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TAUT FOLIATIONS AND THE ACTION OF THE FUNDAMENTAL GROUP ON LEAF SPACES AND UNIVERSAL CIRCLES

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Let $\mathcal{F}$ be a leafwise hyperbolic taut foliation of a closed 3-manifold M and let L be the leaf space of the pullback of $\mathcal{F}$ to the universal cover of M. We show that if $\mathcal{F}$ has branching, then the natural action of $\pi_1(M)$ on L is faithful. We also show that if $\mathcal{F}$ has a finite branch locus B whose stabilizer acts on B nontrivially, then the stabilizer is an infinite cyclic group generated by an indivisible element of $\pi_1(M)$.

1. Introduction

Unless otherwise specified, we assume throughout this article that M is a closed oriented 3-manifold and $\mathcal{F}$ a codimension-one transversely oriented, leafwise hyperbolic, taut foliation of M. Here we say that $\mathcal{F}$ is *leafwise hyperbolic* if there is a transversely continuous leafwise Riemannian metric on M where the leaves are locally isometric to the hyperbolic plane, and that $\mathcal{F}$ is *taut* if there is a loop in M which intersects every leaf of $\mathcal{F}$ transversely. Note that by [Candel 1993], if M is irreducible and atoroidal, then every taut foliation of M is leafwise hyperbolic.

Leafwise hyperbolic taut foliations have been extensively investigated by many people in connection with the theory of 3-manifolds (see, for example, Calegari's book [2007]). One of the most powerful methods of analyzing the structure of such foliations is to consider canonical actions of $\pi_1(M)$ on 1-manifolds naturally associated with $\mathcal{F}$. Two kinds of such 1-manifolds are known. The first one, denoted L, is the leaf space of $\widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M. The action of $\pi_1(M)$ on $\widetilde{M}$ induces an action of $\pi_1(M)$ on L. In the sequel we refer to it as the *natural action*. The second one is a universal circle. By unifying circles at infinity of all the leaves of a given $\widetilde{\mathcal{F}}$, Thurston [1998] (see also [Calegari and Dunfield 2003]) constructs a universal circle with a canonical $\pi_1(M)$ action.

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We say that $\mathcal{F}$ has *branching* if L is non-Hausdorff. The first result of this article is the following:

Theorem 3.2. *If $\mathcal{F}$ has branching, then the natural action on L is faithful.*

This result is obtained from an investigation of both actions of $\pi_1(M)$ on the leaf space and on the universal circle (see Section 3). Notice that the hypothesis that $\mathcal{F}$ has branching is indispensable. In fact, just consider a surface bundle over S^1 foliated by fibers. Notice also that, by Theorem 7.10 of [Calegari and Dunfield 2003], any taut foliation can be modified by suitable Denjoy-like insertions so that the natural action associated with the resulting foliation becomes faithful. In the case where the foliation is leafwise hyperbolic and has branching, our result is stronger in that we assure faithfulness without performing any modifications.

Next we consider the stabilizer of a branch locus of $\mathcal{F}$. We call a subset B of L a *branch locus* if B contains at least two points and can be expressed in the form $B = \lim_{t\to 0} \nu_t$ for some interval $\{\nu_t \in L \mid 0 < t < \epsilon\}$ embedded in L. Furthermore, if the parameter t of the interval is incompatible (resp. compatible) with the orientation of L, we call B a *positive* (resp. *negative*) branch locus. (Note that L has a natural orientation induced from the transverse orientation of $\widetilde{\mathcal{F}}$.) Branch loci have been studied, for example, in [Fenley 1998; Shields 2002]. For a branch locus B we define the *stabilizer* of B by $\mathrm{Stab}(B) = \{\alpha \in \pi_1(M) \mid \alpha(B) = B\}$.

In the case where a branch locus B is finite, we obtain the following results about the action of $\mathrm{Stab}(B)$ on B (see Section 5 for details).

Theorem 5.2. *Let B be a finite branch locus of L. If an element of* $\mathrm{Stab}(B)$ *fixes some point of B then it fixes all the points of B.*

We remark that for Anosov foliations, Theorem D of [Fenley 1998] contains results related to this theorem.

Let $\pi : \widetilde{M} \to M$ be the covering projection. For a leaf λ of $\widetilde{\mathcal{F}}$, we denote by $\underline{\lambda}$ the projected leaf $\pi(\lambda)$ of $\mathcal{F}$.

Theorem 5.3. *Let B be a branch locus of L. Then,*

(1) *if* $\mathrm{Stab}(B)$ *is trivial, $\underline{\lambda}$ is diffeomorphic to a plane, and*

(2) *if B is finite and* $\mathrm{Stab}(B)$ *is nontrivial, $\underline{\lambda}$ is diffeomorphic to a cylinder*

for any $\lambda \in B$.

Theorem 5.6. *Let B be a finite branch locus of L with a nontrivial stabilizer. Then the stabilizer* $\mathrm{Stab}(B)$ *is isomorphic to $\mathbb{Z}$.*

We say that $\alpha \in \pi_1(M)$ is *divisible* if there is some $\beta \in \pi_1(M)$ and an integer $k \geq 2$ such that $\alpha = \beta^k$. Otherwise we say α is *indivisible*.

Theorem 5.7. *Let B be a finite branch locus of L such that* $\mathrm{Stab}(B)$ *acts on B nontrivially. Then a generator of* $\mathrm{Stab}(B)$ $(\cong \mathbb{Z})$ *is indivisible.*

For an oriented loop γ in M, we say that γ is *tangentiable* if γ is freely homotopic to a leaf loop (a loop contained in a single leaf) of $\mathcal{F}$, and that γ is positively (resp. negatively) *transversable* if γ is freely homotopic to a loop positively (resp. negatively) transverse to $\mathcal{F}$. As a final topic of this article, we study relations between the infiniteness of branch loci and the existence of a nontransversable leaf loop in M (see Section 6). One of the results we obtain is the following:

Theorem 6.5. *Suppose $\mathcal{F}$ has branching. If there is a noncontractible leaf loop in M which is not freely homotopic to a loop transverse to $\mathcal{F}$, then $\mathcal{F}$ has an infinite branch locus.*

This article is organized as follows. In Section 2, we briefly review the Calegari–Dunfield construction of a universal circle. Using their construction, we prove the faithfulness of the natural action of $\pi_1(M)$ on L in Section 3. In Section 4, we introduce a notion of comparable sets and give several basic properties of such sets, which are applied in Section 5 to the investigation of the structure of finite branch loci and their stabilizers. In Section 6, we study how the nontransversability of leaf loops in M is related to the infiniteness of branch loci in L.

2. Universal circles

The theory of universal circles was originally developed in [Thurston 1998], and was written up carefully in [Calegari and Dunfield 2003]. In this section we briefly recall the definition of a universal circle after the latter reference.

Let M, $\mathcal{F}$ and L be as in the introduction. Here the topology of L is the quotient topology from $\widetilde{M}$; that is, there is a canonical projection map $q : \widetilde{M} \to L$ sending a point to the leaf containing it. The topology of L is the quotient topology from the map q.

For $\lambda, \mu \in L$ we write $\lambda < \mu$ if there is an oriented path in $\widetilde{M}$ from λ to μ which is positively transverse to $\widetilde{\mathcal{F}}$. We say that λ and μ are *comparable* if either $\lambda \leq \mu$ or $\lambda \geq \mu$. For a leaf λ of $\widetilde{\mathcal{F}}$, the *endpoint map* $e : T_p\lambda - \{0\} \to S^1_\infty(\lambda)$ from the tangent space of λ at p to the ideal boundary of λ takes a vector v to the endpoint at infinity of the geodesic ray γ with $\gamma(0) = p$ and $\gamma'(0) = v$. The *circle bundle at infinity* is the disjoint union $E_\infty = \bigcup_{\lambda \in L} S^1_\infty(\lambda)$ with the finest topology such that the endpoint map $e : T\widetilde{\mathcal{F}} \setminus (\text{zero section}) \to E_\infty$ is continuous. A continuous map $\phi : X \to Y$ between oriented 1-manifolds homeomorphic to S^1 is *monotone* if it is of mapping degree one and if the preimage of any point of Y is contractible. A *gap* of ϕ is the interior in X of such a preimage. The *core* of ϕ is the complement of the union of gaps.

