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AN ALMOST-SCHUR TYPE LEMMA FOR SYMMETRIC (2,0) TENSORS AND APPLICATIONS

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In a previous paper, we generalized the almost-Schur lemma of De Lellis and Topping for closed manifolds with nonnegative Ricci curvature to any closed manifolds. In this paper, we generalize the above results to symmetric (2, 0)-tensors and give the applications for r-th mean curvatures of closed hypersurfaces in space forms and k scalar curvatures for closed locally conformally flat manifolds.

1. Introduction

Recall that an n-dimensional Riemannian manifold (M, g) is said to be Einstein if its traceless Ricci tensor $\mathring{\mathrm{Ric}} = \mathrm{Ric} - (R/n)g$ is identically zero. Here Ric and R denote Ricci curvature and scalar curvature respectively. Schur's lemma states that the scalar curvature of an Einstein manifold of dimension $n \geq 3$ must be constant. De Lellis and Topping [2012] discussed the quantitative version, or the stability of Schur's lemma for closed manifolds, and proved the following almost-Schur lemma, as they called it.

Theorem 1.1 [De Lellis and Topping 2012]. *If (M, g) is a closed Riemannian manifold of dimension n with nonnegative Ricci curvature $n \geq 3$,*

$$\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left| \mathrm{Ric} - \frac{R}{n} g \right|^2 \tag{1-1}$$

and, equivalently,

$$\int_M \left| \mathrm{Ric} - \frac{\bar{R}}{n} g \right|^2 \leq \frac{n^2}{(n-2)^2} \int_M \left| \mathrm{Ric} - \frac{R}{n} g \right|^2, \tag{1-2}$$

where $\bar{R} = (1/\operatorname{Vol} M) \int_M R\, dv$ is the average of R over M. Equality holds in (1-1) or (1-2) if and only if M is Einstein.

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B. Andrews also obtained the above inequalities in an unpublished paper under the assumption that the Ricci curvature is positive. De Lellis and Topping also proved their estimates are sharp. First, the constants are optimal in (1-1) and (1-2) [De Lellis and Topping 2012, Section 2]. Second, the curvature condition $\mathrm{Ric} \geq 0$ cannot simply be dropped (see the examples in the proof of Propositions 2.1 and 2.2 in their paper). Without the condition of nonnegativity of the Ricci curvature, the same type of inequalities as (1-1) and (1-2) cannot hold if the constants in these inequalities only depend on the lower bound of the Ricci curvature.

In the case of closed manifolds without the hypothesis of nonnegativity of Ricci curvature, we have:

Theorem 1.2 [Cheng 2013]. *If (M, g) is a closed Riemannian manifold of dimension $n \geq 3$, then*

$$\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2}\left(1 + \frac{nK}{\lambda_1}\right)\int_M \left|\mathrm{Ric} - \frac{R}{n}g\right|^2; \tag{1-3}$$

equivalently,

$$\int_M \left|\mathrm{Ric} - \frac{\bar{R}}{n}g\right|^2 \leq \frac{n^2}{(n-2)^2}\left[1 + \frac{4(n-1)K}{n\lambda_1}\right]\int_M \left|\mathrm{Ric} - \frac{R}{n}g\right|^2, \tag{1-4}$$

where λ_1 denotes the first nonzero eigenvalue of the Laplace operator on (M, g) and K is a nonnegative constant such that the Ricci curvature of (M, g) satisfies $\mathrm{Ric} \geq -(n-1)K$.

Equality holds in (1-3) or (1-4) if and only if M is an Einstein manifold.

Observe that Theorem 1.1 is a particular case of Theorem 1.2 ($K = 0$). After the work of De Lellis and Topping, in the case of dimension $n = 3, 4$, Y. Ge and G. Wang [2012; 2011] proved that Theorem 1.1 holds under the weaker condition of nonnegative scalar curvature. However, as pointed out in [De Lellis and Topping 2012], this is surely not possible for $n \geq 5$; this can be shown using constructions similar to the one in [De Lellis and Topping 2012, Section 3]. Also, Ge, Wang, and Xia [Ge et al. 2013] proved the case of equalities in (1-1) and (1-2) by a different method and generalized De Lellis and Topping's inequalities for k-Einstein tensors and Lovelock curvature.

On the other hand, there is a similar phenomenon in submanifold theory. In differential geometry, a classical theorem states that a closed totally umbilical surface in the Euclidean space $\mathbb{R}^3$ must be a round sphere $\mathbb{S}^2$ and thus its second fundamental form A is a constant multiple of its metric. This theorem is also true for hypersurfaces in $\mathbb{R}^{n+1}$. It is interesting to discuss the stability of this theorem. De Lellis and Müller [2005] obtained some L^2 inequalities for closed surfaces in $\mathbb{R}^3$ with universal constants. For convex hypersurfaces in $\mathbb{R}^{n+1}$ we have:

