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We establish a relationship between the known parametrization of a family of irreducible representations of a Weyl group and Springer's correspondence.

Introduction

0.1. Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic p. Let W be the Weyl group of G; let $\mathrm{Irr}W$ be a set of representatives for the isomorphism classes of irreducible representations of W over $\bar{\mathbb{Q}}_l$, an algebraic closure of the field of l-adic numbers (l is a fixed prime number other than p).

Now $\mathrm{Irr}W$ is partitioned into subsets called *families* as in [Lusztig 1979b, § 9; 1984a, 4.2]. Moreover to each family $\mathcal{F}$ in $\mathrm{Irr}W$, a certain set $X_{\mathcal{F}}$, a pairing $\{\ ,\ \} : X_{\mathcal{F}} \times X_{\mathcal{F}} \to \bar{\mathbb{Q}}_l$, and an imbedding $\mathcal{F} \to X_{\mathcal{F}}$ was canonically attached in [Lusztig 1979b; 1984a, Chapter 4]. (The set $X_{\mathcal{F}}$ with the pairing $\{\ ,\ \}$, which can be viewed as a nonabelian analogue of a symplectic vector space, plays a key role in the classification of unipotent representations of a finite Chevalley group [Lusztig 1984a] and in that of unipotent character sheaves on G.) In [Lusztig 1979b; 1984a] it is shown that $X_{\mathcal{F}} = M(\mathcal{G}_{\mathcal{F}})$ where $\mathcal{G}_{\mathcal{F}}$ is a certain finite group associated to $\mathcal{F}$ and, for any finite group Γ, $M(\Gamma)$ is the set of all pairs (g, ρ) where g is an element of Γ defined up to conjugacy and ρ is an irreducible representation over $\bar{\mathbb{Q}}_l$ (up to isomorphism) of the centralizer of g in Γ; moreover $\{\ ,\ \}$ is given by the "nonabelian Fourier transform matrix" of [Lusztig 1979b, § 4] for $\mathcal{G}_{\mathcal{F}}$.

In the remainder of this paper we assume that p is not a bad prime for G. In this case a uniform definition of the group $\mathcal{G}_{\mathcal{F}}$ was proposed in [Lusztig 1984a, 13.1] in terms of special unipotent classes in G and the Springer correspondence, but the fact that this leads to a group isomorphic to $\mathcal{G}_{\mathcal{F}}$ as defined in [Lusztig 1984a, Chapter 4] was stated in [Lusztig 1984a, (13.1.3)] without proof. One of the aims of this paper is to supply the missing proof.

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To state the results of this paper we need some definitions. For $E \in \mathrm{Irr} W$ let $a_E \in \mathbb{N}$, $b_E \in \mathbb{N}$ be as in [Lusztig 1984a, 4.1]. As noted in [Lusztig 1979a], for $E \in \mathrm{Irr} W$ we have

(a) $$a_E \leq b_E;$$

we say that E is *special* if $a_E = b_E$.

For $g \in G$ let $Z_G(g)$ or $Z(g)$ be the centralizer of g in G and let $A_G(g)$ or $A(g)$ be the group of connected components of $Z(g)$. Let C be a unipotent conjugacy class in G and let $u \in C$. Let $\mathcal{B}_u$ be the variety of Borel subgroups of G that contain u; this is a nonempty variety of dimension, say, e_C. The conjugation action of $Z(u)$ on $\mathcal{B}_u$ induces an action of $A(u)$ on $\boldsymbol{S}_u := H^{2e_C}(\mathcal{B}_u, \bar{\mathbb{Q}}_l)$. Now W acts on $\boldsymbol{S}_u$ by Springer's representation [Springer 1976]; however here we adopt the definition of the W-action on $\boldsymbol{S}_u$ given in [Lusztig 1984b] which differs from Springer's original definition by tensoring by sign. The W-action on $\boldsymbol{S}_u$ commutes with the $A(u)$-action. Hence we have canonically $\boldsymbol{S}_u = \oplus_{E \in \mathrm{Irr} W} E \otimes \mathcal{V}_E$ (as $W \times A(u)$-modules) where $\mathcal{V}_E$ are finite dimensional $\bar{\mathbb{Q}}_l$-vector spaces with $A(u)$-action. Let $\mathrm{Irr}_C W = \{E \in \mathrm{Irr} W; \mathcal{V}_E \neq 0\}$; this set does not depend on the choice of u in C. By [Springer 1976], the sets $\mathrm{Irr}_C W$ (for C variable) form a partition of $\mathrm{Irr} W$; also, if $E \in \mathrm{Irr}_C W$ then $\mathcal{V}_E$ is an irreducible $A(u)$-module and, if $E \neq E'$ in $\mathrm{Irr}_C W$, then the $A(u)$-modules $\mathcal{V}_E, \mathcal{V}_{E'}$ are not isomorphic. By [Borho and MacPherson 1981] we have

