NONFIBERED L-SPACE KNOTS

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We construct an infinite family of knots in rational homology spheres with irreducible, nonfibered complements, for which every nonlongitudinal filling is an L-space.

The Heegaard Floer homology of a rational homology three-sphere Y is an abelian group $\widehat{HF}(Y)$ satisfying rk $\widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|$ [Ozsváth and Szabó 2004]. When equality is realized in this bound Y is called an L-space and any knot in Y admitting a nontrivial L-space surgery is called an L-space knot [Ozsváth and Szabó 2005]. A result of Ghiggini [2008] and Ni [2007] shows that L-space knots in the three-sphere must be fibered. Since manifolds with finite fundamental group provide examples of L-spaces,[1] this result implies that a knot K in S^3 admitting a finite filling must be fibered. This observation should be compared with other restrictions related to finite fillings such as the Cyclic Surgery Theorem [Culler et al. 1987] and its extensions [Boyer and Zhang 2001].

The restriction to knots in S^3 is not necessary. It is shown in [Boileau et al. 2012] that a primitive knot[2] in an irreducible L-space admitting a nontrivial L-space surgery must be fibered. Irreducibility of the complement is required: removing an unknot from an embedded three-ball in any L-space produces a nonfibered manifold with nontrivial L-space fillings. Even in the general setting of knots in rational homology spheres with irreducible complements fibered is not a necessary condition:

Theorem 1. *There exist infinitely many irreducible, nonfibered knot complements such that all nonlongitudinal Dehn fillings are L-spaces. Moreover, these examples arise as knots in manifolds with finite fundamental group.*

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[1]Ozsváth and Szabó [2005] have shown that manifolds admitting elliptic geometry are L-spaces; Perelman's Geometrization Theorem (see [Kleiner and Lott 2008], for example) implies that three-manifolds with finite fundamental group admit elliptic geometry.

[2]Recall that a knot K is primitive in Y if $[K] \in H_1(Y; \mathbb{Z})$ is a generator.

In particular, our examples are nonprimitive knots in L-spaces.

Before turning to the construction, we fix some terminology. Fibrations will always be locally trivial surface bundles over a circle and we say the total space *fibers*. To avoid confusion, we will refer to Seifert fibrations as Seifert structures; these are foliations of a manifold by circles. The base orbifold is the leaf space of such a foliation, where the (possibly empty) collection of cone points records the multiplicities of the exceptional fibers in the Seifert structure. A circle bundle is a Seifert structure for which there are no exceptional fibers.

Given a three-manifold M with torus boundary, a slope α is a primitive class in $H_1(\partial M; \mathbb{Z})/\{\pm 1\}$. We use $M(\alpha)$ to denote Dehn filling along α. If $\partial M = T_1 \cup T_2$, for tori T_i, then we denote α-filling on T_1 (respectively T_2) by $M(\alpha, -)$ (respectively $M(-, \alpha)$). When M admits a Seifert structure, the slope given by a regular fiber in the boundary is called the fiber slope. For background on Seifert structures and Dehn filling we refer the reader to [Boyer 2002]. A key fact is that Dehn filling a Seifert manifold with torus boundary along any slope α other than the fiber slope results in a Seifert manifold with a possible additional singular fiber. The multiplicity of this new fiber is $\Delta(\alpha, \varphi)$, the distance between the slopes α and ϕ [Heil 1974].

Finally, for knots in rational homology three-spheres recall that there is a preferred slope given by the rational longitude. This slope is characterized by the property that some number of like-oriented parallel copies in the boundary of the knot complement bounds a properly embedded surface. We will refer to this slope as the longitude. Note that an oriented three-manifold M with torus boundary for which $H_1(M; \mathbb{Q}) \cong \mathbb{Q}$ always arises (nonuniquely) as the complement of a knot in a rational homology three-sphere.

1. The twisted I-bundle over the Klein bottle

Let N denote the twisted I-bundle over the Klein bottle. As this orientable three-manifold with torus boundary plays a central role in our construction we will study its construction in depth.