Definition 2.1. A *universal circle* S^1_{univ} for $\mathcal{F}$ is a circle together with a homomorphism $\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}})$ and a family of monotone maps $\phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda)$, $\lambda \in \widetilde{\mathcal{F}}$, satisfying the following conditions:

1. For every $\alpha \in \pi_1(M)$, the following diagram commutes:

$$\begin{tikzcd} S^1_{\mathrm{univ}} \arrow[r, "\rho_{\mathrm{univ}}(\alpha)"] \arrow[d, "\phi_\lambda"'] & S^1_{\mathrm{univ}} \arrow[d, "\phi_{\alpha(\lambda)}"] \\ S^1_\infty(\lambda) \arrow[r, "\alpha"] & S^1_\infty(\alpha(\lambda)). \end{tikzcd}$$

2. If λ and μ are incomparable, then the core of ϕ_λ is contained in the closure of a single gap of ϕ_μ and vice versa.

Calegari and Dunfield's construction for a universal circle is as follows. Let $I = [0, 1]$ be the unit interval. A *marker* for $\mathscr{F}$ is a continuous map $m : I \times \mathbb{R}^+ \to \widetilde{M}$ with the following properties:

- For each $s \in I$, the image $m(s \times \mathbb{R}^+)$ is a geodesic ray in a leaf of $\widetilde{\mathscr{F}}$. We call these the *horizontal rays* of m.
- For each $t \in \mathbb{R}^+$, the image $m(I \times t)$ is transverse to $\widetilde{\mathscr{F}}$ and of length smaller than some constant depending only on $\widetilde{\mathscr{F}}$.

We use the interval notation $[\lambda, \mu]$ to represent the oriented image of an injective continuous map $c : I \to L$ such that $c(0) = \lambda$ and $c(1) = \mu$. We call this the *interval* from λ to μ. Here, notice that the orientation of such an interval is induced from that of I (not from that of L).

Let $J = [\lambda, \mu]$ be an interval in L and let m be a marker which intersects only leaves of $\widetilde{\mathscr{F}}|_J$. Then the endpoints of the horizontal rays of m form an interval in $E_\infty|_J$ which is transverse to the circle fibers. By abuse of notation we refer to such an interval as a marker.

For each $\nu \in J$, the intersection of $S^1_\infty(\nu)$ with the union of all markers is dense in $S^1_\infty(\nu)$. If two markers m_1, m_2 in $E_\infty|_J$ are not disjoint, their union $m_1 \cup m_2$ is also an interval transverse to the circle fibers. It follows that a maximal such union of markers is still an interval. Again by abuse of notation we call such an interval a marker.

A continuous section $\tau : J \to E_\infty|_J$ is *admissible* if the image of τ does not cross (but might run into) any marker. The *leftmost section* $\tau(p, J) : J \to E_\infty|_J$ starting at $p \in S^1_\infty(\lambda)$ is an admissible section which is clockwisemost among all such sections if the order of J is compatible with that of L, and anticlockwisemost otherwise. Here, the meaning of "(anti-)clockwisemost" is the following: Consider the universal cover $\widetilde{E_\infty|_J} \cong \mathbb{R} \times J$ of $E_\infty|_J$ and take a lift $\tilde{p} \in \mathbb{R} \times J$ of p. Then, we say that τ is clockwisemost (resp. anticlockwisemost) if for any admissible section τ' the lifts $\tilde{\tau}, \tilde{\tau}'$ of τ, τ' to $\mathbb{R} \times J$ based at $\tilde{p}$ satisfy $\tilde{\tau}(\nu) \leq \tilde{\tau}'(\nu)$ (resp. $\tilde{\tau}'(\nu) \leq \tilde{\tau}(\nu)$) for any $\nu \in J$. For any p the leftmost section starting at p exists.

Let $B = \lim_{t\to 0} \nu_t$ be a branch locus and let $\mu_1, \mu_2 \in B$. For each $t > 0$, let $\alpha_t = [\mu_1, \nu_t]$ and $\beta_t = [\nu_t, \mu_2]$. Then, we can define a map $r_t : S^1_\infty(\mu_1) \to S^1_\infty(\mu_2)$

by $r_t(p) = \tau(\tau(p, \alpha_t)(\nu_t), \beta_t)(\mu_2)$. As t tends to 0, r_t converges to a constant map. We denote the image of the constant map by $r(\mu_1, \mu_2) \in S^1_\infty(\mu_2)$.

Definition 2.2. We call $r(\mu_1, \mu_2)$ the *turning point from μ_1 to μ_2*.

Given a pair $\lambda, \mu \in L$, we define a *geodesic spine* from λ to μ to be a disjoint union of finitely many intervals $[\hat{\nu}_{i-1}, \check{\nu}_i]$, $1 \le i \le n$, in L (some of them may degenerate to singletons), with the following properties:

(1) $\hat{\nu}_0 = \lambda$ and $\check{\nu}_n = \mu$,

(2) $\check{\nu}_i$ and $\hat{\nu}_i$ belong to a common branch locus for each $1 \le i \le n-1$, and

(3) n is minimal under the conditions (1) and (2).

Note that a geodesic spine connecting any two points in L exists and is unique. Geodesic spines have been extensively used in [Barbot 1996; 1998; Fenley 2003; Roberts et al. 2003].

For a point p in $S^1_\infty(\lambda)$, the *special section* $\sigma_p : L \to E_\infty$ at p is defined as follows. First, set $\sigma_p(\lambda) = p$. Next, pick any point $\mu \in L$. We define $\sigma_p(\mu)$ as follows: When μ is comparable with λ, then σ_p is defined on $[\lambda, \mu]$ to be the leftmost section starting at p. When μ is incomparable with λ, let $\coprod_{i=1}^{n}[\hat{\nu}_{i-1}, \check{\nu}_i]$ $(n > 1)$ be the geodesic spine from λ to μ. We then put $r = r(\check{\nu}_{n-1}, \hat{\nu}_{n-1}) \in S^1_\infty(\hat{\nu}_{n-1})$ and define σ_p on the interval $[\hat{\nu}_{n-1}, \check{\nu}_n]$ by $\sigma_p = \sigma_r$. This completes the definition of σ_p.

Let $\mathfrak{S}$ be the union of the special sections σ_p as p varies over all points in all circles $S^1_\infty(\lambda)$ of points λ in L. By [Calegari and Dunfield 2003, Lemma 6.25], the set $\mathfrak{S}$ admits a natural circular order. The universal circle S^1_{univ} will be derived from $\mathfrak{S}$ as a quotient of the order completion of $\mathfrak{S}$ with respect to the circular order. Remark that limits of special sections are also sections, hence that any element of S^1_{univ} is represented by a section $L \to E_\infty$.

3. Faithfulness of the action

We now show that if $\mathcal{F}$ has branching, the natural action of $\pi_1(M)$ on the leaf space L is faithful.

As explained in Section 2, every element σ of S^1_{univ} can be described as a section $\sigma : L \to E_\infty = \bigcup_{\lambda \in L} S^1_\infty(\lambda)$ and that the maps $\phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda)$ are defined by $\phi_\lambda(\sigma) = \sigma(\lambda)$. For a point x in $S^1_\infty(\lambda)$, we define a (possibly degenerate) closed interval I_x in S^1_{univ} by $I_x = \{\sigma \in S^1_{\text{univ}} \mid \sigma(\lambda) = x\}$. Then, for any x the interval I_x is nonempty because the special section σ_x at x belongs to I_x. From the definition of a turning point, we have the following fact: If μ_1, μ_2 are in a branch locus and if z is in $S^1_\infty(\mu_2)$, then $\phi_{\mu_1}(\sigma_z) = r(\mu_2, \mu_1)$; that is, $\sigma_z \in I_{r(\mu_2,\mu_1)}$.

