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**ALGEBRAIC INVARIANTS, MUTATION, AND
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We construct a family of hyperbolic link complements by gluing tangles along totally geodesic four-punctured spheres, then investigate the commensurability relation among its members. Those with different volume are incommensurable, distinguished by their scissors congruence classes. Mutation produces arbitrarily large finite subfamilies of nonisometric manifolds with the same volume and scissors congruence class. Depending on the choice of mutation, these manifolds may be commensurable or incommensurable, distinguished in the latter case by cusp parameters. All have trace field $\mathbb{Q}(i, \sqrt{2})$; some have integral traces while others do not.

1. Introduction

Manifolds are *commensurable* if they have a common cover, of finite degree over each. W. P. Thurston first studied the commensurability relation among hyperbolic knot and link complements in S^3 , describing commensurable and incommensurable examples in his notes [Thurston 1979, Chapter 6]. The families of chain link complements [Neumann and Reid 1992], two-bridge knot complements [Reid and Walsh 2008], and certain pretzel knot complements [Macasieb and Mattman 2008] have since been further explored. Here we construct another infinite family of hyperbolic link complements and explore the commensurability relation among its members.

We compute the following invariants on members of our family. For $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ the *trace field* of $M = \mathbb{H}^3 / \Gamma$ is the smallest field containing the traces of elements of Γ . If each such trace is an algebraic integer we say M has *integral traces*. The *cusp parameters* of M , used in [Thurston 1979; Neumann and Reid 1992], are algebraic invariants of the Euclidean structures on horospherical cross sections of the cusps of M . The *Bloch invariant* [Neumann and Yang 1999] is determined by a polyhedral decomposition.

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Theorem 1. *For each $n \in \mathbb{N}$, there is a link $L_n \subset S^3$ such that $M_n = S^3 - L_n$ is hyperbolic with trace field $\mathbb{Q}(i, \sqrt{2})$ and integral traces. If $m \neq n$ then M_m and M_n are incommensurable, distinguished by their Bloch invariants and cusp parameters.*

Having integral traces is commensurability-invariant [Maclachlan and Reid 2003, §5.2], and the trace field is a commensurability invariant of link complements [Maclachlan and Reid 2003, Corollary 4.2.2]. Commensurable manifolds have \mathbb{Q} -dependent Bloch invariants and $\mathrm{PGL}_2(\mathbb{Q})$ -dependent cusp parameters, but the M_n have neither (see Proposition 4.7 and Lemma 4.18).

Figure 1 depicts L_2 . The gray lines there indicate the presence of 2-spheres that each meet L_2 in four points, separating it left-to-right into a tangle S in the three-ball B^3 , two copies of a tangle $T \subset S^2 \times I$, and the mirror image \bar{S} of S . For arbitrary $n \in \mathbb{N}$, the link L_n is constructed analogously, using S , \bar{S} , and n copies of T . We number the corresponding 2-spheres for L_n as $S^{(i)}$ for $0 \leq i \leq n$, so that $S^{(0)}$ bounds S , $S^{(n)}$ bounds \bar{S} , and $S^{(i)}$ bounds a copy of T with $S^{(i-1)}$ for $0 < i \leq n$.

We also describe the commensurability relation among the complements of links related to the L_n by *mutation* along the $S^{(i)}$: cutting along $S^{(i)}$ and regluing by an order-two mapping class that preserves $S^{(i)} \cap L_n$ and acts on it as an even permutation. With L_n projected as in Figure 1, for each i we mark the points of $S^{(i)} \cap L_n$ by 2, 3, 4, and 1, reading top to bottom, and refer to a mutation homeomorphism of $S^{(i)}$ by its permutation representation.

Below, for $n \in \mathbb{N}$ and $I \in \{0, 1, 2\}^{n+1}$ let L_I be the link obtained from L_n by the mutation (13)(24) (respectively, (12)(34)) along $S^{(i)}$, for each i such that the i -th entry of I is 1 (respectively, 2). Write $M_I = S^3 - L_I$ for each such I .

Theorem 2. *For $n \in \mathbb{N}$ and $I = (a_0, \dots, a_n) \in \{0, 1\}^{n+1}$, M_I is commensurable to M_n . For $J = (b_0, \dots, b_n) \in \{0, 1\}^{n+1}$, M_J is isometric to M_I if and only if either $b_i = a_i$ for each $i \in \{1, \dots, n-1\}$ or $b_i = a_{n-i}$ for each such i .*

We will show in a future paper that Theorem 2 reflects the fact that M_n has a *hidden symmetry* (see, for example, [Neumann and Reid 1992]) arising from a *hidden extension* of the mutation (13)(24), an extension of a lift of (13)(24) over a finite cover of $(S^2 \times I) - T$.

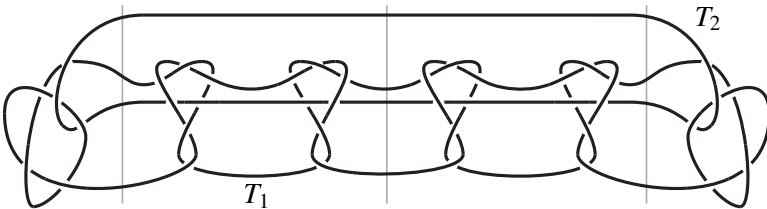


Figure 1. The link L_2 .

Here we prove [Theorem 2](#) more directly, identifying an orbifold O_n jointly covered by M_n and the M_I ; see [Proposition 6.4](#). The key advantage of this approach is that we can also prove the isometry classification above (see [Proposition 6.6](#)) using the fact that O_n is minimal in the commensurability class of M_n ([Corollary 6.5](#)).

[Corollary 6.5](#) is proved following an idea of Goodman, Heard, and Hodgson [[Goodman et al. 2008](#)]. The key step, for each n , is to construct a tiling \mathcal{T}_n of \mathbb{H}^3 by convex polyhedra that is *canonical* in the sense of [[Goodman et al. 2008](#), §2]. See [Theorem 4](#). This is of independent interest, as there are few infinite families for which canonical tilings have been identified.

The mutation (12)(34) has a very different effect than (13)(24).

Theorem 3. *For $n \in \mathbb{N}$, let $\mathcal{L}_n = \{L_I \mid I \in \{0, 2\}^{n+1}\}$. Then:*

- (1) *For each $I \in \{0, 2\}^{n+1} - \{(0, \dots, 0)\}$, M_I has the same volume, Bloch invariant, and trace field as M_n , but has a nonintegral trace.*
- (2) *There is a subfamily of \mathcal{L}_n with at least $n/2$ mutually incommensurable members, distinguished by their cusp parameters.*
- (3) *There is a subfamily of \mathcal{L}_n with n members which all share cusp parameters.*

Remarks. 1. Mutation along 4-punctured spheres preserves hyperbolic volume [[Ruberman 1987](#)], the trace field [[Neumann and Reid 1991](#)], and the Bloch invariant [[Neumann 2011](#), Theorem 2.13]. While unaware of the Bloch invariant reference we proved our case directly; see [Proposition 7.2](#).

2. $L_n = L_{(0, \dots, 0)}$, which accounts for the gap in statement (1) of the theorem.
3. Corollaries [7.4](#) and [7.5](#) describe the subfamilies from (2) and (3) above. We do not know the commensurability relation among members of the latter subfamily.

Theorems [2](#) and [3](#) comprise the first study (to our knowledge) of commensurability among an infinite family of link complements related by mutation. Mutants have a longstanding reputation for being difficult to distinguish, although the algorithm of [[Goodman et al. 2008](#)] can now be used to check particular examples. (For instance, the complement of the Kinoshita–Terasaka knot, 11n42 in the knot tables, is incommensurable with that of its mutant, the Conway knot, 11n34.)

[Theorem 2](#) further gives some evidence counter to the following conjecture of Reid and Walsh [[2008](#)]: the commensurability class of a hyperbolic knot complement in S^3 contains at most two others. This implies in particular that any hyperbolic knot complement is incommensurable with all but two of its (nonisometric) mutants.

We now outline the remainder of the paper. We name the tangle complements $M_S \doteq B^3 - S$ and $M_T \doteq S^2 \times I - T$, and note that M_T is the double of $M_{T_0} \doteq M_T \cap (S^2 \times [0, 1/2])$ across a single boundary component. [Section 2](#) describes hyperbolic structures with totally geodesic boundary on M_S and M_{T_0} as identification

spaces of the regular ideal octahedron and the right-angled ideal cuboctahedron, respectively.

The totally geodesic boundary ∂M_S is isometric to the component of ∂M_{T_0} contained in ∂M_T , and the reflective symmetry of M_T ensures that its totally geodesic boundary components are orientation-reversing isometric. In forming M_n we glue $\partial B^3 - S$ to $S^2 \times \{0\} - T$ by a map isotopic to an isometry, so that the separating four-punctured spheres $F^{(i)} = S^{(i)} - T$ are totally geodesic in M_n for $0 \leq i \leq n$. [Section 3](#) describes this assembly.

Because the $F^{(i)}$ are totally geodesic, each copy of M_S and M_T in M_n inherits its structure with totally geodesic boundary from the ambient hyperbolic structure. This in turn makes it possible to compute the commensurability invariants of [Theorem 1](#) on the M_n . We carry this out in [Section 4](#). Few other link complements are known to contain a surface that is totally geodesic without some topological constraint forcing it so; see, for example, [[Maclachlan and Reid 1991](#); [Aitchison and Rubinstein 1997](#)]. For related results see [[Menasco and Reid 1992](#); [Adams and Schoenfeld 2005](#); [Leininger 2006](#); [Adams et al. 2008](#)].

Our method of construction owes a debt to one that Adams [[1985](#)] and Neumann and Reid used to produce families of hyperbolic 3-manifolds, gluing together manifolds with 3-punctured sphere boundary. (However unlike the 4-punctured sphere, a 3-punctured sphere is totally geodesic in any hyperbolic 3-manifold that contains it [[Adams 1985](#), Theorem 3.1].) The work of Neumann and Reid (see [[Maclachlan and Reid 2003](#), §5.6]) can be combined with an argument like the one in [Proposition 4.2](#) to show that for each imaginary biquadratic extension k of \mathbb{Q} , there are infinitely many commensurability classes of hyperbolic 3-manifolds with trace field k .

In every hyperbolic 4-punctured sphere, each mutation determines a homeomorphism properly isotopic to an isometry [[Ruberman 1987](#)]. In [Section 5](#) we describe the isometries determined by (13)(24) and (12)(34) and the hyperbolic structures on mutants of the M_n . We prove [Theorem 2](#) in [Section 6](#) and [Theorem 3](#) in [Section 7](#).

2. A pair of tangles

This section is devoted to describing hyperbolic structures with totally geodesic boundary on the complements of the tangles S , in B^3 , and T_0 , in $S^2 \times I$, depicted in [Figure 2](#). For a manifold M with boundary, we refer by a *tangle* in M to a pair (M, T) , where T is the image of a disjoint union of circles and closed intervals, embedded in M by a map taking each circle into the interior of M and restricting on each interval to a proper embedding.

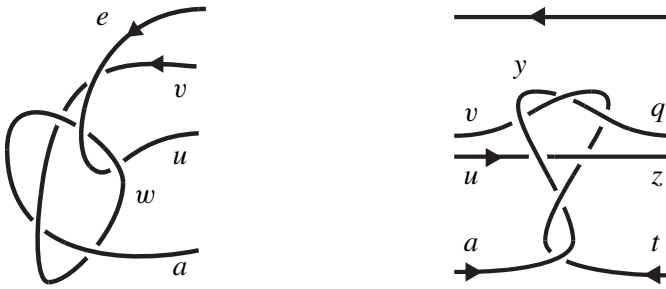


Figure 2. Tangles S and T_0 , labeled with Wirtinger generators.

We will prove there is a homeomorphism taking $B^3 - S$ to a hyperbolic manifold with totally geodesic boundary which is an identification space of an ideal octahedron by pairing certain faces. This was previously known, and it follows from results in [Paoluzzi and Zimmermann 1996] upon taking a geometric limit, but we do not know a reference for a direct proof. We also prove there is a homeomorphism taking $S^2 \times I - T_0$ to a certain identification space of the right-angled ideal cuboctahedron. As far as we are aware, this description was not previously known.

We prove existence of homeomorphisms using faithful representations, from the fundamental groups of tangle complements onto Kleinian groups generated by face pairings. Our main tools drawing connections between the geometric, algebraic, and topological objects involved are Lemma 2.1, which relates a hyperbolic 3-manifold with totally geodesic boundary produced by pairing some faces of a right-angled polyhedron to the Kleinian group generated by the face-pairing isometries, and Lemma 2.6, which describes a homeomorphism from a pared manifold M and the convex core of \mathbb{H}^3/Γ , where Γ is a Kleinian group isomorphic to $\pi_1(M)$.

In the remainder of the paper, we will let $\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C}, t \in (0, \infty)\}$, the upper half-space model of hyperbolic space, equipped with the complete Riemannian metric of constant sectional curvature -1 . In this model, the group of orientation-preserving isometries, $\mathrm{PSL}_2(\mathbb{C})$, acts by extending its action by Möbius transformations on the *ideal boundary* or *sphere at infinity* $\mathbb{C} \cup \{\infty\}$.

The *horosphere of height t centered at ∞* is $\mathbb{C} \times \{t\} \subset \mathbb{H}^3$. This inherits the Euclidean metric, scaled by $1/t$, from the ambient hyperbolic metric. For $v \in \mathbb{C} \times \{0\}$, a *horosphere centered at v* is a Euclidean sphere in $\mathbb{C} \times \mathbb{R}$ centered at a point in \mathbb{H}^3 and tangent to $\mathbb{C} \times \{0\}$ at $(v, 0)$. It is a standard fact that isometries of \mathbb{H}^3 take horospheres to horospheres.

A *hyperplane* of \mathbb{H}^3 is a totally geodesic subspace of the form $\ell \times \mathbb{R}^+$ for a line $\ell \subset \mathbb{C}$, or the intersection with \mathbb{H}^3 of a Euclidean sphere centered at a point in $\mathbb{C} \times \{0\}$. A *half-space* is the closure of a component of the complement in \mathbb{H}^3 of a hyperplane, and a *polyhedron* is the nonempty intersection of a collection of half-spaces with

the property that the corresponding collection of defining hyperplanes is locally finite. A *face* of a polyhedron is its intersection with one of its defining hyperplanes. A polyhedron is *right angled* if its defining hyperplanes meet at right angles (if at all) and *ideal* if any point at which more than two of its defining hyperplanes meet is on the sphere at infinity. Such points are *ideal vertices*.

We say a polyhedron $\mathcal{P} \subset \mathbb{H}^3$ is *checkered* if its set of faces is partitioned into sets \mathcal{S}_i and \mathcal{S}_e of *internal* and *external* faces, respectively, so that each $f \in \mathcal{S}_i$ intersects only faces in \mathcal{S}_e and vice versa. For a face f of a checkered, right-angled ideal polyhedron \mathcal{P} , let \mathcal{H}_f be the geodesic hyperplane in \mathbb{H}^3 containing f and let U_f be the half-space bounded by \mathcal{H}_f that contains \mathcal{P} . Let the *expansion* of \mathcal{P} be

$$E(\mathcal{P}) = \bigcap_{f \in \mathcal{S}_i} U_f.$$

The expansion $E(\mathcal{P})$ is a polyhedron of infinite volume that contains \mathcal{P} , and the components of the frontier of \mathcal{P} in $E(\mathcal{P})$ are the external faces of \mathcal{P} .

An *internal face pairing* for a checkered polyhedron $\mathcal{P} \subset \mathbb{H}^3$ is a collection $\{\phi_f \mid f \in \mathcal{S}_i\}$ of isometries, such that for each $f \in \mathcal{S}_i$ there exists $f' \in \mathcal{S}_i$ with $\phi_f(f) = f'$, $\phi_f(\mathcal{P}) \cap \mathcal{P} = f'$, and $\phi_{f'} = \phi_f^{-1}$. It is *proper* if $f' \neq f$ for all $f \in \mathcal{S}_i$. A proper internal face pairing determines a *proper* $\text{Isom}(\mathbb{H}^3)$ -*side-pairing* of the expansion $E(\mathcal{P})$, in the sense of [Ratcliffe 1994, §10.1]. (In [Ratcliffe 1994], *faces* are called *sides*.)

Given a proper internal face pairing $\{\phi_f\}$ of a checkered polyhedron \mathcal{P} , [Ratcliffe 1994, Theorem 10.1.2] implies the identification space $E(\mathcal{P})/\{\phi_f\}$, determined by setting $x \sim \phi_f(x)$ for all $f \in \mathcal{S}_i$ and $x \in f$, is a hyperbolic manifold. The inclusion $\mathcal{P} \hookrightarrow E(\mathcal{P})$ induces an inclusion from $M_{\mathcal{P}} \doteq \mathcal{P}/\{\phi_f\}$ to $E(\mathcal{P})/\{\phi_f\}$. For each edge e of each $g \in \mathcal{S}_e$, there is a unique $f \in \mathcal{S}_i$ such that $e \subset f \cap g$. Since $f' = \phi_f(f)$ intersects a unique $g' \in \mathcal{S}_i$ along $\phi_f(e)$, the internal face pairing for \mathcal{P} determines an edge pairing for the disjoint union of external faces of \mathcal{P} . Thus $M_{\mathcal{P}} \doteq \mathcal{P}/\{\phi_f\}$ is an isometrically embedded submanifold of $E(\mathcal{P})/\{\phi_f\}$, where $\partial M_{\mathcal{P}}$ is the quotient of the disjoint union of the external faces by the edge pairing induced by $\{\phi_f\}$.

Given an edge pair $\{e, e'\}$ for $\partial M_{\mathcal{P}}$, the total angle around this edge in $M_{\mathcal{P}}$ is the sum of the dihedral angles for e and e' in \mathcal{P} . Therefore, if \mathcal{P} is right angled, $\partial M_{\mathcal{P}}$ is totally geodesic.

If Γ is a Kleinian group, we refer to the convex core of \mathbb{H}^3/Γ as $C(\Gamma)$. This is the convex submanifold of \mathbb{H}^3/Γ , minimal with respect to inclusion, with the property that the inclusion-induced homomorphism $\pi_1 C(\Gamma) \rightarrow \mathbb{H}^3/\Gamma$ is surjective. (See [Morgan 1984] for background on Kleinian groups. The beginning of §6 therein covers convex cores.)

Lemma 2.1. *Let $\mathcal{P} \subset \mathbb{H}^3$ be a finite-sided, checkered right-angled ideal polyhedron, with a proper internal face pairing $\{\phi_f \mid f \in \mathcal{S}_i\}$. Then $\Gamma \doteq \langle \phi_f \mid f \in \mathcal{S}_i \rangle$ is a free*

Kleinian group, and the inclusion $\mathcal{P} \hookrightarrow \mathbb{H}^3$ induces an isometry $p: M_{\mathcal{P}} \rightarrow C(\Gamma)$. If \mathcal{H} is the hyperplane containing $g \in \mathcal{I}_e$ then $\mathcal{H} \rightarrow \mathbb{H}^3$ induces an isometric embedding of $\mathcal{H}/\text{Stab}_{\Gamma}(\mathcal{H})$ to the component of $\partial C(\Gamma)$ containing $p(g)$.

Proof. We will continue to use some terminology and results from [Ratcliffe 1994]. With these hypotheses the inclusion $\mathcal{P} \rightarrow E(\mathcal{P})$ induces an isometric embedding $M_{\mathcal{P}} \rightarrow E(\mathcal{P})/\{\phi_f\}$, and $\partial M_{\mathcal{P}}$ is totally geodesic. If $E(\mathcal{P})/\{\phi_f\}$ is complete as a hyperbolic 3-manifold, then by Poincaré's polyhedron theorem (see, for example, [Ratcliffe 1994, Theorem 11.2.2]), $\Gamma = \langle \phi_f \mid f \in \mathcal{I}_i \rangle$ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ .

By [Ratcliffe 1994, Theorem 11.1.6], to show completeness it suffices to check that the link of any cusp is a complete Euclidean surface. Let $[v] = \{v_0, v_1, \dots, v_{n-1}\}$ be an equivalence class of ideal vertices of \mathcal{P} under the relation generated by $x \sim \phi_f(x)$, $f \in \mathcal{I}_i$, enumerated so that for each j there exists $f_j \in \mathcal{I}_i$ with $\phi_{f_j}(v_j) = v_{j+1}$ (taken modulo n). In particular, v_j is an ideal vertex of f_j and also of $f'_j \doteq \phi_{j-1}(f_{j-1})$.

For each j , let \mathcal{B}_j be a horosphere centered at v_j , chosen small enough that $\mathcal{B}_j \cap \mathcal{B}_{j'} = \emptyset$ for $j \neq j'$. Since \mathcal{P} is right angled, $\mathcal{B}_j \cap \mathcal{P}$ is a Euclidean rectangle for each j . We may assume, by renumbering if necessary, that $\mathcal{B}_0 \cap f_0$ has the shortest length of all the arcs $\mathcal{B}_j \cap f_j$. Then since $\phi_0(\mathcal{B}_0) \cap f'_1$ is parallel to $\phi_0(\mathcal{B}_0) \cap f_1$ in $\phi_0(\mathcal{B}_0) \cap \mathcal{P}$, they have the same length: that of $\mathcal{B}_0 \cap f_0$. Since this is less than the length of $\mathcal{B}_1 \cap f_1$, we have $\phi_0(\mathcal{B}_0) \subset \mathcal{B}_1$.

We may replace \mathcal{B}_1 by $\phi_0(\mathcal{B}_0)$, then replace \mathcal{B}_2 with $\phi_1(\mathcal{B}_1)$, and so on, yielding a new collection of horospheres which are pairwise disjoint and have the additional property that they are interchanged by the face pairings of \mathcal{P} . Equivalence classes of ideal vertices of $E(\mathcal{P})$ are the same as those of \mathcal{P} ; thus this collection satisfies the hypotheses of [Ratcliffe 1994, Theorem 11.1.4], and the link of $[v]$ is complete. It follows that $E(\mathcal{P})/\{\phi_f\}$ is a complete hyperbolic 3-manifold.

Now by the polyhedron theorem, Γ is discrete and $E(\mathcal{P})$ is a fundamental domain for Γ . It follows from a ping-pong argument that Γ is free, since the fact that \mathcal{P} is right angled implies that the hyperplanes containing its internal faces are mutually disjoint. The inclusion $E(\mathcal{P}) \rightarrow \mathbb{H}^3$ induces an isometry $E(\mathcal{P})/\{\phi_f\} \rightarrow \mathbb{H}^3/\Gamma$, so the inclusion $\mathcal{P} \rightarrow \mathbb{H}^3$ induces an isometric embedding $p: M_{\mathcal{P}} \rightarrow \mathbb{H}^3/\Gamma$.

That $p(M_{\mathcal{P}}) \subseteq C(\Gamma)$ will follow from the fact that \mathcal{P} is contained in the convex hull of the limit set $\text{Hull}(\Gamma)$ of Γ , since this is well known to be the universal cover of $C(\Gamma)$. Fixed points of parabolic elements of Γ lie in $\text{Hull}(\Gamma)$, so since \mathcal{P} is the convex hull of its ideal vertices we show that it is in $\text{Hull}(\Gamma)$ by observing that each such vertex is a parabolic fixed point of Γ . Indeed, if $\{v_0, v_1, \dots, v_{n-1}\}$ is an equivalence class of ideal vertices enumerated as we described above, then v_0 is fixed by $\phi_{f_{n-1}} \circ \dots \circ \phi_{f_1} \circ \phi_{f_0} \in \Gamma$.

Since $p(M_{\mathcal{P}})$ has totally geodesic boundary it is convex (see [Canary et al. 2006, Corollary I.1.3.7]). Thus if $p(M_{\mathcal{P}})$ carries $\pi_1(\mathbb{H}^3/\Gamma)$ then $C(\Gamma) \subseteq p(M_{\mathcal{P}})$. To show this we use the nearest-point retraction $r : E(\mathcal{P}) \rightarrow \mathcal{P}$ to produce a homeomorphism $M_{\mathcal{P}} \cup_{\partial M_{\mathcal{P}}} (\partial M_{\mathcal{P}} \times [0, \infty)) \rightarrow \mathbb{H}^3/\Gamma$ that restricts to p on $M_{\mathcal{P}}$. The closure of each component of $E(\mathcal{P}) - \mathcal{P}$ intersects \mathcal{P} in a unique $g \in \mathcal{S}_e$, and the map $x \mapsto (r(x), d(x, r(x)))$ determines a homeomorphism to $g \times [0, \infty)$. The inverses of these homeomorphisms, taken over the disjoint union of all $g \in \mathcal{S}_e$, combine to induce the map in question.

The two paragraphs above combine to prove that $C(\Gamma) = p(M_{\mathcal{P}})$. In particular, $C(\Gamma)$ has totally geodesic boundary, so its preimage in \mathbb{H}^3 under the universal cover is a disjoint union of geodesic planes. Since p takes $g \in \mathcal{S}_e$ to $\partial C(\Gamma)$, the hyperplane \mathcal{H} containing g is a component of the preimage of $\partial C(\Gamma)$. The final claim of the lemma follows. □

Corollary 2.2. *Let \mathcal{P}_1 be the regular ideal octahedron in \mathbb{H}^3 , embedded as indicated in Figure 3, and checkered by declaring the face A to be external. The collection $\{s^{\pm 1}, t^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_1 , where*

$$s = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 2i & 2-i \\ i & 1-i \end{pmatrix}.$$

Let $M_S = \mathcal{P}_1/\{s^{\pm 1}, t^{\pm 1}\}$, and let $\Gamma_S = \langle s, t \rangle$. Then the inclusion $\mathcal{P}_1 \rightarrow \mathbb{H}^3$ induces an isometry $p_S : M_S \rightarrow C(\Gamma_S)$.