(b) $$e_C \leq b_E \quad \text{for any } E \in \mathrm{Irr}_C W,$$

and the equality $b_E = e_C$ holds for exactly one $E \in \mathrm{Irr}_C W$ which we denote by E_C (for this E, $\mathcal{V}_E$ is the unit representation of $A(u)$).

Following [Lusztig 1984a, (13.1.1)] we say that C is *special* if E_C is special. (This concept was introduced in [Lusztig 1979a, § 9] although the word "special" was not used there.) From (b) we see that C is special if and only if $a_{E_C} = e_C$.

Now assume that C is special. We denote by $\mathcal{F} \subset \mathrm{Irr} W$ the family that contains E_C. (Note that $C \mapsto \mathcal{F}$ is a bijection from the set of special unipotent classes in G to the set of families in $\mathrm{Irr} W$.) We set $\mathrm{Irr}^*_C W = \{E \in \mathrm{Irr}_C W;\ E \in \mathcal{F}\}$ and

$$\mathcal{K}(u) = \{a \in A(u);\, a \text{ acts trivially on } \mathcal{V}_E \text{ for any } E \in \mathrm{Irr}^*_C W\}.$$

This is a normal subgroup of $A(u)$. We set $\bar{A}(u) = A(u)/\mathcal{K}(u)$, a quotient group of $A(u)$. Now, for any $E \in \mathrm{Irr}^*_C W$, $\mathcal{V}_E$ is naturally an (irreducible) $\bar{A}(u)$-module. Another definition of $\bar{A}(u)$ is given in [Lusztig 1984a, (13.1.1)]. In that definition $\mathrm{Irr}^*_C W$ is replaced by $\{E \in \mathrm{Irr}_C W;\, a_E = e_C\}$ and $\mathcal{K}(u)$, $\bar{A}(u)$ are defined as above but in terms of this modified $\mathrm{Irr}^*_C W$. However the two definitions are equivalent in view of the following result.

Proposition 0.2. *Assume that C is special. Let $E \in \mathrm{Irr}_C W$.*

(a) *We have $a_E \leq e_C$.*

(b) *We have $a_E = e_C$ if and only if $E \in \mathcal{F}$.*

This follows from [Lusztig 1992, 10.9]. Note that (a) was stated without proof in [Lusztig 1984a, (13.1.2)] (the proof I had in mind at the time of [Lusztig 1984a] was combinatorial).

0.3. The following result is equivalent to a result stated without proof in [Lusztig 1984a, (13.1.3)].

Theorem 0.4. *Let C be a special unipotent class of G, let $u \in C$ and let $\mathcal{F}$ be the family that contains E_C. Then we have canonically $\boldsymbol{X}_{\mathcal{F}} = M(\bar{A}(u))$ so that the pairing $\{\ ,\ \}$ on $\boldsymbol{X}_{\mathcal{F}}$ coincides with the pairing $\{\ ,\ \}$ on $M(\bar{A}(u))$. Hence $\mathcal{G}_{\mathcal{F}}$ can be taken to be $\bar{A}(u)$.*

This is equivalent to the corresponding statement in the case where G is adjoint, which reduces immediately to the case where G is adjoint simple. It is then enough to prove the theorem for one G in each isogeny class of semisimple, almost simple algebraic groups; this will be done in Section 3 after some combinatorial preliminaries in Sections 1 and 2. The proof uses the explicit description of the Springer correspondence: for type A_n, G_2 in [Springer 1976]; for type B_n, C_n, D_n in [Shoji 1979a; 1979b] (as an algorithm) and in [Lusztig 1984b] (by a closed formula); for type F_4 in [Shoji 1980]; for type E_n in [Alvis and Lusztig 1982; Spaltenstein 1982].