Let $\lambda \in L$ and $\alpha \in \pi_1(M)$ be such that $\alpha(\lambda) = \lambda$. Then α, as the restriction of a covering transformation of $\widetilde{M}$ to λ, induces an isometry of the hyperbolic plane λ, (hence also a projective transformation of $S^1_\infty(\lambda)$). We notice that this isometry is a hyperbolic element (meaning that its trace is greater than 2). In fact, since it has no fixed points in λ, it is not elliptic. If it were parabolic, then it would yield in M a noncontractible loop whose length can be made arbitrarily small, contradicting the compactness of M.

The following is a key lemma.

Lemma 3.1. *Let $B = \lim_{t\to 0} \nu_t$ be a branch locus of L. If $\alpha \in \pi_1(M)$ fixes two distinct points μ_1 and μ_2 in B and also fixes the interval $\{\nu_t \mid 0 < t < \epsilon\}$ pointwise, then α is trivial in $\pi_1(M)$.*

Proof. Suppose α is nontrivial. Let $p_1, q_1 \in S^1_\infty(\mu_1)$ and $p_2, q_2 \in S^1_\infty(\mu_2)$ be the fixed points of α, and let $r_1 \in S^1_\infty(\mu_1)$ be the turning point from μ_2 to μ_1. Without loss of generality, we assume that $p_1 \neq r_1$. Note that by construction of the universal circle, the special sections σ_{p_i} and σ_{q_i} in S^1_{univ} are fixed by $\rho_{\text{univ}}(\alpha)$ for $i = 1, 2$; therefore the images $\phi_{\nu_t}(\sigma_{p_i})$ and $\phi_{\nu_t}(\sigma_{q_i})$ are fixed by α for any $t \in (0, \epsilon)$.

We claim that if t is sufficiently close to 0, then $\phi_{\nu_t}(\sigma_{p_1})$ and $\phi_{\nu_t}(I_{r_1})$ are disjoint in $S^1_\infty(\nu_t)$. Take two distinct points x and y in $S^1_\infty(\mu_1) - \{p_1, r_1\}$ so that the 4-tuple (p_1, x, r_1, y) lies in circular order. Because of the density of markers, for sufficiently small $t > 0$ the 4-tuple $(\sigma_{p_1}(\nu_t), \sigma_x(\nu_t), \sigma_{r_1}(\nu_t), \sigma_y(\nu_t))$ lies in $S^1_\infty(\nu_t)$ also in circular order. Let K_t be the closed interval in $S^1_\infty(\nu_t)$ with boundary points $\sigma_x(\nu_t)$ and $\sigma_y(\nu_t)$ and containing $\sigma_{r_1}(\nu_t)$. Since I_{r_1} contains σ_{r_1} but not σ_{p_1}, σ_x and σ_y, and since special sections cannot cross, $\phi_{\nu_t}(I_{r_1})$ is contained in K_t. In particular, $\phi_{\nu_t}(\sigma_{p_1})$ and $\phi_{\nu_t}(I_{r_1})$ are disjoint. This shows the claim.

For t sufficiently close to 0, the two points $\sigma_{p_2}(\nu_t)$ and $\sigma_{q_2}(\nu_t)$ are distinct. Since both σ_{p_2} and σ_{q_2} pass through the turning point r_1 from μ_2 to μ_1, it follows that $\phi_{\mu_1}(\sigma_{p_2}) = \phi_{\mu_1}(\sigma_{q_2}) = r_1$; that is, σ_{p_2} and σ_{q_2} are contained in I_{r_1}. Therefore the 3 points $\sigma_{p_1}(\nu_t)$, $\sigma_{p_2}(\nu_t)$ and $\sigma_{q_2}(\nu_t)$ are also mutually distinct. Thus, we find at least 3 fixed points of α in $S^1_\infty(\nu_t)$, contradicting the fact that α is a nontrivial orientation preserving isometry of the hyperbolic plane ν_t. □

Now, the first main result of this article is the following:

Theorem 3.2. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M. If $\mathcal{F}$ has branching, then the natural action of $\pi_1(M)$ on the leaf space of $\widetilde{\mathcal{F}}$ is faithful.*

Proof. This is a direct consequence of Lemma 3.1. □

4. Comparable sets

In this section we do not assume leafwise hyperbolicity of $\mathcal{F}$. For $\alpha \in \pi_1(M)$, we define the *comparable set* C_α for α to be the subset of L consisting of points λ such that λ and $\alpha(\lambda)$ are comparable. Below we collect some basic properties of comparable sets.

Obviously, $\alpha(C_\alpha) = C_\alpha$, $C_\alpha = C_{\alpha^{-1}}$ and $C_\alpha \subset C_{\alpha^k}$ for every $k > 0$.

We say that $\mathcal{F}$ has *one-sided branching in the positive* (resp. *negative*) *direction* if L has positive (resp. negative) branch loci but has no negative (resp. positive) ones. If L has both positive loci and negative loci, then we say $\mathcal{F}$ has *two-sided branching*.

Lemma 4.1. *Let $\mathcal{F}$ have one-sided branching in the positive direction, and let $\alpha \in \pi_1(M)$. Suppose λ and μ are points in L such that λ is a common lower bound of μ and $\alpha(\mu)$, meaning that $\lambda \le \mu$ and $\lambda \le \alpha(\mu)$. Then $\lambda \in C_\alpha$.*

Proof. Since the natural action preserves the order of L, the inequality $\lambda \le \mu$ implies $\alpha(\lambda) \le \alpha(\mu)$. Thus, by the hypothesis, $\alpha(\mu)$ is a common upper bound of λ and $\alpha(\lambda)$. Since $\mathcal{F}$ has no branching in the negative direction, it follows that λ and $\alpha(\lambda)$ are comparable. □

From this lemma we see the following fact: Let $\mathcal{F}$ and α be as above. Then, there is $\lambda \in L$ such that $\{\mu \in L \mid \mu < \lambda\} \subset C_\alpha$.

Lemma 4.2. *Let $\alpha \in \pi_1(M)$ and let $\lambda, \mu \in C_\alpha$. Then the geodesic spine γ from λ to μ is entirely contained in C_α. Furthermore, if γ is written as $\gamma = \coprod_{i=1}^n [\hat{\nu}_{i-1}, \check{\nu}_i]$ $(\hat{\nu}_0 = \lambda,\ \check{\nu}_n = \mu)$ by using a union of intervals, then $\check{\nu}_i$, $\hat{\nu}_i$ are fixed by α for each $1 \le i \le n-1$.*

Proof. Without loss of generality we may assume that $\lambda \le \check{\nu}_1$. We may also assume that $\alpha(\lambda) \le \lambda$, because if $\alpha^{-1}(\lambda) \le \lambda$ we may just consider α^{-1} instead of α.

We first treat the case when $n = 1$ (that is, the case when λ and μ are comparable). Suppose $\nu \notin C_\alpha$ for some $\nu \in [\lambda, \mu]$. Then we have $\nu \in [\alpha(\lambda), \mu]$ and $\alpha(\nu) \in [\alpha(\lambda), \alpha(\mu)]$. Since ν and $\alpha(\nu)$ are incomparable, it follows that μ and $\alpha(\mu)$ are also incomparable, which is a contradiction. Therefore, we have $[\lambda, \mu] \subset C_\alpha$.