Proof. With the indicated embedding, \mathcal{P}_1 is a tile of the $\mathrm{PSL}_2(\mathbb{O}_1)$ -invariant tessellation \mathcal{T}_1 constructed in [Hatcher 1983]. Here $\mathbb{O}_1 = \mathbb{Z}[i]$ is the ring of integers of the

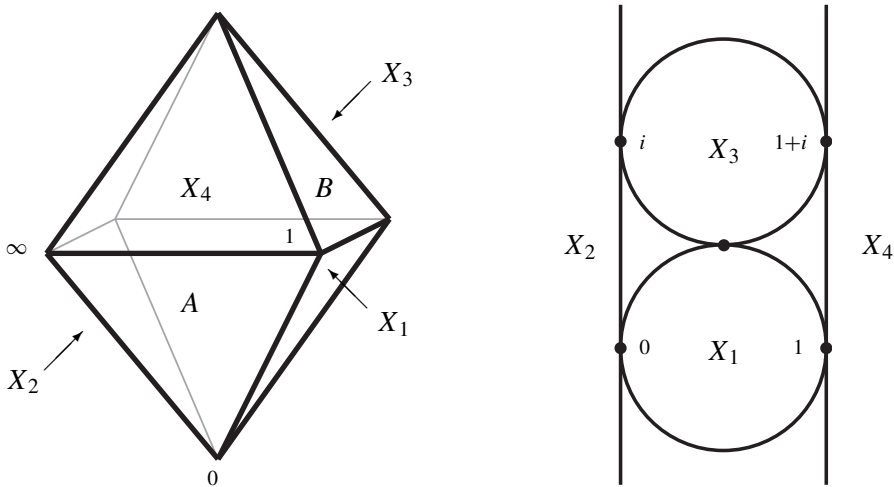


Figure 3. The regular ideal octahedron \mathcal{P}_1 , and its expansion $E(\mathcal{P}_1)$.

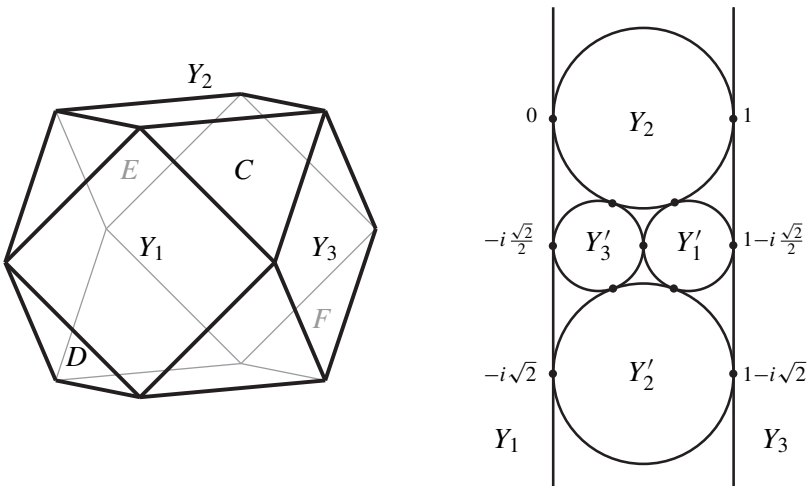


Figure 4. The right-angled ideal cuboctahedron \mathcal{P}_2 , and $E(\mathcal{P}_2)$.

field $\mathbb{Q}(i)$. In particular, the face A shown on the left in [Figure 3](#) has ideal vertices 0 , 1 , and ∞ , and all other ideal vertices of \mathcal{P}_1 have positive imaginary part.

Since A is external, the faces X_1 , X_2 , X_3 , and X_4 of \mathcal{P}_1 indicated on the left in [Figure 3](#) are internal. Direct computation reveals that s takes X_1 to X_2 , fixing the ideal vertex they share, and t takes X_3 to X_4 so that the vertex they share goes to the vertex shared by X_4 and X_2 . Hence $\{s^{\pm 1}, t^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_1 . The corollary now follows from [Lemma 2.1](#). \square

The external faces of \mathcal{P}_1 triangulate ∂M_S , and their images under p_S determine a triangulation of $\partial C(\Gamma_S)$, which we will denote by Δ_S .

Corollary 2.3. *Let \mathcal{P}_2 be the right-angled ideal cuboctahedron in \mathbb{H}^3 , embedded as indicated in [Figure 4](#), and checkered by declaring triangular faces external. The collection $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_2 , where*

$$f = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} -1+i\sqrt{2} & 1-2i\sqrt{2} \\ -2 & 3-i\sqrt{2} \end{pmatrix}, \quad h = \begin{pmatrix} 2i\sqrt{2} & -3-i\sqrt{2} \\ -3+i\sqrt{2} & -3i\sqrt{2} \end{pmatrix}.$$

Let $M_{T_0} = \mathcal{P}_2 / \{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$, and let $\Gamma_{T_0} = \langle f, g, h \rangle$. The inclusion $\mathcal{P}_2 \rightarrow \mathbb{H}^3$ induces an isometry $p_{T_0}: M_{T_0} \rightarrow C(\Gamma_{T_0})$.

Proof. With the indicated embedding, \mathcal{P}_2 is a tile of the $\mathrm{PSL}_2(\mathbb{O}_2)$ -invariant tessellation \mathcal{T}_2 of \mathbb{H}^3 defined in [[Hatcher 1983](#)], where $\mathbb{O}_2 = \mathbb{Z}[i\sqrt{2}]$ is the ring of integers of $\mathbb{Q}(i\sqrt{2})$. In particular, the face C labeled in the figure has ideal vertices 0 , 1 , and ∞ .

Label the internal faces Y_i as indicated on the left in [Figure 4](#), and label the square face opposite Y_i as Y'_i . Direct computation reveals that f takes Y_2 to Y_1 ,

fixing the ideal vertex they share, g takes Y_3 to Y'_1 , fixing the ideal vertex they share, and h takes Y'_2 to Y'_3 , taking the vertex they share to the opposite vertex on Y'_3 . Hence $\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$ is an internal face pairing for \mathcal{P}_2 . The corollary now follows from [Lemma 2.1](#). \square

The external faces of \mathcal{P}_2 triangulate ∂M_{T_0} . This has two components that we will call $\partial_+ M_{T_0}$ and $\partial_- M_{T_0}$, with the latter triangulated by the letter-labeled faces of [Figure 4](#). Let $\partial_{\pm} C(\Gamma_{T_0}) = p_{T_0}(\partial_{\pm} M_{T_0})$ and let $\Delta_{T_0}^{\pm}$ refer to the triangulation for $\partial_{\pm} C(\Gamma_{T_0})$ determined by the images under p_{T_0} of the external faces of \mathcal{P}_2 .

In the remainder of the paper, if g and h are elements of a group and G is a subgroup, we let g^h denote the conjugate of g by h , hgh^{-1} , and $G^h = hGh^{-1}$. We describe parabolic isometries p_1, p_2 , and p_3 which lie in $\Gamma_S \cap \Gamma_{T_0}$:

$$\begin{aligned} p_1 = s^{-1} &= f^{-1} &&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ p_2 = stst^{-2} &= fg^{-1}f^{-1}h^{-1}g &&= \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}, \\ p_3 = (s^{-1})^{tst} &= (g^{-1})^{g^{-1}f^{-1}h} &&= \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}. \end{aligned}$$

Since these are in $\mathrm{PSL}_2(\mathbb{R})$, they stabilize the hyperplane \mathcal{H} with boundary $\mathbb{R} \cup \{\infty\}$.

Lemma 2.4. *The polygon \mathcal{F} of [Figure 5](#) is a fundamental domain for the action of $\Lambda \doteq \langle p_1, p_2, p_3 \rangle < \mathrm{PSL}_2(\mathbb{R})$ on \mathcal{H} , and $F^{(0)} = \mathcal{H}/\Lambda$ is a 4-punctured sphere. Also:*

- (1) $\Lambda = \mathrm{Stab}_{\Gamma_S}(\mathcal{H}) = \mathrm{Stab}_{\Gamma_{T_0}}(\mathcal{H})$,
- (2) *the inclusion $\mathcal{H} \hookrightarrow \mathbb{H}^3$ induces an isometry $\iota_-^{(0)}: F^{(0)} \rightarrow \partial C(\Gamma_S)$ and an isometry $\iota_+^{(0)}: F^{(0)} \rightarrow \partial_- C(\Gamma_{T_0})$, and*
- (3) *the triangulation of \mathcal{F} pictured in [Figure 5](#) projects to a triangulation Δ_F of $F^{(0)}$ taken by $\iota_-^{(0)}$ and $\iota_+^{(0)}$, respectively, to Δ_S and $\Delta_{T_0}^-$.*

Proof. With \mathcal{P}_1 and \mathcal{P}_2 embedded as prescribed in [Figures 3](#) and [4](#), respectively, their faces A and C coincide and lie in \mathcal{H} as described in [Figure 5](#). As noted in the proofs of [Corollaries 2.2](#) and [2.3](#), Γ_S -translates of \mathcal{P}_1 lie in the tessellation \mathcal{T}_1 described in [\[Hatcher 1983\]](#), and Γ_{T_0} -translates of \mathcal{P}_2 lie in \mathcal{T}_2 . The Farey tessellation is $\mathcal{T}_1 \cap \mathcal{H} = \mathcal{T}_2 \cap \mathcal{H}$, so this contains any Γ_S -translate of any face of \mathcal{P}_1 and any Γ_{T_0} -translate of any face of \mathcal{P}_2 .

Let A' be the external face of \mathcal{P}_1 which shares the ideal vertex 0 with A , and let B' be the external face which shares the vertex ∞ with A and $1+i$ with B . Since t takes X_3 to X_4 , with the edge $X_3 \cap B'$ taken to $X_4 \cap A$, it follows that $t(B')$ lies in \mathcal{H} , abutting A along the geodesic between 1 and ∞ . Since $t(B')$ is a Farey triangle it has its other ideal vertex at 2 . It follows similarly that $g^{-1}(E) = t(B')$ (where E

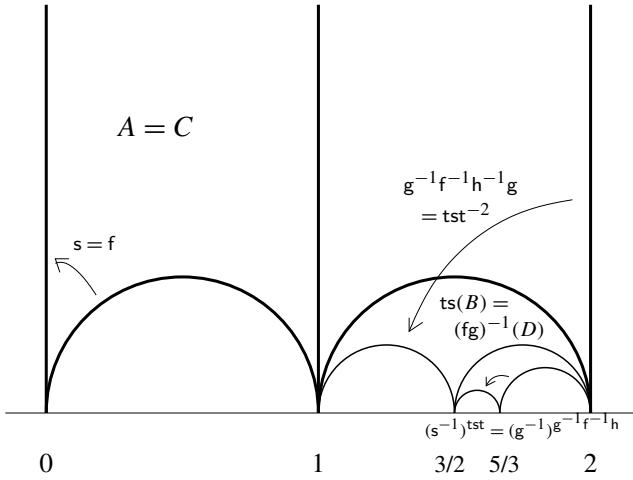


Figure 5. A triangulated fundamental domain \mathcal{F} for the action of Λ on \mathcal{H} , and side pairings.

is as labeled in Figure 4), that $\text{ts}(B) = (\text{fg})^{-1}(D)$, as indicated in Figure 5, and that $\text{tst}(A') = g^{-1}f^{-1}h(F)$ has vertices at $3/2$, 2 , and $5/3$.

Combinatorial considerations or direct calculation establish that $s = f$, $\text{tst}^{-2} = g^{-1}f^{-1}h^{-1}g$, and $(s^{-1})^{\text{tst}} = (g^{-1})g^{-1}f^{-1}h$, and that each stabilizes \mathcal{H} and pairs edges of \mathcal{F} as indicated in Figure 5. By inspection the quotient is a 4-punctured sphere $F^{(0)}$. By the polyhedron theorem \mathcal{F} is a fundamental domain for the group that they generate, which acts on \mathcal{H} with quotient $F^{(0)}$. Since p_1 , p_2 , and p_3 are easily obtained from the edge pairings above and vice versa, it follows that

$$\Lambda = \langle s, \text{tst}^{-2}, (s^{-1})^{\text{tst}} \rangle = \langle f, g^{-1}f^{-1}h^{-1}g, (g^{-1})g^{-1}f^{-1}h \rangle.$$

Therefore \mathcal{F} is a fundamental domain for Λ , and $\mathcal{H}/\Lambda = F^{(0)}$.

It is easy to see from its combinatorics that ∂M_S is a four-punctured sphere, as is the component of ∂M_{T_0} containing C . Thus by Corollaries 2.2 and 2.3 the same holds true for $\partial C(\Gamma_S)$ and the component of $\partial C(\Gamma_{T_0})$ containing the image of C . Lemma 2.1 implies that $\partial C(\Gamma_S)$ is the image of $\mathcal{H}/\text{Stab}_{\Gamma_S}(\mathcal{H})$ under the inclusion-induced map. Since it is clear from the above that $\Lambda < \text{Stab}_{\Gamma_S}(\mathcal{H})$, and since \mathcal{H}/Λ is itself a four-punctured sphere, the conclusions of assertions (1) and (2) above follow for Γ_S . A similar argument implies the same for Γ_{T_0} . The conclusions of (3) follow from the description above of the triangulation of \mathcal{F} . \square

Remarks. 1. The parabolic elements of Λ fixing the ideal points 0 , ∞ , and $5/3$ of \mathcal{H} are p_1 , p_2 , and p_3 . The final conjugacy class of parabolic elements in Λ

is represented by

$$p_4 = p_1 p_2 p_3^{-1} = (\text{stst}^{-2})^{\text{tst}^{-1}} = \begin{pmatrix} 29 & -45 \\ 20 & -31 \end{pmatrix}.$$

Evidently p_1 and p_3 are conjugate in Γ_S , as are p_2 and p_4 . The combinatorial considerations of [Lemma 4.13](#) will show that $C(\Gamma_S)$ has exactly two cusps, each of rank one, so every parabolic element of Γ_S is conjugate to one of p_1 or p_2 .

2. There exists $k \in \text{PSL}_2(\mathbb{C})$, with order 2, which normalizes Γ_{T_0} :

$$(1) \quad k = \begin{pmatrix} i & i - \sqrt{2} \\ 0 & -i \end{pmatrix}.$$

The action of k on the generators f , g , and h is given by

$$f^k = g^{fg^{-1}}, \quad g^k = f^{fg^{-1}}, \quad \text{and} \quad h^k = (h^{-1})^{fg^{-1}}.$$

If Γ is a Kleinian group and $u \in \text{Isom}(\mathbb{H}^3)$, we write $\phi_u: C(\Gamma) \rightarrow C(\Gamma^u)$ for the restriction to $C(\Gamma)$ of the isometry $\mathbb{H}^3/\Gamma \rightarrow \mathbb{H}^3/\Gamma^u$ induced by u . Since k normalizes Γ_{T_0} , $\phi_k: C(\Gamma_{T_0}) \rightarrow C(\Gamma_{T_0})$ is an orientation-preserving involution. The elements p_i^k , $i \in \{1, 2, 3\}$, all preserve the geodesic hyperplane $k(\mathcal{H})$, which lies over the line $\mathbb{R} - i\sqrt{2}$ and contains an external face of \mathcal{P}_2 projecting to $\partial_+ C(\Gamma_{T_0})$. The lemma below follows and, together with [Lemma 2.4](#), completely describes $\partial C(\Gamma_{T_0})$.

Lemma 2.5. $\Lambda^k = \text{Stab}_{\Gamma_T}(k(\mathcal{H}))$, and the inclusion $k(\mathcal{H}) \rightarrow \mathbb{H}^3$ induces an isometry from $F' \doteq k(\mathcal{H})/\Lambda^k$ to $\partial_+ C(\Gamma_{T_0})$.

It is easy to see that p_1^k is conjugate in Γ_{T_0} to p_3 and that $p_2^k = p_2^{-1}$. The combinatorial considerations of [Lemma 4.14](#) will imply that M_{T_0} has four cusps. Hence, by [Lemma 2.5](#), each of the cusps of $C(\Gamma_{T_0})$ joins $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$, and each parabolic in Γ_{T_0} is conjugate to exactly one p_i , $i \in \{1, 2, 3, 4\}$.

Our second main tool in this section is [Lemma 2.6](#). We refer to [[Morgan 1984](#), Definition 4.8] for the definition of a pared manifold.

Lemma 2.6. *Let (M, P) be a pared manifold, and suppose that $\rho: \pi_1 M \rightarrow \Gamma < \text{PSL}_2(\mathbb{C})$ is a faithful representation onto a non-Fuchsian geometrically finite Kleinian group Γ , where $C(\Gamma)$ has totally geodesic boundary. If ρ determines a one-to-one correspondence between conjugacy classes of subgroups of $\pi_1(M)$ corresponding to components of P and conjugacy classes of maximal parabolic subgroups of Γ , then ρ is induced by a homeomorphism of $M - P$ to $C(\Gamma)$.*

This is well known to experts in Kleinian groups, but we do not know of a reference for a written proof. It seems worth writing down as it may fail if $C(\Gamma)$

does not have totally geodesic boundary (see [Canary and McCullough 2004] for a thorough exploration of this phenomenon). The proof follows easily from results in [Canary and McCullough 2004], for example, but requires introduction of the characteristic submanifold machinery. Since this falls outside the scope of the rest of the paper, we defer the proof to the [Appendix](#).

Let (B^3, S) be the tangle pictured on the left in [Figure 2](#). Take a base point for $\pi_1(B^3 - S)$ on $\partial(B^3 - S)$ high above the projection plane, and let its Wirtinger generators correspond in the usual way to labeled arcs of the diagram.

Proposition 2.7. *There is a homeomorphism $f_S: B^3 - S \rightarrow C(\Gamma_S)$, such that*

$$f_{S*}: \pi_1(B^3 - S) \rightarrow \Gamma_S$$

is given by $f_{S}(a) = p_1^{-1}$, $f_{S*}(e) = p_2$, and $f_{S*}(v) = p_3^{-1}$.*

Proof. Reducing a standard Wirtinger presentation for $\pi_1(B^3 - S)$, we obtain

$$\langle a, w, e \mid ewe^{-1}a = awaw^{-1} \rangle = \langle a, w, e \mid w(e^{-1}aw) = (e^{-1}aw)a \rangle.$$

Thus, taking $b = e^{-1}aw$, one finds that $\pi_1(B^3 - S)$ is freely generated by a and b .

By [Lemma 2.1](#) and [Corollary 2.2](#), Γ_S is free on the generators s and t . Hence, the map $f_{S*}: \pi_1(B^3 - S) \rightarrow \Gamma_S$ given by $a \mapsto s$ and $b \mapsto t$ is an isomorphism. Notice that the subgroup of $\pi_1(B^3 - S)$ corresponding to the 4-punctured sphere $\partial B^3 - \partial S$ is freely generated by a , v , and e . It is easily checked that

$$f_{S*}(v) = \begin{pmatrix} 16 & -25 \\ 9 & -14 \end{pmatrix} = p_3^{-1} \quad \text{and} \quad f_{S*}(e) = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} = p_2.$$

The map f_{S*} takes $\pi_1(\partial B^3 - S)$ isomorphically to Λ . Since a , v , and e generate $\pi_1(\partial B^3 - S)$ and their images in Γ_S generate Λ , Since any meridian of S is conjugate in $\pi_1(B^3 - S)$ to either a or e , and these are taken to p_1 and p_2 , respectively, meridians are taken to parabolic elements of Γ_S .

Now let $N(S)$ be a small open tubular neighborhood of S in B^3 . Then $B^3 - N(S)$ is a compact manifold with genus-2 boundary, and the pair $(B^3 - N(S), \partial N(S))$ is a pared manifold. The proposition follows from [Lemma 2.6](#), after noting that $(B^3 - N(S)) - \partial N(S)$ is homeomorphic to $B^3 - S$. \square

Let $(S^2 \times I, T_0)$ be the tangle pictured on the right side of [Figure 2](#), where I is oriented so that $\partial_- T_0 \doteq T_0 \cap S^2 \times \{0\}$ contains the endpoints labeled a , u , and v . Take a base point for $\pi_1(S^2 \times I - T_0)$ on $S^2 \times \{0\}$ high above the projection plane and let Wirtinger generators correspond to the labeled arcs of [Figure 2](#).

The next proposition is the analog of [Proposition 2.7](#) for T_0 .

Proposition 2.8. *There is a homeomorphism $f_{T_0}: S^2 \times I - T_0 \rightarrow C(\Gamma_{T_0})$ such that*

$$f_{T_0*}: \pi_1(S^2 \times I - T_0) \rightarrow \Gamma_{T_0}$$

is given by $f_{T_0*}(a) = p_1^{-1}$, $f_{T_0*}(e) = p_2$, and $f_{T_0*}(v) = p_3^{-1}$.

Proof. $(S^2 \times I - N(T_0))$ may be isotoped in S^3 to a standard embedding of a genus-3 handlebody. Thus $\pi_1(S^2 \times I - T_0)$ is free on three generators. We claim that the group is generated by a , e , and t . This follows after noticing that $v = y^{-1}xy$, where $y = (ta)^{-1}a(ta)$ and $x = (azq)^{-1}t(azq) = (ate)^{-1}t(ate)$. (The relation $zq = te$ used in the last equality comes from the relationship between four peripheral elements in a 4-punctured sphere group.) So far, we have established that $v, y \in \langle a, e, t \rangle$. Now using the other punctured sphere relation, we have $u = a^{-1}ev \in \langle a, e, t \rangle$. Finally, $z = yuy^{-1}$ and $q = z^{-1}te$. Therefore a , e , and t generate the group as claimed.

By Lemma 2.1 and Corollary 2.3, Γ_{T_0} is freely generated by f , g , and h . For our purposes, a more convenient free generating set for Γ_{T_0} is $\{f, fgf^{-1}, p_2\}$. Note that all of these generators are parabolic and peripheral, and conjugation by k interchanges the first two and takes the third to its inverse. The representation of $\pi_1(S^2 \times [0, 1/2] - T_0)$ given by

$$a \mapsto f, \quad t \mapsto fgf^{-1}, \quad e \mapsto p_2$$

is clearly faithful, and it is easily checked that v maps to p_3^{-1} . Because $u = a^{-1}ev$ is mapped to $p_1p_2p_3^{-1} = p_4$, we conclude that meridians are mapped to parabolic elements and that $\pi_1(S^2 \times \{0\} - \partial_-T_0)$ is taken to Λ . The result now follows from Lemma 2.6 as previously. □

There is a visible involution of $S^2 \times I - T_0$ which is a rotation by π around a circle in $S^2 \times \{1/2\}$. This involution exchanges the two boundary components. With a proper choice of path between our basepoint and its image under this involution, the corresponding action on $\pi_1(S^2 \times I - T_0)$ is given by

$$a \leftrightarrow t, \quad e \leftrightarrow e^{-1}.$$

This commutes with the action of the element k defined in (1) on Γ_{T_0} , under the representation f_{T_0*} . Hence this involution is isotopic to the pullback of ϕ_k by f_{T_0} .

Recall from Lemma 2.4 that $\Lambda = \text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$, and from Lemma 2.5 that $\Lambda^k = \text{Stab}_{\Gamma_{T_0}}(k(\mathcal{H}))$. By its definition in Proposition 2.8, it is clear that f_{T_0*} maps $\pi_1(S^2 \times \{0\} - \partial_-T_0)$ isomorphically to Λ . Since \mathcal{H} projects to $\partial_-C(\Gamma_{T_0})$, using the involution equivariance of f_{T_0} we obtain the corollary below.

Corollary 2.9. *Let $\partial_+T_0 = T_0 \cap S^2 \times \{1\}$. Then $f_{T_0}(S^2 \times \{0\} - \partial_-T_0) = \partial_-C(\Gamma_{T_0})$, and $f_{T_0}(S^2 \times \{1\} - \partial_+T_0) = \partial_+C(\Gamma_{T_0})$.*

3. Combination

In this section, we will describe how to join copies of the tangles S and T_0 to construct links in S^3 whose complements are uniformized by combinations of

Γ_S and Γ_{T_0} . The main tool in this section is a corollary of Maskit's combination theorem for free products with amalgamation [Maskit 1971]. Denote the convex hull of the limit set for a Kleinian group Γ by $\text{Hull}(\Gamma)$.

Definition 3.1. Kleinian groups Γ_0 and Γ_1 *meet cute* along a hyperplane $\mathcal{H} \subset \mathbb{H}^3$ if $\mathcal{H} = \text{Hull}(\Gamma_0) \cap \text{Hull}(\Gamma_1)$ and $\text{Stab}_{\Gamma_0}(\mathcal{H}) = \text{Stab}_{\Gamma_1}(\mathcal{H})$.

The fact below follows easily from this definition, and accounts for its utility.

Fact. *If Γ_0 and Γ_1 meet cute along \mathcal{H} then $\text{Stab}_{\Gamma_0}(\mathcal{H}) = \Gamma_0 \cap \Gamma_1 = \text{Stab}_{\Gamma_1}(\mathcal{H})$. Furthermore, \mathcal{H} divides \mathbb{H}^3 into \mathcal{B}_0 and \mathcal{B}_1 such that for $i \in \{0, 1\}$, if $g_i \in \Gamma_i$ satisfies $g_i(\mathcal{B}_{1-i}) \cap \mathcal{B}_{1-i} \neq \emptyset$, then $g_i \in \Gamma_0 \cap \Gamma_1$.*

In general, if Θ is a subgroup of Γ , the limit set of Θ is contained in that of Γ , and so the covering map $\mathbb{H}^3/\Theta \rightarrow \mathbb{H}^3/\Gamma$ maps $C(\Theta)$ into $C(\Gamma)$ — we will call this restriction the *natural map* $C(\Theta) \rightarrow C(\Gamma)$. When Γ_0 and Γ_1 meet cute along \mathcal{H} then the natural map $C(\Gamma_0 \cap \Gamma_1) \rightarrow C(\Gamma_i)$ restricts to an embedding of the 2-orbifold $\mathcal{H}/(\Gamma_0 \cap \Gamma_1)$.

The lemma below is a geometric combination theorem for Kleinian groups which meet cute along a hyperplane. It follows from Maskit's combination theorem and observations on the geometry of Kleinian groups that go back at least to J. Morgan's [1984] account of geometrization for Haken manifolds.

Lemma 3.2. *Suppose Γ_0 and Γ_1 meet cute along a plane \mathcal{H} . Let $E = \mathcal{H}/\Theta$, where $\Theta = \Gamma_0 \cap \Gamma_1$, and for $i = 0, 1$ let $\iota_i: E \rightarrow C(\Gamma_i)$ be the natural embedding. Then $\langle \Gamma_0, \Gamma_1 \rangle$ is a Kleinian group, and the inclusions $\Gamma_i \rightarrow \langle \Gamma_0, \Gamma_1 \rangle$ determine an isomorphism $\Gamma_0 *_{\Theta} \Gamma_1 \rightarrow \langle \Gamma_0, \Gamma_1 \rangle$ as abstract groups. The natural maps $C(\Gamma_i) \rightarrow C(\langle \Gamma_0, \Gamma_1 \rangle)$ determine an isometry $C(\Gamma_0) \cup_{\iota_0^{-1}} C(\Gamma_1) \rightarrow C(\langle \Gamma_0, \Gamma_1 \rangle)$.*

In using Lemma 3.2, we often write $C(\Gamma_0) \cup_E C(\Gamma_1)$ when the maps ι_i are clear.

