 2\dim\psi$. This shows that $u(K) \geq 2u(\kappa_v)$. Using Springer's Theorem for complete discretely valued fields (see [Lam 2005, Chapter VI, §1]), we further obtain that $u(K^v) = 2u(\kappa_v)$. □

A generalization of Proposition 5.2 for arbitrary nondyadic valuations is given in [Becher and Leep 2013, Theorem 5.2].

Corollary 5.3. *Let k be the residue field of a nondyadic $\mathbb{Z}$-valuation on K. For every algebraic function field F/K there exists an algebraic function field E/k such that $u(F) \geq 2u(E)$.*

Proof. Let T denote the discrete valuation ring with field of fractions K and residue field k. Let F/K be an algebraic function field. Choose $x \in F$ transcendental over K. Consider the valuation ring T' in $K(x)$ described in Proposition 3.3. Note that T' is a discrete valuation ring. Since $F/K(x)$ is a finite extension, there exists a $\mathbb{Z}$-valuation v on F dominating T'. The residue field E of v is a finite extension of $k(\bar{x})$, hence an algebraic function field over k. By Proposition 5.2 we obtain that $u(F) \geq 2u(E)$. □

We define

$$\hat{u}(K) = \tfrac{1}{2}\sup\{u(F) \mid F/K \text{ algebraic function field}\}.$$

For nonreal fields $\hat{u}$ coincides with the *strong u-invariant* defined in [Harbater et al. 2009, Definition 1.2], by the following result.

Corollary 5.4. *For any algebraic extension L/K we have*

$$u(L) \leq \tfrac{1}{2}u(K(X)) \leq \hat{u}(K).$$

Proof. If L is a field of odd characteristic p, then the Frobenius homomorphism given by $x \longmapsto x^p$ shows that any quadratic form over L is obtained by scalar extension from a quadratic form defined over L^p. Therefore every torsion form defined over an algebraic extension of K comes from a torsion form defined over a finite separable extension of K. Since any finite separable extension of K is the residue field of a $\mathbb{Z}$-valuation v on $K(X)$, the first inequality now follows from Proposition 5.2. The second inequality is obvious. □

6. Function fields over complete discretely valued fields

In this section we assume that K is the field of fractions of a complete discrete valuation ring T with residue field k of characteristic different from 2. We want to apply the following reformulation of the local-global principle in [Colliot-Thélène et al. 2012, Theorem 3.1] to the study of the u-invariant and the Pythagoras number of algebraic function fields over K.

Theorem 6.1 (Colliot-Thélène, Parimala, Suresh). *Let F be an algebraic function field over K. A regular quadratic form over F of dimension at least 3 is isotropic if and only if it is isotropic over F^v for every $v \in \Omega(F)$.*

Proof. This slightly more general version of the result cited follows from [Harbater et al. 2013, Proposition 9.10]. □

We will apply Theorem 6.1 to obtain upper bounds for the two mentioned field invariants. We have to distinguish two types of $\mathbb{Z}$-valuations on algebraic function fields over K.

Proposition 6.2. *Let F/K be an algebraic function field and $v \in \Omega(F)$. Then either v is trivial on K or it dominates T.*

Proof. This follows from Proposition 2.2. □

The lower bounds that we will obtain are based on more elementary arguments:

Lemma 6.3. *Let F/K be an algebraic function field and v a $\mathbb{Z}$-valuation on F that is trivial on K. Then $p(F^v) = p'(\kappa_v) \le p(k(X))$ and $u(F^v) = 2u(\kappa_v) \le u(k(X))$.*