Next, we assume $n \ge 2$. We claim that $\alpha(\check{\nu}_1) = \check{\nu}_1$ and $\alpha(\hat{\nu}_1) = \hat{\nu}_1$. Note that

$$[\alpha(\lambda), \lambda] \cup \gamma = [\alpha(\lambda), \check{\nu}_1] \cup \left(\coprod_{i=2}^n [\hat{\nu}_{i-1}, \check{\nu}_i] \right)$$

is the geodesic spine from $\alpha(\lambda)$ to μ, and that

$$\alpha(\gamma) = [\alpha(\lambda), \alpha(\check{\nu}_1)] \cup \left(\coprod_{i=2}^n [\alpha(\hat{\nu}_{i-1}), \alpha(\check{\nu}_i)] \right)$$

is the geodesic spine from $\alpha(\lambda)$ to $\alpha(\mu)$. Then the reader can work through the several possibilities ($\alpha(\check{\nu}_1) < \check{\nu}_1$, $\alpha(\check{\nu}_1) > \check{\nu}_1$, or $\alpha(\check{\nu}_1)$ and $\check{\nu}_1$ are incomparable) to deduce that any point $\nu \in \coprod_{i=2}^n [\hat{\nu}_{i-1}, \check{\nu}_i]$ is incomparable with $\alpha(\nu)$, contrary to the hypothesis that $\mu \in C_\alpha$. Similarly, if $\alpha(\hat{\nu}_1) \ne \hat{\nu}_1$, we also obtain that $\coprod_{i=2}^n [\hat{\nu}_{i-1}, \check{\nu}_i] \cap C_\alpha = \varnothing$, and therefore $\mu \notin C_\alpha$, which is a contradiction. The claim is proven. Since $\lambda, \check{\nu}_1 \in C_\alpha$, by arguing just as in the case of $n = 1$ we have that $[\lambda, \check{\nu}_1] \subset C_\alpha$. Now, since $\hat{\nu}_1 \in C_\alpha$, the induction on n proves the lemma. $\square$

Lemma 4.3. *Let $\alpha \in \pi_1(M)$ and let B be an α-invariant branch locus. If $\{\nu_t\}_{0<t<\epsilon}$ is an embedded interval such that $B = \lim_{t\to 0} \nu_t$, then there exists $0 < \epsilon' < \epsilon$ such that ν_t is in C_α for any $t \in (0, \epsilon')$.*

Proof. Let $\{\nu_t\}_{0<t<\epsilon}$ be an embedded interval as described above. Then we have $\alpha(B) = \lim_{t\to 0} \alpha(\nu_t)$. Since $B = \alpha(B)$, the two intervals $\{\nu_t\}_{0<t<\epsilon}$ and $\{\alpha(\nu_t)\}_{0<t<\epsilon}$ are both asymptotic to B from the same direction as t tends to 0. This with the fact that L is a 1-manifold implies that the two intervals coincide near B. Thus, the conclusion of the lemma follows. $\square$

Proposition 4.4. *For any $\alpha \in \pi_1(M)$, C_α is connected and open.*

Proof. First, we will show connectedness. Let λ and μ be any points in C_α, and $\gamma = \coprod_{i=1}^n [\hat{\nu}_{i-1}, \check{\nu}_i]$ ($\hat{\nu}_0 = \lambda$, $\check{\nu}_n = \mu$) the geodesic spine from λ to μ. By Lemma 4.2, we have that $\gamma \subset C_\alpha$, and that $\check{\nu}_i$ and $\hat{\nu}_i$ are fixed by α for each $1 \le i \le n-1$. Now let B_i $(1 \le i \le n-1)$ denote the branch locus which contains both $\check{\nu}_i$ and $\hat{\nu}_i$. Then B_i is α-invariant. Therefore, by Lemma 4.3, there is an interval $\{\nu_t^i\}_{0<t<\epsilon} \subset C_\alpha$ such that $B_i = \lim_{t\to 0} \nu_t^i$. It follows that $\check{\nu}_i$ and $\hat{\nu}_i$ can be joined by a path in $\{\nu_t^i\}_{0<t<\epsilon} \subset C_\alpha$, hence that λ and μ can be joined by some path.

Next, we will prove openness. Let λ be any point in C_α. If $\alpha(\lambda) \neq \lambda$ then the open interval bounded by $\alpha^{-1}(\lambda)$ and $\alpha(\lambda)$ is contained in C_α and contains λ. Thus, λ is an interior point of C_α. If $\alpha(\lambda) = \lambda$, take any point $\mu \in L$ with $\lambda < \mu$. Then the interval $[\lambda, \mu]$ is mapped by α orientation preservingly onto the interval $[\lambda, \alpha(\mu)]$. Since L is an oriented 1-manifold, there must exist $\nu \in (\lambda, \mu]$ such that $[\lambda, \nu)$ is contained in $[\lambda, \mu] \cap [\lambda, \alpha(\mu)]$. This implies that $[\lambda, \nu)$ is contained in C_α. Similarly, we can find $\eta < \lambda$ such that $(\eta, \lambda]$ is contained in C_α. Consequently, we have $\lambda \in (\eta, \nu) \subset C_\alpha$, which means λ is an interior point of C_α. This proves the proposition. $\square$

Here we give some definitions. For a geodesic spine $\gamma = \coprod_{i=1}^n [\hat{\nu}_{i-1}, \check{\nu}_i]$, we call n the *length* of γ and denote it by $l(\gamma)$. Let $\lambda, \mu \in L$. As in [Barbot 1998], we set $d(\lambda, \mu) = l(\gamma) - 1$, where γ is the geodesic spine from λ to μ. Moreover, we define the *fundamental axis* A_α of α by $A_\alpha = \{\lambda \in L \mid d(\lambda, \alpha(\lambda)) \text{ is even}\}$. Notice that $C_\alpha = \{\lambda \in L \mid d(\lambda, \alpha(\lambda)) = 0\}$, and therefore, $C_\alpha \subset A_\alpha$.

Proposition 4.5. *Let $\alpha \in \pi_1(M)$. Suppose there is $\lambda \in L$ such that $d(\lambda, \alpha(\lambda))$ is nonzero and even. Then $C_{\alpha^k} = \varnothing$ for any $k > 0$.*

Proof. Let γ be the geodesic spine joining λ to $\alpha(\lambda)$. Since $d(\lambda, \alpha(\lambda))$ is even and since α preserves the orientation on L, there are no nontrivial overlappings in composing k geodesic spines $\gamma, \alpha(\gamma), \ldots, \alpha^{k-1}(\gamma)$ successively, and the result $\gamma \cup \alpha(\gamma) \cup \cdots \cup \alpha^{k-1}(\gamma)$ is the geodesic spine from λ to $\alpha^k(\lambda)$. Then we have

$d(\lambda, \alpha^k(\lambda)) = kd(\lambda, \alpha(\lambda))$, and therefore $d(\lambda, \alpha^k(\lambda))$ is nonzero and even. By Corollary 2.20 of [Barbot 1998], α^k fixes no points, and stabilizes no branch loci.

By Proposition 2.10 of the same reference, we have that $A = \bigcup_{i\in\mathbb{Z}} \alpha^i(\gamma)$ is the fundamental axis of α^k. Then A can be expressed as a union of intervals $A = \coprod_{i\in\mathbb{Z}}[\mu_i, \nu_i]$ where ν_i and μ_{i+1} belong to a common branch locus. By [Barbot 1998, Corollary 2.11], there is an integer $m \neq 0$ such that $\alpha^k([\mu_i, \nu_i]) = [\mu_{i+m}, \nu_{i+m}]$. Since $d(\mu, \alpha^k(\mu)) = m \neq 0$ for any $\mu \in A$, it follows that $\mu \notin C_{\alpha^k}$. Therefore, we have $C_{\alpha^k} = \varnothing$, because $C_{\alpha^k} \subset A$. □

Lemma 4.6. *Let $\alpha \in \pi_1(M)$ and $\lambda \in L$ be such that $\lambda \notin C_\alpha$ and that $\lambda \in C_{\alpha^k}$ for some $k > 1$. Let $\gamma = \coprod_{i=1}^n [\hat{\nu}_{i-1}, \check{\nu}_i]$ ($\hat{\nu}_0 = \lambda$, $\check{\nu}_n = \alpha(\lambda)$) be the geodesic spine from λ to $\alpha(\lambda)$. Then $\alpha(\check{\nu}_m) = \hat{\nu}_m$ and $\alpha^k(\check{\nu}_m) = \check{\nu}_m$ where $m = l(\gamma)/2$ (which is an integer by the above proposition).*

Proof. Let γ_j be the geodesic spine from λ to $\alpha^j(\lambda)$, and let δ_0 and δ_1 be the geodesic spines from λ to $\check{\nu}_m$, and from $\hat{\nu}_m$ to $\alpha(\lambda)$, respectively. By reversing the transverse orientation of $\mathscr{F}$ if necessary, we can assume that $\check{\nu}_m$ and $\hat{\nu}_m$ belong to a common positive branch locus.