Proof. We will use Theorem 8.2 of [Morgan 1984], a version of Maskit's combination theorem. Its hypotheses are satisfied by any Γ_0 and Γ_1 that meet cute along a hyperplane \mathcal{H} , as a consequence of the Fact above; the group-theoretic conclusions follow. That the desired isometry exists follows from the remarks given below Theorem 8.2 of [Morgan 1984], which have since been considerably fleshed out in [Anderson and Canary 2001].

The function $\tilde{f}: \mathbb{H}^3 \rightarrow [0, 1]$ described in [Morgan 1984] is the harmonic extension of the characteristic function of \mathcal{B}_0 : $\tilde{f}(y)$ is the visual measure of the set of vectors pointing from y toward \mathcal{B}_0 . See [Anderson and Canary 2001, §2] for a precise analytic definition. It is not hard to see that here $\mathcal{H} = \tilde{f}^{-1}(\frac{1}{2})$ (see [Anderson and Canary 2001, Proposition 2.2]), whence our E is Morgan's $X = f^{-1}(\frac{1}{2})$.

Our ι_i is Morgan's p_i , mapping to $N_i = \mathbb{H}^3/\Gamma_i$ for $i \in \{0, 1\}$. For each i , $\iota_i(E)$ is a convex core boundary component of N_i , so $p_0(N_-) \cap C(N_0) = p_0(E)$ and $p_1(N_+) \cap C(N_1) = p_1(E)$, and the result follows from the equation at the bottom

of [Morgan 1984, p. 76]. See [Anderson and Canary 2001, Proposition 5.2 and remarks after 5.3] for related results. \square

We first apply Lemma 3.2 to join $C(\Gamma_{T_0})$ to a copy of itself across $\partial_+ C(\Gamma_{T_0})$. Recall from the discussion above Lemma 2.4 that we have defined \mathcal{H} to be the geodesic hyperplane of \mathbb{H}^3 with ideal boundary $\mathbb{R} \cup \{\infty\}$. Let $r \in \text{Isom}(\mathbb{H}^3)$ be the reflection through \mathcal{H} . This acts on $\mathbb{C} \cup \{\infty\}$ by complex conjugation; thus if $q \in \Gamma < \text{PSL}_2(\mathbb{C})$, then $q^r = \bar{q}$, where $\bar{q} \in \text{PSL}_2(\mathbb{C})$ is the element whose entries are the conjugates of the entries of q . Hence, we let $\bar{\Gamma}$ denote Γ^r .

Lemma 3.3. *Define*

$$c = \begin{pmatrix} 1 & i\sqrt{2} \\ 0 & 1 \end{pmatrix}.$$

*Then $\Gamma_T \doteq \langle \Gamma_{T_0}, \bar{\Gamma}_{T_0}^{c^{-2}} \rangle$ is a Kleinian group, and there is an inclusion-induced isomorphism $\Gamma_{T_0} *_{\Lambda^k} \bar{\Gamma}_{T_0}^{c^{-2}} \rightarrow \Gamma_T$ and an isometry $C(\Gamma_{T_0}) \cup_{F'} C(\bar{\Gamma}_{T_0}^{c^{-2}}) \rightarrow C(\Gamma_T)$ determined by the natural maps. Furthermore, $c^{-2}r$ normalizes Γ_T , and $\phi_{c^{-2}r}: C(\Gamma_T) \rightarrow C(\Gamma_T)$ is an orientation-reversing involution fixing F' and exchanging its complementary components.*

Proof. Recall from Lemmas 2.4 and 2.5 that Λ and Λ^k are the stabilizers in Γ_{T_0} of the geodesic planes \mathcal{H} and $k(\mathcal{H})$, respectively, and that these planes project to the components of $\partial C(\Gamma_{T_0})$. It follows that \mathcal{H} and $k(\mathcal{H})$ are components of the boundary of $\text{Hull}(\Gamma_{T_0})$, so $\text{Hull}(\Gamma_{T_0})$ is contained in the region between them.

With c as defined in the statement of the lemma, note that $c(\mathcal{H}) = (\mathbb{R} + i\sqrt{2}) \times \mathbb{R}^+$ and that $ck(\mathcal{H}) = \mathcal{H}$. Since $\text{Hull}(\Gamma_{T_0}^c)$ has boundary components $c(\mathcal{H})$ and $ck(\mathcal{H}) = \mathcal{H}$, and $\Lambda^{ck} = \text{Stab}_{\Gamma_{T_0}^c}(\mathcal{H})$ is invariant under conjugation by r , $\Gamma_{T_0}^c$ and $\bar{\Gamma}_{T_0}^c$ meet cuspally along \mathcal{H} . Applying Lemma 3.2, we obtain an isomorphism $\Gamma_{T_0}^c *_{\Lambda^{ck}} \bar{\Gamma}_{T_0}^c \rightarrow \langle \Gamma_{T_0}^c, \bar{\Gamma}_{T_0}^c \rangle$ and an isometry

$$C(\Gamma_{T_0}^c) \cup_{\phi_c(F')} C(\bar{\Gamma}_{T_0}^c) \rightarrow C(\langle \Gamma_{T_0}^c, \bar{\Gamma}_{T_0}^c \rangle)$$

induced by the natural maps. It is clear that r normalizes $\langle \Gamma_{T_0}^c, \bar{\Gamma}_{T_0}^c \rangle$, exchanging amalgamands, hence ϕ_r acts as an orientation-reversing involution of $C(\langle \Gamma_{T_0}^c, \bar{\Gamma}_{T_0}^c \rangle)$, fixing F' and exchanging $C(\Gamma_{T_0}^c)$ with $C(\bar{\Gamma}_{T_0}^c)$.

Observe that $\bar{c} = c^{-1}$. It follows that $\bar{\Gamma}_{T_0}^c = \bar{\Gamma}_{T_0}^{c^{-1}}$, and hence $\Gamma_T = \langle \Gamma_{T_0}^c, \bar{\Gamma}_{T_0}^c \rangle^{c^{-1}}$. Conjugating the groups of the paragraph above by c^{-1} , we obtain an inclusion-induced isomorphism $\Gamma_{T_0} *_{\Lambda^k} \bar{\Gamma}_{T_0}^{c^{-2}} \rightarrow \Gamma_T$, and an isometry

$$C(\Gamma_{T_0}) \cup_{F'} C(\bar{\Gamma}_{T_0}^{c^{-2}}) \rightarrow C(\Gamma_T)$$

induced by the natural maps. Furthermore, $c^{-1}rc = c^{-2}r$ normalizes Γ_T and induces an orientation-reversing involution $\phi_{c^{-2}r}$, fixing F' and exchanging its sides. \square

If M is an oriented manifold with a boundary component F , the *double* of M across F is $M \cup_F \bar{M}$, where \bar{M} is a copy of M with orientation reversed, and the gluing map $F \rightarrow \bar{F} \subset \bar{M}$ is the identity map.

Corollary 3.4. *There is an isometry $p_T : M_T \rightarrow C(\Gamma_T)$, where M_T is the double of M_{T_0} across F' , that is the natural map following p_{T_0} from [Corollary 2.3](#) on M_{T_0} .*

The advantage that Γ_T has over Γ_{T_0} for our purposes is that the components of $\partial C(\Gamma_T)$ are naturally orientation-reversing isometric, since they are exchanged by $\phi_{c^{-2}r}$. Recall from [Lemma 2.4](#) that $\Lambda = \text{Stab}_{\Gamma_{T_0}}(\mathcal{H})$; thus by [Lemma 3.3](#), $\Lambda = \text{Stab}_{\Gamma_T}(\mathcal{H})$. We will again refer by $i_+^{(0)}$ to the natural map $F^{(0)} \rightarrow C(\Gamma_T)$. Then the lemma below follows from [Lemma 3.3](#).

Lemma 3.5. *Let $F^{(1)} = c^{-2}\mathcal{H}/\Lambda^{c^{-2}}$, and let $\phi_{c^{-2}} : F^{(0)} \rightarrow F^{(1)}$ and $\iota_-^{(1)} : F^{(1)} \rightarrow C(\Gamma_T)$ be the natural maps. Then $\partial C(\Gamma_T) = \partial_- C(\Gamma_T) \sqcup \partial_+ C(\Gamma_T)$, where $\partial_- C(\Gamma_T) \doteq \iota_+^{(0)}(F^{(0)})$ and $\partial_+ C(\Gamma_T) \doteq \iota_-^{(1)}(F^{(1)})$, and $\iota_-^{(1)}\phi_{c^{-2}} = \phi_{c^{-2}r}\iota_+^{(0)}$.*

$C(\Gamma_T)$ is a geometric model for the double of $(S^2 \times I, T_0)$ across $(S^2 \times \{1\}, \partial_+ T_0)$. Note that the double of $S^2 \times I$ across $S^2 \times \{1\}$ is again homeomorphic to $S^2 \times I$, by a map taking $(p, t) \in S^2 \times I$ to $(p, t/2)$ and $(p, t) \in \bar{S}^2 \times \bar{I}$ to $(p, 1 - t/2)$.

Definition 3.6. Let $(S^2 \times I, T)$ be the double of $(S^2 \times I, T_0)$ across $(S^2 \times \{1\}, \partial_+ T_0)$. We will identify $(S^2 \times I, T_0) \subset (S^2 \times I, T)$ with its image under the map discussed above, so that $T_0 = T \cap S^2 \times [0, 1/2]$. In particular, we have $\partial_- T = \partial_- T_0 = T \cap S^2 \times \{0\}$ and $\partial_+ T_0 = T \cap S^2 \times \{1/2\}$, and we will take $\partial_+ T = T \cap S^2 \times \{1\}$.

The tangle $(S^2 \times I, T)$ is pictured in [Figure 6](#), with $T_0 \subset T$ visible to the left of the gray vertical line representing $S^2 \times \{1/2\}$. There is a mirror symmetry of $(S^2 \times I, T)$, visible in the figure as reflection through the gray vertical line:

$$r_T : (S^2 \times I, T) \rightarrow (S^2 \times I, T)$$

given by $r_T(p, x) = (p, 1 - x)$, hence fixing $(S^2 \times \{1/2\}, \partial_+ T_0)$.

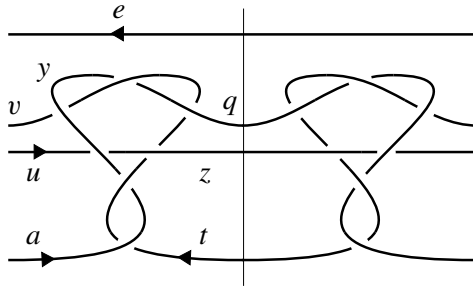


Figure 6. The tangle $T \subset S^2 \times I$ with labeled Wirtinger generators for T_0 .

Proposition 3.7. *There is a homeomorphism $f_T: S^2 \times I - T \rightarrow C(\Gamma_T)$, which restricts on $S^2 \times [0, 1/2] - T_0$ to f_{T_0} followed by the natural map, such that the following diagram commutes:*

$$\begin{array}{ccc} S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T) \\ r_T \downarrow & & \downarrow \phi_{c^{-2}r} \\ S^2 \times I - T & \xrightarrow{f_T} & C(\Gamma_T). \end{array}$$

Furthermore, f_T takes $S^2 \times \{0\} - \partial_- T$ to $\partial_- C(\Gamma_T)$ and $S^2 \times \{1\} - \partial_+ T$ to $\partial_+ C(\Gamma_T)$.

Proof. We define f_T using the properties described in the statement of the proposition. Namely, we first require f_T to restrict on $S^2 \times [0, 1/2] - T_0$ to the homeomorphism f_{T_0} defined in Proposition 2.8, followed by the natural map $C(\Gamma_{T_0}) \rightarrow C(\Gamma_T)$. For $x \in S^2 \times [1/2, 1] - T$, we define $f_T(x) = \phi_{c^{-2}r} f_T r_T(x)$. The resulting map is well defined, since r_T fixes $S^2 \times \{1/2\} - \partial_+ T_0$ and $\phi_{c^{-2}r}$ fixes F' . It is a homeomorphism, since r_T , f_{T_0} , and $\phi_{c^{-2}r}$ are. By Corollary 2.9, f_{T_0} takes $S^2 \times \{0\} - \partial_- T_0$ to $\partial_- C(\Gamma_{T_0})$; it therefore follows from the definitions and Lemma 3.5 that $f_T(S^2 \times \{0\} - \partial_- T) = \partial_- C(\Gamma_T)$. The conclusion thus follows from the reflection equivariance of f_T . \square

Definitions 3.8. (1) Let $j: (\partial B^3, \partial S) \rightarrow (S^2 \times \{0\}, \partial_- T)$ be the homeomorphism such that $(B^3, S) \cup_j (S^2 \times I, T)$ is the tangle pictured in Figure 7.

(2) Define $h: S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}$ by $h(p, x) = (p, x + 1)$, and with $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, let $T^{(i)} = h^{i-1}(T)$ (so $T^{(1)} = T$ in particular). For $n \in \mathbb{N}$, define

$$(S^3, L_n) = (B^3, S) \cup_j \left(S^2 \times [0, n], \bigcup_{i=1}^n T^{(i)} \right) \cup_{j_n} (\bar{B}^3, \bar{S}).$$

For $i \in \{0, 1, \dots, n\}$, let $S^{(i)}$ be the image in (S^3, L_n) of $S^2 \times \{i\} \subset S^2 \times [0, n]$. Above, $(\bar{B}^3, \bar{S}) = r_S(B^3, S)$, where r_S is the reflective involution of S^3 fixing the boundary of an embedding of B^3 and exchanging its sides, and $j_n = r_S j^{-1} r_T h^{-n+1}: (S^{(n)}, \partial_+ T^{(n)}) \rightarrow (\partial \bar{B}^3, \partial \bar{S})$.

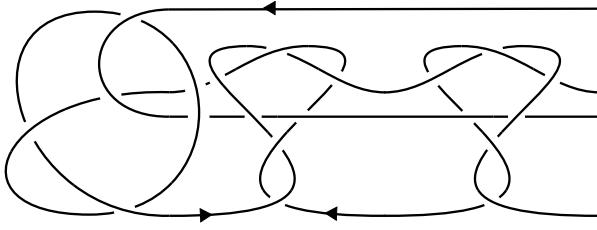
(3) Using Figure 6 and taking $T \subset S^2 \times I \subset S^2 \times \mathbb{R}$, label the points of $S^{(0)} \cap L_n = S^2 \times \{0\} \cap T$ by 2, 3, 4, and 1 top-to-bottom, so that, for example, 2 is the terminal point of the tangle string labeled e and 1 is the initial point of the string labeled a . Label each point of $S^{(i)} \cap L_n$ by its image under h^{-i} .

Remark. With Wirtinger generators for $\pi_1(B^3 - S)$ and $\pi_1(S^2 \times I - T)$ as labeled in Figures 2 and 6, we have $j_*(a) = a$, $j_*(u) = u$, and $j_*(v) = v$.

We now construct a geometric model of $S^3 - L_n$.

Definitions 3.9. (1) For $i \geq 0$, let $\Lambda^{(i)} = \Lambda c^{-2i}$ and $F^{(i)} = c^{-2i}(\mathcal{H})/\Lambda^{(i)}$.

(2) For $i \geq 1$, let $\Gamma_T^{(i)} = \Gamma_T c^{-2(i-1)}$, and define $\phi_i = \phi_{c^{-2(i-1)}}: C(\Gamma_T) \rightarrow C(\Gamma_T^{(i)})$.

**Figure 7.** $S \cup T$.

The definitions above of $F^{(0)}$ and $F^{(1)}$ above agree with our previous definitions. Also, $\Gamma_T^{(1)} = \Gamma_T$, and since $\Gamma_T^{(i)} = c^{-2(i-1)} \Gamma_T c^{2(i-1)}$, [Lemma 3.5](#) implies that

$$\Lambda^{(i-1)} = \text{Stab}_{\Gamma_T^{(i)}}(c^{-2(i-1)}(\mathcal{H})) \quad \text{and} \quad \Lambda^{(i)} = \text{Stab}_{\Gamma_T^{(i)}}(c^{-2i}(\mathcal{H})),$$

and the resulting natural maps $\iota_+^{(i-1)}: F^{(i-1)} \rightarrow C(\Gamma_T^{(i)})$ and $\iota_-^{(i)}: F^{(i)} \rightarrow C(\Gamma_T^{(i)})$ map to the components of its totally geodesic boundary.

Proposition 3.10. *For $n \in \mathbb{N}$, define $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\bar{\Gamma}_S)$, using gluing maps defined as follows:*

$$\begin{aligned} \iota_+^{(0)}(\iota_-^{(0)})^{-1}: \quad \partial C(\Gamma_S) &\rightarrow \partial_- C(\Gamma_T^{(1)}), \\ \iota_+^{(i)}(\iota_-^{(i)})^{-1}: \quad \partial_+ C(\Gamma_T^{(i)}) &\rightarrow \partial_- C(\Gamma_T^{(i+1)}) \quad \text{for } 1 \leq i < n, \\ \phi_r \iota_-^{(0)} \phi_{n+1}^{-1}(\iota_-^{(n)})^{-1}: \quad \partial_+ C(\Gamma_T^{(n)}) &\rightarrow \partial C(\bar{\Gamma}_S). \end{aligned}$$

There is a homeomorphism $f_n: S^3 - L_n \rightarrow M_n$ which restricts on $B^3 - S$ to f_S , on $S^2 \times [i-1, i] - T^{(i)}$ to $\phi_i f_T h^{-i+1}$ for $1 \leq i \leq n$, and on $\bar{B}^3 - \bar{S}$ to $\phi_r f_{Sr}$.

Proof. We will use the description of f_n above as its definition. Then, by [Proposition 2.7](#) and the definitions, f_n restricts on $B^3 - S$ and $\bar{B}^3 - \bar{S}$ to homeomorphisms to $C(\Gamma_S)$ and $C(\bar{\Gamma}_S)$, respectively. By [Proposition 3.7](#) and definitions, for each i between 1 and n it restricts on $S^2 \times [i-1, i] - T^{(i)}$ to a homeomorphism to $C(\Gamma_T^{(i)})$. Thus in order to show that f_n is a homeomorphism, we must only show that it is well defined on the spheres $S^{(i)} - \{1, 2, 3, 4\}$ that separate these tangle complements.

We first check the case $i = 1$, showing that f_n is well defined on $S^{(0)} - \{1, 2, 3, 4\}$. Since $T^{(1)} = T$ and $\Gamma_T^{(1)} = \Gamma_T$, and h^0 and ϕ_1 are each the identity map, in this case we must only show that on $\partial B^3 - \partial S$, $f_T \circ j = \iota_+^{(0)}(\iota_-^{(0)})^{-1} \circ f_S$.

By their definitions above, f_S and $f_T \circ j$ induce the same isomorphism from $\pi_1(\partial B^3 - \partial S)$ to $\Lambda = \Gamma_S \cap \Gamma_T$. Recall from [Lemma 2.4](#) and the remarks above [Lemma 3.5](#) that the $\iota_\pm^{(0)}$ are induced by the inclusions of Λ into Γ_S and Γ_T . Therefore at the level of fundamental group, $(\iota_+^{(0)}(\iota_-^{(0)})^{-1} \circ f_S)_* = (f_T \circ j)_*$. Since any two homeomorphisms between 4-punctured spheres that induce the same map on

fundamental groups are properly isotopic, we may isotope j so that f_S and $f_T j$ agree on $S^{(0)}$. The conclusion thus follows in this case.

For $1 \leq i < n$, we may use the fact that $\Gamma_T^{(i)}$ and $\Gamma_T^{(i+1)}$ are conjugates of Γ_T to obtain the following model descriptions for $\iota_+^{(i)}$ and $\iota_-^{(i)}$:

$$(2) \quad \iota_+^{(i)} = \phi_{i+1} \iota_+^{(0)} \phi_{i+1}^{-1}, \quad \iota_-^{(i)} = \phi_i \iota_-^{(1)} \phi_i^{-1}.$$

Here $\iota_+^{(0)}: F^{(0)} \rightarrow \partial_- C(\Gamma_T)$ and $\iota_-^{(1)}: F^{(1)} \rightarrow \partial_+ C(\Gamma_T)$ are the natural maps of [Lemma 3.5](#). Using the reflection-invariance property described there, we thus obtain

$$(3) \quad \iota_+^{(i)} (\iota_-^{(i)})^{-1} = \phi_{i+1} \iota_+^{(0)} (\iota_-^{(1)} \phi_2)^{-1} \phi_i^{-1} = \phi_{i+1} \phi_{c^{-2}r}^{-1} \phi_i^{-1}.$$

Then, by the reflection-equivariance property of f_T from [Proposition 3.7](#), we have

$$\iota_+^{(i)} (\iota_-^{(i)})^{-1} \circ \phi_i f_T h^{-i+1} = \phi_{i+1} \phi_{c^{-2}r}^{-1} f_T h^{-i+1} = \phi_{i+1} f_T r_T h^{-i+1}.$$

It follows directly from the definitions that $r_T h^{-i+1} = h^{-i}$ on $S^{(i)}$, whence f_n is well defined on $S^{(i)} - \{1, 2, 3, 4\}$ for $1 \leq i < n$.

To show f_n is well defined on $S^{(n)}$ requires another definition chase, this time to check

$$\phi_r f_S r_S \circ j_n = \phi_r \iota_-^{(0)} \phi_{n+1}^{-1} (\iota_-^{(n)})^{-1} \circ \phi_n f_T h^{-n+1}.$$

By [Definitions 3.8\(2\)](#), $j_n = r_S j^{-1} r_T h^{-n+1}$; therefore, simplifying the left-hand side above yields $\phi_r f_S j^{-1} r_T h^{-n+1}$. On the other hand, using the model description of $\iota_-^{(n)}$ from (2), the right-hand side above simplifies to $\phi_r \iota_-^{(0)} \phi_2^{-1} (\iota_-^{(1)})^{-1} f_T h^{-n+1}$. The reflection-invariance property of [Lemma 3.5](#) and an appeal to the case $i = 0$ establish the desired equation. \square

Corollary 3.11. *For $0 \leq i < n$, refer again by $F^{(i)}$ to the image of $\iota_+^{(i)}(F^{(i)}) \subset C(\Gamma_T^{(i)})$ under its inclusion into M_n , and refer by $F^{(n)}$ to the image of $\iota_-^{(n)}(F^{(n)})$. For each i , $F^{(i)}$ is totally geodesic in M_n and $f_n(S^{(i)} - \{1, 2, 3, 4\}) = F^{(i)}$.*

This follows immediately from [Proposition 3.10](#), since the maps $\iota_{\pm}^{(i)}$ are isometric embeddings. The following proposition describes an algebraic model for M_n .

Proposition 3.12. *For $n \in \mathbb{N}$, define $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(n)}, \bar{\Gamma}_S^{c^{-2n}} \rangle$. There is an isometry $M_n \rightarrow \mathbb{H}^3 / \Gamma_n$ which restricts on $C(\Gamma_S)$ and each $C(\Gamma_T^{(i)})$ to the natural map, and on $C(\bar{\Gamma}_S)$ to ϕ_{n+1} followed by the natural map.*

Proof. We first recall from [Lemma 2.4](#) that the plane \mathcal{H} with ideal boundary $\mathbb{R} \cup \{\infty\}$ projects to $\partial C(\Gamma_S)$ under the quotient map $\mathbb{H}^3 \rightarrow \mathbb{H}^3 / \Gamma_S$, so it is a component of $\partial \text{Hull}(\Gamma_S)$. Because the octahedron \mathcal{P}_1 is contained in $\text{Hull}(\Gamma_S)$ and all its ideal vertices have nonnegative imaginary part, it follows that

$$\text{Hull}(\Gamma_S) \subset \{z \in \mathbb{C} \mid \Im z \geq 0\} \cup \{\infty\}.$$

Similarly, from [Lemma 3.5](#) and the positioning of \mathcal{P}_2 we find that

$$\text{Hull}(\Gamma_T) \subset \{z \in \mathbb{C} \mid 0 \geq \Im z \geq -2\sqrt{2}\} \cup \{\infty\}.$$

Then, inspecting the action of c on $\mathbb{C} \cup \{\infty\}$, we find that for each $i \in \mathbb{N}$, any point of $\text{Hull}(\Gamma_T^{(i)})$ has imaginary part between $-2(i-1)\sqrt{2}$ and $-2i\sqrt{2}$ for $i \in \mathbb{N}$.

The following claim builds an inductive picture of a family of isometrically embedded, codimension-0 submanifolds of M_n with totally geodesic boundary.

Claim. *For $1 \leq i \leq n$, define $\Gamma_-^{(i)} = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(i)} \rangle$. There is an isometry $C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(i)}) \rightarrow C(\Gamma_-^{(i)})$, where the gluing maps for the domain are as in [Proposition 3.10](#), which restricts on $C(\Gamma_S)$ and each $C(\Gamma_T^{(j)})$, $j < i$, to the natural map. Furthermore:*

- (1) $\Lambda^{(i)} = \text{Stab}_{\Gamma_-^{(i)}}(c^{-2n}(\mathcal{H}))$, and the resulting natural map $F^{(i)} \rightarrow \partial C(\Gamma_-^{(i)})$ factors as $\iota_-^{(i)}$ followed by the natural map $C(\Gamma_T^{(i)}) \rightarrow C(\Gamma_-^{(i)})$.
- (2) $\text{Hull}(\Gamma_-^{(i)}) \subset \{z \in \mathbb{C} \mid \Im z \geq -i\sqrt{2}\} \cup \{\infty\}$.

Proof of claim. We will prove the claim by induction. If it holds for some $i < n$, then (1) and (2) above, together with the observations above the claim, imply that $\Gamma_-^{(i)}$ and $\Gamma_T^{(i+1)}$ meet cute along $c^{-2i}(\mathcal{H})$. Then, by [Lemma 3.2](#), the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_T^{(i+1)}) \rightarrow C(\Gamma_-^{(i+1)})$, where by the inductive hypothesis and the observation above [Proposition 3.10](#), the gluing map for the domain is $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ following the inverse of the natural map.