Theorem 1.3 [Perez 2011]. *Let Σ be a smooth, closed and connected hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, with induced Riemannian metric g and nonnegative Ricci curvature. Then*

$$\int_\Sigma \left| A - \frac{\overline{H}}{n} g \right|^2 \leq \frac{n}{n-1} \int_\Sigma \left| A - \frac{H}{n} g \right|^2 \tag{1-5}$$

and, equivalently,

$$\int_\Sigma (H - \overline{H})^2 \leq \frac{n}{n-1} \int_\Sigma \left| A - \frac{H}{n} g \right|^2, \tag{1-6}$$

where A and $H = \operatorname{tr} A$ denote the second fundamental form and the mean curvature of Σ, respectively, and $\overline{H} = (1/\operatorname{Vol}_n \Sigma) \int_\Sigma H$. In particular, the above estimate holds for smooth, closed hypersurfaces which are the boundary of a convex set in $\mathbb{R}^{n+1}$.

As pointed out in [De Lellis and Topping 2012], Perez's theorem holds even for closed hypersufaces with nonnegative Ricci curvature when the ambient space is Einstein. Indeed, a slight modification of the proof of Theorem 1.3 gives the following.

Theorem 1.4. *Inequalities (1-5) and (1-6) hold under the same assumptions as in Theorem 1.3 except that the ambient space $(N^{n+1}, \tilde{g})$, $n \geq 2$, is supposed to be an Einstein manifold.*

Regarding the conditions for equality in (1-5) and (1-6), we have:

Theorem 1.5 [Cheng and Zhou 2012]. *Under the assumptions of Theorem 1.3, equality holds in (1-5) or (1-6) if and only if Σ is a totally umbilical hypersurface, that is, Σ is a distance sphere S^n in $\mathbb{R}^{n+1}$.*

We also studied the general case for hypersurfaces without a convexity hypothesis (that is, $A \geq 0$, which is equivalent to $\operatorname{Ric} \geq 0$ when Σ is a closed hypersurface in $\mathbb{R}^{n+1}$). We mention the following result (more details in the reference given):

Theorem 1.6 [Cheng and Zhou 2012]. *Let $(N^{n+1}, \tilde{g})$ be an Einstein manifold, $n \geq 2$. Let Σ be a smooth, connected, oriented and closed hypersurface immersed in N with induced metric g. Then*

$$\int_\Sigma \left| A - \frac{\overline{H}}{n} g \right|^2 \leq \frac{n}{n-1} \left(1 + \frac{K}{\lambda_1} \right) \int_\Sigma \left| A - \frac{H}{n} g \right|^2 \tag{1-7}$$

and, equivalently,

$$\int_\Sigma (H - \overline{H})^2 \leq \frac{n}{n-1} \left(1 + \frac{nK}{\lambda_1} \right) \int_\Sigma \left| A - \frac{H}{n} g \right|^2, \tag{1-8}$$

where λ_1 *is the first nonzero eigenvalue of the Laplacian operator on* Σ, $K \geq 0$ *is a nonnegative constant so that the Ricci curvature of* Σ *satisfies* $\mathrm{Ric} \geq -K$.

When N^{n+1} *is the Euclidean space* $\mathbb{R}^{n+1}$, *the hyperbolic space* $\mathbb{H}^{n+1}(-1)$, *or the closed hemisphere* $\mathbb{S}^{n+1}_{+}(1)$, *equality holds in* (1-7) *and* (1-8) *if and only if* Σ *is a totally umbilical hypersurface, that is,* Σ *is a distance sphere* S^n *in* N^{n+1}.

From [De Lellis and Topping 2012; Ge and Wang 2012; 2011; Ge et al. 2013; Cheng 2013; Perez 2011; Cheng and Zhou 2012], we observe that the inequalities mentioned above may be generalized to symmetric $(2,0)$ tensor fields. Applying such unified inequalities for symmetric $(2,0)$ tensors, we may obtain inequalities besides those in the papers mentioned above. For this purpose, we prove the following.

Theorem 1.7. *Let* (M, g) *be a closed Riemannian manifold of dimension* $n \geq 2$. *Let* T *be a symmetric* $(2,0)$*-tensor field on* M. *If the divergence* $\operatorname{div} T$ *and the trace* $B = \operatorname{tr} T$ *satisfy* $\operatorname{div} T = c\nabla B$, *where* c *is a constant, then*

$$(1\text{-}9)\qquad (nc-1)^2 \int_M (B-\bar{B})^2 \leq n(n-1)\left(1+\frac{nK}{\lambda_1}\right)\int_M \left|T-\frac{B}{n}g\right|^2$$

and, equivalently,

$$(1\text{-}10)\qquad (nc-1)^2 \int_M \left|T-\frac{\bar{B}}{n}g\right|^2 \leq \left[(nc-1)^2+(n-1)\left(1+\frac{nK}{\lambda_1}\right)\right]\int_M \left|T-\frac{B}{n}g\right|^2,$$

where $\bar{B} = (1/\operatorname{Vol} M)\int_M B\,dv$ *denotes the average of* B *over* M *and* λ_1 *and the constant* $K \geq 0$ *are as in Theorem 1.2.*

Assume the Ricci curvature Ric *of* M *is positive. If* $c \neq 1/n$, *statements* (i), (ii) *and* (iii) *below are equivalent. If* $c = 1/n$, *then* (i) *and* (ii) *are equivalent.*

(i) *Equality holds in* (1-9) *and in* (1-10).