First, we show that $\check{\nu}_m \notin C_\alpha$. Suppose on the contrary that $\check{\nu}_m \in C_\alpha$. Note that the length of the geodesic spine $\alpha(\delta_0)$ joining $\alpha(\lambda)$ to $\alpha(\check{\nu}_m)$ is $l(\gamma)/2$. So if $\check{\nu}_m$ and $\alpha(\check{\nu}_m)$ are comparable, the intersection $\gamma \cap \alpha(\delta_0)$ must coincide with δ_1 as a set. In particular, $\alpha(\delta_0)$ cannot contain $\check{\nu}_m$. Therefore $\check{\nu}_m > \alpha(\check{\nu}_m)$. See Figure 1. Then $\check{\nu}_m > \alpha^{k-1}(\check{\nu}_m)$, and we have

$$\gamma_k = \delta_0 \cup [\check{\nu}_m, \alpha^{k-1}(\check{\nu}_m)] \cup \alpha^{k-1}(\delta_1).$$

Since γ_k passes through $\alpha^{k-1}(\check{\nu}_m)$ and $\alpha^{k-1}(\hat{\nu}_m)$, it follows that λ and $\alpha^k(\lambda)$ are incomparable, which contradicts the choice of λ.

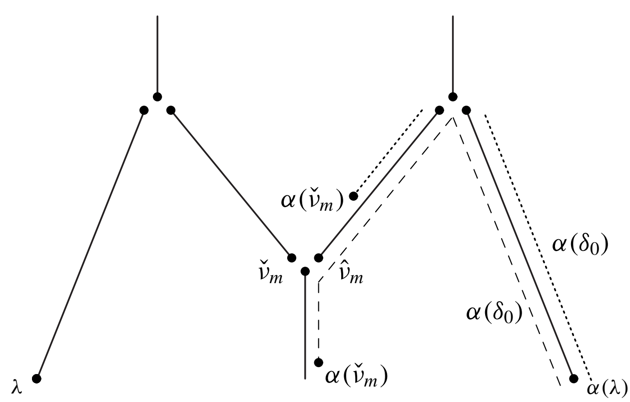

Figure 1. $\alpha(\delta_0)$ is shown as a broken line in the case $\check{\nu}_m \in C_\alpha$, and as a dotted line in the case $\alpha(\check{\nu}_m) \in (\hat{\nu}_m, \check{\nu}_{m+1}]$.

Next, we show that $\alpha(\check{\nu}_m) \notin (\hat{\nu}_m, \check{\nu}_{m+1}]$. Suppose not. Then $\alpha(\check{\nu}_m)$ is in the interval $(\hat{\nu}_m, \check{\nu}_{m+1}]$; that is, the branch locus obtained from the embedded interval $(\hat{\nu}_m, \alpha(\check{\nu}_m))$ contains $\alpha(\hat{\nu}_m)$. It follows that $\hat{\nu}_m$ and $\alpha(\hat{\nu}_m)$ are comparable. See Figure 1. Since we are assuming that $\check{\nu}_m$ and $\hat{\nu}_m$ belong to a common positive branch locus, we have $\hat{\nu}_m < \alpha(\hat{\nu}_m)$. Then $\hat{\nu}_m < \alpha^{k-1}(\hat{\nu}_m)$, and therefore

$$\gamma_k = \delta_0 \cup [\hat{\nu}_m, \alpha^{k-1}(\hat{\nu}_m)] \cup \alpha^{k-1}(\delta_1).$$

Since γ_k passes through $\check{\nu}_m$ and $\hat{\nu}_m$, it follows that λ and $\alpha^k(\lambda)$ are incomparable, which is a contradiction.

Finally, we consider other cases. If $\alpha(\check{\nu}_m) \neq \hat{\nu}_m$, we have

$$l(\alpha^{j+1}(\gamma) - \alpha^j(\gamma)) > l(\gamma)/2 \quad \text{for all } 0 \le j < k.$$

Therefore, we have

$$1 < l(\gamma_1) < l(\gamma_2) < \cdots < l(\gamma_k) = 1.$$

This contradiction shows that $\alpha(\check{\nu}_m) = \hat{\nu}_m$. In particular, $\alpha(\check{\nu}_m)$ is nonseparated from $\check{\nu}_m$ on the negative side. So $\alpha^k(\check{\nu}_m)$ is also nonseparated from $\check{\nu}_m$ on the negative side. We also have that $\alpha^k(\check{\nu}_m) = \check{\nu}_m$. Otherwise, $\gamma_k = \delta_0 \cup \alpha^k(\delta_0)$, and therefore γ_k passes through $\check{\nu}_m$ and $\alpha^k(\check{\nu}_m)$, which belong the common branch locus. It follows that $\lambda \notin C_{\alpha^k}$, which is a contradiction. □

5. Branch loci and their stabilizers

In this section we focus on a branch locus of the leaf space L. We consider the case where a branch locus is a finite set and clarify the structure of the stabilizer of such a locus.

Lemma 5.1. *Let B be a finite branch locus and let $\alpha \in \mathrm{Stab}(B)$. If $\rho_{\mathrm{univ}}(\alpha)$ has a fixed point in S^1_{univ}, then α fixes B pointwise.*

Proof. Let $\alpha \in \mathrm{Stab}(B)$ be a nontrivial element satisfying the hypothesis of the lemma, and let λ be any point of B. Then, since B is finite, there exists some $k \in \mathbb{N}$ such that $\alpha^k(\lambda) = \lambda$. Notice here that α^k is nontrivial in $\pi_1(M)$, because by tautness of $\mathscr{F}$ and by Novikov's theorem [1965] our manifold M is aspherical and hence has no torsion in $\pi_1(M)$ (see [Hempel 1976, Corollary 9.9]).

Now, let us suppose by contradiction that $\alpha(\lambda) \neq \lambda$. Let $r \in S^1_\infty(\lambda)$ be the turning point from $\alpha(\lambda)$ to λ and let $p \in S^1_\infty(\lambda)$ be one of the two fixed points of α^k which is different from r. Then the special section σ_p in S^1_{univ} is fixed by $\rho_{\mathrm{univ}}(\alpha^k)$. This with the hypothesis that $\rho_{\mathrm{univ}}(\alpha)$ has a fixed point implies that σ_p must be fixed by $\rho_{\mathrm{univ}}(\alpha)$ itself. So we have $\rho_{\mathrm{univ}}(\alpha)(\sigma_p) \in I_p$. On the other hand, since $\alpha(p) \in S^1_\infty(\alpha(\lambda))$, it follows from the definition of turning point that $\rho_{\mathrm{univ}}(\alpha)(\sigma_p) = \sigma_{\alpha(p)} \in I_r$. This is a contradiction because I_p and I_r are disjoint. □

Theorem 5.2. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M. Suppose $\mathcal{F}$ has a finite branch locus B. If an element of* $\mathrm{Stab}(B)$ *fixes some point of B then it fixes all the points of B.*

Proof. Let λ be the α-fixed point in B, and let $p, q \in S^1_\infty(\lambda)$ be the fixed points of α. Then $\sigma_p, \sigma_q \in S^1_{\text{univ}}$ are fixed by $\rho_{\text{univ}}(\alpha)$. The result follows from Lemma 5.1. □

The next result gives information on topological types of leaves in a finite branch locus.

Theorem 5.3. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M. Let B be a branch locus of L. Then,*

(1) *if* $\mathrm{Stab}(B)$ *is trivial, $\underline{\lambda}$ is diffeomorphic to a plane, and*

(2) *if B is finite and if* $\mathrm{Stab}(B)$ *is nontrivial, $\underline{\lambda}$ is diffeomorphic to a cylinder,*

for any $\lambda \in B$.