Proof. By Corollary 4.4 and Proposition 5.2 we have $p(F^v) = p'(\kappa_v)$ and $u(F^v) = 2u(\kappa_v)$. As κ_v is a finite extension of K and T is a complete discrete valuation ring of K, there is a unique $\mathbb{Z}$-valuation w on κ_v with $\mathbb{O}_w \cap K = T$. Then κ_w is a finite extension of k, and κ_v is complete with respect to w, in particular henselian. By Corollary 5.4 and Proposition 4.3 we obtain that $p'(\kappa_v) = p'(\kappa_w)$ and $u(\kappa_v) = 2u(\kappa_w)$. We choose $\alpha \in \kappa_w$ such that $\kappa_w/k(\alpha)$ is purely inseparable. Since k is of characteristic different from 2, it follows that every element of κ_w is a product of a square and an element from $k(\alpha)$. This yields that $p'(\kappa_w) \le p'(k(\alpha))$ and $u(\kappa_w) \le u(k(\alpha))$. Since $k(\alpha)$ is the residue field of a $\mathbb{Z}$-valuation on $k(X)$, we obtain from Proposition 4.3 and Corollary 5.4 that $p'(k(\alpha)) \le p(k(X))$ and $2u(k(\alpha)) \le u(k(X))$. □

We can now extend Theorem 4.10 of [Harbater et al. 2009] to the current setting, thus covering real function fields. C. Scheiderer [2009, Theorem 3] independently gave a more geometric proof.

Theorem 6.4. *We have $\hat{u}(K) = 2\hat{u}(k)$.*

Proof. For any algebraic function field E/k, by Proposition 3.4 there exists an algebraic function field F/K and a $\mathbb{Z}$-valuation on F with residue field E, and using Proposition 5.2 we obtain that $u(E) \le \frac{1}{2}u(F) \le \hat{u}(K)$. This yields that $2\hat{u}(k) \le \hat{u}(K)$.

To prove the converse inequality, we need to show for an arbitrary algebraic function field F/K that $u(F) \le 4\hat{u}(k)$ holds. Fix F/K. By Theorem 6.1, any anisotropic form over F remains anisotropic over F^v for some $v \in \Omega(F)$. It thus suffices to show that $u(F^v) \le 4\hat{u}(k)$ for every $v \in \Omega(F)$. Fix $v \in \Omega(F)$. As

$u(F^v) = 2u(\kappa_v)$, it suffices to show that $u(\kappa_v) \leq 2\hat{u}(k)$. If v is trivial on K, we obtain by Lemma 6.3 that $2u(\kappa_v) \leq u(k(X)) \leq 2\hat{u}(k)$. Assume that v is nontrivial on K. Then $\mathcal{O}_v \cap K = T$ by Proposition 6.2. If κ_v/k is an algebraic function field then $u(\kappa_v) \leq 2\hat{u}(k)$ by the definition of $\hat{u}(k)$. Otherwise κ_v/k is an algebraic extension and then $u(\kappa_v) \leq \hat{u}(k)$ by Corollary 5.4. □

Corollary 6.5. *Let $m \in \mathbb{N}$. If $u(E) = m$ for every algebraic function field E/k, then $u(F) = 2m$ for every algebraic function field F/K.*

Proof. Let F/K be an algebraic function field over K. Using Theorem 6.4 we obtain that $u(F) \leq 2\hat{u}(K) = 4\hat{u}(k)$. By Corollary 5.3 there exists an algebraic function field E/k with $u(F) \geq 2u(E)$. If we assume that $u(E) = m$ holds for every algebraic function field E/k, we obtain that $2\hat{u}(k) = m$ and conclude that $u(F) = 2m$. □

Theorem 6.6. *We have that*

$$u(K(X)) = 2 \cdot \sup \{u(\ell(X)) \mid \ell/k \textit{ finite separable field extension}\}.$$

Proof. Let $F = K(X)$. As $u(F) \geq 2$, it follows from Theorem 6.1 that

$$u(F) \leq \sup \{u(F^v) \mid v \in \Omega(F)\}.$$

Consider $v \in \Omega(F)$. If v is trivial on K then $u(F^v) \leq 2u(k(X))$ by Lemma 6.3. If v is nontrivial on K, then by Proposition 2.2 and Theorem 3.2 κ_v/k is either an algebraic extension or algebrorational. In any case we obtain that $u(\kappa_v) \leq u(\ell(X))$ and thus $u(F^v) = 2u(\kappa_v) \leq 2u(\ell(X))$ for a finite extension ℓ/k. Let ℓ'/k be the separable subextension of ℓ/k such that ℓ/ℓ' is purely inseparable. Then $\ell(X)/\ell'(X)$ is purely inseparable and of odd degree, so every element of $\ell(X)$ is a product of a square in $\ell(X)$ with an element of $\ell'(X)$, whereby $u(\ell(X)) \leq u(\ell'(X))$. This together shows that

$$u(F) \leq 2 \cdot \sup \{u(\ell(X)) \mid \ell/k \text{ finite separable field extension}\}.$$

On the other hand, given a finite separable field extension ℓ/k, it follows from Theorem 3.5 that there exists a $\mathbb{Z}$-valuation on F with residue field $\ell(X)$, which by Proposition 5.2 implies that $u(F) \geq 2u(\ell(X))$. This shows the claimed equality. □

We turn to the study of sums of squares and the Pythagoras number.