First consider the group G generated by $f, g \colon \mathbb{R}^3 \to \mathbb{R}^3$, where

$$f(x, y, z) = (x + 1, y, z),$$
$$g(x, y, z) = (-x, y + 1, -z),$$

and consider the noncompact, orientable three-manifold $N^\circ = \mathbb{R}^3/G$. Note that the z-component of $\mathbb{R}^3$ gives N° the structure of a line bundle, the zero-section of which is a Klein bottle; this is the unique line bundle over the Klein bottle with orientable total space. By restricting the action of G to $\widetilde{N} = \mathbb{R}^2 \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathbb{R}^3$ we obtain the twisted I-bundle over the Klein bottle $N = \widetilde{N}/G$.

From this description two Seifert structures on N become apparent: the x- and y-components of $\widetilde{N}$ both determine foliations of N by circles. (This is essentially the observation that the Klein bottle is foliated by circles in two ways.) The leaf space of the foliation described by the x-components is a Möbius strip without cone points. Denote a regular fiber in this Seifert structure by ϕ_0. The base orbifold of the foliation determined by y-components is $D^2(2, 2)$, with regular fiber denoted ϕ_1; this follows readily from a natural Heegaard decomposition which we now describe.

From the preceding construction, a fundamental domain for N is obtained by taking $\left[-\frac{1}{2}, \frac{1}{2}\right)^2 \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \mathbb{R}^3$. Then given a disk D^2 of radius less than $\frac{1}{2}$ (and centered at the origin of the xy-plane), the result of removing $D^2 \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ is a genus-two handlebody. This gives rise to a Heegaard decomposition for N; a Heegaard diagram corresponding to this decomposition is described in Figure 1, from which the fundamental group $\pi_1(N) = \langle a, b \mid a^2b^2 \rangle$ may be calculated. Note that since $fgfg^{-1}$ is trivial in the group G, the homomorphism determined by $a \mapsto fgf^{-1}$ and $b \mapsto fg^{-1}$ is well-defined and gives an isomorphism $G \cong \langle a, b \mid a^2b^2 \rangle$. Further, by considering a separating disk decomposing the handlebody into solid
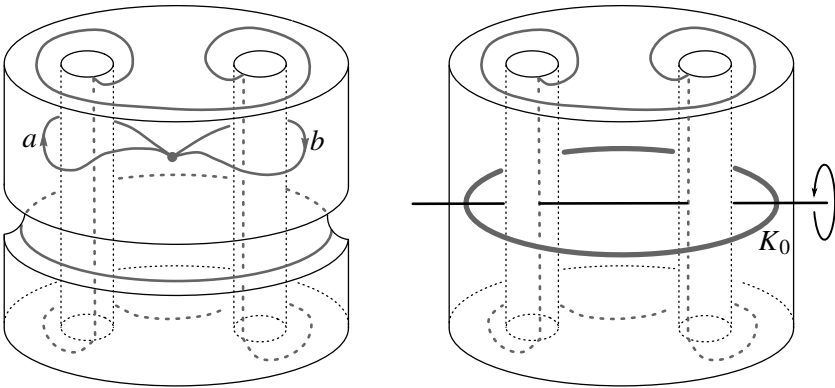
Figure 1. Two views of the Heegaard diagram for the twisted I-bundle over the Klein bottle N. With a and b generating the fundamental group of the genus-two handlebody, N is obtained by attaching a handle along a curve in the boundary representing a^2b^2 so that $\phi_0 \simeq ab$ and $\phi_1 \simeq b^2$. On the left, an annulus in the boundary with core representing the element $\phi_0 \simeq ab$ may be used to find the fundamental group of M, the complement of a regular fiber in the interior of N, via HNN extension. On the right, the axis of rotational symmetry shows that the hyperelliptic involution on the handlebody induces a strong inversion on the pair (N, K_0) where K_0 is a knot in N isotopic to a regular fiber ϕ_0 in the interior of N.

tori, it is immediate that N is the union of two solid tori along essential annuli in the boundary. By fixing Seifert structures on each of these solid tori with base orbifolds $D^2(2)$, these annuli are foliated by regular fibers. The identification along these essential annuli therefore extends to a Seifert structure on N with base orbifold $D^2(2, 2)$ as claimed.