(ii) $T = (B/n)g$ *on* M.

(iii) $T = (\bar{B}/n)g$ *on* M.

Take $K = 0$ in Theorem 1.7. We obtain corresponding inequalities with universal constants.

Theorem 1.8. *Let* (M, g) *be a closed Riemannian manifold of dimension* $n \geq 2$ *with nonnegative Ricci curvature. With the same notation as in Theorem 1.7, we have*

$$(1\text{-}11)\qquad (nc-1)^2 \int_M (B-\bar{B})^2 \leq n(n-1)\int_M \left|T-\frac{B}{n}g\right|^2$$

and, *equivalently*,

$$(1\text{-}12)\qquad (nc-1)^2\int_M\left|T-\frac{\bar{B}}{n}g\right|^2\le[(nc-1)^2+1]\int_M\left|T-\frac{B}{n}g\right|^2.$$

Assume the Ricci curvature Ric *of M is positive. If $c\neq 1/n$, statements* (i), (ii) *and* (iii) *below are equivalent. If $c=1/n$, then* (i) *and* (ii) *are equivalent.*

(i) *Equality holds in* (1-11) *and* (1-12).

(ii) $T=(B/n)g$ *on* M.

(iii) $T=(\bar{B}/n)g$ *on* M.

It is a known fact that, for (M^n, g), $n\ge 2$, if $T=(B/n)g$ and $\operatorname{div}T=c\nabla B$ with constant $c\neq 1/n$, then B is constant on M and thus T is a constant multiple of its metric g (see Proposition 2.1). Theorems 1.7 and 1.8 discuss the stability and rigidity of this fact for closed manifolds. Especially, take $T=\mathrm{Ric}$, A, etc. in Theorems 1.7 and 1.8. We obtain the corresponding inequalities mentioned before 1.7. In this paper, we obtain two other applications as follows.

First we deal with r-th mean curvatures of closed hypersurfaces in space forms. Assume (Σ, g) is a connected oriented closed hypersurface immersed in a space form with induced metric g. Associated with the second fundamental form A of Σ, we have r-th mean curvatures H_r of Σ and the Newton transformations P_r, $0\le r\le n$, (see their definition and related notation in Section 4). Since Reilly [1973] introduced them, there has been much work in studying high-order r-mean curvatures (see, for instance, [Rosenberg 1993; Barbosa and Colares 1997; Cheng and Rosenberg 2005; Alías et al. 2006]). It can be verified that if the Newton transformations P_r satisfy $P_r=(\operatorname{tr}P_r/n)g$ on Σ, Σ has constant r-th mean curvature and thus P_r is a constant multiple of its metric g (see Proposition 2.1 and Section 4). In this paper, we discuss the stability of this fact.

In addition, although it is true that a closed immersed totally umbilical hypersurface Σ (that is, Σ satisfies $P_1=(\operatorname{tr}P_1/n)g$) in $\mathbb{R}^{n+1}$ must be a round sphere $\mathbb{S}^n$, it is unknown, to the best of our knowledge, if it is true that a closed immersed hypersurface Σ satisfying $P_r=(\operatorname{tr}P_r/n)g$ in $\mathbb{R}^{n+1}$ must be a round sphere $\mathbb{S}^n$ for $r\ge 2$. When Σ is embedded, Ros [1988; 1987] showed that the round spheres are the only closed embedded hypersurfaces with constant r-th mean curvature in $\mathbb{R}^{n+1}$, $2\le r\le n$ (recall that the Alexandrov theorem says [Aleksandrov 1958] that the round spheres are the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature). Hence the round spheres are the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with $P_r=(\operatorname{tr}P_r/n)g$, $2\le r\le n$.

In Section 4, we prove the following.

Theorem 1.9. *Let $(N_a^{n+1}, \tilde{g})$ be a space form with constant sectional curvature a, $n \geq 2$. Assume that Σ is a smooth connected oriented closed hypersurface immersed in N with induced metric g. Then, for $2 \leq r \leq n$,*

$$(n-r)^2 \int_\Sigma (s_r - \bar{s}_r)^2 \leq n(n-1)\left(1 + \frac{nK}{\lambda_1}\right) \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2 \tag{1-13}$$

and, equivalently,

$$\int_\Sigma \left| P_r - \frac{(n-r)\bar{s}_r}{n} g \right|^2 \leq n\left[1 + \frac{(n-1)K}{\lambda_1}\right] \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2, \tag{1-14}$$

where $s_r = \operatorname{tr} P_r = \binom{n}{r} H_r$, $\bar{s}_r = (1/\operatorname{Vol}\Sigma)\int_\Sigma s_r \, dv$, and λ_1 and the constant $K \geq 0$ are as in Theorem 1.6. Moreover:

(1) *If the Ricci curvature* Ric *of Σ is positive, these three statements are equivalent:*
  (i) *Equality holds in (1-13) and (1-14).*
  (ii) *$P_r = ((n-r)s_r/n)g$ holds on Σ.*
  (iii) *$P_r = ((n-r)\bar{s}_r/n)g$ holds on Σ.*

(2) *If Σ is embedded in the Euclidean space $\mathbb{R}^{n+1}$ and the Ricci curvature* Ric *of Σ is positive, equality holds in (1-13) and (1-14) if and only if Σ is a round sphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+1}$.*

Taking $K = 0$ in Theorem 1.9, we obtain the following inequalities.