Theorem 6.7. *Let F/K be an algebraic function field. For any $m \geq 2$ we have that $D_F(m) = F^\times \cap (\bigcap_{v \in \Omega(F)} D_{F^v}(m))$. Moreover, $p(F) = \sup\{p'(\kappa_v) \mid v \in \Omega(F)\}$.*

Proof. Applying Theorem 6.1 to the quadratic forms $m \times \langle 1 \rangle \perp \langle -a \rangle$ for $a \in F^\times$ shows for any $m \geq 2$ the claimed equality of sets. Note that $\Omega(F)$ contains a nonreal valuation v, and we have that $p(F^v) = s(\kappa_v) + 1 \geq 2$. Since $p(F) \geq 2$ by Corollary 4.8, we obtain that

$$\begin{aligned} p(F) &= \inf\{m \geq 2 \mid D_F(m) = D_F(m+1)\} \\ &\leq \inf\{m \geq 2 \mid D_{F^v}(m) = D_{F^v}(m+1) \text{ for all } v \in \Omega(F)\} \\ &= \sup\{p(F^v) \mid v \in \Omega(F)\}. \end{aligned}$$

Moreover, by Proposition 4.3 we have $p(F^v) = p'(\kappa_v)$ for every $v \in \Omega(F)$. □

Theorem 6.8. *Let F/K be an algebraic function field. There exists an algebraic function field E/k such that $p'(E) \geq p'(F)$. Moreover, if F/K is algebrorational, then one may choose E/k to be algebrorational.*

Proof. If $p(F) = \infty$, then as F is a finite extension of a rational function field, we conclude with [Pfister 1995, Chapter 7, Proposition 1.13] that $p(K(X)) = \infty$ and then with Theorem 4.14 we obtain that $p(k(X)) = \infty$, so that for $E = k(X)$ we have $p'(E) = \infty = p'(F)$.

We now suppose that $p(F) < \infty$. By Theorem 6.7 there exists $v \in \Omega(F)$ such that $p(F) = p'(\kappa_v)$.

Assume first that $p'(F) \neq p(F)$. Then F is nonreal with $p(F) = s(F)$, and by Corollary 4.6 F is not algebrorational. It follows that κ_v is nonreal with $s(\kappa_v) = p'(\kappa_v) - 1 = p(F) - 1 = s(F) - 1$, and as $s(\kappa_v)$ and $s(F)$ are both powers of two, we conclude that $s(F) = 2$. Then $s(k) \geq 2$ and for $E = k(X)(\sqrt{-(1+X^2)})$ we have that $s(E) = 2$ and thus $p'(E) = 3 = p'(F)$.

Suppose now that $p'(F) = p(F) = p'(\kappa_v)$. If $v|_K$ is trivial, then we have $p(k(X)) \geq p'(\kappa_v) = p(F)$ by Lemma 6.3 and further $s(k(X)) = s(k) = s(K) \geq s(F)$, so we may choose $E = k(X)$ to have $p'(E) \geq p'(F)$. Suppose that $v|_K$ is nontrivial. By Proposition 6.2 then v dominates T, and the residue extension κ_v/k is either algebraic or it is an algebraic function field. If κ_v/k is an algebraic function field, we may choose $E = \kappa_v$ and have that $p'(E) \geq p'(F)$. Moreover, by Theorem 3.2, if F/K is algebrorational, then so is E/k. If κ_v/k is algebraic, then as $p'(\kappa_v) = p'(F) < \infty$ there exists a finite extension ℓ/k contained in κ_v/k with $p'(\ell) \geq p'(\kappa_v)$, and we may thus choose $E = \ell(X)$ to have $p'(E) \geq p'(\ell) \geq p'(\kappa_v) = p'(F)$. □