Both Seifert structures induce foliations on the torus ∂N. Let ϕ_0 and ϕ_1 be regular fibers in ∂N, and notice that $\Delta(\phi_0, \phi_1) = 1$. (These conventions are consistent with [Boyer et al. 2013, Section 3].) The longitude of N is homotopic to the element ab (this element has order two in the abelianization of $\pi_1(N)$). That is, ϕ_0 represents the longitude of N. Any filling $N(\alpha)$ for which $\alpha \neq \phi_0, \phi_1$ admits a pair of Seifert structures with base orbifolds $\mathbb{R}P^2(\Delta(\alpha, \phi_0))$ and $S^2(2, 2, \Delta(\alpha, \phi_1))$. We point out that these manifolds always admit elliptic geometry [Scott 1983].

Now consider a knot K_0 in N that is isotopic to a regular fiber ϕ_0 in the interior of N. Define M by removing a neighborhood of K_0 from N; by construction M inherits a Seifert structure (the base orbifold is a punctured Möbius band). Now $\partial M = T_1 \cup T_2$ where T_2 denotes the boundary of a regular neighborhood of K_0.

The fundamental group of M is presented by

$$\pi_1(M) = \langle a, b, t \mid a^2b^2, [t, ab] \rangle.$$

To see this, consult Figure 1 and notice that M may be constructed by identifying (disjoint neighborhoods of) each boundary component of the annulus with core ab in ∂N. This gives rise to the HNN extension presented above. Notice that $M(-, \mu) \cong N$ for any slope on T_2 satisfying $\Delta(\mu, \phi_0) = 1$. A preferred choice for μ is given by a representative of the homotopy class of t in the above presentation.

A final observation pertains to a natural strong inversion on (N, K_0) that descends to an involution on M with one-dimensional fixed point set. Recall that a strong inversion on (N, K_0) is an orientation preserving involution on N that reverses orientation on K_0; such a symmetry is illustrated in Figure 1. The involution on N is induced by the hyperelliptic involution on the genus-two handlebody since the attaching curve is fixed (as a set) by this involution. A fundamental domain for this involution is a three-ball, with one dimensional fixed point set. That is, N is
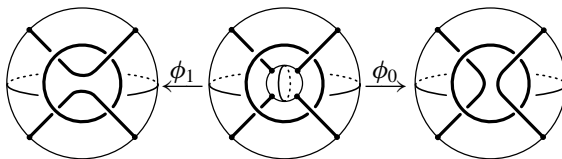
Figure 2. The branch set for the manifold $M = M(-, -)$ with branch sets for the fillings $M(\phi_1, -) = N$ and $M(\phi_0, -)$. Notice that $M(\phi_0, -)$ is reducible, containing an $S^2 \times S^1$ summand.

the twofold branched cover of a two-tangle; this is the leftmost tangle in Figure 2. We leave the following step to the reader: the genus-two handlebody is the twofold branched cover of a three-tangle, and attaching the handle closes one of the arcs (the arc meeting the attaching curve) to an unknotted curve in the branch set. The same construction may be applied to the complement of K_0 in N, to see that M is the twofold branched cover of a tangle in $S^2 \times I$. This tangle is shown in Figure 2.

Towards a proof of Theorem 1, our interest is in the family of manifolds

$$\{M(-, \alpha) \mid \text{for any slope } \alpha \text{ with } \Delta(\alpha, \phi_0) > 1\}.$$

Notice that each manifold in this set admits a Seifert structure with base orbifold a Möbius band with a single cone point of order $\Delta(\alpha, \phi_0)$. Since $M(\phi_1, \alpha)$ admits a Seifert structure with base orbifold $S^2(2, 2, n)$ it follows that $M(-, \alpha)$ is the complement of a knot in an elliptic manifold for all α.

2. The proof of Theorem 1

Let M be the complement of K_0 in the twisted I-bundle over the Klein bottle N. We assume all of the notation introduced in the previous section.