Proof. Let $\lambda \in B$. Since $\mathcal{F}$ is taut, by Novikov's theorem the inclusion map of each leaf of $\mathcal{F}$ into M is π_1-injective. So, if $\underline{\lambda}$ is not a plane, there exists a nontrivial element $\alpha \in \pi_1(M)$ such that $\alpha(\lambda) = \lambda$. This α must belong to $\mathrm{Stab}(B)$, showing the first statement of the theorem.

To prove the second statement, suppose that B is finite and that $\mathrm{Stab}(B)$ is nontrivial. Then, we can first observe that $\underline{\lambda}$ is not a plane. In fact, let γ be any nontrivial element of $\mathrm{Stab}(B)$. Since B is finite, $\gamma^n(\lambda) = \lambda$ for some $n \in \mathbb{N}$. By the same argument as in the proof of Lemma 5.1 we see that γ^n nontrivial in $\pi_1(M)$. This shows the observation.

Now, by way of contradiction, let us assume $\underline{\lambda}$ is not a cylinder, either. Then, again by π_1-injectivity of the inclusion $\underline{\lambda} \to M$, we can find elements $\alpha, \beta \in \pi_1(M)$ generating a free subgroup of rank 2 such that $\alpha(\lambda) = \beta(\lambda) = \lambda$. These two elements are hyperbolic as isometries of λ and having no common fixed point on $S^1_\infty(\lambda)$. Let μ be another leaf in B, and let $r \in S^1_\infty(\lambda)$ be the turning point from μ to λ. By exchanging α and β if necessary, we may assume $\alpha(r) \neq r$. Then, $\alpha^k(r) \neq \alpha^l(r)$ for any $k \neq l \in \mathbb{Z}$. Pick a point $s \in S^1_\infty(\mu)$ and consider the special section σ_s at s. Then, $\rho_{\text{univ}}(\alpha^k)(\sigma_s) = \sigma_{\alpha^k(s)}$ is the special section at $\alpha^k(s)$. Since $\sigma_{\alpha^k(s)}(\lambda) = \phi_\lambda \circ \rho_{\text{univ}}(\alpha^k)(\sigma_s) = \alpha^k \circ \phi_\lambda(\sigma_s) = \alpha^k(r)$, it follows that $\alpha^k(r)$ is the turning point from $\alpha^k(\mu)$ to λ. In particular, $\alpha^k(\mu) \neq \alpha^l(\mu)$ for $k \neq l$; hence, B contains infinitely many elements $\alpha^k(\mu)$, $k \in \mathbb{Z}$, contradicting the finiteness of B. □

Remark 5.4. The author does not know whether or not there exists a branch locus which has a trivial stabilizer.

Proposition 5.5. *Let $B = \{\lambda_1, \dots, \lambda_n\}$ be a finite branch locus which has a nontrivial stabilizer and let $r_i^j \in S^1_\infty(\lambda_i)$ be the turning point from λ_j to λ_i. Then there exists $1 \leq k \leq n$ such that the set of turning points $\{r_k^j \mid j \neq k\}$ is a single point in $S^1_\infty(\lambda_k)$.*

Proof. By Theorem 5.3, each $\underline{\lambda_i}$ is a cylindrical leaf. Let γ be a generator of $\mathrm{Stab}(\lambda_1)=\{\alpha\in\pi_1(M)\mid\alpha(\lambda_1)=\lambda_1\}$. By Theorem 5.2, γ fixes all points in B. Let $p_i, q_i\in S^1_\infty(\lambda_i)$ be the fixed points of γ acting on $S^1_\infty(\lambda_i)$. Note that $r_i^j\in\{p_i,q_i\}$ for any i, j. Otherwise, B cannot be finite by the same argument as in the proof of Theorem 5.3.

We suppose that $\{r_1^j\mid j\neq 1\}=\{p_1,q_1\}$. After renumbering the indices if necessary, we can assume that $r_1^j=p_1$ for $2\le j<n_1$ and $r_1^j=q_1$ for $n_1\le j\le n$, where $3\le n_1\le n$. Then, we claim that $r_{n_1}^j=r_{n_1}^1$ for $1\le j<n_1$. In fact, let $2\le j<n_1$, and take 4 points x, y, z, w as follows: x, y are in $S^1_\infty(\lambda_1)-\{p_1,q_1\}$ such that the 4-tuple (p_1,x,q_1,y) is circularly ordered, $z\in S^1_\infty(\lambda_j)$ and $w\in S^1_\infty(\lambda_{n_1})-\{r_{n_1}^1\}$. Then, $\sigma_z\in I_{p_1}$, $\sigma_w\in I_{q_1}$ and the 4-tuple $(I_{p_1},\sigma_x,I_{q_1},\sigma_y)$ is circularly ordered in S^1_{univ}. Furthermore, $\sigma_x,\sigma_y\in I_{r_{n_1}^1}$ and $\sigma_w\notin I_{r_{n_1}^1}$. It follows that $\sigma_z\in I_{r_{n_1}^1}$; that is, $r_{n_1}^1$ is the turning point from λ_j to λ_{n_1}. This proves the claim.

Now, if $\{r_{n_1}^j\mid j\neq n_1\}=\{r_{n_1}^1\}$ we can put $k=n_1$. Otherwise, by renumbering the indices again, we can assume that $r_{n_1}^j=r_{n_1}^1=p_{n_1}$ for $1\le j<n_2$ $(j\neq n_1)$, and $r_{n_1}^j=q_{n_1}$ for $n_2\le j\le n$, where $n_1<n_2\le n$. Similarly, we have $r_{n_2}^j=r_{n_2}^1$ for $1\le j<n_2$. Since B is finite, we can find a desired k after repeating this process finitely many times. □

Theorem 5.6. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M. Let B be a finite branch locus of L with a nontrivial stabilizer. Then the stabilizer* $\mathrm{Stab}(B)$ *is isomorphic to* $\mathbb{Z}$.

Proof. Let $B=\{\lambda_1,\dots,\lambda_n\}$, and let $r_i^j\in S^1_\infty(\lambda_i)$ be the turning point from λ_j to λ_i for $i\neq j$. By Proposition 5.5, without loss of generality we can assume that $\{r_1^j\mid j\neq 1\}$ is a single point. Let γ be a generator of $\mathrm{Stab}(\lambda_1)$.

Now, if $\mathrm{Stab}(B)$ acts on B trivially, then each $\alpha\in\mathrm{Stab}(B)$ fixes λ_1. It follows that there exists an integer k such that $\alpha=\gamma^k$; that is, γ is a generator of $\mathrm{Stab}(B)$.

So we assume that $\mathrm{Stab}(B)$ acts on B nontrivially. By Theorem 5.2, γ fixes every point λ_i in B. Let $p_i, q_i\in S^1_\infty(\lambda_i)$ be the fixed points of γ acting on $S^1_\infty(\lambda_i)$. Put $\mathrm{Stab}(B)(\lambda_1)=\{\alpha(\lambda_1)\mid\alpha\in\mathrm{Stab}(B)\}=\{\lambda_1,\dots,\lambda_m\}$ where $1<m\le n$. Since the natural action preserves the set of turning points, $\{r_i^j\mid 1\le j\le n, j\neq i\}$ is also a single point for any $i\le m$. Let us denote this single point by p_i. It follows that the subset $\{\sigma_{p_i}\mid 1\le i\le m\}$ of S^1_{univ} is kept invariant by homeomorphisms $\rho_{\mathrm{univ}}(\alpha)$ for $\alpha\in\mathrm{Stab}(B)$. After renumbering indices if necessary, we can assume that the m-tuple $(\sigma_{p_1},\dots,\sigma_{p_m})$ is circularly ordered in S^1_{univ}. Let $\beta\in\mathrm{Stab}(B)$ be such that $\rho_{\mathrm{univ}}(\beta)(\sigma_{p_1})=\sigma_{p_2}$; that is, $\beta(\lambda_1)=\lambda_2$. Since $\rho_{\mathrm{univ}}(\beta)$ preserves the circular order on S^1_{univ}, we have $\beta(\lambda_i)=\lambda_{i+1}$ where the indices i are taken modulo m.