Theorem 1.10. *Besides the same assumptions as in Theorem 1.9, assume that Σ has nonnegative Ricci curvature. Then, for $2 \leq r \leq n$,*

$$(n-r)^2 \int_\Sigma (s_r - \bar{s}_r)^2 \leq n(n-1) \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2 \tag{1-15}$$

and, equivalently,

$$\int_\Sigma \left| P_r - \frac{(n-r)\bar{s}_r}{n} g \right|^2 \leq n \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2. \tag{1-16}$$

Second, we consider the k-scalar curvatures of locally conformally flat closed manifolds (see their definition in Section 5). Since they were first introduced in [Viaclovsky 2000], k-scalar curvatures have been much studied; see, for instance, [Guan 2002; Viaclovsky 2006]. When M is locally conformally flat, we obtain an almost-Schur type lemma for k-scalar curvatures, $k \geq 2$, as follows.

Theorem 1.11. *Let (M^n, g) be an n-dimensional closed locally conformally flat manifold, $n \geq 3$. Then, for $2 \leq k \leq n$, the k-scalar curvature $\sigma_k(S_g)$ and the Newton transformation T_k associated with the Schouten tensor S_g satisfy*

$$(1\text{-}17)\quad (n-k)^2\int_M(\sigma_k(S_g)-\overline{\sigma}_k(S_g))^2 \le n(n-1)\left(1+\frac{nK}{\lambda_1}\right)\int_M\left|T_k-\frac{(n-k)\sigma_k(S_g)}{n}g\right|^2$$

and, equivalently,

$$(1\text{-}18)\quad \int_M\left|T_k-\frac{(n-k)\overline{\sigma}_k(S_g)}{n}g\right|^2\le n\left[1+\frac{(n-1)K}{\lambda_1}\right]\int_M\left|T_k-\frac{(n-k)\sigma_k(g)}{n}g\right|^2,$$

where $\overline{\sigma}_k(S_g)=(1/\operatorname{Vol}M)\int_M\sigma_k(S_g)\,dv$ *and* λ_1 *and the constant* $K\ge 0$ *are as in Theorem 1.2.*

If the Ricci curvature Ric *of* M *is positive, these three statements are equivalent:*

(i) *Equality holds in (1-17) and (1-18).*

(ii) $T_k=((n-k)\sigma_k(S_g)/n)g$ *on* M.

(iii) $T_k=((n-k)\overline{\sigma}_k(S_g)/n)g$ *on* M.

As for Theorem 1.10, taking $K=0$ in Theorem 1.11, one obtains the corresponding inequalities with the universal constants.

The rest of this paper is organized as follows. In Section 2, we prove Theorems 1.7 and 1.8. In Section 3, we recall the definitions of the Newton transformation and the r-th symmetric function associated with a symmetric endomorphism of an n-dimensional vector space. In Section 4, we prove Theorem 1.9 by applying Theorem 1.7. In Section 5, we prove Theorem 1.11 by applying Theorem 1.7.

2. Proof of theorems on symmetric (2, 0)-tensors

First we give some notation. Assume (M, g) is an n-dimensional closed, that is, compact and without boundary, Riemannian manifold. Let ∇ denote the Levi-Civita connection on (M, g) and also the induced connections on tensor bundles on M. Let T denote a symmetric $(2,0)$-tensor field on M. Let tr denote the trace of a tensor. $B=\operatorname{tr}T=T_i^i=g^{ij}T_{ij}$ denotes the trace of T. Hereafter we use the Einstein summation convention. Denote by $\overline{B}=(1/\operatorname{Vol}M)\int_M B$ the average of B over M and set $\mathring{T}=T-(B/n)g$. Denote by div the divergence of a tensor field. For T, $\operatorname{div}T=\operatorname{tr}\nabla T$ is a $(1,0)$-tensor. Under the local coordinates $\{x_i\}$ on M, $\operatorname{div}T=g^{ij}(\nabla_iT_{jk})\,dx^k$, where $\nabla_iT_{jk}=(\nabla_{\partial_i}T)(\partial_j,\partial_k)$.

The following fact, already mentioned in the introduction, can be proved directly by noting that $T=(B/n)g$ implies $\operatorname{div}T=\nabla B/n$.

Proposition 2.1. *Assume* (M^n, g), $n\ge 2$, *is a connected Riemannian manifold of dimension* n. *If* $T=(B/n)g$ *and* $\operatorname{div}T=c\nabla B$, *where* $c\ne 1/n$ *is a constant, then* $B=\text{const}$ *on* M *and* T *is a constant multiple of its metric* g.