Furthermore, since $C(\Gamma_-^{(i)})$ has a unique totally geodesic boundary component, which is isometrically identified with $\partial_- C(\Gamma_T^{(i+1)})$ in the isometry to $C(\Gamma_-^{(i+1)})$ described above, the unique totally geodesic boundary component of $C(\Gamma_-^{(i+1)})$ is the isometric image of $\partial_+ C(\Gamma_T^{(i+1)})$. Therefore the observations above [Proposition 3.10](#) imply that this boundary component is the image of $\iota_-^{(i+1)}(F^{(i+1)})$ under the natural map. Assertion (1) of the claim thus follows for $\Gamma_-^{(i+1)}$. It follows that $\text{Hull}(\Gamma_-^{(i+1)})$ is entirely on one side or the other of the boundary at infinity of $c^{-2(i+1)}(\mathcal{H})$. Since $\Gamma_T^{(i+1)} < \Gamma_-^{(i+1)}$, assertion (2) now follows.

By our definition of “natural map” above [Lemma 3.2](#), the composition of the natural map $C(\Gamma_T^{(j)}) \rightarrow C(\Gamma_-^{(i)})$ with the natural map $C(\Gamma_-^{(i)}) \rightarrow C(\Gamma_-^{(i+1)})$ is itself natural, for $j \leq i$. Hence if the claim holds for $\Gamma_-^{(i)}$, $i < n$, it holds for $\Gamma_-^{(i+1)}$. The claim will therefore hold by induction if it is true in the base case $i = 1$. But this follows from the fact that Γ_S and $\Gamma_T^{(1)}$ meet cute along \mathcal{H} . This follows in turn from [Lemmas 2.4](#) and [3.5](#), which establish that $\Lambda^{(0)} = \text{Stab}_{\Gamma_S}(\mathcal{H}) = \text{Stab}_{\Gamma_T}(\mathcal{H})$, and the first paragraph of the proof. \square

Using the claim, it now follows that $\Gamma_-^{(n)}$ and $\bar{\Gamma}_S^{-2n}$ meet cute along $c^{-2n}(\mathcal{H})$; hence a final application of [Lemma 3.2](#) implies that the natural maps determine an isometry $C(\Gamma_-^{(n)}) \cup C(\bar{\Gamma}_S^{-2n}) \rightarrow C(\Gamma_n)$. Since each of $C(\Gamma_-^{(n)})$ and $C(\bar{\Gamma}_S^{-2n})$ has

a unique boundary component, $C(\Gamma_n)$ is boundaryless and hence equal to \mathbb{H}^3/Γ_n . The conclusion of the proposition follows. \square

The result below follows from [Proposition 3.12](#), or, really, its proof.

Corollary 3.13. *For fixed n and $0 \leq i \leq n$, define*

$$\Gamma_-^{(i)} = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(i)} \rangle, \quad \Gamma_+^{(i)} = \langle \Gamma_T^{(i+1)}, \dots, \Gamma_T^{(n)}, \bar{\Gamma}_S c^{-2n} \rangle.$$

Then $\Gamma_+^{(i)}$ and $\Gamma_-^{(i)}$ meet cuspally along $c^{-2i}(\mathcal{H})$ and the natural maps determine an isometry $C(\Gamma_-^{(i)}) \cup C(\Gamma_+^{(i)}) \rightarrow \mathbb{H}^3/\Gamma_n$. The isometry of [Proposition 3.12](#) factors through this map, so that the component of $M_n - F^{(i)}$ containing $C(\Gamma_S)$ is taken isometrically to its image in $C(\Gamma_-^{(i)})$.

In the remainder of the paper, we will frequently take the isometry above for granted and refer to the components obtained by splitting M_n along $F^{(i)}$ by $C(\Gamma_{\pm}^{(i)})$.

4. Invariants

4.1. Traces. If $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a discrete group, its trace field $\mathbb{Q}(\mathrm{tr} \Gamma)$ is obtained by adjoining to \mathbb{Q} the traces of elements of Γ . If the hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ has finite volume, Mostow rigidity implies that this is a topological invariant of M . It follows from the local rigidity theorems of Garland and Prasad that in this case the trace field is a number field; that is, a finite extension of \mathbb{Q} . The trace field is not generally an invariant of the commensurability class of M , however, and to obtain one we pass to the *invariant trace field* $k\Gamma$. This is obtained by adjoining to \mathbb{Q} the traces of squares of elements of Γ . When M is the complement of a link in a \mathbb{Z}_2 -homology sphere, its trace field and invariant trace field coincide (see [\[Maclachlan and Reid 2003\]](#)).

Proposition 4.1. *We have $k(\Gamma_S) = \mathbb{Q}(i)$, $k(\Gamma_T) = \mathbb{Q}(i\sqrt{2})$, and $k(M_n) = \mathbb{Q}(i, i\sqrt{2})$ for all $n \in \mathbb{N}$. In particular, M_n is not arithmetic for any $n \in \mathbb{N}$.*

Proof. Its definition in [Corollary 2.2](#) immediately implies $\Gamma_S < \mathrm{PSL}_2(\mathbb{Q}(i))$. The description in [Corollary 2.3](#), of Γ_{T_0} , and [Lemma 3.3](#) imply that $\Gamma_T < \mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$. Thus $k\Gamma_S \subseteq \mathbb{Q}(i)$, and $k\Gamma_T \subseteq \mathbb{Q}(i\sqrt{2})$. That equality holds is clear upon noting that $\mathrm{Tr}(h) = \pm i\sqrt{2}$ and $\mathrm{Tr}(t) = \pm(1+i)$. Since Γ_S and Γ_T are in Γ_n we have $\mathbb{Q}(i, i\sqrt{2}) \subseteq k(\Gamma_n)$. For the other containment we note that c from [Lemma 3.3](#) lies in $\mathrm{PSL}_2(\mathbb{Q}(i\sqrt{2}))$, and Γ_n is contained in the group generated by Γ_S , Γ_T , and c .

It is well known that any noncompact arithmetic manifold M has $k(M) \subset \mathbb{Q}(i\sqrt{d})$ for some $d \in \mathbb{N}$ (see, for example, [\[Maclachlan and Reid 2003, Theorem 8.2.3\]](#)), so M_n is not arithmetic. \square

We say $M = \mathbb{H}^3/\Gamma$ has *integral traces* if for each $\gamma \in \Gamma$, $\mathrm{tr} \gamma$ is an algebraic integer. Otherwise we say M_n has a nonintegral trace. M has integral traces if and

only if all manifolds commensurable to M do as well (see [Maclachlan and Reid 2003]).

Proposition 4.2. *For each n , M_n has integral traces.*

Proof. As in the proposition above, this follows from the fact that each Γ_n is contained in the group generated by Γ_S , Γ_T , and c . It is easy to see that the *entries* of the generators for Γ_S and Γ_{T_0} are algebraic integers. Since c has integral entries as well, all elements of Γ_n have integral entries, and hence integral traces. \square

Remark. Bass [1984] showed that if $M = \mathbb{H}^3 / \Gamma$ where Γ has an element with a nonintegral trace, there are closed essential surfaces in M associated to this trace. We say that such surfaces are *detected by the trace ring*. For fixed n and $1 \leq i \leq n$, closed essential surfaces in M_n can be obtained by “tubing” $S^{(i)}$ through $B^3 - L_-^{(i)}$. More precisely, let \mathcal{N}_i be a regular neighborhood of $L_-^{(i)}$ in $(B^3, L_-^{(i)}) \subset (S^3, L_n)$, let $A_i = \mathcal{N}_i \cap B^3 - \mathcal{N}_i$, and let

$$\hat{S}_i = \overline{S^{(i)} - (S^{(i)} \cap \mathcal{N}_i)} \cup A_i.$$

Then \hat{S}_i is a closed surface of genus 2 which is incompressible in M_n . We will show below that certain mutants have nonintegral traces, and one easily finds surfaces analogous to \hat{S}_i in the mutants. It is interesting to note that although these surfaces are present in all of these link complements, the trace ring does not detect any closed surfaces in the M_n .

4.2. Scissors congruence and the Bloch invariant. In Proposition 4.7 we will prove that the *Bloch invariant* distinguishes the commensurability class of M_m from that of M_n for $m \neq n$. This is an invariant of a polyhedral decomposition which by construction is invariant under *scissors congruence*: cutting the constituent polyhedra apart and reassembling them in new ways. Its deep connection to algebraic k -theory is what makes the Bloch invariant useful, though. For background and an account of the connection to scissors congruence we refer the reader to [Dupont 2001] and [Neumann 1998], our main source for the expository material here.

Definition 4.3. For a field $k \subset \mathbb{C}$, define the *pre-Bloch group* $\mathcal{P}(k)$ to be the quotient of the free \mathbb{Z} -module on $k - \{0, 1\}$ by all instances of the following relations:

- (4) $[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0, \quad x \neq y \in k - \{0, 1\},$
- (5) $[z] = \left[1 - \frac{1}{z}\right] = \left[\frac{1}{1-z}\right] = -\left[\frac{1}{z}\right] = -\left[\frac{z}{z-1}\right] = -[1-z], \quad z \in k - \{0, 1\}.$

There is a map $\delta: \mathcal{P}(k) \rightarrow k^* \wedge k^*$ given by $[z] \mapsto 2(z \wedge (1-z))$. (Here k^* is considered a \mathbb{Z} -module with multiplication as the group operation and \mathbb{Z} -action given by $a \cdot x = x^a$, $a \in \mathbb{Z}$.) The *Bloch group* is $\mathcal{B}(k) = \ker \delta$.

Remark. If k is algebraically closed, relation (4) above, called the *five term relation*, implies (5). For instance, taking \sqrt{z} and $\sqrt{z^{-1}}$ as x and y , respectively, in (4), then interchanging their roles and summing the results yields $[z] + [1/z] = 0$.

For any ideal tetrahedron T in \mathbb{H}^3 , there is an orientation-preserving isometry of \mathbb{H}^3 taking its ideal vertices to 0 , 1 , ∞ , and a complex number z with nonnegative imaginary part. Let the *cross ratio parameter* of T be $[z] \in \mathcal{P}(\mathbb{C})$. This is well defined because any other isometry answering the description above fixes z or replaces it by one of $1 - 1/z$ or $1/(1 - z)$.

For $k' \subset k$, inclusion induces a map $\mathcal{P}(k') \rightarrow \mathcal{P}(k)$. Although this is not injective in general, a theorem of Borel that we record below implies that if k' is a number field then $\mathcal{B}(k')$ does inject, modulo torsion. We offer this observation to excuse occasional imprecision about the precise location of our invariants.

Definition 4.4. Let $M = T_1 \cup \cdots \cup T_n$ be a triangulated complete, orientable hyperbolic 3-manifold of finite volume (with or without boundary); that is, with each T_i isometric to an ideal hyperbolic tetrahedron and $T_i \cap T_j$ either empty, an edge of each, or a face of each for $i \neq j$. Define the *Bloch invariant* of M as

$$\beta(M) = [z_1] + [z_2] + \cdots + [z_n] \in \mathcal{P}(\mathbb{C}),$$

where $[z_i]$ is the cross ratio parameter of T_i for each i in $\{1, \dots, n\}$.

Remark. If $\partial M = \emptyset$ then $\beta(M) \in \mathcal{B}(\mathbb{C})$ by a geometric interpretation of the Bloch invariant, and by work of Neumann and Yang [1999] it does not vary with triangulation.

We will obtain a triangulation of M_n by subdividing the decomposition below.

Lemma 4.5. *The members of $\mathcal{S} = \{\mathcal{P}_1, \mathcal{P}_2, c^{-1}\mathcal{P}_2, \dots, c^{-2n+1}\mathcal{P}_2, c^{-2n}r\mathcal{P}_1\}$ project under $\mathbb{H}^3 \rightarrow \mathbb{H}^3 / \Gamma_n$ to the cells of an ideal polyhedral decomposition of M_n .*

Proof. By Corollary 2.2, \mathcal{P}_1 projects under $\mathbb{H}^3 \rightarrow \mathbb{H}^3 / \Gamma_S$ to an ideal polyhedral decomposition of $C(\Gamma_S)$: it maps onto $C(\Gamma_S)$ with internal faces identified in pairs. Corollary 2.3 implies the same for $\mathcal{P}_2 \rightarrow C(\Gamma_{T_0})$ under $\mathbb{H}^3 \rightarrow \mathbb{H}^3 / \Gamma_{T_0}$, and hence also for $c^{-2}r\mathcal{P}_2 \rightarrow C(\bar{\Gamma}_{T_0}^{c^{-2}})$ (see the paragraph above Lemma 3.3).

It is easy to see that $r\mathcal{P}_2 = c\mathcal{P}_2$, for instance, by comparing sets of ideal vertices, so $c^{-2}r\mathcal{P}_2 = c^{-1}\mathcal{P}_2$. Therefore Lemma 3.3 implies that $\mathcal{P}_2 \cup c^{-1}\mathcal{P}_2$ projects to an ideal polyhedral decomposition of $C(\Gamma_T)$ under $\mathbb{H}^3 \rightarrow \mathbb{H}^3 / \Gamma_T$. In particular, this projection identifies the external faces of \mathcal{P}_2 that map to F' with external faces of $c^{-1}\mathcal{P}_2$ pairwise, since their images are fixed by the doubling involution $\phi_{c^{-2}r}$.

It follows from the above that $c^{-2(i-1)}\mathcal{P}_2 \cup c^{-2i+1}\mathcal{P}_2$ projects to a decomposition of $C(\Gamma_T^{(i)})$ for any $i \in \mathbb{N}$ (recall Definitions 3.9), and from the first paragraph that the same holds for $c^{-2n}r\mathcal{P}_1 \rightarrow C(\bar{\Gamma}_S^{c^{-2n}})$. By Proposition 3.12, it remains only to show that the gluings producing M_n preserve induced triangulations of boundaries. These

are defined in [Proposition 3.10](#). [Lemma 2.4\(3\)](#) implies that $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ preserves triangulations, and (3) does the same for $\iota_+^{(i)}(\iota_-^{(i)})^{-1}$ for $1 \leq i \leq n$. They combine to imply that the final map does as well. \square

Lemma 4.6. *M_n has Bloch invariant $\beta_1 - \bar{\beta}_1 + n\beta_2 \in \mathcal{B}(\mathbb{Q}(i, i\sqrt{2}))$ for any n , where $\beta_1 = 4[(1+i)/2] \in \mathcal{P}(\mathbb{Q}(i))$, $\bar{\beta}_1 = 4[(1-i)/2]$, and $\beta_2 \in \mathcal{P}(\mathbb{Q}(i\sqrt{2}))$.*

Proof. We will produce a triangulation of M_n by subdividing the polyhedral decomposition from [Lemma 4.5](#). \mathcal{P}_1 divides into a collection of 4 tetrahedra by the addition of a single edge γ joining the ideal vertices $(1+i)/2$ and ∞ , and four ideal triangular faces that share γ . One has ideal vertices $0, 1, \infty$, and $(1+i)/2$ and thus a parameter of $[(1+i)/2]$. Since the others are its image under rotation about γ they have identical cross ratio parameters. Their union projects to a triangulation of $C(\Gamma_S)$ with Bloch invariant $\beta_1 = 4[(1+i)/2]$.

Any ideal tetrahedron with its vertex set contained in that of \mathcal{P}_2 has cross ratio parameter in $\mathcal{P}(\mathbb{Q}(i\sqrt{2}))$, since \mathcal{P}_2 has ideal vertices in $\mathbb{Q}(i\sqrt{2}) \cup \{\infty\}$. We leave it to the reader to divide \mathcal{P}_2 into ideal tetrahedra in such a way that the resulting division of square faces, each into two ideal triangles, is preserved by the face-pairings that produce M_{T_0} . Such a triangulation projects to one of $C(\Gamma_{T_0})$, and its image under $c^{-2}r$ projects to one of $C(\Gamma_{T_0})$.

Above it is important to use $c^{-2}r$ and not c^{-1} , since the face pairings of $c^{-1}\mathcal{P}_2$ project it to $C(\bar{\Gamma}_{T_0}^{c^{-2}})$. Recall that r is a reflection, extending to \mathbb{C} as complex conjugation. One checks using (5) that if a tetrahedron has cross ratio parameter $[z]$ then its mirror image has parameter $[-\bar{z}]$. Since $\mathbb{Q}(i\sqrt{2})$ is preserved by complex conjugation, using the triangulations from the paragraph above gives $C(\Gamma_T)$ a Bloch invariant $\beta_2 \in \mathcal{P}(\mathbb{Q}(i\sqrt{2}))$.

For each i with $1 \leq i \leq n$, $C(\Gamma_T^{(i)})$ inherits a triangulation with Bloch invariant β_2 from $c^{-2(i-1)}\mathcal{P}_2 \cup c^{-2i+1}\mathcal{P}_2 = c^{-2(i-1)}(\mathcal{P}_2 \cup c^{-1}\mathcal{P}_2)$, and $C(\bar{\Gamma}_S)$ inherits one with invariant $\bar{\beta}_1$ from $r(\mathcal{P}_1)$. [Lemma 4.5](#) implies that these combine to triangulate M_n , so its Bloch invariant is as described above. \square

Below we record a standard formula for the *Bloch–Wigner dilogarithm function* $D_2: \mathbb{C} - \{0, 1\} \rightarrow \mathbb{R}$ in terms of the *dilogarithm*, $\psi(z) = \sum_{i=1}^{\infty} (z^i/i^2)$ (for $|z| < 1$):

$$D_2(z) = \Im \psi(z) + \log|z| \arg(1-z).$$

For z in the upper half plane, the ideal tetrahedron with ideal vertices $0, 1, \infty$, and z has volume $D_2(z)$; note also that $D_2(\bar{z}) = -D_2(z)$. D_2 determines a well-defined functional on $\mathcal{P}(\mathbb{C})$, and this in turn produces the *Borel regulator*, B_k .

Theorem [Borel 1977]. *For a number field k fix embeddings $\sigma_1, \dots, \sigma_{r_2}$ to \mathbb{C} , one representing each complex-conjugate pair. The map $B_k: \mathcal{P}(k) \rightarrow \mathbb{R}^{r_2}$ extending $[z] \mapsto (D_2(\sigma_1(z)), \dots, D_2(\sigma_{r_2}(z)))$ takes $\mathcal{B}(k)$ onto a lattice in \mathbb{R}^{r_2} , with kernel consisting entirely of torsion elements.*

We use the Borel regulator B_k to show that Bloch invariants distinguish the commensurability class of M_m from that of M_n for $m \neq n$.

Proposition 4.7. *For $m \neq n$, M_m is not commensurable with M_n .*

Remark. We thank the referee on an earlier version of this paper for describing the argument below. (Our original proof used cusp parameters; see [Lemma 4.18](#).)

Proof. It is clear that $k = \mathbb{Q}(i, i\sqrt{2})$ has two pairs of complex conjugate embeddings, each determined by its action on i and $i\sqrt{2}$. We will take $\sigma_1 = id_k$, and $\sigma_2(i) = i$, $\sigma_2(i\sqrt{2}) = -i\sqrt{2}$, in defining the Borel regulator B_k on k . Since each σ_i restricts on $\mathbb{Q}(i)$ to the identity, B_k takes each of β_1 and $-\bar{\beta}_1$ to $(v_1, v_1) \in \mathbb{R}^2$, where v_1 is the volume of \mathcal{P}_1 . On the other hand, $B_k(\beta_2) = 2(v_2, -v_2)$, where $v_2 = \text{vol}(\mathcal{P}_2)$.

For any n , a covering space $\tilde{M} \rightarrow M_n$ of degree k has $\beta(\tilde{M}) = k\beta(M_n)$. This is because the preimage in \tilde{M} of each tetrahedron T from the triangulation of M_n described in [Lemma 4.6](#) is a nonoverlapping union of k isometric copies of T . Thus if $\tilde{M} \rightarrow M_m$ with degree p and $\tilde{M} \rightarrow M_n$ with degree q it would follow that

$$p[\beta_1 - \bar{\beta}_1 + m\beta_2] = q[\beta_1 - \bar{\beta}_1 + n\beta_2].$$

Applying B_k to each side of the equation above, we find that since (v_1, v_1) and $(v_2, -v_2)$ are linearly independent in \mathbb{R}^2 we must have $p = q$ and $m = n$. \square

4.3. Cusp parameters. Following Neumann and Reid [[1992](#), §2.3], for a cusp of a complete hyperbolic 3-manifold M we will call the *cusp parameter* the complex modulus (or the conformal parameter) of a horospherical cusp cross section, a Euclidean torus. Thurston [[1979](#), Chapter 6] also used this invariant to distinguish hyperbolic manifolds.

Definition 4.8. Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, where $\Lambda \subset \mathbb{C}$ is a lattice. Define the *complex modulus* of T as $m(T) = \alpha/\beta$, where $\Lambda = \langle \alpha, \beta \rangle$.

Remark. The complex modulus is not really an invariant of a Euclidean torus; rather, it is an invariant of a particular basis for π_1 . However, we have:

Lemma 4.9. *The $\text{PGL}_2(\mathbb{Z})$ -orbit of the complex modulus is a similarity invariant of Euclidean tori. The $\text{PGL}_2(\mathbb{Q})$ -orbit is a commensurability invariant.*

Here we say T and T' are *commensurable* if T has a finite cover which is similar to a cover of T' .

Proof. The complex modulus is clearly scale-invariant.

Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, where $\Lambda = \langle \alpha, \beta \rangle$. For a different generating pair $\gamma = p\alpha + q\beta$, $\delta = r\alpha + s\beta$ the change-of-basis matrix

$$m = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \text{PSL}_2(\mathbb{Z})$$

has an inverse there as well, since α and β are linear combinations of γ and δ . Computing the modulus with γ and δ yields

$$\frac{p\alpha + q\beta}{r\alpha + s\beta} = \frac{p(\alpha/\beta) + q}{r(\alpha/\beta) + s} = m^T(m(T)).$$

If γ and λ generate a finite-index sublattice then, since they are linearly independent, m has a nonzero determinant. This implies the commensurability-invariance assertion. \square

It will prove useful here to understand the complex modulus of a torus by decomposing it into annuli using a family of parallel geodesics.

Definition 4.10. For a Euclidean annulus A with core of length ℓ and distance d between geodesic boundary components, let the *real modulus* of A be $m(A) = d/\ell$.

If $T = \mathbb{C}/\Lambda$, and $\Lambda = \langle \alpha, \beta \rangle$, then α and β determine isotopy classes of simple closed geodesics on T with representatives which intersect once. These are the projections to T of the line segments in \mathbb{C} joining 0 to α and β , respectively. Below let A_β denote the Euclidean annulus with geodesic boundary obtained as the path completion of the metric on $T - \beta$ inherited from T .

Lemma 4.11. *Let $T = \mathbb{C}/\Lambda$ be a Euclidean torus, and suppose α, β is a generating pair for Λ . Decompose $m(T)$ into real and imaginary parts:*

$$m(T) = \tau_\beta + i \cdot \mu_\beta,$$

where $\tau_\beta = \Re(\alpha/\beta)$ and $\mu_\beta = \Im(\alpha/\beta) \in \mathbb{R}$. Then $\tau_\beta = (\|\alpha\|/\|\beta\|) \cos \theta$, where θ is the angle between the geodesics α and β on T , and $|\mu_\beta| = m(A_\beta)$.

Proof. Write $\alpha = \|\alpha\|e^{i\theta_1}$ and $\beta = \|\beta\|e^{i\theta_2}$. Then $\theta = \theta_1 - \theta_2$ is the angle between the geodesics corresponding to α and β , and $\alpha/\beta = (\|\alpha\|/\|\beta\|)e^{i\theta}$. Writing $e^{i\theta} = \cos \theta + i \sin \theta$ yields the first assertion immediately.

To establish the second, consider the strip \tilde{A}_β in \mathbb{C} bounded by the line containing 0 and β and its translate by α , containing α and $\alpha + \beta$. The quotient of \tilde{A}_β induced by the action of β is the universal covering $\tilde{A}_\beta \rightarrow A_\beta$. The distance between boundary components of \tilde{A}_β is $\|\alpha\||\sin \theta|$, and the length of the core of A_β is the translation length of β , which is $\|\beta\|$. \square

Lemma 4.11 provides a convenient means for understanding the modulus of a Euclidean torus in terms of “Fenchel–Nielsen” coordinates (μ_β, τ_β) associated to a simple closed geodesic β . We regard μ_β as a length parameter for the annulus A_β , and τ_β as a twist parameter.

Lemma 4.12. *Suppose T is a Euclidean torus decomposed into annuli A_1, \dots, A_n by simple closed geodesics parallel to β . Then*

$$|\mu_\beta| = m(A_1) + m(A_2) + \dots + m(A_n).$$

Proof. By isotoping β if necessary, we may assume that it is one of the geodesics determining the A_i ; hence $A_\beta = A_1 \cup A_2 \cup \cdots \cup A_n$. Then if α_0 is an arc perpendicular to ∂A_β , joining one component to the other, for each i , $\alpha_0 \cap A_i$ is an arc perpendicular to ∂A_i joining one component to the other. This is because ∂A_i is parallel to β . Since $\ell(\alpha_0) = \sum_i \ell(\alpha_0 \cap A_i)$ and the core of each A_i has length $\ell(\beta)$, the result follows. \square

The annuli we are concerned with arise as horospherical cross sections of the cusps of M_S and M_T . Recall from [Lemma 2.4](#) that $\text{Stab}_{\Gamma_S}(\mathcal{H})$ is a group Λ generated by parabolic isometries p_1 , p_2 , and p_3 . Furthermore, as pointed out in Remark 1 on page [351](#), p_1 and p_3 are conjugate in Γ_S , as are p_2 and $p_4 = p_1 p_2 p_3^{-1}$. We asserted there that $C(\Gamma_S)$ has two cusps, one corresponding to p_1 and one to p_2 . This follows from the lemma below.

In what follows, we let $\mathcal{V}_1 = \{\infty, 0, 1, i, 1+i, (1+i)/2\}$, the set of ideal vertices of the ideal octahedron \mathcal{P}_1 . Let $\{h_v \mid v \in \mathcal{V}_1\}$ be a collection of horospheres invariant under the action of the symmetry group of \mathcal{P}_1 , such that h_v is centered at v for each $v \in \mathcal{V}_1$ and h_∞ is at height 2.