Now, since $\beta\gamma\beta^{-1}(\lambda_1)=\lambda_1$, it follows that $\beta\gamma\beta^{-1}=\gamma^k$ for some $k\neq 0$. Moreover, there is $l\neq 0$ such that $\beta^m=\gamma^l$. It follows that $\beta^{km}=\gamma^{kl}=\beta\gamma^l\beta^{-1}=\beta^m$; that is, $\beta^{(k-1)m}$ is trivial. If $k\neq 1$, β is a torsion element in $\pi_1(M)$, which is a contradiction. Therefore $k = 1$ and we have that γ and β commute. Since $\pi_1(M)$ is torsion-free, the subgroup $\langle \gamma, \beta \mid \gamma^l \beta^{-m} \rangle$ must be isomorphic to $\mathbb{Z}$. It follows that there is $\delta \in \pi_1(M)$ such that $\gamma = \delta^i$ and $\beta = \delta^j$ where $i \neq 0$ and $j \neq 0$. Let α be any element in $\mathrm{Stab}(B)$. Then $\alpha(\lambda_1) = \lambda_i$ for some $1 \leq i \leq m$. By the choice of γ and β, we have that α can be represented as a word in γ and β, and hence in δ. It follows that $\mathrm{Stab}(B)$ is isomorphic to $\mathbb{Z}$. □

We say that $\alpha \in \pi_1(M)$ is *infinitely divisible* if for any integer ℓ, there are $k > \ell$ and $\beta \in \pi_1(M)$ such that $\alpha = \beta^k$.

Theorem 5.7. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M. Let B be a finite branch locus of L such that* $\mathrm{Stab}(B)$ *acts on B nontrivially. Then a generator of* $\mathrm{Stab}(B)$ $(\cong \mathbb{Z})$ *is indivisible.*

Proof. Let $B = \{\lambda_1, \ldots, \lambda_n\}$. By Theorem 5.6, $\mathrm{Stab}(B)$ is generated by some single element α. We assume by contradiction that α is divisible. Since M is aspherical (as was noted in the proof of Lemma 5.1), $\pi_1(M)$ has no infinitely divisible elements (see [Friedl 2011, Theorem 4.1]). Hence, there exists an indivisible element β in $\pi_1(M)$ such that $\alpha = \beta^k$ for some $k > 1$.

Note that since $\beta \notin \mathrm{Stab}(B)$, the points $\lambda_i \in B$ and $\beta(\lambda_i) \in \beta(B)$ are distinct for any i. Moreover, we see that they are incomparable for any i. In fact, if λ_i and $\beta(\lambda_i)$ were comparable, say, $\lambda_i < \beta(\lambda_i)$, then $\lambda_i < \beta^k(\lambda_i) = \alpha(\lambda_i)$, contradicting the assumption that $\alpha \in \mathrm{Stab}(B)$.

Let $\{\nu_t\}_{0<t<\epsilon}$ be an embedded interval such that $B = \lim_{t\to 0} \nu_t$. Since B is α-invariant, it follows from Lemma 4.3 that there is some $\nu \in \{\nu_t\}_{0<t<\epsilon}$ such that $\nu \in C_\alpha = C_{\beta^k}$. We can (and do) take such ν so that ν also satisfies that $\nu \notin C_\beta$. Let $\coprod_{i=1}^{l} [\hat{\nu}_{i-1}, \check{\nu}_i]$ $(l > 1)$ be the geodesic spine joining ν to $\beta(\nu)$. By the choice of ν and by Lemma 4.6, we have $\beta^k(\check{\nu}_m) = \check{\nu}_m$ where $m = l/2$. It follows that $\rho_{\mathrm{univ}}(\beta^k)$ has a fixed point in S^1_{univ}. By Lemma 5.1, $\beta^k = \alpha$ fixes all points in B, which is a contradiction to the hypothesis, as α generates $\mathrm{Stab}(B)$ and $\mathrm{Stab}(B)$ acts on B nontrivially. □

Remark 5.8. The author does not know whether or not there is a finite branch locus B such that $\mathrm{Stab}(B)$ acts on B trivially and is generated by a divisible element.

We will give an example of a tautly foliated compact 3-manifold admitting a finite branch locus whose stabilizer acts on the locus nontrivially. We remark that a recipe how to construct such a locus has already been provided in [Calegari and Dunfield 2003, Example 3.7], and our construction follows it.

Example 5.9. Let $P = D^2 \setminus (E_1 \cup E_2)$ be the unit disk in $\mathbb{C}$ with two open disks removed, where E_1, E_2 are disks centered in $-\frac{1}{2}, \frac{1}{2}$ with radius $\frac{1}{4}$ respectively. Put $S_0 = \partial D^2$, $S_1 = \partial E_1$ and $S_2 = \partial E_2$. On P we consider a standard singular foliation $\mathcal{G}$ (see Figure 2) satisfying the following properties:

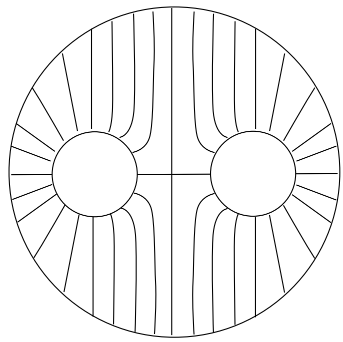

Figure 2. A singular foliation $\mathcal{G}$ of P.

(1) $\mathcal{G}$ has the origin as its unique singular point, which is of saddle type.

(2) $\mathcal{G}$ is transverse to ∂P.

(3) All leaves of $\mathcal{G}$ (except the 4 separatrices) are compact.

(4) $\mathcal{G}$ is symmetric with respect to both the x-axis and y-axis.

(5) The holonomy maps $h_1 : S_1 \setminus \{-\frac{1}{4}\} \to S_0$ and $h_2 : S_2 \setminus \{\frac{1}{4}\} \to S_0$ of $\mathcal{G}$ are given by

$$\begin{aligned} h_1\big(\tfrac{1}{4}e^{2\pi i\theta} - \tfrac{1}{2}\big) &= e^{\pi i(\theta+\frac{1}{2})} && \text{if } 0 < \theta < 1, \\ h_2\big(\tfrac{1}{4}e^{2\pi i\theta} + \tfrac{1}{2}\big) &= e^{\pi i\theta} && \text{if } -\tfrac{1}{2} < \theta < \tfrac{1}{2}. \end{aligned}$$

Let $(P', \mathcal{G}')$ be a copy of $(P, \mathcal{G})$, and let $c : P \to P'$ be the map induced by the identity. We construct a double $\Sigma = P \cup P'$ using diffeomorphisms $g_i : S_i \to c(S_i)$ (for $i = 0, 1, 2$) to glue S_i to $c(S_i)$, where $c^{-1} \circ g_0$ is given by

$$c^{-1} \circ g_0(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$$

for some $\alpha \in \mathbb{R} - \mathbb{Q}$, and $c^{-1} \circ g_i$ is the antipodal map of S_i for $i = 1, 2$. Since h_1, h_2 preserve rational (with respect to θ) points in S_1, S_2 and S_0, it follows that $\mathcal{G}$ and $\mathcal{G}'$ induce a singular foliation $\mathcal{G}''$ of Σ with two saddle singularities and without any saddle connection. By construction, the homeomorphism ρ of Σ which is defined to be the rotation by π in both P and P' preserves $\mathcal{G}''$.

Fix a hyperbolic structure on Σ. Then each leaf of $\mathcal{G}''$ except the singular points and the separatrices is isotopic to a unique embedded geodesic, and the closure of the union of these geodesics constitutes a geodesic lamination, say, λ, on Σ. Note that the two complementary regions Q_1 and Q_2 to λ are ideal open squares. There exists a λ-preserving homeomorphism ψ of Σ isotopic to ρ. Let M be the mapping torus of ψ, that is, $M = \Sigma \times [0, 1]/(s, 1) \sim (\psi(s), 0)$. Then λ induces a surface lamination Λ of M whose complementary regions R_i are Q_i-bundles over S^1 for $i = 1, 2$. Denote by $p_i : R_i \to S^1$ the bundle projection.