The argument of Theorem 1.7 is similar to that of Theorem 1.2 (that is, [Cheng 2013, Theorem 1.2]) and, in the case of $K = 0$, that of Theorem 1.1 (that is, [De Lellis and Topping 2012, Theorem 0.1]).

Proof of Theorem 1.7. Obviously, it suffices to prove the case $c \neq 1/n$. By the assumption $\operatorname{div} T = c\nabla B$,

$$\operatorname{div} \mathring{T} = \operatorname{div} T - \operatorname{div}\Big(\frac{B}{n} g\Big) = \operatorname{div} T - \frac{\nabla B}{n} = \frac{nc-1}{n}\nabla B. \tag{2-1}$$

Let f be the unique solution of the following Poisson equation on M:

$$\Delta f = B - \bar{B}, \qquad \int_M f = 0. \tag{2-2}$$

By (2-1), (2-2), and Stokes' formula,

$$\begin{aligned}
\int_M (B-\bar{B})^2 &= \int_M (B-\bar{B})\Delta f = -\int_M \langle \nabla B, \nabla f\rangle \\
&= -\frac{n}{nc-1}\int_M \langle \operatorname{div}\mathring{T}, \nabla f\rangle \\
&= \frac{n}{nc-1}\int_M \langle \mathring{T}, \nabla^2 f\rangle \\
&= \frac{n}{nc-1}\int_M \langle \mathring{T}, \nabla^2 f - \frac{1}{n}(\Delta f) g\rangle \\
&\leq \frac{n}{|nc-1|}\Big(\int_M |\mathring{\mathrm{T}}|^2\Big)^{1/2}\Big[\int_M |\nabla^2 f - \frac{1}{n}(\Delta f) g|^2\Big]^{1/2} \\
&= \frac{n}{|nc-1|}\Big(\int_M |\mathring{\mathrm{T}}|^2\Big)^{1/2}\Big[\int_M |\nabla^2 f|^2 - \frac{1}{n}\int_M (\Delta f)^2\Big]^{1/2}.
\end{aligned} \tag{2-3}$$

Recall the Bochner formula

$$\tfrac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\Delta f)\rangle,$$

and integrate it. By Stokes' formula, we have

$$\int_M |\nabla^2 f|^2 = \int_M (\Delta f)^2 - \int_M \operatorname{Ric}(\nabla f, \nabla f). \tag{2-4}$$

By (2-3) and (2-4),

$$\int_M (B-\bar{B})^2 \leq \frac{n}{|nc-1|}\Big(\int_M |\mathring{\mathrm{T}}|^2\Big)^{1/2}\Big[\frac{n-1}{n}\int_M (\Delta f)^2 - \int_M \operatorname{Ric}(\nabla f, \nabla f)\Big]^{1/2}. \tag{2-5}$$

By (2-2), $f \equiv 0$ if and only if $B - \bar{B} \equiv 0$ on M. In this case, (1-9) and (1-10) obviously hold. In the following we only consider that f is not identically zero.

Since the Ricci curvature has $\mathrm{Ric} \geq -(n-1)K$ on M,

$$\int_M \mathrm{Ric}(\nabla f, \nabla f) \geq -(n-1)K \int_M |\nabla f|^2. \tag{2-6}$$

By (2-6), (2-5) turns into

$$\int_M (B-\bar{B})^2 \leq \frac{n}{|nc-1|}\left(\int_M |\mathring{T}|^2\right)^{1/2}\left[\frac{n-1}{n}\int_M (\Delta f)^2 + (n-1)K\int_M |\nabla f|^2\right]^{1/2}. \tag{2-7}$$

Since the first nonzero eigenvalue λ_1 of the Laplace operator on M satisfies

$$\lambda_1 = \inf\left\{\frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2} : \varphi \in C^\infty(M) \text{ is not identically zero and } \int_M \varphi = 0\right\}$$

and

$$\begin{aligned}\int_M |\nabla f|^2 &= -\int_M f\Delta f = -\int_M f(B-\bar{B})\\ &\leq \left(\int_M f^2\right)^{1/2}\left[\int_M (B-\bar{B})^2\right]^{1/2}\\ &\leq \left(\frac{\int_M |\nabla f|^2}{\lambda_1}\right)^{1/2}\left[\int_M (B-\bar{B})^2\right]^{1/2},\end{aligned}$$

we have

$$\int_M |\nabla f|^2 \leq \frac{1}{\lambda_1}\int_M (B-\bar{B})^2. \tag{2-8}$$

Substitute (2-8) into (2-7) and note that $K \geq 0$. We have

$$\begin{aligned}\int_M (B-\bar{B})^2 &\leq \frac{n}{|nc-1|}\left(\int_M |\mathring{\mathrm{T}}|^2\right)^{1/2}\left[\frac{n-1}{n}\int_M (B-\bar{B})^2 + \left(\frac{(n-1)K}{\lambda_1}\right)\int_M (B-\bar{B})^2\right]^{1/2}\\ &= \frac{n^{1/2}(n-1)^{1/2}}{|nc-1|}\left(1+\frac{nK}{\lambda_1}\right)^{1/2}\left[\int_M |\mathring{T}|^2\right]^{1/2}\left[\int_M (B-\bar{B})^2\right]^{1/2},\end{aligned} \tag{2-9}$$

which implies that

$$\int_M (B-\bar{B})^2 \leq \frac{n(n-1)}{(nc-1)^2}\left(1+\frac{nK}{\lambda_1}\right)\int_M |\mathring{T}|^2. \tag{2-10}$$