An immediate consequence of (the proof of) Theorem 0.4 is the following result which answers a question of R. Bezrukavnikov and which plays a role in [Losev and Ostrik 2012].

Corollary 0.5. *In the setup of Theorem 0.4 let $E \in \mathrm{Irr}^*_C W$ and let $\mathcal{V}_E$ be the corresponding $A(u)$-module viewed as an (irreducible) $\bar{A}(u)$-module. The image of E under the canonical imbedding $\mathcal{F} \to \boldsymbol{X}_{\mathcal{F}} = M(\bar{A}(u))$ is represented by the pair $(1, \mathcal{V}_E) \in M(\bar{A}(u))$. Conversely, if $E \in \mathcal{F}$ and the image of E under $\mathcal{F} \to \boldsymbol{X}_{\mathcal{F}} = M(\bar{A}(u))$ is represented by the pair $(1, \rho) \in M(\bar{A}(u))$ where ρ is an irreducible representation of $\bar{A}(u)$, then $E \in \mathrm{Irr}^*_C W$ and $\rho \cong \mathcal{V}_E$.*

0.6. Corollary 0.5 has the following interpretation. Let Y be a (unipotent) character sheaf on G whose restriction to the regular semisimple elements is $\neq 0$; assume that in the usual parametrization of unipotent character sheaves by $\bigsqcup_{\mathcal{F}'} \boldsymbol{X}_{\mathcal{F}'}$, Y corresponds to $(1, \rho) \in M(\bar{A}(u))$ where C is the special unipotent class corresponding to a family $\mathcal{F}$, $u \in C$ and ρ is an irreducible representation of $\bar{A}(u)$. Then $Y|_C$ is (up to shift) the irreducible local system on C defined by ρ.

0.7. ***Notation.*** If A, B are subsets of $\mathbb{N}$ we denote by $A \dot{\cup} B$ the union of A and B regarded as a multiset (each element of $A \cap B$ appears twice). For any set $\mathscr{X}$, we denote by $\mathscr{P}(\mathscr{X})$ the set of subsets of $\mathscr{X}$ viewed as an F_2-vector space with sum given by the symmetric difference. If $\mathscr{X} \neq \varnothing$ we note that $\{\varnothing, \mathscr{X}\}$ is a line in $\mathscr{P}(\mathscr{X})$ and we set $\bar{\mathscr{P}}(\mathscr{X}) = \mathscr{P}(\mathscr{X})/\{\varnothing, \mathscr{X}\}$, $\mathscr{P}_{\mathrm{ev}}(\mathscr{X}) = \{L \in \mathscr{P}(\mathscr{X}); |L| = 0 \bmod 2\}$; let $\bar{\mathscr{P}}_{\mathrm{ev}}(\mathscr{X})$ be the image of $\mathscr{P}_{\mathrm{ev}}(\mathscr{X})$ under the obvious map $\mathscr{P}(\mathscr{X}) \to \bar{\mathscr{P}}(\mathscr{X})$ (thus $\bar{\mathscr{P}}_{\mathrm{ev}}(\mathscr{X}) = \bar{\mathscr{P}}(\mathscr{X})$ if $|\mathscr{X}|$ is odd and $\bar{\mathscr{P}}_{\mathrm{ev}}(\mathscr{X})$ is a hyperplane in $\bar{\mathscr{P}}(\mathscr{X})$ if $|\mathscr{X}|$ is even). Now if $\mathscr{X} \neq \varnothing$, the assignment $L, L' \mapsto |L \cap L'| \bmod 2$ defines a symplectic form on $\mathscr{P}_{\mathrm{ev}}(\mathscr{X})$ which induces a nondegenerate symplectic form $(\ ,\ )$ on $\bar{\mathscr{P}}_{\mathrm{ev}}(\mathscr{X})$ via the obvious linear map $\mathscr{P}_{\mathrm{ev}}(\mathscr{X}) \to \bar{\mathscr{P}}_{\mathrm{ev}}(\mathscr{X})$.

For $g \in G$ let g_s and g_ω be the semisimple and unipotent parts of g.