Corollary 6.9. *We have $p(K(X)) = \sup\{p(\ell(X)) \mid \ell/k \text{ finite field extension}\}$.*

Proof. The statement is trivial if k is nonreal. Assume that k is real. Given an arbitrary finite extension ℓ/k, by Theorem 3.5 there is a $\mathbb{Z}$-valuation on $K(X)$ with residue field $\ell(X)$, whereby Proposition 4.3 yields that $p'(\ell(X)) \leq p'(K(X))$. On the other hand, by Theorem 6.8, there exists a finite extension ℓ/k with $p'(K(X)) \leq p'(\ell(X))$. Since $p'(K(X)) = p(K(X))$ and $p'(k(X)) = p(k(X))$ the statement follows. □

Note that Corollary 6.9 shows the equivalence of Conjectures 4.15 and 4.16.

Theorem 6.10. *Let $n \in \mathbb{N}$. Assume that $p(k(X)) \le 2^n$ and that $\sum E^2/D_E(2^n)$ is finite for every algebraic function field E/k. Then $p(K(X)) \le 2^n$ and $\sum F^2/D_F(2^n)$ is finite for every algebraic function field F/K.*

Proof. By Theorem 4.14 we have $p(K(X)) \le 2^n$. Consider an algebraic function field F/K. By Theorem 6.7 the natural homomorphism

$$\sum F^2/D_F(2^n) \longrightarrow \prod_{v\in\Omega(F)} \sum (F^v)^2/D_{F^v}(2^n)$$

is injective. To prove that $\sum F^2/D_F(2^n)$ is finite, it thus suffices to show that the set

$$S = \{v \in \Omega(F) \mid p(F^v) > 2^n\}$$

is finite and that $\sum (F^v)^2/D_{F^v}(2^n)$ is finite for each $v \in S$. Consider $v \in \Omega(F)$. If v is trivial on K, then $p(F^v) \le p(k(X)) \le 2^n$ by Lemma 6.3. Otherwise $\mathcal{O}_v \cap K = T$ by Proposition 6.2 and κ_v is an extension of k. If the extension κ_v/k is algebraic, then $p(F^v) = p'(\kappa_v) \le p(k(X)) \le 2^n$. If κ_v/k is an algebraic function field which is algebrorational, then using Corollary 6.9 we obtain that $p(F^v) = p'(\kappa_v) \le p(K(X)) \le 2^n$. This proves that, for any $v \in S$, we have $\mathcal{O}_v \cap K = T$ and κ_v/k is an algebraic function field that is not algebrorational. The finiteness of S thus follows from Corollary 3.9, and for any $v \in S$ we have $\left|\sum (F^v)^2/D_{F^v}(2^n)\right| \le 2 \cdot \left|\sum (\kappa_v)^2/D_{\kappa_v}(2^n)\right|$, which is finite by the hypothesis. □

Theorem 6.11. *Assume that $n \in \mathbb{N}$ is such that $p(E) \le 2^n$ for any algebraic function field E/k. Let F/K be an algebraic function field. Then $p(F) \le 2^n + 1$ and the set $S = \{v \in \Omega(F) \mid s(\kappa_v) = 2^n\}$ is finite with $\left|\sum F^2/D_F(2^n)\right| = 2^{|S|}$. Moreover, $\Phi_S : F^\times \to \mathbb{Z}^S$ induces an isomorphism $\sum F^2/D_F(2^n) \to (\mathbb{Z}/2\mathbb{Z})^S$.*

Proof. Consider $v \in \Omega(F)$. If $v|_K$ is trivial, then $p'(\kappa_v) \le p(k(X)) \le 2^n$ by Lemma 6.3 and the hypothesis. Suppose now that $v|_K$ is nontrivial. By Proposition 6.2 then $\mathcal{O}_v \cap K = T$ and the residue field extension κ_v/k is either algebraic or it is an algebraic function field. If κ_v/k is algebraic, then $p'(\kappa_v) \le 2^n$. Suppose that κ_v/k is an algebraic function field. Then $p(\kappa_v) \le 2^n$ by the hypothesis. Moreover, if κ_v/k is algebrorational, then Corollary 4.6 yields that $p'(\kappa_v) = p(\kappa_v) \le 2^n$.