Thus we have inequality (1-9):

$$(nc-1)^2\int_M (B-\bar{B})^2 \leq n(n-1)\left(1+\frac{nK}{\lambda_1}\right)\int_M \left|T-\frac{B}{n}g\right|^2.$$

From the identity

$$|T-(\bar{B}/n)g|^2=|T-(B/n)g|^2+(1/n)(B-\bar{B})^2,$$

we obtain (1-10):

$$(nc-1)^2\int_M\left|T-\frac{\bar{B}}{n}g\right|^2\le\left[(nc-1)^2+(n-1)\left(1+\frac{nK}{\lambda_1}\right)\right]\int_M\left|T-\frac{B}{n}g\right|^2.$$

Now, assuming positivity of the Ricci curvature Ric of M, we may prove the case of equalities in (1-9) and (1-10). Obviously, if $T=(B/n)g$ on M, the equalities in (1-9) and (1-10) hold. On the other hand, suppose the equality in (1-9) (or, equivalently, (1-10)) holds. If $c=1/n$, it is obvious that $T=(B/n)g$ on M. If $c\neq 1/n$, we may take $K=0$. By the proof of (1-9), the equality in (1-9) holds if and only if

(1) $\mathrm{Ric}(\nabla f,\nabla f)=0$ on M and

(2) $T-B/ng$ and $\nabla^2 f-1/n(\Delta f)g$ are linearly dependent.

Note that Ric >0 and (1) holds. $\nabla f\equiv 0$ on M must hold. Then $f\equiv 0$. Thus $B=\bar{B}$ on M. By (1-9), we obtain that $T=(B/n)g$ on M. Hence conclusions (i) and (ii) are equivalent. Obviously (iii) implies (ii). When $c\neq 1/n$, if (ii) holds, by the above argument, (ii) implies $B=\bar{B}$ on M. Thus (iii) also holds. □

Corollary 2.2. *Besides the assumptions and notation of Theorem 1.7, suppose the constant c satisfies $c\neq 1/n$. Then*

$$\int_M(B-\bar{B})^2\le C_{(Kd^2)}\int_M\left|T-\frac{B}{n}g\right|^2 \tag{2-11}$$

and

$$\int_M\left|T-\frac{\bar{B}}{n}g\right|^2\le\bar{C}_{(Kd^2)}\int_M\left|T-\frac{B}{n}g\right|^2, \tag{2-12}$$

where d denotes the diameter of M and $C_{(Kd^2)}$ and $\bar{C}_{(Kd^2)}$ are constants only depending on Kd^2.

Proof. When Ric $\ge -(n-1)K$, where the constant $K>0$, Li and Yau [1980] proved that the first nonzero eigenvalue λ_1 has the lower bound

$$\lambda_1\ge\alpha=\frac{1}{(n-1)d^2\exp[1+\sqrt{1+4(n-1)^2Kd^2}]},$$

where d denotes the diameter of M. So

$$\frac{K}{\lambda_1} \le \frac{K}{\alpha} = (n-1)Kd^2 \exp[1+\sqrt{1+4(n-1)^2Kd^2}\,].$$

By Theorem 1.7, we obtain inequality (2-11) with the constant

$$C_{(Kd^2)} = \frac{n(n-1)}{(nc-1)^2}\big(1+n(n-1)Kd^2 \exp[1+\sqrt{1+4(n-1)^2Kd^2}\,]\big).$$

Inequality (2-11) implies (2-12). □

Remark 2.3. There are other lower estimates α of λ_1 using the diameter d and negative lower bound $-(n-1)K$ of the Ricci curvature (see, for example, [Kalka et al. 1997]). Hence we may have other values of constants $C_{(Kd^2)}$ and $\overline{C}_{(Kd^2)}$.