Lemma 4.13. *The projection to M_S of $\bigcup (h_v \cap \mathcal{P}_1)$ is a disjoint union of Euclidean annuli A_1 and A_2 with geodesic boundary, such that $p_S(A_1)$ is a horospherical cross section of the cusp of $C(\Gamma_S)$ corresponding to p_1 , $p_S(A_2)$ is a cross section of the cusp corresponding to p_2 , and $m(A_1) = 1$, $m(A_2) = 1/5$.*

Proof. Since h_∞ is at height 2 and our embedding of \mathcal{P}_1 is as in [Figure 3](#), $h_\infty \cap \mathcal{P}_1$ is a square with sides of length $1/2$. Since the symmetry group of \mathcal{P}_1 acts transitively on vertices, this holds for all $h_v \cap \mathcal{P}_1$, $v \in \mathcal{V}_1$. We will call a side of $h_v \cap \mathcal{P}_1$ *internal* if it is contained in an internal face of \mathcal{P}_1 and *external* otherwise. The face-pairing s has the property that if v and v' are ideal vertices of \mathcal{P}_1 and $s(v) = v'$, then $s(h_v) = h_{v'}$, and $s(h_v \cap \mathcal{P}_1)$ abuts $h_{v'} \cap \mathcal{P}_1$ along an internal side. The analogous property holds for t .

Each of s and t identifies a pair of internal faces of \mathcal{P}_1 , yielding M_S . The isometry p_S of [Corollary 2.2](#) is induced by the inclusion $\mathcal{P}_1 \rightarrow \mathbb{H}^3$. Since $p_1 = s^{-1}$ fixes the ideal vertex of \mathcal{P}_1 at 0, it identifies the opposite internal sides of $h_0 \cap \mathcal{P}_1$. This square thus projects to a cusp cross section A_1 of M_S , mapped by p_S to one of the cusp of $C(\Gamma_S)$ corresponding to p_1 . This is depicted on the left side of [Figure 8](#).

The other cusp cross section of M_S , the annulus A_2 , is the identification space of the collection

$$\{h_v \cap \mathcal{P}_1 \mid v \in \mathcal{V}_1 - \{0\}\},$$

shown on the right side of [Figure 8](#). In this figure, each square is the projection to M_S of $h_v \cap \mathcal{P}_1$ for the ideal vertex v , by which it is labeled. The combinatorics can be verified by considering the action of s and t on \mathcal{V}_1 .

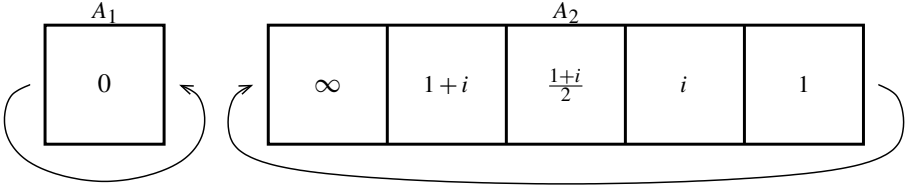


Figure 8. Cross sections of the cusps of M_S .

By assumption each square in [Figure 8](#) has side length $1/2$, and so the cores of A_1 and A_2 have lengths $1/2$ and $5/2$, respectively. For any square in [Figure 8](#), a vertical side projects to an arc joining the distinct boundary components of the corresponding A_i , hence the distance between them is $1/2$. Thus it follows directly from the definition that $m(A_1) = 1$ and $m(A_2) = 1/5$. \square

The following lemma describes the moduli of the cusps of $C(\Gamma_{T_0})$. We asserted below [Lemma 2.5](#) that $C(\Gamma_{T_0})$ has four cusps, one corresponding to each p_i , $i \in \{1, 2, 3, 4\}$, and each joining $\partial_- C(\Gamma_{T_0})$ to $\partial_+ C(\Gamma_{T_0})$. This follows from [Lemma 4.14](#). Let \mathcal{V}_2 be the set of ideal vertices of \mathcal{P}_2 , and consider a collection of horospheres $\{h_v \mid v \in \mathcal{V}_2\}$, invariant under the symmetry group of \mathcal{P}_2 , such that h_v is centered at v for each $v \in \mathcal{V}$ and h_∞ is at height 2.

Lemma 4.14. *The projection of $\bigcup (h_v \cap \mathcal{P}_2)$ to M_{T_0} is a collection of disjoint Euclidean annuli B_j with geodesic boundary, $j \in \{1, 2, 3, 4\}$, such that $p_{T_0}(B_j)$ is a cross section of the cusp of $C(\Gamma_{T_0})$ corresponding to $p_j \in \Lambda$, and $m(B_1) = m(B_3) = \sqrt{2}$ and $m(B_2) = m(B_4) = \sqrt{2}/5$.*

Proof. For $v \in \mathcal{V}_2$, we again call a side of $h_v \cap \mathcal{P}_2$ external if it is contained in an external face of \mathcal{P}_2 and internal otherwise. Each cusp cross section of M_{T_0} is the projection of a subcollection of the $h_v \cap \mathcal{P}_2$, identified along their internal faces. From [Figure 4](#), we find that $h_\infty \cap \mathcal{P}_2$ is a Euclidean rectangle with two opposite internal sides and two external. Since the symmetry group of \mathcal{P}_2 is transitive on its set of ideal vertices, this holds for the other h_v as well. It follows that each cusp cross section of M_{T_0} is a Euclidean annulus with geodesic boundary.

In [Figure 9](#), the lower rectangles of each annulus DB_j are labeled by vertices v such that $h_v \cap \mathcal{P}_2$ projects to a subrectangle of the cross section of the cusp of M_{T_0} whose image under p_{T_0} corresponds p_j . Then B_j is the lower half of DB_j . The reasons for this picture will become clear after the current proof.

The isometries f , g , and h defined in [Corollary 2.3](#) identify the internal sides of \mathcal{P}_2 in pairs, yielding the manifold M_{T_0} with totally geodesic boundary. The parabolic $p_1 = f^{-1}$ fixes 0, identifying the internal sides of \mathcal{P}_2 sharing this ideal vertex. Thus in M_{T_0} , B_1 consists of $h_0 \cap \mathcal{P}_2$ with its internal sides identified. The description of the p_i in terms of f , g , and h above [Lemma 2.4](#) shows that p_3 is a

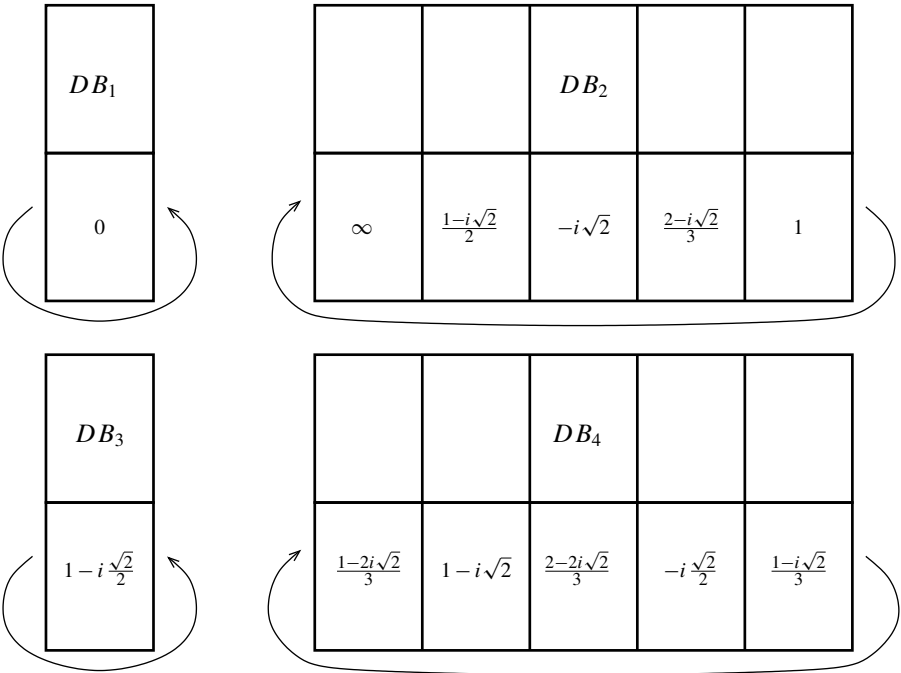


Figure 9. Cross sections of the cusps of M_T .

conjugate of g^{-1} . Since g^{-1} fixes $1 - i\sqrt{2}/2$, identifying the internal edges of \mathcal{P}_2 which about it, $h_{1-i\sqrt{2}/2}$ projects to B_3 in M_{T_0} . This justifies the depictions of B_1 and B_3 in Figure 9.

Since p_2 fixes ∞ , $h_\infty \cap \mathcal{P}_2$ projects to a subrectangle of B_2 . Since g takes the internal side Y_3 to Y'_1 and ∞ to $(1 - i\sqrt{2})/2$, in B_2 the projection of $h_\infty \cap \mathcal{P}_2$ meets the projection of $h_{(1-i\sqrt{2})/2} \cap \mathcal{P}_2$ along a side contained in the projection of Y_3 to M_{T_0} . Since the internal face of \mathcal{P}_2 meeting Y'_1 at $(1 - i\sqrt{2})/2$ is Y'_3 , and this is taken to Y'_2 by h^{-1} , the rectangle meeting the projection of $h_{(1-i\sqrt{2})/2}$ in B_2 on the internal side opposite its intersection with h_∞ is $h_{-i\sqrt{2}}$. Carrying this line of argument to completion yields the depictions of B_2 and B_4 in the figure.

From Figure 4, we find that the internal sides of $h_\infty \cap \mathcal{P}_2$ have length $\sqrt{2}/2$ and the external sides length $1/2$. Since the symmetry group of \mathcal{P}_2 is transitive on its ideal vertices, the same holds for each rectangle $h_v \cap \mathcal{P}_2$. Thus the cores of B_1 and B_3 have length $1/2$, and the cores of B_2 and B_4 have length $5/2$. For any square $h_v \cap \mathcal{P}_2$, an internal side projects to a perpendicular arc joining opposite sides of the cusp cross section in M_{T_0} containing $h_v \cap \mathcal{P}_2$. The moduli are thus as described. \square

By Corollary 3.4, $p_{T_0}: M_{T_0} \rightarrow C(\Gamma_{T_0})$ determines a reflection-invariant map from the double M_T of M_{T_0} across $\partial_+ M_{T_0}$ to $C(\Gamma_T)$. Furthermore, as we remarked

below [Lemma 2.5](#), each cusp of $C(\Gamma_{T_0})$ joins one component of $\partial C(\Gamma_{T_0})$ to the other. Therefore taking $DB_j \subset M_T$, $j \in \{1, 2, 3, 4\}$, to be the double of B_j across its component of intersection with $\partial_+ M_{T_0}$, we have:

Lemma 4.15. *For each $j \in \{1, 2, 3, 4\}$, the image in $C(\Gamma_T)$ of DB_j is a cross section of the cusp corresponding to p_j , and $m(DB_1) = m(DB_3) = 2\sqrt{2}$ and $m(DB_2) = m(DB_4) = 2\sqrt{2}/5$.*

It is a well-known consequence of the Margulis lemma that each cusp C of a hyperbolic manifold $M = \mathbb{H}^3 / \Gamma$ of finite volume is foliated by similar Euclidean tori, the projections to M of horospheres in \mathbb{H}^3 centered at the fixed point of a parabolic subgroup of Γ corresponding to C .

Definition 4.16. The *parameter* of a cusp C of a finite-volume complete hyperbolic manifold is the complex modulus of a horospherical cross section of C .

By [Lemma 4.9](#) the $\mathrm{PSL}_2(\mathbb{Z})$ orbit of the cusp parameter is an invariant of the *cusp shape*, the Euclidean similarity class of a cross section.

Proposition 4.17. *For $j = 1, 2$, let T_j be a cusp cross section of M_n containing the annular cusp cross section A_j of M_S (see [Lemma 4.13](#)). Then $m(T_1) = i(2 + 4n\sqrt{2})$, and $m(T_2)$ is $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to $m(T_1)$.*

Remarks. 1. It is not hard to show that $m(T_2) = i(2 + 4n\sqrt{2})/5$, but this is not necessary for our purposes and requires more work.

2. The cusps T_1 and T_2 are labeled in [Figure 1](#).

Proof. By [Proposition 3.10](#) $M_n = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\bar{\Gamma}_S)$, with gluing maps that factor through the inclusion-induced isometries $\iota_{\pm}^{(i)}$ defined on $F^{(i)}$ for $0 \leq i \leq n-1$, and final gluing $\phi_r \iota_-^{(0)} \phi_{n+1}^{-1} (\iota_-^{(n)})^{-1}: \partial_+ C(\Gamma_T^{(n)}) \rightarrow \partial C(\bar{\Gamma}_S)$.

For $j \in \{1, 2, 3, 4\}$ and $i \in \mathbb{N}$, define $DB_j^{(i)} = \phi_i \circ p_T(DB_j) \subset C(\Gamma_T^{(i)})$, with ϕ_i as in [Definitions 3.9](#). We also refer by $DB_j^{(i)}$ to its image in $C(\Gamma_n)$ under the natural map, or in M_n under inclusion. Let $\partial_{\pm} DB_j^{(i)} = DB_j^{(i)} \cap \partial_{\pm} C(\Gamma_T^{(i)})$.

By [Lemma 4.13](#), $p_S(A_1)$ is a cross section of the cusp of $C(\Gamma_S)$ corresponding to p_1 , and by [Lemma 4.15](#), $DB_1^{(1)}$ is a cross section of the cusp of $C(\Gamma_T)$ corresponding to p_1 . [Lemma 2.4](#) thus implies that $\iota_+^{(0)} (\iota_-^{(0)})^{-1}$ takes one component of $p_S(\partial A_1)$ to $\partial_- DB_1^{(0)}$. In [Remark 1](#) on page 351, we note that p_1 and p_3 are conjugate in Γ_S . It follows that the other component of $p_S(\partial A_1)$ is a cross section of the cusp of $\partial C(\Gamma_S)$ corresponding to p_3 , so $\iota_+^{(0)} (\iota_-^{(0)})^{-1}$ takes this component to $\partial_- DB_3^{(1)}$.

The doubling involution of M_T preserves DB_j by construction, exchanging its boundary components. Therefore by [Corollary 3.4](#), ϕ_{c-2r} preserves $p_T(DB_j)$ and exchanges boundary components. It follows that $\iota_+^{(i)} (\iota_-^{(i)})^{-1}$ takes $\partial_+ DB_j^{(i)}$ to $\partial_- DB_j^{(i+1)}$ for each i between 1 and $n-1$, upon recalling the identity (3):

$$\iota_+^{(i)} (\iota_-^{(i)})^{-1} = \phi_{i+1} \iota_+^{(0)} \phi_2^{-1} (\iota_-^{(1)})^{-1} \phi_i^{-1} = \phi_{i+1} \phi_{c-2r} \phi_i^{-1}.$$

One finds that $\phi_r \iota_-^{(0)} \phi_{n+1}^{-1} (\iota_-^{(n)})^{-1}$ takes $\partial_+ DB_1^{(n)} \sqcup \partial_+ DB_3^{(n)}$ to the components of $\phi_r \circ p_S(\partial A_1)$, arguing as above and applying (2). Therefore T_1 is decomposed by its intersection in M_n with the separating spheres $F^{(i)}$ into the following collection of Euclidean annuli with geodesic boundary:

$$p_S(A_1) \cup DB_1^{(1)} \cup \dots \cup DB_1^{(n)} \cup \phi_r \circ p_S(A_1) \cup DB_3^{(n)} \cup \dots \cup DB_3^{(1)}.$$

Similarly, we find that T_2 decomposes into the union of $p_S(A_2)$, $\phi_r \circ p_S(A_2)$, and $DB_j^{(i)}$ for $1 \leq i \leq n$ and $j = 2, 4$. We may take β_1 to be the geodesic $\partial_- DB_1^{(1)}$ on T_1 and $\beta_2 = \partial_- DB_2^{(1)} \subset T_2$. Then we obtain the following from Lemma 4.12, applying Lemmas 4.13 and 4.15:

$$\Re(m(T_1)) = \pm(2 + 4n\sqrt{2}), \quad \Re(m(T_2)) = \pm \frac{2 + 4n\sqrt{2}}{5}.$$

We will show $m(T_1)$ and $m(T_2)$ have real part equal to 0 by describing geodesics α_j , $j = 1, 2$, which meet the β_j once, perpendicularly. Let a_1 be the arc in A_1 which is the projection of the internal edges of $h_0 \cap \mathcal{P}_1$ (the vertical arcs on the left-hand square in Figure 8). Recall that the internal edges of $h_0 \cap \mathcal{P}_1$. In particular, $p_S(\partial a_1)$ is the intersection of $p_S(A_1)$ with the one-skeleton of the triangulation Δ_S defined below Corollary 2.2.

Let $b_1 \subset B_1$ and $b_3 \subset B_3$ similarly be projections of internal edges of $h_0 \cap \mathcal{P}_2$ and $h_{1-i\sqrt{2}/2} \cap \mathcal{P}_2$, respectively (see Figure 9), and let db_1 and db_3 be the geodesic arcs of DB_1 and DB_3 containing them. Let $db_j^{(i)} = \phi_i \circ p_T(db_j)$, and let $\partial_\pm db_j^{(i)} = db_j^{(i)} \cap \partial_\pm DB_j^{(i)}$, $j = 1, 3$ and $i \in \mathbb{N}$. Let Δ_T^- be the image of the triangulation $\Delta_{T_0}^-$ defined below Corollary 2.3 under the inclusion $M_{T_0} \rightarrow M_T$, and let Δ_T^+ be its image under the doubling involution of M_T . Then $\partial_\pm db_j^{(i)}$ is the intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i(\Delta_T^\pm)$.

By Lemma 2.4, $\iota_+^{(0)} (\iota_-^{(0)})^{-1}$ preserves triangulations, and the discussion above implies that the other gluing maps do as well. From Figure 5 it is apparent that the cusps of $F^{(0)}$ corresponding to p_1 and p_3 each contain only one end of an edge of the triangulation that $F^{(0)}$ inherits from the pictured fundamental domain \mathcal{F} . Therefore $\iota_0(\partial a_1) = \partial_- db_1^{(1)} \cup \partial_- db_3^{(1)}$. It then follows as before that

$$\alpha_1 = p_S(a_1) \cup db_1^{(1)} \cup \dots \cup db_1^{(n)} \cup \phi_r \circ p_S(a_1) \cup db_3^{(n)} \cup \dots \cup db_3^{(1)}$$

is a closed geodesic on T_1 which meets β_1 once, at right angles. Therefore by Lemma 4.11 $\Re(m(T_1)) = 0$, so $m(T_1) = i(2 + 4n\sqrt{2})$.

A similar argument will give $m(T_2)$. Let \mathcal{A}_2 be the collection of arcs in A_2 which are the projections of internal edges of the squares h_v , $v \in \mathcal{V}_1 - \{0\}$. From Figure 8, \mathcal{A}_2 consists of five arcs evenly spaced around A_2 , each joining one component of ∂A_2 to the other and perpendicular to ∂A_2 at each endpoint. For $j = 2, 4$, we define a collection of arcs $D\mathcal{B}_j \subset DB_j$ analogously, and take $D\mathcal{B}_j^{(i)} = \phi_i \circ p_T(D\mathcal{B}_j)$. Let

$\partial_{\pm} D\mathcal{B}_j^{(i)} = D\mathcal{B}_j^{(i)} \cap \partial_{\pm} DB_j^{(i)}$, and note that the points of $\partial_{\pm} D\mathcal{B}_j^{(i)}$ are the points of intersection of $\partial DB_j^{(i)}$ with the one-skeleton of $\phi_i(\Delta_T^{\pm})$.

For the same reasons as above, $i_+^{(0)}(i_-^{(0)})^{-1}$ takes $p_S(\partial\mathcal{A}_2)$ to $\partial_- D\mathcal{B}_2^{(1)} \cup \partial_- D\mathcal{B}_4^{(1)}$, and the other gluing maps take the $\partial_+ D\mathcal{B}_j^{(i)}$ to $\partial_- D\mathcal{B}_j^{(i+1)}$ for the appropriate i and j . Then the collection

$$p_S(\mathcal{A}_2) \cup D\mathcal{B}_2^{(1)} \cup \dots \cup D\mathcal{B}_2^{(n)} \cup \phi_r \circ p_S(\mathcal{A}_2) \cup D\mathcal{B}_4^{(n)} \cup \dots \cup D\mathcal{B}_4^{(1)}$$

consists of a disjoint union of up to five closed geodesics, each meeting β_2 perpendicularly in at most five points.

Fix a component α_2 of the collection above, let k be the intersection number of α_2 with β_2 , and let \tilde{T}_2 be the k -fold cover of T_2 dual to α_2 . Then β_2 lifts to \tilde{T}_2 , and any lift intersects the preimage $\tilde{\alpha}_2$ of α once, perpendicularly. Computing the modulus of \tilde{T}_2 using this pair, we obtain $\pm k \cdot i(2 + 4n\sqrt{2})/5$. This is $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to $m(T_1)$, so the result follows from [Lemma 4.9](#). \square

Lemma 4.18. *Suppose $z = i(m + n\sqrt{2})$ is $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to $z' = i(m + n'\sqrt{2})$, where $m, n, n' \in \mathbb{Q}$ and $m \neq 0$. Then $n' = \pm n$.*

Remark. Since commensurable hyperbolic manifolds have commensurable cusps, the collection of $\mathrm{PGL}_2(\mathbb{Q})$ -orbits of cusp parameters is a commensurability invariant (see [Lemma 4.9](#)). Thus [Proposition 4.17](#) and [Lemma 4.18](#) imply [Proposition 4.7](#).

Proof. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$ takes z to z' . After clearing denominators (which does not change the action by Möbius transformations), we may assume that $a, b, c, d \in \mathbb{Z}$. We have

$$\frac{ai(m + n\sqrt{2}) + b}{ci(m + n\sqrt{2}) + d} = i(m + n'\sqrt{2}).$$

Multiplying by the denominator on the left, and collecting the real and imaginary parts, we find

$$m(a - d) + (an - dn')\sqrt{2} = 0, \quad b + c(m^2 + 2nn') + cm(n' + n)\sqrt{2} = 0.$$

Since 1 and $\sqrt{2}$ are linearly independent over \mathbb{Q} , the left-hand equation above implies that $m(a - d) = 0$ and $an - dn' = 0$. Since $m \neq 0$, the first equation implies $a = d$. Then the second equation implies $n = n'$ unless $a = d = 0$. But in this case, $c \neq 0$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$. Hence, using the coefficient of $\sqrt{2}$ in the right-hand equation above, we find $n' = -n$. \square

5. Mutants

In the remaining sections, we will consider links obtained from L_n by mutation along the separating spheres $S^{(i)}$, $0 \leq i \leq n$, from [Definitions 3.8\(3\)](#).

Definition 5.1. For marked points $1, 2, 3, 4 \in S^2$, a *mutation* of $(S^2, \{1, 2, 3, 4\})$ is a mapping class of order 2 which acts on $\{1, 2, 3, 4\}$ by an even permutation.

Above a *mapping class* is the isotopy class, $\text{rel } \{1, 2, 3, 4\}$, of an orientation-preserving self-homeomorphism of the pair $(S^2, \{1, 2, 3, 4\})$. The set $\text{Mod}_{0,4}$ of such classes inherits the structure of a group from its bijection with the quotient of the group of orientation-preserving homeomorphisms by its identity component. See, for example, [Farb and Margalit 2012] for an introduction to the study of mapping class groups; here we need only the following fact on recognizing mutations using the symmetric group S_4 :

Proposition 5.2. *The homomorphism $\theta: \text{Mod}_{0,4} \rightarrow S_4$ that records the action on $\{1, 2, 3, 4\}$ takes the set of mutations bijectively to $\{(12)(34), (13)(24), (14)(23)\}$.*

Proof. Here we will embed S^2 as the unit sphere in \mathbb{R}^3 and take

$$\{1, 2, 3, 4\} = \left\{ \left(\frac{\pm 1}{\sqrt{2}}, \frac{\pm 1}{\sqrt{2}}, 0 \right), \left(\frac{\pm 1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}, 0 \right) \right\}.$$

The definitions imply that θ takes any mutation into the subset of S_4 listed above, and the 180-degree rotations m_x , m_y , and m_z in the three coordinate axes of \mathbb{R}^3 determine mutations of $(S^2, \{1, 2, 3, 4\})$ taken by θ to each of its distinct elements.

The kernel of θ is the *pure mapping class group* $\text{PMod}_{0,4}$. This group is free on two generators: Dehn twists in essential simple closed curves $\alpha, \beta \subset S^2 - \{1, 2, 3, 4\}$ that intersect exactly twice. See the beginning of [Farb and Margalit 2012, §4.2.4] for a proof of this fact, and for the definition of a Dehn twist see [Farb and Margalit 2012, §3.1.1]. With S^2 as above we can take α to be its intersection with the xz -plane and β the intersection with the yz -plane; then it is clear that each of m_x , m_y , and m_z takes each of α and β to itself. It follows that m_x , m_y , and m_z centralize $\text{PMod}_{0,4}$ (see [Farb and Margalit 2012, Fact 3.8]).

For an arbitrary mutation $m \in \text{Mod}_{0,4}$ we have $\theta(m) = \theta(m_x)$, $\theta(m) = \theta(m_y)$, or $\theta(m) = \theta(m_z)$. Assuming (without loss of generality) that the first case holds, it follows that $m = m_x h$ for some $h \in \text{PMod}_{0,4}$. Since m has order 2 we have:

$$id = m^2 = (m_x h)^2 = m_x^2 h^2 = h^2.$$

Thus since $\text{PMod}_{0,4}$ is a free group, $h = id$ and $m = m_x$. □

It is easy to see that every mutation of $(S^2, \{1, 2, 3, 4\})$ is isotopic to the identity as a self-homeomorphism of S^2 , so cutting S^3 along a smoothly embedded copy and regluing by a mutation recovers S^3 . This motivates:

Definition 5.3. For a link $L \subset S^3$ and a smoothly embedded two-sphere $S \subset S^3$ intersecting L in four points, let B^\pm be the closures of the components of $S^3 - S$ and $T^\pm = L \cap B^\pm$. For a mutation m of $(S, S \cap L)$, we define $(S^3, L') = (B^-, T^-) \cup_m (B^+, T^+)$ and say L' is *obtained from L by mutation along S* .

The lemma below describes the change in projection from L to a link L' obtained from it by mutation. Below we refer to mutations by their images under θ .