For $z \in \frac{1}{2}\mathbb{Z}$ we set $\lfloor z \rfloor = z$ if $z \in \mathbb{Z}$ and $\lfloor z \rfloor = z - \frac{1}{2}$ if $z \in \mathbb{Z} + \frac{1}{2}$.

Errata to [Lusztig 1984a]. On page 86, on line -6 delete "$b' < b$" and on line -4 before "In the language..." insert "The array above is regarded as identical to the array obtained by interchanging its two rows."

On page 343, line -5, after "respect to M" insert "and where the group $\mathscr{G}_{\mathscr{F}}$ defined in terms of (u', M) is isomorphic to the group $\mathscr{G}_{\mathscr{F}}$ defined in terms of (u, G)".

Erratum to [Lusztig 1984b]. In the definition of A_α, B_α in [Lusztig 1984b, 11.5], the condition $I \in \alpha$ should be replaced by $I \in \alpha'$ and the condition $I \in \alpha'$ should be replaced by $I \in \alpha$.

1. Combinatorics

1.1. Let N be an even integer ≥ 0. Let $a := (a_0, a_1, a_2, \dots, a_N) \in \mathbb{N}^{N+1}$ be such that $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$, $a_0 < a_2 < a_4 < \cdots$, $a_1 < a_3 < a_5 < \cdots$. Let $\mathscr{J} = \{i \in [0, N]; a_i$ appears exactly once in $a\}$. We have $\mathscr{J} = \{i_0, i_1, \dots, i_{2M}\}$ where $M \in \mathbb{N}$ and $i_0 < i_1 < \cdots < i_{2M}$ satisfy $i_s = s \bmod 2$ for $s \in [0, 2M]$. Hence for any $s \in [0, 2M-1]$ we have $i_{s+1} = i_s + 2m_s + 1$ for some $m_s \in \mathbb{N}$. Let $\mathscr{E}$ be the set of $b := (b_0, b_1, b_2, \dots, b_N) \in \mathbb{N}^{N+1}$ such that $b_0 < b_2 < b_4 < \cdots$, $b_1 < b_3 < b_5 < \cdots$ and such that $[b] = [a]$ (we denote by $[b]$, $[a]$ the multisets $\{b_0, b_1, \dots, b_N\}$, $\{a_0, a_1, \dots, a_N\}$). We have $a \in \mathscr{E}$. For $b \in \mathscr{E}$ we set

$$\begin{aligned}\hat{b} &= (\hat{b}_0, \hat{b}_1, \hat{b}_2, \dots, \hat{b}_N)\\ &= (b_0, b_1 + 1, b_2 + 1, b_3 + 2, b_4 + 2, \dots, b_{N-1} + N/2, b_N + N/2).\end{aligned}$$

Let $[\hat{b}]$ be the multiset $\{\hat{b}_0, \hat{b}_1, \hat{b}_2, \dots, \hat{b}_N\}$. For $s \in \{1, 3, \dots, 2M-1\}$ we define $a^{\{s\}} = (a_0^{\{s\}}, a_1^{\{s\}}, a_2^{\{s\}}, \dots, a_N^{\{s\}}) \in \mathscr{E}$ by

$$(a_{i_s}^{\{s\}}, a_{i_s+1}^{\{s\}}, a_{i_s+2}^{\{s\}}, a_{i_s+3}^{\{s\}}, \dots, a_{i_s+2m_s}^{\{s\}}, a_{i_s+2m_s+1}^{\{s\}})$$
$$= (a_{i_s+1}, a_{i_s}, a_{i_s+3}, a_{i_s+2}, \dots, a_{i_s+2m_s+1}, a_{i_s+2m_s})$$

and $a_i^{\{s\}} = a_i$ if $i \in [0, N] - [i_s, i_{s+1}]$. More generally, for $X \subset \{1, 3, \dots, 2M-1\}$ we define $a^X = (a_0^X, a_1^X, a_2^X, \dots, a_N^X) \in \mathcal{E}$ by $a_i^X = a_i^{\{s\}}$ if $s \in X$, $i \in [i_s, i_{s+1}]$, and $a_i^X = a_i$ for all other $i \in [0, N]$. Note that $[\widehat{a^X}] = [\hat{a}]$. Conversely, we have the following result.