Lemma 2. *Fix a slope α on T_2 with $\Delta(\alpha, \phi_0) = p$. Then*

$$M(\phi_0, \alpha) = \begin{cases} S^2 \times S^1 \,\#\, S^2 \times S^1 & \text{if } p = 0, \\ S^2 \times S^1 \,\#\, L(p, q) & \text{if } p > 1, \\ S^2 \times S^1 & \text{if } p = 1. \end{cases}$$

Proof. Since $\pi_1(M) \cong \langle a, b, t \mid a^2 b^2, [t, ab] \rangle$ and $\phi_0 \simeq ab$, we have that

$$\pi_1(M(\phi_0, -)) \cong \langle a, b, t \mid a^2 b^2, [t, ab] \rangle / \langle\langle ab \rangle\rangle \cong \langle a, b, t \mid ab \rangle.$$

In other words, $\pi_1(M(\phi_0, -)) \cong \mathbb{Z} * \mathbb{Z}$. If $\alpha = p\mu + q\phi_0$, then

$$\pi_1(M(\phi_0, \alpha)) \cong \langle a, b, t \mid ab \rangle / \langle\langle t^p (ab)^q \rangle\rangle \cong \mathbb{Z} * \mathbb{Z}/p.$$

By Whitehead's proof of Kneser's conjecture [Whitehead 1958], $M(\phi_0, \alpha)$ is a connect-sum of closed manifolds Y_1 and Y_2 with $\pi_1(Y_1) \cong \mathbb{Z}$ and $\pi_1(Y_2) \cong \mathbb{Z}/p$. Geometrization now establishes the lemma. $\qquad\square$

Remark 3. Alternatively, Lemma 2 follows from considering $M(\phi_0, -)$ as the double branched cover of a tangle as in Figure 2. The unknotted component gives rise to the $S^2 \times S^1$ summand. Dehn filling corresponds to attaching a rational tangle, which (ignoring the unknotted component) produces a two-bridge link and exhibits the lens space connect-summand.

Proposition 4. *For any α on T_2 with $\Delta(\alpha, \phi_0) > 1$, the manifold $M(-, \alpha)$ does not fiber over the circle.*

Proof. Suppose that $M(-, \alpha)$ fibers. Since ϕ_0 is the longitude, this is the only filling that extends the fibration on $M(-, \alpha)$ as any other filling of $M(-, \alpha)$ results in a rational homology sphere. By Lemma 2, $M(\phi_0, \alpha) \cong S^2 \times S^1 \# L(p, q)$ for $p = \Delta(\phi_0, \alpha) \geq 2$. Since $M(\phi_0, \alpha)$ is fibered and $\pi_2(M(\phi_0, \alpha)) \neq 0$, the fiber surface F must also have $\pi_2(F) \neq 0$ by the long exact sequence for a fibration. Hence F must be S^2 or $\mathbb{R}P^2$. However, $\pi_1(M(\phi_0, \alpha))$ is not the fundamental group of such a fibration, since it does not admit a surjective homomorphism onto $\mathbb{Z}$ with finite kernel. $\square$

Proof of Theorem 1. Fix α with $\Delta(\alpha, \phi_0) \geq 2$. As the fiber slope of the Seifert structure on $M(-, \alpha)$ is the longitude, all nonlongitudinal fillings will extend the Seifert structure, yielding a base orbifold $\mathbb{R}P^2$ with two cone points. By [Boyer et al. 2013, Proposition 5], such manifolds are always L-spaces. Proposition 4 shows that $M(-, \alpha)$ is not fibered. Furthermore, $M(-, \alpha)$ is irreducible, since the only orientable, reducible Seifert manifolds are $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ (and in particular, are closed). Finally, $M(-, \alpha)$ is the complement of a knot in an elliptic manifold as observed in Section 1. $\square$

Remark 5. Further examples may be constructed in an analogous way by removing a regular fiber from any manifold which has a Seifert structure with base orbifold $\mathbb{R}P^2$ with any positive number of singular fibers. It is also possible to construct examples, in a similar manner, admitting Sol geometry. The main observation is that every Sol rational homology sphere is an L-space [Boyer et al. 2013, Theorem 2]. Since every such L-space arises by identifying two twisted I-bundles along the boundary tori, one may consider the complement of the knot K_0 in one of the twisted I-bundles. In this setting, our construction goes through almost verbatim, having noticed that the obvious essential torus must be horizontal to the purported fibration of the exterior of K_0.

Question 6. All of our examples relied on the presence of an essential annulus, and have nonhyperbolic exterior. Do there exist examples of hyperbolic, nonfibered knots for which every nonlongitudinal surgery is an L-space?

Acknowledgments

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