Lemma 5.4. *For a link L projected to \mathbb{R}^2 , if a two-sphere $S \subset S^3$ intersects \mathbb{R}^2 in a vertical line and L in four points, label them 2, 3, 4, and 1, reading top-to-bottom. The link obtained by the mutation (13)(24) (respectively, (12)(34)) along S has projection obtained by cutting L along S and inserting the braid on the left (resp. right) of Figure 10.*

Proof. We may assume L is arranged so that there is an axis in \mathbb{R}^2 intersecting S perpendicularly, midway between points 3 and 4 and so that points 2 and 1 are also equidistant from it. The 180-degree rotation in this axis restricts on S to an involution acting on the marked points by the permutation (12)(34).

There is a homeomorphism $R: S \times I \rightarrow S \times I$, which preserves slices $S \times \{t\}$ and restricts on each to rotation by $-180 \cdot t$ degrees in the horizontal axis. This interpolates between the identity, on $S \times \{0\}$, and the inverse of (12)(34) on $S \times \{1\}$, although it does not preserve marked points for $0 < t < 1$.

Let (B^\pm, T^\pm) be as in Definition 5.3, and let C be a collar of S in B^- that is small enough that it intersects T^- in the collection of horizontal arcs $\{\{j\} \times I \mid j \in \{1, 2, 3, 4\}\}$. There is a homeomorphism $h: B^- \cup_{(12)(34)} B^+ \rightarrow S^3$ defined as the identity on B^+ and the complement of C in B^- , and as R on C . By the definition of R , the image of $T^- \cap C$ under R is as pictured on the right-hand side of Figure 10, thus the image of L' in S^3 under h is as stated in the lemma.

Note that the braid on the left-hand side of Figure 10 is the conjugate of the braid on the right by a left-handed half-twist exchanging the points 2 and 3. This reflects the fact that the conjugate of (12)(34), by any homeomorphism of $(S, \{1, 2, 3, 4\})$ which exchanges 2 and 3 and fixes 1 and 4, is a mapping class of order 2 acting on the marked points as (13)(24); hence such a conjugate is (13)(24). The conjugating braid in Figure 10 tracks the marked points under an isotopy $S \times I \rightarrow S$ taking the simplest such conjugator to the identity. The conclusion for (13)(24) thus follows as it did above for (12)(34). \square

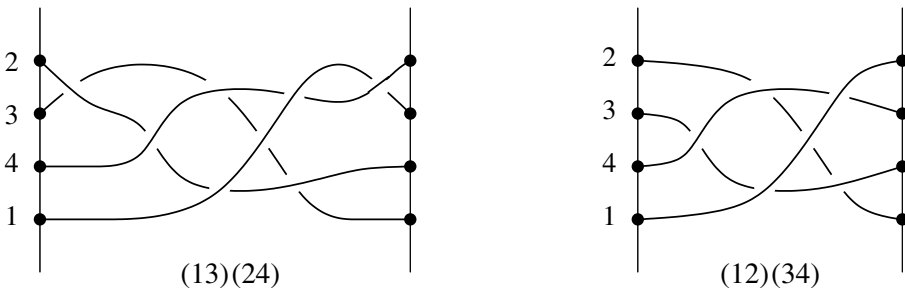


Figure 10. The mutations as braids.

The numbering of marked points from [Lemma 5.4](#) and [Figure 10](#) agrees with the numbering of the $S^{(i)} \cap L_n$ from [Definitions 3.8\(3\)](#). This, in turn, was chosen to agree with the numbering of parabolics of Λ from [Lemma 2.4](#). To be more precise:

Let S be the sphere obtained by compactifying each cusp of $F^{(0)} = \mathcal{H}/\Lambda$ with a single point. Label each new point with a number between 1 and 4, according to the parabolic p_i corresponding to the cusp it compactifies. With the points of $S^{(0)} \cap L_n$ numbered as in [Definitions 3.8\(3\)](#), it follows from [Proposition 2.8](#) that the restriction of f_T (as in [Proposition 3.7](#)) to $S^{(0)} - T$ extends to a map $S^{(0)} \rightarrow S$ that preserves numbering. [Corollary 3.11](#) and the definition of $F^{(i)}$ (see [Definitions 3.9](#)) now imply that for each i between 0 and n , $\phi_{c^{2i}} \circ \iota_i^{-1} \circ f_n$ extends to a homeomorphism $S^{(i)} \rightarrow S$ that takes marked points to marked points preserving numbering.

By [\[Ruberman 1987, Theorem 2.2\]](#), each mutation of $(S, \{1, 2, 3, 4\})$ is *realized* by an isometry of $F^{(0)}$; that is, there exists an isometry of $F^{(0)}$ whose extension to $(S, \{1, 2, 3, 4\})$ represents the mutation mapping class. The following lemma identifies lifts to $\mathrm{PSL}_2(\mathbb{R})$ of the isometries realizing (13)(24) and (12)(34).

Lemma 5.5. *Define*

$$m_1 = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & \sqrt{5} \\ -1/\sqrt{5} & 0 \end{pmatrix}.$$

Each of m_1 and m_2 normalizes Λ (from [Lemma 2.4](#)), and the induced isometries ϕ_{m_1} and ϕ_{m_2} of $F^{(0)}$ realize (13)(24) and (12)(34), respectively.

Proof. Since each of m_1 and m_2 has trace equal to zero, it has order 2 in $\mathrm{PSL}_2(\mathbb{C})$. Their actions by conjugation on the generators p_1 , p_2 , and p_3 for Λ defined above [Lemma 2.4](#) are given by

$$p_1^{m_1} = p_3^{-1}, \quad p_2^{m_1} = p_4^{-1}, \quad p_1^{m_2} = p_2, \quad p_3^{m_2} = p_1^{-1},$$

as may be verified by direct computation. Here $p_4 = p_1 p_2 p_3^{-1}$ is as described in [Remark 1](#) on page [351](#). Therefore m_1 and m_2 normalize Λ and induce isometries ϕ_{m_1} and ϕ_{m_2} , respectively, of $F^{(0)} = C(\Lambda)$.

Each of ϕ_{m_1} and ϕ_{m_2} has order 2, since m_1 and m_2 have order 2, and their extensions to S act on the set of marked points as described in the statement of the lemma. Its conclusion therefore follows from [Proposition 5.2](#). \square

Corollary 5.6. *For $j = 1, 2$ and $i \in \mathbb{Z}$, let $m_j^{(i)} = c^{-2i} m_j c^{2i}$. Each of $m_1^{(i)}$ and $m_2^{(i)}$ normalizes $\Lambda^{(i)}$ (from [Definitions 3.9](#)), and the induced isometries of $F^{(i)}$ realize (13)(24) and (12)(34), respectively.*

[Lemma 5.4](#) gives a prescription for describing links obtained from L_n by the mutations (13)(24) and (12)(34). The result below describes hyperbolic manifolds to which their complements are homeomorphic, analogous to [Proposition 3.10](#).

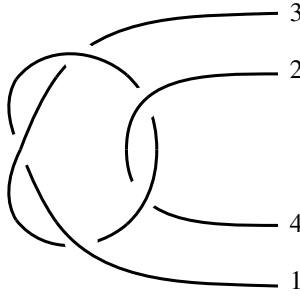


Figure 11. The tangle S admits an order-2 rotational symmetry which restricts to the mutation (13)(24) on its boundary.

Proposition 5.7. For $I = (a_0, a_1, \dots, a_n) \in \{0, 1, 2\}^{n+1}$, let L_I be the link obtained from L_n by the following prescription: for $0 \leq i \leq n$, if $a_i = 0$, do not mutate along $S^{(i)}$; if $a_i = 1$, mutate by (13)(24); and if $a_i = 2$, mutate by (12)(34). Let $M_I = C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\bar{\Gamma}_S)$, where for each i such that $a_i = 0$ the gluing is as in [Proposition 3.10](#), and otherwise is given by

$$\begin{aligned} & \iota_+^{(i)} \phi_{m_j^{(i)}} (\iota_-^{(i)})^{-1} && \text{for } 0 \leq i < n, \text{ where } a_i = j \in \{1, 2\}, \text{ and} \\ & \phi_r \iota_-^{(0)} \phi_{c^{2n} m_j^{(n)}} (\iota_-^{(n)})^{-1} && \text{if } a_n = j \in \{1, 2\}. \end{aligned}$$

Then there is a homeomorphism $f_I: S^3 - L_I \rightarrow M_I$ whose restriction to each complementary component of the collection $\{S^{(i)}\}$ agrees with that of f_n .

[Proposition 5.7](#) follows immediately from [Proposition 3.10](#) and [Corollary 5.6](#). Below we note a couple of “obvious” isometry relations on the $\{M_I\}$.

Lemma 5.8. For fixed $(a_1, \dots, a_n) \in \{0, 1, 2\}^n$ let $I_0 = (0, a_1, \dots, a_n)$ and $I_1 = (1, a_1, \dots, a_n)$. M_{I_0} is isometric to M_{I_1} .

Proof. It is evident from [Figure 11](#) that the mutation (13)(24) extends to a homeomorphism on $B^3 - S$. Thus (S^3, L_{I_0}) is homeomorphic to (S^3, L_{I_1}) , and the result follows from Mostow rigidity. \square

Lemma 5.9. For $I = (a_0, a_1, \dots, a_n) \in \{0, 1, 2\}^{n+1}$, let $\bar{I} = (a_n, a_{n-1}, \dots, a_0)$. There is an orientation-reversing isometry $M_I \rightarrow M_{\bar{I}}$ that, for each $i \in \{1, \dots, n\}$, takes the image of $C(\Gamma_T^{(i)})$ in M_I to the image of $C(\Gamma_T^{(n-i)})$ in $M_{\bar{I}}$.

Proof. It is straightforward to check that the braids in [Figure 10](#) are isotopic (in $S^2 \times I$) to their mirror images. Therefore, there is an orientation-reversing homeomorphism $L_I \rightarrow L_{\bar{I}}$. By composing this homeomorphism with f_I^{-1} and $f_{\bar{I}}$ we get a homeomorphism $M_I \rightarrow M_{\bar{I}}$. The result follows by Mostow rigidity. \square

Below we describe the change effected at the level of Kleinian groups by cutting a hyperbolic manifold along an embedded, separating totally geodesic surface and regluing by an isometry.

Lemma 5.10. *Suppose Γ_0 and Γ_1 meet cute along a plane \mathcal{H} , and take $\Theta = \Gamma_0 \cap \Gamma_1$, $E = \mathcal{H}/\Theta$, and ι_0 and ι_1 as in [Lemma 3.2](#). If n normalizes Θ and preserves components of $\mathbb{H}^3 - \mathcal{H}$, then $\langle \Gamma_0, \Gamma_1^n \rangle$ is a Kleinian group, and there is an isometry*

$$C(\Gamma_0) \cup_{\iota_1 \phi_n^{-1} \iota_0^{-1}} C(\Gamma_1) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$$

which restricts on $C(\Gamma_0)$ to the natural map, and on $C(\Gamma_1)$ to $\phi_n: C(\Gamma_1) \rightarrow C(\Gamma_1^n)$ followed by the natural map.

Proof. Since n normalizes Θ , it preserves \mathcal{H} ; hence our hypotheses ensure that Γ_0 and Γ_1^n meet cute along \mathcal{H} , and [Lemma 3.2](#) applies. Thus $\langle \Gamma_0, \Gamma_1^n \rangle$ is a Kleinian group, and in particular, the natural maps $C(\Gamma_0) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$ and $C(\Gamma_1^n) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle)$ determine an isometry

$$C(\Gamma_0) \cup_{n\iota_1 \iota_0^{-1}} C(\Gamma_1^n) \rightarrow C(\langle \Gamma_0, \Gamma_1^n \rangle).$$

Here we are using $n\iota_1: E \rightarrow C(\Gamma_1^n)$ to refer to the natural map. It is now an exercise in definition-chasing to show that $n\iota_1 \circ \phi_n = \phi_n \circ \iota_1$, whence the map

$$C(\Gamma_0) \cup_{\iota_1 \phi_n^{-1} \iota_0^{-1}} C(\Gamma_1) \rightarrow C(\Gamma_0) \cup_{n\iota_1 \iota_0^{-1}} C(\Gamma_1^n),$$

defined as the identity on $C(\Gamma_0)$ and ϕ_n on $C(\Gamma_1)$, is well defined. The lemma follows. \square

Since m_1 and m_2 have order 2, $\phi_{m_i} = \phi_{m_i}^{-1}$ for $i = 1, 2$. [Lemma 5.10](#) thus yields the result below, which describes how the algebraic model for M_n from [Proposition 3.12](#) changes under mutation.

Proposition 5.11. *For $I = (a_0, a_1, \dots, a_n) \in \{0, 1, 2\}^{n+1}$ let $q_{i+1} = m_{a_0}^{(0)} \cdots m_{a_i}^{(i)}$ for $0 \leq i \leq n$, with $m_0^{(j)} := id$, and $m_1^{(j)}$ and $m_2^{(j)}$ as in [Corollary 5.6](#) for every j . Define*

$$\Gamma_I = \langle \Gamma_S, (\Gamma_T^{(1)})^{q_1}, \dots, (\Gamma_T^{(n)})^{q_n}, (\bar{\Gamma}_S^{-2n})^{q_{n+1}} \rangle.$$

There is an isometry $M_I \rightarrow C(\Gamma_I)$ that restricts on $C(\Gamma_S)$ to the natural map, and on $C(\Gamma_T^{(i)})$ to $\phi_{q_i} \circ \phi_i$ followed by the natural map, for $1 \leq i \leq n$.

The proof of [Proposition 5.11](#) follows the inductive approach of that of [Proposition 3.12](#), but at each stage appeals to [Lemma 5.10](#) for instructions on how to change the construction. We will not write the details, as it is very similar.

6. Commensurable mutants

Here we show that M_n is commensurable to each of its mutants by (13)(24) and with this fact classify the M_I up to isometry for $I \in \{0, 1\}^{n+1}$, proving [Theorem 2](#). In the process, we show that our polyhedral decomposition of M_n is “canonical” in the sense of [[Goodman et al. 2008](#), §2]; that is, produced by a construction of Epstein and Penner [[1988](#)]. This allows us to identify the commensurator for Γ_n and the minimal orbifold quotient of M_n . In practice, it is a challenge to find Epstein–Penner decompositions, commensurators, and commensurator quotients. Infinite families where these are known are rare.

Below, let \mathcal{B}_0 be the open half-ball in \mathbb{H}^3 bounded by the Euclidean hemisphere of unit radius centered at $0 \in \mathbb{C}$, and let $\mathcal{B}_j = c^{-j}(\mathcal{B}_0)$, where c is as defined in [Lemma 3.3](#). Recall that we have defined \mathcal{H} as the geodesic hyperplane of \mathbb{H}^3 with ideal boundary $\mathbb{R} \cup \{\infty\}$. If w and z are complex numbers, we will take $w\mathcal{H} + z$ to be the hyperplane with ideal boundary $(w\mathbb{R} + z) \cup \{\infty\}$.

Definitions 6.1. (1) Let f_0 be obtained by first reflecting in $i\mathcal{H}$ and then in $i\mathcal{H} + 1/2$.
 (2) Let b_0 be obtained by first reflecting in $\mathcal{H} + i/2$ and then in $\partial\mathcal{B}_0$.
 (3) For $j \geq 0$, let a_j be obtained by reflecting in $i\mathcal{H} + 1/2$ and then in $\partial\mathcal{B}_j$.

Since $i\mathcal{H}$ and $i\mathcal{H} + 1/2$ are parallel and share the ideal point ∞ , f_0 is a parabolic isometry fixing ∞ . $\mathcal{H} + i/2$ meets $\partial\mathcal{B}_0$ at an angle of $\pi/3$, so b_0 is an elliptic isometry of order 3 rotating around the geodesic of intersection. For the same reason, a_i is elliptic of order 3, rotating around the geodesic $i\mathcal{H} + 1/2 \cap \partial\mathcal{B}_i$, for each $i \geq 0$.

Lemma 6.2. *Let G_n be the group generated by reflections in the face of P_n , where*

$$P_n = \{(z, t) \in \mathbb{H}^3 \mid 0 \leq \Re(z) \leq 1/2, -n\sqrt{2} \leq \Im(z) \leq 1/2\} - \bigcup_{k=0}^n \mathcal{B}_k.$$

Then G_n contains a_i for $0 \leq i \leq 2n$, as well as f_0 and b_0 , and $O_n \doteq \mathbb{H}^3/G_n$ is a one-cusped hyperbolic orbifold.

Proof. By its definition, P_n is cut out by $\mathcal{H} + i/2$, $i\mathcal{H}$, $i\mathcal{H} + 1/2$, $\mathcal{H} - n \cdot i\sqrt{2}$, and the $\partial\mathcal{B}_k$, $0 \leq k \leq n$. It is not hard to show directly that the dihedral angle between any two of these planes that intersect is an integer submultiple of π , whence by the Poincaré polyhedron theorem G_n is discrete and \mathbb{H}^3/G_n is an orbifold isometric to P_n with mirrored sides (see [[Ratcliffe 1994](#), Theorem 13.5.1]). In particular, since P_n has a single ideal point \mathbb{H}^3/G_n has one cusp.

One finds that G_n contains f_0 , b_0 , and the a_i , for $0 \leq i \leq n$, by direct appeal to [Definitions 6.1](#). It remains to establish that G_n contains a_i for $n < i \leq 2n$. Note that $\mathcal{H} - n \cdot i\sqrt{2}$ is the image of \mathcal{H} under c^{-n} , so reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is given

by $c^{-n}rc^n$, where r is the reflection through \mathcal{H} . By the property of r observed above [Lemma 3.3](#), conjugating an element $x \in \mathrm{PSL}_2(\mathbb{C})$ by reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ gives

$$(6) \quad c^{-n}rc^n \times c^{-n}rc^n = c^{-2n}\bar{x}c^{2n}.$$

We further observe that c conjugates a_i to a_{i-1} for $i \geq 1$, since $c(i\mathcal{H} + 1/2) = i\mathcal{H} + 1/2$ and $c(\mathcal{B}_i) = \mathcal{B}_{i-1}$, and we note that $\bar{a}_0 = a_0$. Thus:

$$(7) \quad \overline{c^i a_i c^{-i}} = \bar{a}_0 = a_0 = c^i a_i c^{-i} \implies c^{-2i} \bar{a}_i c^{2i} = a_i.$$

For $0 \leq i \leq n$, it follows that the conjugate of a_i by reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ is

$$c^{-2n} \bar{a}_i c^{2n} = c^{-2(n-i)} a_i c^{2(n-i)} = a_{2n-i} \in G_n.$$

Therefore G_n contains a_i for $n \leq i \leq 2n$ as well, and the lemma is proved. \square

Since \mathcal{H} meets both $\partial\mathcal{B}_0$ and $i\mathcal{H} + 1/2$ at right angles, it does the same for the fixed geodesic of a_0 and is therefore preserved by a_0 . In fact, the following description of $a_0 \in \mathrm{PSL}_2(\mathbb{C})$ is easily obtained from its definition:

$$a_0 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

In particular, a_0 acts on the ideal points of $\mathcal{P}_1 \cap \mathcal{P}_2$ by $0 \mapsto 1 \mapsto \infty \mapsto 0$. Similarly, it is easy to see that $f_0(z, t) = (z + 1, t)$.

Then the face pairings f (defined in [Corollary 2.3](#)) and s (defined in [Corollary 2.2](#)), which are equal, are obtained from f_0 by conjugating by a_0 :

$$(8) \quad s = f = a_0 f_0 a_0^{-1}.$$

One may use similar analyses to establish the following:

$$(9) \quad t = f_0 a_0 b_0, \quad g = (a_0^{-1} a_1) f_0^{-1} (a_0^{-1} a_1)^{-1}, \quad h = a_1 a_0 f_0^{-1} a_1.$$

The main group-theoretic fact of this section extends these observations.

Proposition 6.3. *For each $n \in \mathbb{N}$, G_n contains Γ_n and $m_1^{(i)}$ for $0 \leq i \leq n$.*

Proof. We recall from [Proposition 3.12](#) that $\Gamma_n = \langle \Gamma_S, \Gamma_T^{(1)}, \dots, \Gamma_T^{(n)}, \bar{\Gamma}_S^{c^{-2n}} \rangle$, where by [Definitions 3.9\(2\)](#), $\Gamma_T^{(i)} \doteq \Gamma_T^{c^{-2(i-1)}}$ for each i between 1 and n . Furthermore, by [Lemma 3.3](#), $\Gamma_T = \langle \Gamma_{T_0}, \bar{\Gamma}_{T_0}^{c^{-2}} \rangle$.

It is a direct consequence of the descriptions (8) and (9) that $\Gamma_S < G_n$ and $\Gamma_{T_0} < G_n$. Furthermore, since f_0 commutes with c and $\bar{f}_0 = f_0$, (7) implies, for instance, that

$$c^{-2} \bar{f} c^2 = c^{-2} (\bar{a}_0 \bar{f}_0 \bar{a}_0^{-1}) c^2 = a_2 f_0 a_2^{-1} \in \Gamma_n,$$

since $\bar{a}_0 = a_0$ and $c^{-2} a_0 c^2 = a_2$. Using the same strategy, we find

$$c^{-2} \bar{g} c^2 = (a_2^{-1} a_1) f_0^{-1} (a_2^{-1} a_1)^{-1} \in G_n \quad \text{and} \quad c^{-2} \bar{h} c^2 = a_1 a_2 f_0^{-1} a_1 \in G_n.$$

Thus G_n contains $\Gamma_T = \Gamma_T^{(1)}$. Since conjugation by c^{-1} takes a_i to a_{i+1} , and $a_i \in \Gamma_n$ for each i between 0 and $2n$, it follows from the descriptions above and in (8) and (9) that $\Gamma_T^{(i)} < G_n$, for $1 \leq i \leq n$. Finally the relation (6) immediately implies that $\bar{\Gamma}_S^{c^{-2n}} < G_n$, and we have established that $\Gamma_n < G_n$.

To show that G_n contains the elements $m_1^{(j)}$ for each j between 0 and n , we observe that the element obtained by reflecting first in $\partial\mathcal{B}_0$ and then in $i\mathcal{H}$ is the rotation of order 2 described by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is well known to generate $\mathrm{PSL}_2(\mathbb{Z})$, along with a_0 . Since $m_1 \in \mathrm{PSL}_2(\mathbb{Z})$, it follows that $m_1 \in G$.

We note that c^{-2j} preserves $i\mathcal{H}$ and takes \mathcal{B}_0 to \mathcal{B}_{2j} , and that \mathcal{B}_{2j} intersects P_n , for $j \leq n/2$, and intersects its image under reflection in $\mathcal{H} - n \cdot i\sqrt{2}$ for $n/2 \leq j \leq n$. Thus for each j between 0 and n , the rotation obtained by reflecting first in $\partial\mathcal{B}_{2j}$ and then in $i\mathcal{H}$ is contained in G_n . If m_1 is expressed as a word in the two elements described in the paragraph above, then $c^{-2j}m_1c^{2j}$ is expressed as the same word in a_{2j} and the rotation obtained from $\partial\mathcal{B}_{2j}$ as above. The lemma follows. \square

It is now easy to prove the first part of Theorem 2, that the complement of each link obtained from L_n using only the mutation (13)(24) is commensurable to M_n .

Proposition 6.4. *M_n branched covers O_n , as does M_I for any $I \in \{0, 1\}^{n+1}$. Hence these are commensurable.*

Proof. Since G_n is a discrete reflection group, it is enough to show that $\Gamma_I \subset G_n$. This is immediate from Propositions 5.11 and 6.3. \square

To finish the proof of Theorem 2 we need an isometry classification of the link complements that fall under the purview of Proposition 6.4. Our first step is to show that G_n is the commensurator of Γ_n .

The *commensurator* of a Kleinian group Γ is the group

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) \mid [\Gamma : g\Gamma g^{-1}] < \infty\}.$$

It follows easily from the definition that since Γ_n is a finite-index subgroup of G_n , G_n is contained in $\mathrm{Comm}(\Gamma_n)$. Since Γ_n is nonarithmetic (by Proposition 4.1), by a famous theorem of Margulis [1991, (1) Theorem] $\mathrm{Comm}(\Gamma_n)$ is discrete.

Let O'_n be the hyperbolic orbifold $\mathbb{H}^3 / \mathrm{Comm}(\Gamma_n)$. Since $G_n < \mathrm{Comm}(\Gamma_n)$, O'_n is finitely covered by O_n . Recall from Lemma 6.2 that O_n , and therefore also O'_n , has exactly one cusp. It is our goal to show that $G_n = \mathrm{Comm}(\Gamma_n)$; hence $O_n = O'_n$.

We use the strategy of [Goodman et al. 2008]. Recall the *hyperboloid* model for \mathbb{H}^3 . The *Lorentz inner product* on \mathbb{R}^4 is the indefinite bilinear form

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4.$$

We let $\mathbb{H}^3 = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{v} \rangle = -1, v_4 > 0\}$ and equip $T_{\mathbf{v}}\mathbb{H}^3$ with the Riemannian metric determined by the Lorentz inner product. The *positive light cone* is the set $L^+ = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{v} \rangle = 0, v_4 \geq 0\}$. The *ideal point* of \mathbb{H}^3 represented by $\mathbf{v} \in L^+$ is the

set $[v]$ of scalar multiples of v in $L^+ - \{0\}$. $\text{Isom}(\mathbb{H}^3)$ is the group of matrices in $\text{GL}_4(\mathbb{R})$ which act on \mathbb{R}^4 preserving the Lorentz inner product and the sign of the last coordinate, hence acting on \mathbb{H}^3 by isometries. Those in $\text{Isom}^+(\mathbb{H}^3) \subset \text{Isom}(\mathbb{H}^3)$ preserve orientation on \mathbb{H}^3 .

For $v \in L^+ - \{0\}$ the set $H_v = \{w \in \mathbb{H}^3 \mid \langle v, w \rangle = -1\}$ is a horosphere centered at the ideal point $[v]$. If $\alpha \in \mathbb{R}^+$ then $H_{\alpha v}$ is a horosphere centered at $[\alpha v] = [v]$, and if $\alpha \leq 1$ then H_v is contained in the horoball $\{w \mid \langle \alpha v, w \rangle \geq -1\}$ determined by αv . This determines a bijective correspondence between vectors in L^+ and horospheres in \mathbb{H}^3 , so we call the vectors in L^+ *horospherical vectors*.