Now we extend Λ to a foliation $\mathcal{F}$ of M by filling R_i (for $i = 1, 2$) with leaves diffeomorphic to Q_i as follows. Denote the boundary components of R_i by C_{i1} and C_{i2}, which are open cylinders. Let γ_i be an oriented loop in R_i such that $p_i|\gamma_i$ is a diffeomorphism onto S^1. Then the composition $\gamma_i^2 = \gamma_i * \gamma_i$ is freely homotopic to a leaf loop γ_{ij} of C_{ij} which is a generator of $\pi_1(C_{ij})$. We foliate R_i as a product by leaves isotopic to the fibers Q_i so that the holonomy along γ_{i1} is contracting and the holonomy along γ_{i2} is expanding. Then the resulting foliation $\mathcal{F}$ is taut and has two-sided branching, and each end of a lift of γ_i to $\widetilde{M}$ gives a branch locus consisting of two points. Let α_i be an element in $\pi_1(M)$ whose conjugacy class corresponds with the free homotopy class of γ_i. Then α_i belongs to the stabilizer of some branch locus and acts on the locus nontrivially, as desired.

6. Loops and actions

Given a loop in a tautly foliated manifold $(M, \mathcal{F})$, it is natural to ask whether it is transversable, or tangentiable, to $\mathcal{F}$. In this section, we observe that these properties of loops are expressed completely in the language of the natural action. Furthermore, we consider relations between such properties and the branching phenomenon of $\widetilde{\mathcal{F}}$.

We do not need to assume leafwise hyperbolicity in the first two propositions below.

Proposition 6.1. *Let γ be a loop in M, and α an element in $\pi_1(M, p)$ whose conjugacy class corresponds with the free homotopy class of γ. Then, γ is tangentiable if and only if the action of α on L has a fixed point. Similarly, γ is positively (resp. negatively) transversable if and only if there is a point λ in L such that $\alpha(\lambda) > \lambda$ (resp. $\alpha(\lambda) < \lambda$).*

Proof. Let λ be a leaf of $\widetilde{\mathcal{F}}$ and suppose that the deck transformation α leaves λ invariant. Take any point x in λ and join x to $\alpha(x)$ by a path in λ. Then it projects down to a leaf loop in M freely homotopic to α. Conversely, suppose γ is a leaf loop in M. Join the base point p to a point of γ by a path c. Then, the loop $c * \gamma * c^{-1}$ represents an element of $\pi_1(M, p)$ conjugate to α. Obviously it has a fixed point, hence so does α. The claim on transversability is also shown easily. □

We remark here that $\pi_1(M)$ can have an element which is neither tangentiable nor transversable. Such an element exists if and only if $\mathcal{F}$ has two-sided branching. This fact is due to Barbot, and also follows from Lemma 4.1 and Proposition 4.5. (Notice that if $\mathcal{F}$ has two-sided branching, there are $\lambda, \mu \in L$ such that $d(\lambda, \mu)$ is nonzero and even. Then by the tautness of $\mathcal{F}$, we can find $\nu \in L$ which satisfies $d(\mu, \nu) = 0$, $d(\lambda, \nu) = d(\lambda, \mu)$, and $\alpha(\lambda) = \nu$ for some $\alpha \in \pi_1(M)$.)

Proposition 6.2. *Let $\alpha \in \pi_1(M)$. Suppose there are points $\lambda, \mu \in L$ such that $\alpha(\lambda) > \lambda$ and $\alpha(\mu) < \mu$. Then there exists a point $\nu \in L$ such that $\alpha(\nu) = \nu$.*

Moreover, if λ and μ are incomparable, then such ν can be found in some branch locus.

Proof. If λ and μ are comparable, then the conclusion follows immediately from the intermediate value theorem. If λ and μ are incomparable, then the conclusion follows from Lemma 4.2. $\square$

This proposition means that if a loop in M is both positively and negatively transversable to $\mathcal{F}$, then it is tangentiable to $\mathcal{F}$.

In the following we assume leafwise hyperbolicity and observe that tangentiability and/or transversability of loops in M and the infiniteness of branch loci are closely related.

Theorem 6.3. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M with one-sided branching. Suppose that there is a noncontractible leaf loop γ in M which is not transversable. Then every branch locus of L is an infinite set.*

Proof. Suppose that there exists a finite (say, positive) branch locus $B = \{\lambda_1, \ldots, \lambda_n\}$. Let α be an element in $\pi_1(M)$ whose conjugacy class corresponds with the free homotopy class of γ. By Proposition 6.1, α has a fixed point in L, and for each $\mu \in L$ if μ is not fixed by α then $\mu \notin C_\alpha$. Let ν be a fixed point of α. By Lemma 4.1, for every η with $\eta \leq \nu$, we have $\eta \in C_\alpha$, and therefore $\alpha(\eta) = \eta$. By replacing B with $\beta(B)$ for some $\beta \in \pi_1(M)$ if necessary, we can assume that $\lambda_1 \leq \nu$ and therefore $\alpha(\lambda_1) = \lambda_1$. This implies in particular that B is α-invariant. Since B is finite, by Theorem 5.2 we have $\alpha(\lambda_i) = \lambda_i$ for any $1 \leq i \leq n$. By Lemma 3.1, α must be trivial, which is a contradiction. $\square$

Corollary 6.4. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M with every leaf dense. Suppose that there is noncontractible leaf loop γ in M which is not transversable. Then every branch locus of L is an infinite set.*

Proof. Suppose there is a finite branch locus B. Let $\alpha \in \pi_1(M)$ be as in the proof of the preceding theorem. By Proposition 4.4, there is an embedded open interval $I \subset L$ such that I is contained in C_α. Since every leaf of $\mathcal{F}$ is dense, there is $\beta \in \pi_1(M)$ such that $\beta(B) \cap I \neq \varnothing$. Then the same argument as in Theorem 6.3 shows the conclusion. $\square$

Theorem 6.5. *Let M be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of M with branching. Suppose that there is a noncontractible leaf loop γ in M which is not transversable. Then L has an infinite branch locus.*

Proof. Let α be as in Theorem 6.3. Then α has a fixed point $\nu \in L$, and for each $\mu \in L$ if μ is not fixed by α then $\mu \notin C_\alpha$. Without loss of generality, we assume that $\mathcal{F}$ has a positive branch locus.

We claim that there exist some $\nu' > \nu$ such that ν' and $\alpha(\nu')$ are incomparable. Put $L' = \{\mu \mid \mu > \nu\}$. Notice that $\alpha(L') = L'$. Then we can observe that L' is a submanifold of L with one-sided branching in the positive direction and contains at least one branch locus. For, by the tautness of $\mathcal{F}$ we can find a positive branch locus B' in L and $\beta \in \pi_1(M)$ such that $\beta(\nu)$ is a common lower bound of all points in B'; that is, $\beta^{-1}(B') \subset L'$. If α fixes all leaves in L', then by applying Lemma 3.1 to a branch locus in L' we obtain that α is trivial in $\pi_1(M)$, which contradicts the hypothesis that α is represented by a noncontractible loop. Therefore, there exists some $\nu' \in L$ which is not fixed by α. Since such ν' does not belong to C_α, the claim is shown.

Since $\nu < \nu'$ and $\alpha(\nu) = \nu$, it follows that ν is a common lower bound for ν' and $\alpha(\nu')$. Thus, the fact that ν' and $\alpha(\nu')$ are incomparable implies that there is a unique $\lambda \in (\nu, \nu']$ such that $\mu \in [\nu, \nu']$ is fixed by α if and only if $\mu \in [\nu, \lambda)$. Evidently, λ belongs to some α-invariant branch locus, say, B. Also note that $\rho_{\text{univ}}(\alpha)$ has a fixed point because α fixes a point in L. We now show B is infinite. Suppose not. Then, by Lemma 5.1, all leaves in B are α-fixed, contradicting $\alpha(\lambda) \neq \lambda$. □

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