Lemma 1.2. *Let $b \in \mathcal{E}$ be such that $[\hat{b}] = [\hat{a}]$. There exists $X \subset \{1, 3, \dots, 2M-1\}$ such that $b = a^X$.*

The proof is given in 1.3–1.5.

1.3. We argue by induction on M. We have

$$a = (y_1 = y_1 < y_2 = y_2 < \cdots < y_r = y_r < a_{i_0} < \dots)$$

for some r. Since $[b] = [a]$, we must have

$$(b_0, b_2, b_4, \dots) = (y_1, y_2, \dots, y_r, \dots), (b_1, b_3, b_5, \dots) = (y_1, y_2, \dots, y_r, \dots).$$

Thus,

(a) $$b_i = a_i \quad \text{for } i < i_0.$$

We have $a = (\cdots < a_{i_{2M}} < y'_1 = y'_1 < y'_2 = y'_2 < \cdots < y'_{r'} = y'_{r'})$ for some r'. Since $[b] = [a]$, we must have

$$(b_0, b_2, b_4, \dots) = (\dots, y'_1, y'_2, \dots, y'_{r'}), (b_1, b_3, b_5, \dots) = (\dots, y'_1, y'_2, \dots, y'_{r'}).$$

Thus,

(b) $$b_i = a_i \quad \text{for } i > i_{2M}.$$

If $M = 0$ we see that $b = a$ and there is nothing further to prove. In the rest of the proof we assume that $M \geq 1$.

1.4. From 1.3 we see that

$$(a_0, a_1, a_2, \dots, a_{i_{2M}}) = (\dots, a_{i_{2M-1}} < x_1 = x_1 < x_2 = x_2 < \cdots < x_q = x_q < a_{i_{2M}})$$

(for some q) has the same entries as $(b_0, b_1, b_2, \dots, b_{i_{2M}})$ (in some order). Hence the pair

$$(\dots, b_{i_{2M}-5}, b_{i_{2M}-3}, b_{i_{2M-1}}), (\dots, b_{i_{2M}-4}, b_{i_{2M}-2}, b_{i_{2M}})$$

must have one of the following four forms.

$$(\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q), (\dots, x_1, x_2, \dots, x_q, a_{i_{2M}}),$$
$$(\dots, x_1, x_2, \dots, x_q, a_{i_{2M}}), (\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q),$$
$$(\dots, x_1, x_2, \dots, x_q), (\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q, a_{i_{2M}}),$$
$$(\dots, a_{i_{2M-1}}, x_1, x_2, \dots, x_q, a_{i_{2M}}), (\dots, x_1, x_2, \dots, x_q).$$

Hence $(\dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}})$ must have one of the following four forms.

(I) $(\dots, a_{i_{2M-1}}, x_1, x_1, x_2, x_2, \dots, x_q, x_q, a_{i_{2M}})$,

(II) $(\dots, x_1, a_{i_{2M-1}}, x_2, x_1, x_3, x_2, \dots, x_q, x_{q-1}, a_{i_{2M}}, x_q)$,

(III) $(\dots, a_{i_{2M-1}}, z, x_1, x_1, x_2, x_2, \dots, x_q, x_q, a_{i_{2M}})$,

(IV) $(\dots, a_{i_{2M-1}}, z', x_1, z'', x_2, x_1, x_3, x_2, \dots, x_q, x_{q-1}, a_{i_{2M}}, x_q)$,

where $a_{i_{2M-1}} > z$, $a_{i_{2M-1}} > z'' \geq z'$ and all entries in $\dots$ are $< a_{i_{2M-1}}$. Correspondingly, $(\dots, \hat{b}_{i_{2M}-2}, \hat{b}_{i_{2M}-1}, \hat{b}_{i_{2M}})$ must have one of the following four forms.