Hence, in any case we have that $p(\kappa_v) \le 2^n$, and thus $p(F^v) = p'(\kappa_v) \le 2^n + 1$ by Corollary 4.4. Furthermore, we conclude that $p(F^v) = 2^n + 1$ if and only if $v \in S$, and in this case the residue field extension κ_v/k is an algebraic function field but not algebrorational.

By Theorem 6.7 we conclude that $p(F) \le p'(F) \le 2^n + 1$ and furthermore

$$\sum F^2 = \Big(\bigcap_{v\in S} D_{F^v}(2^n{+}1)\Big) \cap \Big(\bigcap_{v\in S^c} D_{F^v}(2^n)\Big) \cap F^\times,$$

where $S^c = \Omega(F) \setminus S$. Moreover, using Corollary 3.9 we obtain that S is finite. By Corollary 4.11 then $\Phi_S : F^\times \to \mathbb{Z}^S$ induces a surjective homomorphism $\sum F^2/D_F(2^n) \to (\mathbb{Z}/2\mathbb{Z})^S$. It remains to show that this homomorphism is also injective. In view of Theorem 6.7 and the above equality for $\sum F^2$, it suffices to verify that $\Phi_S^{-1}((2\mathbb{Z})^S) \subseteq \bigcap_{v\in S} D_{F^v}(2^n)$. Consider $x \in \sum F^2$ and $v \in S$ with $v(x) \in 2\mathbb{Z}$. Then $x = t^2 y$ with $t \in F^\times$ and $y \in \mathcal{O}_v^\times \cap \left(\sum F^2\right)$, whereby $y + \mathfrak{m}_v \in \sum {\kappa_v}^2$. Since F^v is complete and $p(\kappa_v) \le 2^n$, it follows that $x = t^2 y \in D_{F^v}(2^n)$. This shows the claim. □

Recall that the field K is said to be *hereditarily quadratically closed* if $L^\times = L^{\times 2}$ for every finite field extension L/K. The following result applies in particular to the situation where R is a real closed field.

Theorem 6.12. *Let $n \in \mathbb{N}$ and $K = R(\!(t_1)\!) \ldots (\!(t_n)\!)$ for a field R such that $R(\sqrt{-1})$ is hereditarily quadratically closed. Let F/K be an algebraic function field. Then $u(F) = 2^{n+1}$, $2 \le p(F) \le 3$, and the group $\sum F^2/D_F(2)$ is finite.*

Proof. We prove this by induction on n. For $n = 0$ we obtain from [Elman and Wadsworth 1987, Theorem] that $u(F) = 2$, and we conclude by [Lam 2005, Chapter XI, Corollary 6.26] and Corollary 4.8 that $p(F) = 2$, whereby $\sum F^2 = D_F(2)$ and $2 \le p(F) \le p'(F) \le 3$. Assume that $n > 0$. Applying the induction hypothesis to all algebraic function fields over $k = R(\!(t_1)\!) \ldots (\!(t_{n-1})\!)$, we obtain by Corollary 6.5 that $u(F) = 2^{n+1}$, by Corollary 4.8 and Theorem 6.8 that $2 \le p(F) \le p'(F) \le 3$, and by Theorem 6.10 that $\sum F^2/D_F(2)$ is finite. □

For certain real function fields over $\mathbb{R}(\!(t)\!)$, it was asked in [Becher and Van Geel 2009, Question 5.15] whether their Pythagoras number is three or four. We can now answer this question:

Corollary 6.13. *Let $h \in \mathbb{R}[X]$ be a nonconstant square-free polynomial with no roots in $\mathbb{R}$. Let F be the function field of the curve $Y^2 = (tX - 1)h$ over $\mathbb{R}(\!(t)\!)$. Then $p(F) = 3$.*

Proof. We have $p(F) \ge 3$ by [Becher and Van Geel 2009, Theorem 5.3 and Corollary 4.2] and $p(F) \le 3$ by Theorem 6.12. □

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