We use the hyperboloid model to construct certain canonical tilings of \mathbb{H}^3 associated to M_n as in [Epstein and Penner 1988]. First, choose a horospherical vector $v \in L^+$ fixed by a peripheral element of Γ_n , so that under the covering map $\mathbb{H}^3 \rightarrow O'_n$ the horosphere H_v projects to a cross section of the cusp. Then $V_n = \text{Comm}(\Gamma_n) \cdot v$ is $\text{Comm}(\Gamma_n)$ -invariant and determines a $\text{Comm}(\Gamma_n)$ -invariant set of horospheres. The convex hull of V_n in \mathbb{R}^4 is called the *Epstein–Penner convex hull*; we denote it as C_n . Epstein and Penner show that ∂C_n consists of a countable set of 3-dimensional faces F_i , where each F_i is a finite-sided Euclidean polyhedron in \mathbb{R}^4 . Furthermore, this decomposition of ∂C_n projects along straight lines through the origin to a $\text{Comm}(\Gamma_n)$ -invariant tiling \mathcal{T}_n of \mathbb{H}^3 by ideal polyhedra [Epstein and Penner 1988, Proposition 3.5 and Theorem 3.6]. We refer to the tiling so obtained as a *canonical tiling*. (It is easy to see that a different choice for the vector v yields a convex hull which differs from C_n by multiplication by a positive scalar. Therefore it projects to the same canonical tiling as C_n .)

Consider the group of symmetries $\text{Sym}(\mathcal{T}_n) < \text{Isom}(\mathbb{H}^3)$. Given that \mathcal{T}_n is $\text{Comm}(\Gamma_n)$ -invariant we have that $\text{Comm}(\mathcal{T}_n) < \text{Sym}(\mathcal{T}_n)$. On the other hand, $\text{Sym}(\mathcal{T}_n)$ is discrete [Goodman et al. 2008, Lemma 2.1] and since Γ_n is nonarithmetic $\text{Comm}(\Gamma_n)$ is the maximal discrete group containing Γ_n . Therefore $\text{Sym}(\mathcal{T}_n) = \text{Comm}(\Gamma_n)$. Below we will first identify the tiling \mathcal{T}_n and then show that $G_n = \text{Sym}(\mathcal{T}_n)$.

Theorem 4. *With \mathcal{S} as in Lemma 4.5, $\mathcal{T}_n = \Gamma_n \cdot \bigcup\{\mathcal{P} \in \mathcal{S}\}$ is the canonical tiling for $\text{Comm}(\Gamma_n)$.*

Proof. We choose matrices

$$M = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & -1 & -2 & -1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 1 & -2 & -1 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

For $X = M, N$, let x_i be the i -th column of X . Each x_i below lies in L^+ and so represents an ideal point of \mathbb{H}^3 . We will call \mathcal{P}_X the convex hull in \mathbb{H}^3 of the $[x_i]$.

Our M and N are such that \mathcal{P}_M is a right-angled ideal cuboctahedron and \mathcal{P}_N a regular ideal octahedron, and, furthermore:

- For $X = M, N$, each member of $\text{Isom}(\mathcal{P}_X)$ fixes $(0, 0, 0, 1)^T \in \mathbb{H}^3$ and the set of columns of X is $\text{Isom}(\mathcal{P}_X)$ -invariant.
- There exists $h \in \text{Isom}^+(\mathbb{H}^3)$ with $h(n_1) = m_1$, $h(n_2) = m_9$, and $h(n_3) = m_4$, so that $h(\mathcal{P}_N) \cap \mathcal{P}_M$ is the face (m_1, m_9, m_4) with ideal vertices at $[m_1]$, $[m_9]$, and $[m_4]$.

Let $\mathcal{P}_1 = h(\mathcal{P}_N)$ and $\mathcal{P}_2 = \mathcal{P}_M$. There is an isometry p from the upper half-space to the hyperboloid model taking \mathcal{P}_i (as in Corollaries 2.2 and 2.3) to \mathcal{P}_i as above for $i = 1, 2$, and ∞ to the center $[m_1]$ of the horosphere H_{m_1} . We again refer by \mathcal{S} to the image under p of the set \mathcal{S} from Lemma 4.5. Also, p conjugates each of the isometries we've used thus far in our constructions to elements of $\text{GL}_4(\mathbb{R})$, to which we'll refer by the same names.

From the explicit description in Lemma 6.2 it is clear that $[m_1]$ is a parabolic fixed point of G_n . Since G_n is discrete, each element fixing $[m_1]$ actually fixes m_1 , so the orbit $V_n = G_n.m_1$ is a G_n -invariant collection of horospherical vectors bijective to the set of parabolic fixed points of G_n . Since $O_n = \mathbb{H}^3/G_n$ has one cusp and the same holds for $O'_n = \mathbb{H}^3/\text{Comm}(\Gamma_n)$, it follows that V_n is also $\text{Comm}(\Gamma_n)$ -invariant.

Lemma 4.5 implies that $\Gamma_n. \bigcup \{\mathcal{P} \in \mathcal{S}\}$ is a Γ_n -invariant tiling of \mathbb{H}^3 . We claim that it is identical to the canonical tiling \mathcal{T}_n , the projection to \mathbb{H}^3 of the boundary of the convex hull of V_n in \mathbb{R}^4 . Note that \mathcal{T}_n is also Γ_n -invariant, since it is G_n -invariant by construction and $\Gamma_n < G_n$.

We will use [Goodman et al. 2008, Proposition 6.1] to prove the claim. The proposition requires for each element of \mathcal{S} that the horospherical vectors representing its vertices be coplanar in \mathbb{R}^4 , and that the angle between this plane and the plane determined by each neighboring tile be convex. Equivalently, if $v_1, \dots, v_k \in V_n$ represent the ideal vertices of an element of \mathcal{S} and $w \in V_n - \{v_1, \dots, v_k\}$ represents a vertex of a neighboring tile, then there exists a vector $\mathbf{n} \in \mathbb{R}^4$ such that

- (1) (coplanarity) $\mathbf{n} \cdot v_i = 1$ for every $i = 1, \dots, k$, and
- (2) (convex angles) $\mathbf{n} \cdot w > 1$.

(See the proof of [Goodman et al. 2008, Proposition 6.1].) Observe that these conditions are invariant under $\text{Isom}(\mathbb{H}^3)$, for if $\mathbf{n} \cdot v = \alpha$ and $A \in \text{Isom}(\mathbb{H}^3)$ then $(\mathbf{n}A^{-1}) \cdot Av = \alpha$.

For each member \mathcal{P} of \mathcal{S} , we note that the subset of V_n representing the set of ideal points of \mathcal{P} contains m_1 and is $\text{Isom}(\mathcal{P})$ -invariant. This is because the members of \mathcal{S} all share the ideal vertex $[m_1]$, and the stabilizer in G_n of any $\mathcal{P} \in \mathcal{S}$ acts transitively on its set of ideal vertices. (The latter assertion can be proved by directly examining $P_n \cap \mathcal{P}$, for P_n as in Lemma 6.2.) In particular, the ideal

vertices of \mathcal{P}_1 are represented in V_n by $\{h(n_i)\}_{i=1}^6$ and those of \mathcal{P}_2 by $\{m_i\}_{i=1}^{12}$, by the properties bulleted above.

Take $\mathbf{n} = (0, 0, 0, 1/2)^T$. Then $\mathbf{n} \cdot m_i = 1$ for $i = 1, \dots, 12$, so the m_i are coplanar. The n_i (and hence also the $h(n_i)$) are also coplanar, since $\sqrt{2}\mathbf{n} \cdot n_i = 1$ for $i = 1, \dots, 6$ and the same \mathbf{n} . Coplanarity follows for the other elements of \mathcal{S} , since, for example, $\{c^{-1}(m_i)\}_{i=1}^{12}$ is an $\text{Isom}(c^{-1}(\mathcal{P}_2))$ -invariant collection of horospherical vectors containing $m_1 = c^{-1}(m_1)$ and representing the ideal vertices of $c^{-1}(\mathcal{P}_2)$.

Consider all pairs $(\mathcal{Q}, \mathcal{P}_X)$ where $X \in \{M, N\}$ and \mathcal{Q} is a regular ideal octahedron or cuboctahedron which meets \mathcal{P}_X in a face. Choose horospherical vectors for \mathcal{Q} to agree with those chosen for \mathcal{P}_X and to be $\text{Isom}(\mathcal{Q})$ -invariant. Since the convexity condition (2) is invariant under isometries, to finish the proof it suffices to check this condition for each possible pair $(\mathcal{Q}, \mathcal{P}_X)$.

If \mathcal{Q} is a cuboctahedron adjacent to \mathcal{P}_M sharing the triangular face (m_1, m_9, m_4) then $w = (7, 1, -5\sqrt{2}, 10)^T$ is a horospherical vector for \mathcal{Q} which is not shared by \mathcal{P}_M . We have $\mathbf{n} \cdot w = 5 > 1$. If \mathcal{Q} is a cuboctahedron adjacent to \mathcal{P}_M sharing the square face (m_1, m_2, m_3, m_4) then $w = (3, 5, -\sqrt{2}, 6)^T$ is a horospherical vector for \mathcal{Q} which is not shared by \mathcal{P}_M . We have $\mathbf{n} \cdot w = 3 > 1$. If \mathcal{Q} is an octahedron adjacent to \mathcal{P}_N sharing the face (n_1, n_2, n_3) then $w = \sqrt{2}(1, 2, 2, 3)^T$ is a horospherical vector for \mathcal{Q} which is not shared by \mathcal{P}_N . We have $\sqrt{2}\mathbf{n} \cdot w = 3 > 1$. By construction, $\mathcal{P}_1 = h(\mathcal{P}_N)$ is an octahedron intersecting \mathcal{P}_M in (m_1, m_9, m_4) . For $w = h(n_1) = (2 + 2\sqrt{2}, 0, -2 - 2\sqrt{2}, 4 + 4\sqrt{2})^T$ we have $\mathbf{n} \cdot w = 2 + \sqrt{2} > 1$.

With coplanarity and convex angles thus established, [Goodman et al. 2008, Proposition 6.1] implies that $\Gamma_n \cup \{\mathcal{P} \in \mathcal{S}\}$ implies the claim; hence the result. \square

By construction G_n is a subgroup of the symmetry group for \mathcal{T}_n . We complete the proof that $G_n = \text{Comm}(\Gamma_n)$ below, showing that it is the full symmetry group.

Corollary 6.5. *G_n is the commensurator of Γ_n and O_n is the minimal orbifold quotient of M_n . If $I \in \{0, 1\}^{n+1}$, then $\text{Comm}(\Gamma_I) = G_n$ and O_n is the minimal quotient of M_I .*

Proof. Proposition 6.4 implies $\text{Comm}(\Gamma_I) = \text{Comm}(\Gamma_n)$. Take $x \in \text{Comm}(\Gamma_n)$. We want to show that $x \in G_n$. Recall that $c^{-n}rc^n \in G_n$ exchanges \mathcal{P}_1 and $c^{-2n}r\mathcal{P}_1$. Therefore the octahedral tiles of \mathcal{T}_n lie in a single G_n -orbit, and we may assume that x fixes \mathcal{P}_1 .

Recall, for instance from Corollary 2.2, that \mathcal{P}_1 is checkered and its face A spanned by the vertices 0 , 1 , and ∞ is external, with $A = \mathcal{P}_1 \cap \mathcal{P}_2$. We have that $a_0, b_0 \in \text{Isom}(\mathcal{P}_1) \cap G_n$. The internal faces of \mathcal{P}_1 are paired by elements of Γ_S , so every internal face of \mathcal{P}_1 meets an octahedron in \mathcal{T}_n . Since \mathcal{P}_2 is a cuboctahedron, $x(A)$ must be an external face of \mathcal{P}_1 .

It follows immediately from the definitions of a_0 and b_0 that $\langle a_0, b_0 \rangle$ acts transitively on the external faces of \mathcal{P}_1 . Hence after multiplying by an element of $\langle a_0, b_0 \rangle < G_n$, we may assume that $x(A) = A$. By construction it is clear that G_n contains the stabilizer of A in $\text{Isom}(\mathcal{P}_1)$, so we have $x \in G_n$ as desired. \square

The second half of [Theorem 2](#) follows from the isometry classification below.

Proposition 6.6. *Suppose $I = (0, a_1, \dots, a_{n-1}, 0)$ and $J = (0, b_1, \dots, b_{n-1}, 0)$ are elements of $\{0, 1\}^{n+1}$. M_I is isometric to M_J if and only if $J = I$ or $J = \bar{I}$.*

Remark. We have assumed that the first and last entries of I and J are all zero to make the proposition easier to state. By [Lemma 5.8](#), changing the first or last entry of either I or J to “1” yields another isometric manifold.

Proof. Any two distinct tiles of \mathcal{T}_n which meet the interior of the fundamental domain P_n from [Lemma 6.2](#) have distinct G_n -orbits. On the other hand, any tile that does not is contained in the orbit of one that does. It follows that G_n has a unique orbit of octahedral tiles (that of \mathcal{P}_1) and exactly n of cuboctahedral tiles, those of $\mathcal{P}_2, c^{-1}(\mathcal{P}_2), \dots, c^{-n+1}(\mathcal{P}_2)$, since P_n has an open subset in each of these and is contained in their union.

The planes $c^{-i}(\mathcal{H})$ meet the interior of P_n for $i \in \{0, 1, \dots, n-1\}$, so their G_n -orbits are also distinct. We note that the G_n -orbit of \mathcal{H} is distinct from that of $i\mathcal{H}$ since \mathcal{H} contains points of the interior of P_n but $i\mathcal{H}$ contains a face. Since $i\mathcal{H} \cap P_n$ is contained in an internal face of \mathcal{P}_1 and $\mathcal{H} \cap P_n$ in an external face, it follows that the G_n -orbit of an internal face of \mathcal{P}_1 is distinct from that of an external face.

For I as in the hypothesis, it follows, as in [Lemma 4.5](#), that the members of

$$\mathcal{S}_I = \{\mathcal{P}_1, \mathcal{P}_2, c^{-1}\mathcal{P}_2, q_2c^{-2}\mathcal{P}_2, \dots, q_n c^{-2n+1}\mathcal{P}_2, q_n c^{-2n}r\mathcal{P}_1\}$$

project to a polyhedral decomposition of M_I , where the q_i are as defined in [Proposition 5.11](#). (In particular, $q_1 = 1$ and $q_{n+1} = q_n$, since I has first and last entries equal to 0.) This is because $q_i(c^{-2(i-1)}\mathcal{P}_2 \cup c^{-2i+1}\mathcal{P}_2)$ projects to a decomposition of $C((\Gamma_T^{(i)})^{q_i})$ for each i (see the proof of [Lemma 4.5](#)), and ϕ_{m_1} preserves the triangulation $\Delta_{\mathcal{T}}$ of [Lemma 2.4](#). Therefore \mathcal{S}_I is in bijective correspondence with the set of Γ_I -orbits of the top-dimensional tiles of \mathcal{T}_n .

Clearly $q_i(c^{-2(i-1)}\mathcal{P}_2)$ is G_n -equivalent to $c^{-2(i-1)}\mathcal{P}_2$ for each i between 1 and n , and $q_i(c^{-2i+1}\mathcal{P}_2)$ to $c^{-2i+1}\mathcal{P}_2$. The reflection u through $\mathcal{H} - n \cdot i\sqrt{2}$, also in G_n , exchanges \mathcal{P}_1 with $c^{-2n}r\mathcal{P}_1$ and $c^{-i}\mathcal{P}_2$ with $c^{-2n+i+1}\mathcal{P}_2$ for each i between 0 and $2n-1$. It follows that each G_n -orbit of top-dimensional tiles of \mathcal{T}_n is the union of exactly two Γ_I -orbits.

Now suppose for some J as in the hypothesis that there is an isometry $M_J \rightarrow M_I$. This lifts to $x \in \text{Isom}(\mathbb{H}^3)$ with the property that $\Gamma_J^x = \Gamma_I$. Since Γ_I and Γ_J are each finite-index subgroups of G_n they are commensurable, by definition $x \in \text{Comm}(\Gamma_J) = G_n$. By the above, $x\mathcal{P}_1$ is Γ_I -equivalent to one of \mathcal{P}_1 or $q_n c^{-2n}r\mathcal{P}_1$.

The reflection isometry of [Lemma 5.9](#) determines $\rho \in \text{Isom}(\mathbb{H}^3)$ that conjugates Γ_I to $\Gamma_{\bar{I}}$ and takes $q_n c^{-2n} r \mathcal{P}_1$ into the $\Gamma_{\bar{I}}$ -orbit of \mathcal{P}_1 , so replacing \times by $\rho \times$ (and I by \bar{I}) if necessary, we may ensure that there exists $\gamma \in \Gamma_I$ with $\gamma \times \mathcal{P}_1 = \mathcal{P}_1$. By the above $\gamma \times$ takes internal faces of \mathcal{P}_1 to internal faces. Because it conjugates Γ_J to Γ_I and \mathcal{P}_1 is contained in a fundamental domain for each, $\gamma \times$ preserves the internal face-pairings induced by the projections to M_I and to M_J .

It follows from [Proposition 5.11](#) that each of these is the pairing described in [Corollary 2.2](#). The combinatorial description there implies that $\gamma \times$ preserves the pairs $\{X_1, X_2\}$ and $\{X_3, X_4\}$ (see [Figure 3](#)), so it is either the identity or 180-degree rotation in the axis joining the ideal vertex 0 (the “intersection” $X_1 \cap X_2$ on the sphere at infinity) to $1+i = X_3 \cap X_4$. However the latter map does not preserve equivalence classes of the ideal vertices of X_3 and X_4 under face pairing, so $\gamma \times = 1$. It follows that $\times \in \Gamma_I$, so $\Gamma_J = \Gamma_I$.

We claim, however, that if $J \neq I$ then $\Gamma_J \neq \Gamma_I$. The key fact here is that $\Gamma_{T_0}^{m_1} \neq \Gamma_{T_0}$: for instance, the face $(\text{fg})^{-1}(Y'_2)$ of $(\text{fg})^{-1}(\mathcal{P}_2)$ is taken by $(\text{fg})^{-1} \text{hfg} \in \Gamma_{T_0}$ to $(\text{fg})^{-1}(Y'_3)$ (see the proof of [Corollary 2.3](#)), but $(\text{fg})^{-1}(Y'_2) = m_1 g^{-1}(Y'_1)$ is taken by $m_1 g^{-1} m_1^{-1} \in \Gamma_{T_0}^{m_1}$ to $m_1 g^{-1}(Y_3)$. (This description follows from the fact that m_1 preserves the polygon \mathcal{F} from [Lemma 2.4](#), acting on it as a rotation exchanging $g^{-1}(E)$ with $(\text{fg})^{-1}(D)$.) In fact, this further implies that no group Γ containing Γ_{T_0} also contains $\Gamma_{T_0}^{m_1}$, as long as the natural map $C(\Gamma_{T_0}) \rightarrow C(\Gamma)$ is embedding.

If $J \neq I$ then for the minimal i such that $b_i \neq a_i$ we have $\Gamma_{T_0}^w < \Gamma_I$ and $(\Gamma_{T_0}^{m_1})^w < \Gamma_J$, where $w = q_i c^{-2(i-1)}$ (see [Proposition 5.11](#)). The claim, and hence also the result, thus follows from [Proposition 5.11](#) and [Lemma 3.3](#). \square

7. Incommensurable mutants

[Lemma 5.5](#) might lead one to suspect that the mutations (13)(24) and (12)(34) of $F^{(0)}$ act very differently at the level of Kleinian groups. Indeed, it follows from [Proposition 7.1](#) below, together with [Proposition 4.2](#), that $S^3 - L_n$ is incommensurable with the complement of any link obtained from it by the mutation (12)(34) along a subcollection of the $S^{(i)}$. In fact, we consider it likely that no two such mutants are commensurable unless they are isometric.

We lack the tools to fully prove this assertion — mutants are notoriously difficult to distinguish — but in this section we will describe large families of mutants whose members have different cusp parameters and are mutually incommensurable. We begin with traces, however. By [\[Neumann and Reid 1991\]](#) the M_I all have trace field $\mathbb{Q}(i, \sqrt{2})$.

Proposition 7.1. *For fixed n and any $I = (a_0, \dots, a_n) \in \{0, 1, 2\}^{n+1}$ such that $a_i = 2$ for some i , Γ_I has a nonintegral trace.*

Proof. Suppose $I = (a_0, \dots, a_n) \in \{0, 1, 2\}^{n+1}$ satisfies the hypothesis, and fix i_0

with $a_{i_0} = 2$. By [Proposition 5.11](#), if $i_0 = 1$ then Γ_I contains the following matrix:

$$\mathrm{tm}_2 \mathrm{gm}_2^{-1} = \begin{pmatrix} \frac{1}{5}(-2 + 12\sqrt{2} + 31i + 4i\sqrt{2}) & -2 + \sqrt{2} + 21i + 2i\sqrt{2} \\ \frac{1}{5}(-1 + 7\sqrt{2} + 16i + 2i\sqrt{2}) & -1 + \sqrt{2} + 11i + i\sqrt{2} \end{pmatrix}.$$

(Recall from [Corollary 5.6](#) that $\mathrm{m}_2^{(0)} = \mathrm{m}_2$.) The trace of $\mathrm{tm}_2 \mathrm{gm}_2^{-1}$ is not an algebraic integer, since the ring of integers of $\mathbb{Q}(i, \sqrt{2})$ is $\mathbb{Z}[i, \sqrt{2}]$. If $i_0 = n$ then Γ_I contains a conjugate of $\bar{\mathrm{g}} \mathrm{m}_2 \bar{\mathrm{t}} \mathrm{m}_2^{-1} = (\mathrm{m}_2 \bar{\mathrm{g}})^{-1} (\mathrm{tm}_2 \mathrm{gm}_2^{-1}) \mathrm{m}_2 \bar{\mathrm{g}}$.

In all other cases, [Proposition 5.11](#) implies that Γ_I contains an element with the same trace as the following matrix:

$$\begin{aligned} \bar{\mathrm{h}}(\mathrm{m}_2 \mathrm{h} \mathrm{m}_2^{-1}) &= \begin{pmatrix} -2i\sqrt{2} & -3 + i\sqrt{2} \\ -3 - i\sqrt{2} & 3i\sqrt{2} \end{pmatrix} \begin{pmatrix} -3i\sqrt{2} & 15 - 5i\sqrt{2} \\ \frac{1}{5}(3 + i\sqrt{2}) & 2i\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} -71/5 & -20 - 30i\sqrt{2} \\ \frac{18}{5}(-2 + 3i\sqrt{2}) & 55 \end{pmatrix}. \end{aligned}$$

The trace of this matrix is evidently not an algebraic integer. □

For fixed n and any $I \in \{0, 1, 2\}^{n+1}$, since M_n and M_I decompose along totally geodesic surfaces into isometric pieces, they have the same volume. (In fact, [\[Ruberman 1987, Theorem 1.3\]](#) asserts that hyperbolic volume is always invariant under mutation.) It would follow from the classical “Dehn invariant sufficiency” conjecture that any two hyperbolic manifolds with the same volume are scissors congruent (again see [\[Neumann 1998\]](#), for instance). In our situation we will verify this explicitly.

Proposition 7.2. *For fixed n and any $I \in \{0, 1, 2\}^{n+1}$, M_n and M_I have the same Bloch invariant.*

Proof. Recall from [Lemma 2.4](#) that $F^{(0)}$ inherits a triangulation Δ_F from the fundamental domain \mathcal{F} for the action of Λ on \mathcal{H} pictured in [Figure 5](#). From the figure, one finds that Δ_F has six edges, each a geodesic arc joining cusps of F . For example, the geodesic joining 0 and ∞ projects to an edge which joins cusp 1 to cusp 2. Of the other five edges, one joins 3 to 4, two join 2 to 4, and for each of 2 and 4 there is an edge joining it to itself.

Since $\mathrm{m}_1 \in \mathrm{PSL}_2(\mathbb{Z})$ it preserves the Farey tessellation of \mathcal{H} , which restricts on \mathcal{F} to the triangulation pictured in [Figure 5](#). Therefore ϕ_{m_1} preserves Δ_F . On the other hand, since ϕ_{m_2} exchanges 1 with 2 and 3 with 4 it does not preserve Δ_F . For instance, if e is the edge joining 2 to itself then $\phi_{\mathrm{m}_2}(e)$ joins 1 to itself.

Fix $I = (a_0, \dots, a_n) \in \{0, 1, 2\}^{n+1}$ and suppose $a_i = 2$ for some $0 < i < n$. The gluing map $C(\Gamma_T^{(i)}) \rightarrow C(\Gamma_T^{(i+1)})$ factors through $\phi_{\mathrm{m}_2^{(i)}}: F^{(i)} \rightarrow F^{(i)}$ by [Proposition 5.7](#). This is conjugate to ϕ_{m_2} by the inverse of ϕ_{i+1} from [Definitions 3.9](#), so the gluing does not preserve the triangulations of $F^{(i)}$ induced by its intersections with external faces of the cuboctahedra on either side (see [Lemma 2.4\(3\)](#)). The cases

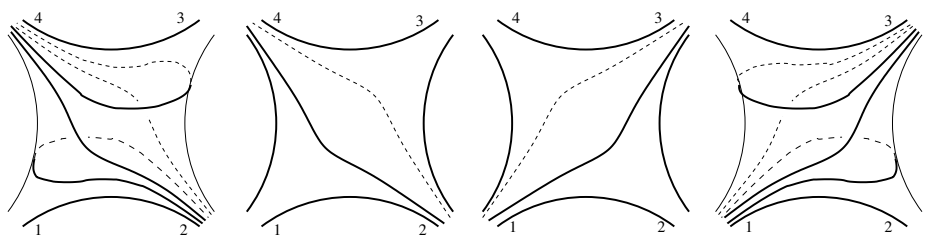


Figure 12. Interpolating between Δ_F , on the left, and $\phi_{m_2}(\Delta_F)$.

$i = 0$ and $i = n$ are analogous, and show that if $a_i = 2$ for any i then the division of M_I into octahedra and cuboctahedra is not a true ideal polyhedral decomposition.

It is possible to rectify this by gluing “flat” tetrahedra between copies of $C(\Gamma_S)$ and/or $C(\Gamma_T)$ joined by the mutation ϕ_{m_2} . If \mathcal{T} is a flat tetrahedron glued to, say, $C(\Gamma_S)$ along two adjacent triangles in $\partial C(\Gamma_S)$, then $C(\Gamma_S) \cup \mathcal{T}$ is homeomorphic to $C(\Gamma_S)$, but in the induced triangulation of the boundary, the edge separating the triangles along which \mathcal{T} is glued has been replaced by an edge joining their two opposite vertices. See [Neumann and Yang 1999, §4] for a more thorough exposition.