(I) $(\dots, a_{i_{2M-1}}+h-q, x_1+h-q, x_1+h-q+1, x_2+h-q+1, x_2+h-q+2, \dots, x_q+h-1, x_q+h, a_{i_{2M}}+h)$,

(II) $(\dots, x_1+h-q, a_{i_{2M-1}}+h-q, x_2+h-q+1, x_1+h-q+1, x_3+h-q+2, x_2+h-q+1, \dots, x_q+h-1, x_{q-1}+h-1, a_{i_{2M}}+h, x_q+h)$,

(III) $(\dots, a_{i_{2M-1}}+h-q-1, z+h-q, x_1+h-q, x_1+h-q+1, x_2+h-q+1, x_2+h-q+2, \dots, x_q+h-1, x_q+h, a_{i_{2M}}+h)$,

(IV) $(\dots, a_{i_{2M-1}}+h-q-1, z'+h-q-1, x_1+h-q, z''+h-q, x_2+h-q+1, x_1+h-q+1, x_3+h-q+2, x_2+h-q+1, \dots, x_q+h-1, x_{q-1}+h-1, a_{i_{2M}}+h, x_q+h)$,

where $h = i_{2M}/2$ and in cases (III) and (IV), $a_{i_{2M-1}}+h-q$ is not an entry of $(\dots, \hat{b}_{i_{2M}-2}, \hat{b}_{i_{2M}-1}, \hat{b}_{i_{2M}})$.

Since $(\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})$ is given by (I) we see that $a_{i_{2M-1}}+h-q$ is an entry of $(\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})$. Using (b) in 1.3 we see that

$$\{\dots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}}\} = (\dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}})$$

as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus we have *either*

(a) $$(b_{i_{2M-1}}, b_{i_{2M-1}+1}, \dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) = (a_{i_{2M-1}}, a_{i_{2M-1}+1}, \dots, a_{i_{2M}-2}, a_{i_{2M}-1}, a_{i_{2M}})$$

or

(b) $$(b_{i_{2M-1}}, b_{i_{2M-1}+1}, \dots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) = (a_{i_{2M-1}+1}, a_{i_{2M-1}}, a_{i_{2M-1}+3}, a_{i_{2M-1}+2}, \dots, a_{i_{2M}}, a_{i_{2M}-1}).$$

1.5. Let $a' = (a_0, a_1, a_2, \dots, a_{i_{2M-1}-1})$, $b' = (b_0, b_1, b_2, \dots, b_{i_{2M-1}-1})$,

$$\hat{a}' = (a_0, a_1+1, a_2+1, a_3+2, a_4+2, \dots, a_{i_{2M-1}-1}+(i_{2M-1}-1)/2),$$
$$\hat{b}' = (b_0, b_1+1, b_2+1, b_3+2, b_4+2, \dots, b_{i_{2M-1}-1}+(i_{2M-1}-1)/2),$$

From $[\hat{b}] = [\hat{a}]$, (b) in 1.3 and (a)+(b) in 1.4 we see that the multiset formed by the entries of $\hat{a}'$ coincides with the multiset formed by the entries of $\hat{b}'$. Using the induction hypothesis we see that there exists $X' \subset \{1, 3, \dots, 2M-3\}$ such that $b' = a'^{X'}$ where $a'^{X'}$ is defined in terms of a', X' in the same way as a^X was defined (see 1.1) in terms of a, X. We set $X = X'$ if we are in case (a) of 1.4 and $X = X' \cup \{2M-1\}$ if we are in case (b). Then we have $b = a^X$ (see again (a) and (b) in 1.4), as required. This completes the proof of 1.2.

1.6. We shall use the notation of 1.1. Let $\mathfrak{T}$ be the set of all unordered pairs $(\mathfrak{A}, \mathfrak{B})$ where $\mathfrak{A}$, $\mathfrak{B}$ are subsets of $\{0, 1, 2, \dots\}$ and $\mathfrak{A} \dot\cup \mathfrak{B} = (a_0, a_1, a_2, \dots, a_N)$ as multisets. For example, setting $\mathfrak{A}_\varnothing = (a_0, a_2, a_4, \dots, a_N)$ and $\mathfrak{B}_\varnothing = (a_1, a_3, \dots, a_{N-1})$, we have $(\mathfrak{A}_\varnothing, \mathfrak{B}_\varnothing) \in \mathfrak{T}$. For any subset $\mathfrak{a}$ of $\mathcal{J}$ we consider

$$\mathfrak{A}_\mathfrak{a} = ((\mathcal{J} - \mathfrak{a}) \cap \mathfrak{A}_\varnothing) \cup (\mathfrak{a} \cap \mathfrak{B}_\varnothing) \cup (\mathfrak{A}_\varnothing \cap \mathfrak{B}_\varnothing),$$
$$\mathfrak{B}_\mathfrak{a} = ((\mathcal{J} - \mathfrak{a}) \cap \mathfrak{B}_\varnothing) \cup (\mathfrak{a} \cap \mathfrak{A}_\varnothing) \cup (\mathfrak{A}_\varnothing \cap \mathfrak{B}_\varnothing).$$