3. Newton transformations and the r-th elementary symmetric function

Let $\sigma_r : \mathbb{R}^r \to \mathbb{R}$ denote the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r(x_1, \dots, x_n) = \sum_{i_1<\cdots<i_r} x_{i_1} \cdots x_{i_r}, \quad 1 \le r \le n.$$

Let V be an n-dimensional vector space and $A : V \to V$ be a symmetric linear transformation. If $\eta_1, \dots, \eta_n$ are the eigenvalues of A corresponding to the orthonormal eigenvectors $\{e_i\}$, $i = 1, \dots, n$, respectively, define the r-th symmetric functions $\sigma_r(A)$ associated with A by

$$\begin{aligned} &\sigma_0(A) = 1, \\ &\sigma_r(A) = \sigma_r(\eta_1, \dots, \eta_n),\ 1 \le r \le n. \end{aligned} \tag{3-1}$$

For convenience of notation, we simply denote $\sigma_r(A)$ by σ_r if there is no confusion. The Newton transformations $P_r : V \to V$ associated with A, $0 \le r \le n$ are defined by

$$\begin{aligned} P_0 &= I, \\ P_r &= \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j = \sigma_r I - \sigma_{r-1} A + \cdots + (-1)^r A^r, \quad r = 1, \dots, n. \end{aligned}$$

By definition, $P_r = \sigma_r I - AP_{r-1}$, $P_n = 0$. It was proved in [Reilly 1973] that P_r has the following basic properties:

(i) $P_r(e_i) = \dfrac{\partial \sigma_{r+1}}{\partial \eta_i} e_i$.

(ii) $\operatorname{tr}(P_r) = (n-r)\sigma_r$.

(iii) $\operatorname{tr}(AP_r) = (r+1)\sigma_{r+1}$.

Clearly, each P_r corresponds to a symmetric $(2,0)$-tensor on V, still denoted by P_r.

4. High-order mean curvatures of hypersurfaces in space forms

Assume $(N, \tilde{g})$ is an $(n+1)$-dimensional Riemannian manifold, $n \geq 2$. Suppose (Σ, g) is a smooth connected oriented closed hypersurface immersed in $(N, \tilde{g})$ with induced metric g. Let ν denote the outward unit normal to Σ, and $A = (h_{ij})$, the second fundamental form $A : T_p\Sigma \otimes_s T_p\Sigma \to \mathbb{R}$, defined by $A(X, Y) = -\langle \widetilde{\nabla}_X Y, \nu\rangle$, where $X, Y \in T_p\Sigma$, $p \in \Sigma$, and $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(N, \tilde{g})$. A determines an equivalent $(1, 1)$-tensor, called the shape operator A of

$$\Sigma : T_p\Sigma \to T_p\Sigma,$$

given by $AX = \widetilde{\nabla}_X \nu$. Σ is called totally umbilical if A is a multiple of its metric g at every point of Σ, that is, $A = (\operatorname{tr} A/n)g$ on Σ. Now we recall the definition of r-th mean curvatures of a hypersurface, which was introduced in [Reilly 1973]; compare [Rosenberg 1993].

Let η_i, $i = 1, \dots, n$ denote the principle curvatures of Σ at p, which are the eigenvalues of A at p corresponding to the orthonormal eigenvectors $\{e_i\}$, $i = 1. \dots, n$, respectively. By Section 3, we have the r-th symmetric functions $\sigma_r(A)$ associated with A, denoted by $s_r = \sigma_r(A)$, and the Newton transformations P_r associated with A at p, $0 \leq r \leq n$.

Definition 4.1. The r-th mean curvature H_r of Σ at p is defined by $s_r = \binom{n}{r} H_r$, $0 \leq r \leq n$.

For instance, $H_1 = s_1/n = H/n$ (in this paper, we also call $H = \operatorname{tr} A$ the mean curvature of Σ, consistent with earlier papers [Perez 2011; Cheng and Zhou 2012], among others). H_n is the Gauss–Kronecker curvature. When the ambient space N is a space form N_a^{n+1} with constant sectional curvature a,

$$\begin{aligned} \operatorname{Ric} &= (n-1)aI + HA - A^2, \\ R = \operatorname{tr} \operatorname{Ric} &= n(n-1)c + H^2 - |A|^2 = n(n-1)a + 2s_2. \end{aligned}$$

Hence H_2 is, modulo a constant, the scalar curvature of Σ.

Lemma 4.2 ([Reilly 1973]; cf. [Rosenberg 1993; Alías et al. 2006]). *When the ambient space is a space form N_a^{n+1}, we have* $\operatorname{div} P_r = 0$, *for* $0 \leq r \leq n$.

Proof of Theorem 1.9. By Section 3, $\operatorname{tr} P_r = (n-r)s_r$. Denote $\bar{s}_r = (1/\operatorname{Vol}\Sigma)\int_\Sigma s_r$. By Lemma 4.2, $\operatorname{div} P_r = 0$. Take $T = P_r$ and $B = (n-r)s_r$ in Theorem 1.7. Then

$$(n-r)^2 \int_\Sigma (s_r - \bar{s}_r)^2 \leq n(n-1)\left(1 + \frac{nK}{\lambda_1}\right) \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2;$$

equivalently,

$$\int_\Sigma \left| P_r - \frac{(n-r)\bar{s}_r}{n} g \right|^2 \leq n\left(1 + \frac{(n-1)K}{\lambda_1}\right) \int_\Sigma \left| P_r - \frac{(n-r)s_r}{n} g \right|^2,$$

which are (1-13) and (1-14), respectively.