Figure 12 illustrates a process by which Δ_F may be changed to its image under ϕ_{m_2} by a sequence of moves on edges. The edges of Δ_F are pictured on the left in bold. Moving left to right, at each stage two edges are replaced by edges transverse to them and disjoint from the remaining edges. After three such moves, the original triangulation has been changed to its image under ϕ_{m_2} .

Now suppose $I = (a_1, \dots, a_n) \in \{0, 1, 2\}^{n+1}$. For each $i < n$ such that $a_i = 2$, replace $C(\Gamma_T^{(i+1)})$ by its union with 6 flat tetrahedra, glued successively along $\partial_- C(\Gamma_T^{(i+1)})$, to realize the change of triangulations illustrated in Figure 12. The result is homeomorphic to $C(\Gamma_T^{(i+1)})$, since adding a flat tetrahedron does not change the homeomorphism type, but the gluing induced by $\phi_{m_2}^{i+1}$ now preserves the triangulation. The case $i = n$ is similar, but $C(\bar{\Gamma}_S)$ is changed instead.

It follows from the above that the Bloch invariant $\beta(M_I)$ may be calculated using the resulting polyhedral decomposition. This differs from the original by the addition of the cross ratio parameters of the flat tetrahedra. Each of these is equal to 2, since the triangulation of F is a projection of the Farey tessellation of \mathbb{H} . But in the Bloch group, $2 \cdot [2] = 0$ is a consequence of the relation $[z] = [z/(z - 1)]$. Since the number of flat tetrahedra is a multiple of 6, the sum of their cross ratio parameters contributes nothing to the Bloch invariant. \square

The following proposition tracks the change of cusp parameters under mutation. To simplify our task, we restrict our attention to complements of links obtained by mutating only with (12)(34) along a subcollection of the $S^{(i)}$ and note in passing

that since those obtained by mutating only with (13)(24) are commensurable with M_n , their cusp parameters are $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to those of M_n .

Proposition 7.3. *For $I = (t_0, t_1, \dots, t_n) \in \{0, 2\}^{n+1}$ and $j \in \{0, 1, \dots, n\}$, define*

$$c_j = \sum_{k=0}^j \frac{t_k}{2} \pmod{2}.$$

Let T_1 be a cross section of the cusp of M_I such that $T_1 \cap C(\Gamma_S) = p_S(A_1)$ (as defined in Lemma 4.13), and let T_2 be a cross section of the cusp of M_I with $T_2 \cap C(\Gamma_S) = p_S(A_2)$. Up to the action of $\mathrm{PGL}_2(\mathbb{Q})$, their complex moduli are:

$$m(T_1) = i \left[1 + 2 \sum_{j=1}^n \frac{2\sqrt{2}}{5^{c_{j-1}}} + \frac{1}{5^{c_n}} \right],$$

$$m(T_2) = i \left[\frac{1}{5} + 2 \sum_{j=1}^n \frac{2\sqrt{2}}{5^{(1-c_{j-1})}} + \frac{1}{5^{(1-c_n)}} \right].$$

Proof. To simplify notation, we will identify A_k with $p_S(A_k)$ and view $A_k \subset C(\Gamma_S)$ for $k = 1, 2$. Recall the decomposition of M_I , along the surfaces $F^{(j)}$, into a union of isometric copies of $C(\Gamma_S)$ and $C(\Gamma_T)$ as described in Proposition 5.7:

$$C(\Gamma_S) \cup C(\Gamma_T^{(1)}) \cup \dots \cup C(\Gamma_T^{(n)}) \cup C(\bar{\Gamma}_S) \rightarrow M_I.$$

We will denote by l_j the gluing map supplied by Proposition 5.7, taking $\partial_+ C(\Gamma_T^{(j)})$ to $\partial_- C(\Gamma_T^{(j+1)})$ when $1 \leq j < n$. The map l_0 takes $\partial C(\Gamma_S)$ to $\partial_- C(\Gamma_T^{(1)})$, and $l_n: \partial_+ C(\Gamma_T^{(n)}) \rightarrow \partial C(\bar{\Gamma}_S)$.

For $1 \leq j \leq n$ and $k \in \{1, 2, 3, 4\}$ we take $DB_k^{(j)} = \phi_j \circ p_T(DB_k)$ as in the proof of Proposition 4.17. DB_k is defined above Lemma 4.15, which implies that $DB_k^{(j)}$ is an annular cross section of the cusp of $C(\Gamma_T^{(j)})$ corresponding to $\rho_k^{\varepsilon-2j}$. Each of T_1 and T_2 meets each of the $C(\Gamma_T^{(j)})$ in a collection of cusp cross sections parallel to a subcollection of the $DB_k^{(j)}$, $k \in \{1, 2, 3, 4\}$. Similarly, each of $T_1 \cap C(\bar{\Gamma}_S)$ and $T_2 \cap C(\bar{\Gamma}_S)$ is parallel to one of the cross sections \bar{A}_1 or \bar{A}_2 .

By the proof of Proposition 4.17, for $1 \leq j < n$, if $t_j = 0$ then $l_j = \iota_+^{(j)}(\iota_-^{(j)})^{-1}$ takes $\partial_+ DB_k^{(j)}$ to $\partial_- DB_k^{(j+1)}$ for each $k \in \{1, 2, 3, 4\}$. However if $t_j = 2$ then l_j acts on the indices k by the permutation (12)(34), since it uses $\phi_{m_2}^{(j)}$. Likewise if $t_0 = 0$ then $l_0(\partial A_k) = \partial_- DB_k^{(1)} \sqcup \partial_- DB_{k+2}^{(1)}$ for $k = 1, 2$, by the proof of Proposition 4.17; hence if $t_0 = 2$, then $l_0(\partial A_k) = \partial_- DB_{3-k}^{(1)} \sqcup \partial_- DB_{5-k}^{(1)}$. A similar dichotomy holds for l_n .

Remark. The definitions of the annular cusp cross sections in Lemmas 4.13 and 4.14 depended on a particular collection of horospheres centered at the ideal vertices of \mathcal{P}_1 and \mathcal{P}_2 . These give rise to a particular collection of horospherical cross sections of the cusps of $F^{(0)}$, which is not preserved by ϕ_{m_2} .

It is more accurate to say, for example, that when $t_j = 2$ and $1 \leq j < n$, $l_j(\partial_+ DB_1^{(j)})$ is a cusp cross section of $\partial_- C(\Gamma_T^{(j+1)})$ *parallel* (and therefore similar) to $\partial_- DB_2^{(j+1)}$. Since the modulus is unaffected by similarities, we have largely ignored this distinction above and will continue to do so below.

Claim. *For each $j \in \{1, \dots, n\}$,*

$$T_1 \cap C(\Gamma_T^{(j)}) = \begin{cases} DB_1^{(j)} \cup DB_3^{(j)} & \text{if } c_{j-1} = 0, \\ DB_2^{(j)} \cup DB_4^{(j)} & \text{if } c_{j-1} = 1. \end{cases}$$

Furthermore, $T_1 \cap C(\bar{\Gamma}_S) = \bar{A}_1$ if $c_n = 0$ and \bar{A}_2 if $c_n = 1$.

Proof of claim. This is proved by induction on j . In the base case $j = 1$, since $c_0 = t_0/2$ and $T_1 \cap C(\Gamma_S) = A_1$, the conclusion in this case follows directly from the dichotomy in the behavior of l_0 recorded above the claim.

Suppose now that the claim holds for some $j < n$, and note that therefore $T_1 \cap M_T^{(j)}$ has components $DB_k^{(j)}$ and $DB_{k'}^{(j)}$, where $k, k' \in \{1, 2, 3, 4\}$ have the same parity, which is opposite that of c_{j-1} . By definition, c_j has the opposite parity from c_{j-1} if and only if $t_j = 2$. Writing $l_j(\partial_+ DB_k^{(j)}) = \partial_- DB_{k''}^{(j+1)}$, the above implies that k'' has parity opposite that of k if and only if $t_j = 2$. A similar assertion holds for k' , and the claim follows for $j + 1$.

By induction, the claim holds for each $j \leq n$. The final statement in the claim follows by an argument that mimics the one used in the inductive step. \square

The moduli of A_1, A_2, \bar{A}_1 , and \bar{A}_2 are described in [Lemma 4.13](#), and those of the $DB_j^{(i)}$ are described in [Lemma 4.15](#). Using these descriptions and [Lemma 4.12](#), the claim above shows that the imaginary part of $m(T_1)$ is as described in the statement of the proposition. The description of the imaginary part of $m(T_2)$ follows similarly.

Now recall the definitions of the arcs a_1 and $db_k^{(j)}$ for $1 \leq j \leq n$ and $k = 1, 3$, and the collections of arcs \mathcal{A}_2 and $D\mathcal{B}_k^{(j)}$ for $1 \leq j \leq n$ and $k = 2, 4$, from the proof of [Proposition 4.17](#). For our purposes here, we additionally define \mathcal{A}_1 to be a collection of five arcs evenly spaced around A_1 , each perpendicular to ∂A_1 at each of its endpoints, such that $a_1 \in \mathcal{A}_1$. We analogously define $D\mathcal{B}_k^{(j)}$ as a collection of evenly spaced arcs in $DB_k^{(j)}$ containing $db_k^{(j)}$ for $1 \leq j \leq n$ and $k = 1, 3$.

Claim. *If $t_0 = 0$ then $l_0(\partial \mathcal{A}_k) = \partial_- D\mathcal{B}_k^{(1)} \cup \partial_- D\mathcal{B}_{k+2}^{(1)}$ for $k = 1, 2$, and if $t_0 = 2$ then $l_0(\partial \mathcal{A}_k) = \partial_- D\mathcal{B}_{3-k}^{(1)} \cup \partial_- D\mathcal{B}_{5-k}^{(1)}$. Similarly, for $1 \leq j \leq n - 1$,*

$$l_j(\partial_+ D\mathcal{B}_k^{(j)}) = \partial_- D\mathcal{B}_k^{(j+1)} \text{ for } k = 1, 2, 3, 4, \text{ if } t_j = 0,$$

$$l_j(\partial_+ D\mathcal{B}_k^{(j)}) = \begin{cases} \partial_- D\mathcal{B}_{3-k}^{(j+1)} & \text{for } k = 1, 2, \\ \partial_- D\mathcal{B}_{7-k}^{(j+1)} & \text{for } k = 3, 4, \end{cases} \quad \text{if } t_j = 2.$$

Also, if $t_n = 0$ then $l_n^{-1}(\partial \bar{\mathcal{A}}_k) = \partial_+ D\mathcal{B}_k^{(n)} \cup \partial_+ D\mathcal{B}_{k+2}^{(n)}$ for $k = 1, 2$, and if $t_n = 2$ then $l_n^{-1}(\partial \bar{\mathcal{A}}_k) = \partial_+ D\mathcal{B}_{3-k}^{(n)} \cup \partial_- D\mathcal{B}_{5-k}^{(n)}$.

In the discussion above the first claim, we recorded the analogous dichotomy to that of the claim above for the action of the gluing maps l_j on boundaries of annular cusp cross sections. The substance of this claim is thus that the gluing maps preserve arc endpoints.

Proof of claim. Suppose first that $t_j = 0$, so by its definition $l_j = \iota_+^{(j)}(\iota_-^{(j)})^{-1}$. The proof of [Proposition 4.17](#) directly addresses the cases of \mathcal{A}_2 , $\bar{\mathcal{A}}_2$, and $D\mathcal{B}_k^{(j)}$, where $k = 2$ or 4 . In the remaining case of \mathcal{A}_1 , the definition implies that $\partial \mathcal{A}_1$ consists of ten points, five evenly spaced around each component of ∂A_1 , with each such collection containing a point of ∂a_1 . Also by definition, $\partial_- D\mathcal{B}_k^{(1)}$ is a collection of five points spaced evenly around $\partial_- DB_k^{(1)}$, one of which is $\partial_- db_k^{(1)}$ for $k = 1, 3$. By the proof of [Proposition 4.17](#), $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ takes ∂a_1 to $\partial_- db_1^{(1)} \cup \partial_- db_3^{(1)}$; hence the entire collection $\partial \mathcal{A}_1$ is taken to $\partial_- D\mathcal{B}_1^{(1)} \cup \partial_- D\mathcal{B}_3^{(1)}$ since $\iota_+^{(0)}(\iota_-^{(0)})^{-1}$ is an isometry. The remaining cases when $t_j = 0$, $j \geq 1$, follow similarly.

To illustrate the case $t_j = 2$ we focus on the subcase $1 \leq j < n$. When $t_0 = 2$, l_j takes $\partial_+ DB_1^{(j)}$ to $\partial_- DB_2^{(j+1)}$, for example. The crucial observation here is that $l_0(\partial_+ db_1^{(j)})$ is in $\partial_- D\mathcal{B}_2^{(j+1)}$. This holds because by definition, $\partial_+ db_1^{(j)}$ is a point in the edge of the triangulation Δ_T which exits the ideal vertex 1. (This is the top edge in [Figure 12](#).) Although ϕ_{m_2} does not preserve Δ_T , it preserves this edge, exchanging its endpoints at 1 and 2. Since $\partial_- D\mathcal{B}_2^{(j+1)}$ has a point in each edge which exits 2, it contains $\phi_{m_2}(\partial_+ db_1^{(j)})$. Since the points of $\partial_+ D\mathcal{B}_1^{(j)}$ are evenly spaced around $\partial_+ DB_1^{(j)}$ and the same is true for $\partial_- D\mathcal{B}_2^{(j+1)}$, it follows that $l_0(\partial_+ D\mathcal{B}_1^{(j)}) = \partial_- D\mathcal{B}_2^{(j+1)}$.

Since ϕ_{m_2} takes the edge of Δ_T to itself and exchanges its endpoints, we have $l_0(\partial_+ db_3^{(j)}) \in \partial_- D\mathcal{B}_4^{(j+1)}$ in this case. Then it follows from “even-spacedness” that $l_0(\partial_+ D\mathcal{B}_3^{(j)}) = \partial_- D\mathcal{B}_4^{(j+1)}$. The same argument implies that $\partial_- db_1^{(j+1)}$ lies in $l_0(\partial_+ D\mathcal{B}_2^{(j+1)})$ and therefore that $l_0(\partial_+ D\mathcal{B}_2^{(j)}) = \partial_- D\mathcal{B}_1^{(j+1)}$, and similarly that $l_0(\partial_+ D\mathcal{B}_4^{(j)}) = \partial_- D\mathcal{B}_3^{(j+1)}$. The same sequence of observations, applied to $\partial \mathcal{A}_k$ and $\partial \bar{\mathcal{A}}_k$, $k = 1, 2$, completes the claim. \square

The second claim implies that the set

$$\mathcal{A}_1 \cup \mathcal{A}_2 \cup \bigcup_{j,k} D\mathcal{B}_k^{(j)} \cup \bar{\mathcal{A}}_1 \cup \bar{\mathcal{A}}_2$$

consists of a disjoint union of closed geodesics, some in T_1 and some in T_2 , each meeting any of the geodesics $F^{(j)} \cap T_1$ or $F^{(j)} \cap T_2$ perpendicularly in up to five points. That $m(T_1)$ and $m(T_2)$ have real part equal to 0 (up to the action of $\mathrm{PGL}_2(\mathbb{Q})$) now follows as in the proof of [Proposition 4.17](#). \square

Proposition 7.3 allows us to describe arbitrarily large subfamilies of the M_I which have $\mathrm{PGL}_2(\mathbb{Q})$ -inequivalent cusp parameters and hence are not commensurable.

Corollary 7.4. *For $0 \leq k \leq n$, let $I_k = (t_0, t_1, \dots, t_n)$ be defined by $t_i = 0$ for $i \neq k$, and $t_k = 2$. The cusp parameters of M_{I_k} are not $\mathrm{PGL}_2(\mathbb{Q})$ -equivalent to those of $M_{I_{k'}}$ for $k \neq k'$, when both are less than $(n+1)/2$.*

Proof. By **Proposition 7.3**, the cusps of M_{I_k} have moduli described as

$$m(T_1) = i\left[\frac{6}{5} + \frac{4}{5}(n+4k)\sqrt{2}\right], \quad m(T_2) = i\left[\frac{6}{5} + \frac{4}{5}(5n-4k)\sqrt{2}\right].$$

Since $m(T_1)$ and $m(T_2)$ are both of the form described in **Lemma 4.18** for any k , if the cusp parameters of M_{I_k} are equivalent to those of $M_{I_{k'}}$, then one of the two following cases holds:

$$\begin{aligned} n+4k &= n+4k', & \text{and} & \quad 5n-4k = 5n-4k', \\ n+4k &= 5n-4k', & \text{and} & \quad 5n-4k = n+4k', \end{aligned}$$

In the first case, $k = k'$, and in the second, $k' = n - k$. Thus as long as k and $k' < (n+1)/2$ are unequal, their cusp parameters are as well. \square

There are also arbitrarily large subfamilies which share cusp parameters, even among complements of links obtained by mutating only with (12)(34). We do not know if these are commensurable, although we suspect they are not.

Corollary 7.5. *For $0 \leq k < n$, let $I_k = (t_0, \dots, t_n)$ be defined by $t_i = 0$ for $i \neq k, k+1$, and $t_k = t_{k+1} = 2$. For each k , the cusp parameters of M_{I_k} are*

$$m(T_1) = i\left[2 + 4\left(n - \frac{4}{5}\right)\sqrt{2}\right], \quad m(T_2) = i\left[\frac{2}{5} + \frac{4}{5}(n+4)\sqrt{2}\right],$$

up to the action of $\mathrm{PGL}_2(\mathbb{Q})$.

Corollaries 7.4 and 7.5 prove parts (2) and (3), respectively, of **Theorem 3**.

Appendix: Proof of **Lemma 2.6**

Following Morgan [1984], we define a *pared manifold* to be a pair (M, P) , where M is a compact, orientable, irreducible 3-manifold with nonempty boundary which is not a 3-ball, and $P \subseteq \partial M$ is the union of a collection of disjoint incompressible annuli and tori satisfying the following properties:

- Every noncyclic abelian subgroup of $\pi_1 M$ is conjugate into the fundamental group of a component of P .
- Every map $\phi: (S^1 \times I, S^1 \times \partial I) \rightarrow (M, P)$ which induces an injection on fundamental groups is homotopic as a map of pairs to a map ψ such that $\psi(S^1 \times I) \subset P$.

This definition describes the topology of the compact manifold obtained by truncating the cusps of the convex core of a geometrically finite hyperbolic 3-manifold by open horoball neighborhoods. Indeed, Corollary 6.10 of [Morgan 1984] asserts that if (M, P) is obtained in this way, where P consists of the collection of boundaries of the truncating horoball neighborhoods, then (M, P) is a pared manifold.

Lemma 2.6 asserts that if (M, P) has the pared homotopy type of a geometrically finite hyperbolic manifold \mathbb{H}^3/Γ where Γ is not Fuchsian and $\partial C(\Gamma)$ is totally geodesic, then $M - P$ is homeomorphic to $C(\Gamma)$. The key point of the proof is that the geometric conditions on Γ ensure that (M, P) is an acylindrical pared manifold. Then Johansson's theorem [Johansson 1979], that pared homotopy equivalences between acylindrical pared manifolds are homotopic to pared homeomorphisms, applies. We expand on this below.

It is worth noting that **Lemma 2.6** fails in more general circumstances. The memoir [Canary and McCullough 2004] gives examples of this; Example 1.4.5, for instance, describes homotopy-equivalent non-Fuchsian geometrically finite manifolds with incompressible convex core boundary which are not homeomorphic. That work is devoted to understanding the ways in which homotopy equivalences of hyperbolic 3-manifolds can fail to be homotopic to homeomorphisms, and **Lemma 2.6** follows quickly from results therein.

The treatment of Canary and McCullough itself uses the theory of *characteristic submanifolds* of manifolds with *boundary pattern* developed in [Johansson 1979]. The characteristic submanifold of a manifold with boundary pattern is a maximal collection of disjoint codimension-zero submanifolds, each an interval bundle or Seifert-fibered space embedded reasonably with respect to the boundary pattern. Rather than attempting to establish all of the notation necessary to define this formally, we refer the interested reader to the two works just cited. Here we simply transcribe the relevant theorem of [Canary and McCullough 2004], which strongly restricts the topology of the characteristic submanifold of a pared manifold with boundary pattern determined by the pared locus.

For the purposes of **Lemma 2.6** we exclude from consideration certain pared manifolds which never arise from convex cores of geometrically finite hyperbolic 3-manifolds. We say (M, P) is *elementary* if it is homeomorphic to one of $(T^2 \times I, T^2 \times \{0\})$, $(A^2 \times I, A^2 \times \{0\})$, or $(A^2 \times I, \emptyset)$, where T^2 and A^2 denote the torus and annulus, respectively; otherwise (M, P) is *nonelementary*. Define $\partial_0 M := \overline{M - P}$. We say an annulus properly embedded in $M - P$ is *essential* in (M, P) if it is incompressible and boundary-incompressible in $M - P$. For a codimension-0 submanifold V embedded in M , we denote by $\text{Fr}(V)$ the *frontier* of V (that is, its topological boundary in M), and note that $\text{Fr}(V) = \overline{\partial V - (V \cap \partial M)}$. With notation thus established, the following theorem combines the definition of the characteristic submanifold with [Canary and McCullough 2004, Theorem 5.3.4].

Theorem. *Let (M, P) be a nonelementary pared manifold with $\partial_0 M$ incompressible. Select the fibering of the characteristic submanifold so that no component is an I -bundle over an annulus or Möbius band.*

- (1) *Suppose V is a component of the characteristic submanifold which is an I -bundle over a surface B . Then each component of the associated ∂I -bundle is contained in $\partial_0 M$, each component of the associated I -bundle over ∂B is either a component of P or a properly embedded essential annulus, and B has negative Euler characteristic.*
- (2) *A Seifert-fibered component V of the characteristic submanifold is homeomorphic either to $T^2 \times I$ or to a solid torus. If V is $T^2 \times I$ then one component of $T^2 \times \partial I$ lies in P and the other components of $V \cap \partial M$ are annuli in $\partial_0 M$. If V is a solid torus, then $V \cap \partial M$ has at least one component, each an annulus either containing a component of P or contained in $\partial_0 M$. In either case, each component of the frontier $\text{Fr}(V)$ of V in M is a properly embedded essential annulus.*

The characteristic submanifold contains regular neighborhoods of all components of P .

The key claim in the proof of [Lemma 2.6](#) is a further restriction on the characteristic submanifold of (M, P) , in the case that M is obtained from the convex core of a non-Fuchsian geometrically finite manifold with totally geodesic convex core boundary by removing horoball neighborhoods of the cusps. P is the union of the boundaries of these neighborhoods.

Claim. *(M, P) as above is nonelementary, and $\partial_0 M$ is incompressible. The characteristic submanifold of (M, P) consists only of (Seifert-fibered) regular neighborhoods of the components of P , each of whose boundary has a unique component of intersection with ∂M .*

We prove the claim below, but assuming it for now, the proof of [Lemma 3](#) proceeds as follows. A representation as given in the statement of the lemma induces a pared homotopy equivalence between (M, P) and the pared manifold (N, Q) obtained by truncating $C(\Gamma)$ with open horoball neighborhoods. Since $C(\Gamma)$ has totally geodesic convex core boundary, (N, Q) is as described by the claim; hence (M, P) is as well (see [\[Canary and McCullough 2004, Theorem 2.11.1\]](#), for example). Johansson's classification theorem (see [\[Canary and McCullough 2004, Theorem 2.9.10\]](#)) implies that the original pared homotopy equivalence is homotopic to one which maps the complement of the characteristic submanifold of (M, P) homeomorphically to the complement of the characteristic submanifold of (N, Q) . It follows from the claim that these are homeomorphic to $M - P$ and $N - Q$, respectively, and the lemma follows.

Proof of claim. As was mentioned above, the elementary pared manifolds do not arise from geometrically finite hyperbolic manifolds. Since (M, P) is obtained from the convex core of a geometrically finite manifold with totally geodesic convex core boundary, the following are known not to occur:

- (1) A compressing disk for $\partial_0 M$. (By definition $\partial_0 M$ lifts to a geodesic hyperplane in \mathbb{H}^3 , hence the induced map $\pi_1 \partial M_0 \rightarrow \pi_1 M$ is injective.)
- (2) An *accidental parabolic*: an essential annulus properly embedded in M with one boundary component in P and one in $\partial_0 M$, which is not parallel to P . (Every essential curve on $\partial_0 M$ that is not boundary-parallel is homotopic to a geodesic, but an element of $\pi_1(M)$ corresponding to an accidental parabolic has translation length 0.)
- (3) A *cylinder*; that is, a properly embedded essential annulus in $M - P$. (The double DM of M across $\partial_0 M$ is a hyperbolic manifold, but the double of a cylinder in M would be an essential torus in DM .)

We show that if the characteristic manifold has any components other than those listed in the claim then at least one of the above facts cannot hold.

For a component V of the characteristic submanifold which is an I -bundle over a surface B , at least one component of the associated I -bundle over ∂B must be properly embedded, since otherwise we would have $M = V$ and it is well known that an I -bundle over a surface does not admit a hyperbolic structure with totally geodesic convex core boundary unless the convex core is a Fuchsian surface. But this annulus violates (2) or (3). Thus there are no I -bundle components of the characteristic submanifold.

If V is a Seifert-fibered component of the characteristic submanifold homeomorphic to $T^2 \times I$, then one component of ∂V is a torus $P_1 \subset P$, and all other components of $\partial V \cap \partial M$ are annuli in $\partial_0 M$. If this second class is nonempty then each component of $\text{Fr}(V)$ is an essential annulus properly embedded in $M - P$, contradicting fact (3). Thus $\partial V \cap \partial M = P_1$ and V is a regular neighborhood of P_1 .

If V is a solid torus and $V \cap \partial M$ contains a component of P , then a similar argument shows that this is the unique component of $\partial V \cap \partial M$, so in this case V is a regular neighborhood of an annular component of P . If, on the other hand, $V \cap \partial M$ does *not* contain any components of P , then it has at least two components, for otherwise a meridional disk of V determines a boundary compression of the annulus $\text{Fr}(V)$ in $M - P$. But then any component of $\text{Fr}(V)$ violates fact (3). \square

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