Now we prove conclusions (1) and (2) in Theorem 1.9. If the Ricci curvature of Σ is positive, by Theorem 1.7, conclusion (1) holds and $s_r = \bar{s}_r$ is constant on Σ. If Σ is also embedded in $\mathbb{R}^{n+1}$, by a theorem of Ros [1987] stating that a closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant r-th mean curvature must be a distance sphere $\mathbb{S}^{n+1}$, $2 \le r \le n$, we obtain conclusion (2). □

Remark 4.3. If $r = 1$, $P_1 = s_1 I - A = HI - A$. P_1 is equivalent to the symmetric $(2,0)$-tensor $P_1 = Hg - A$. So (1-13) turns into

$$\int_\Sigma (H-\overline{H})^2 \le \frac{n}{n-1}\left(1+\frac{nK}{\lambda_1}\right)\int_\Sigma \left|Hg - A - \frac{(n-1)H}{n}g\right|^2 = \frac{n}{n-1}\left(1+\frac{nK}{\lambda_1}\right)\int_\Sigma \left|A - \frac{H}{n}g\right|^2. \tag{4-1}$$

In particular, if $K = 0$,

$$\int_\Sigma (H-\overline{H})^2 \le \frac{n}{n-1}\int_\Sigma \left|A - \frac{H}{n}g\right|^2. \tag{4-2}$$

Equations (4-1) and (4-2) are (1-8) and (1-6), respectively, which were proved in [Cheng and Zhou 2012] and [Perez 2011], respectively, if Σ is a closed hypersurface immersed in an Einstein manifold. This is because $\operatorname{div} P_1 = 0$ even if the ambient space is Einstein.

When $r = 2$, we have $2s_2 = R - n(n-1)a$,

$$P_2 = s_2 I - s_1 A + s_0 A^2 = \frac{R-(n-2)(n-1)a}{2} I - \mathrm{Ric},$$

and, by direct computation,

$$P_2 - \frac{(n-2)s_2}{n} g = \frac{R}{n} I - \mathrm{Ric}\,.$$

As a symmetric $(2,0)$-tensor, $P_2 = (R/n)g - \mathrm{Ric}$. Hence (1-13) turns into

$$\int_\Sigma (s_2-\bar{s}_2)^2 \le \frac{n(n-1)}{(n-2)^2}\left(1+\frac{nK}{\lambda_1}\right)\int_\Sigma \left|P_2 - \frac{(n-2)s_2}{n}g\right|^2,$$

which is

$$\int_\Sigma (R-\overline{R})^2 \le \frac{4n(n-1)}{(n-2)^2}\left(1+\frac{nK}{\lambda_1}\right)\int_\Sigma \left|\mathrm{Ric} - \frac{R}{n}g\right|^2. \tag{4-3}$$

Equation (4-3) was proved in [Cheng 2013], and, in the case of $K = 0$, in [De Lellis and Topping 2012].

If $r = n$, (1-13) is trivial.

5. k-scalar curvature of locally conformal flat manifolds

We first recall the definition of the k-scalar curvatures of a Riemannian manifold, introduced in [Viaclovsky 2000]. If (M^n, g) is an n-dimensional Riemannian manifold, $n \geq 3$, the Schouten tensor of M is

$$S_g = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{1}{2(n-1)} R g\right).$$

By definition, $S_g : TM \to TM$ is a symmetric (1, 1)-tensor field. By Section 3, we have the symmetric k-th function $\sigma_k(S_g)$ and the Newton transformations $T_k(S_g) = T_k$ associated with S_g, $1 \leq k \leq n$. We call $\sigma_k(S_g)$ the k-scalar curvatures of M

Lemma 5.1 [Viaclovsky 2000]. *If* (M, g) *is locally conformally flat*, *then*, *for* $1 \leq k \leq n$, $\operatorname{div} T_k(S_g) = 0$.

Because of Lemma 5.1, we can applying Theorem 1.7 to $T_k(S_g)$ to obtain Theorem 1.11.

Remark 5.2. When $k = 1$, $\sigma_1(S_g) = \operatorname{tr} S_g = R/(2(n-1))$ and $T_1 = \sigma_1(S_g)I - S_g$. As a symmetric (2, 0)-tensor, $T_1 = -(1/(n-2))(\mathrm{Ric} - Rg/2)$. Hence (1-17) turns into (1-3),

$$\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2}\left(1 + \frac{nK}{\lambda_1}\right) \int_M \left|\mathrm{Ric} - \frac{R}{n} g\right|^2,$$

and, in particular, if $K = 0$, (1-17) turns into (1-1),

$$\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left|\mathrm{Ric} - \frac{R}{n} g\right|^2.$$

Equations (1-3) and (1-1) were proved in [Cheng 2013] and [De Lellis and Topping 2012], respectively, without the hypothesis that M is locally conformally flat. The reason is that $\operatorname{div} T_1 = 0$ (the contracted second Bianchi identity) holds on any Riemannian manifold.

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Xu Cheng
Instituto de Matemática
Universidade Federal Fluminense
24020-140 Niterói, RJ
Brazil

xcheng@impa.br

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