Then $(\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a}) \in \mathfrak{T}$ and the map $\mathfrak{a} \mapsto (\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a})$ induces a bijection $\bar{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathfrak{T}$. (Note that if $\mathfrak{a} = \varnothing$ then $(\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a})$ agrees with the earlier definition of $(\mathfrak{A}_\varnothing, \mathfrak{B}_\varnothing)$.)

Let $\mathfrak{T}'$ be the set of all $(\mathfrak{A}, \mathfrak{B}) \in \mathfrak{T}$ such that $|\mathfrak{A}| = |\mathfrak{A}_\varnothing|$ and $|\mathfrak{B}| = |\mathfrak{B}_\varnothing|$.

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}_{\mathrm{ev}}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \dots, \{a_{i_{2M-2}}, a_{i_{2M-1}}\}$$

of $\mathcal{J}$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}_{\mathrm{ev}}(\mathcal{J})$ spanned by the 2-element subsets

$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \dots, \{a_{i_{2M-1}}, a_{i_{2M}}\}$$

of $\mathcal{J}$.

Let $\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$ be the images of $\mathcal{P}(\mathcal{J})_0$ and $\mathcal{P}(\mathcal{J})_1$ under the obvious map $\mathcal{P}(\mathcal{J}) \to \bar{\mathcal{P}}(\mathcal{J})$. Then:

(a) $\bar{\mathcal{P}}(\mathcal{J})_0$ *and* $\bar{\mathcal{P}}(\mathcal{J})_1$ *are opposed Lagrangian subspaces of the symplectic vector space* $\bar{\mathcal{P}}(\mathcal{J})$, $(\ ,\ )$ (see 0.7); *hence* $(\ ,\ )$ *defines an identification*

$$\bar{\mathcal{P}}(\mathcal{J})_0 = \bar{\mathcal{P}}(\mathcal{J})_1^*,$$

where $\bar{\mathcal{P}}(\mathcal{J})_1^*$ *is the vector space dual to* $\bar{\mathcal{P}}(\mathcal{J})_1$.

Let $\mathfrak{T}_0$ and $\mathfrak{T}_1$ be the subsets of $\mathfrak{T}$ corresponding to $\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$, respectively, under the bijection $\bar{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathfrak{T}$. Note that $\mathfrak{T}_0 \subset \mathfrak{T}'$, $\mathfrak{T}_1 \subset \mathfrak{T}'$, and $|\mathfrak{T}_0| = |\mathfrak{T}_1| = 2^M$.

For any $X \subset \{1, 3, \dots, 2M-1\}$ we set $\mathfrak{a}_X = \bigcup_{s \in X} \{a_{i_s}, a_{i_{s+1}}\} \in \mathcal{P}(\mathcal{J})$. Then $(\mathfrak{A}_{\mathfrak{a}_X}, \mathfrak{B}_{\mathfrak{a}_X}) \in \mathfrak{T}_1$ is related to a^X in 1.1 as follows:

$$\mathfrak{A}_{\mathfrak{a}_X} = \{a_0^X, a_2^X, a_4^X, \dots, a_N^X\}, \qquad \mathfrak{B}_{\mathfrak{a}_X} = \{a_1^X, a_3^X, \dots, a_{N-1}^X\}.$$

1.7. We shall use the notation of 1.1. Let T be the set of all ordered pairs (A, B) where A is a subset of $\{0, 1, 2, \dots\}$, B is a subset of $\{1, 2, 3, \dots\}$, A contains no consecutive integers, B contains no consecutive integers, and $A \dot{\cup} B = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ as multisets. For example, setting $A_\varnothing = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \dots, \hat{a}_N)$ and $B_\varnothing = (\hat{a}_1, \hat{a}_3, \dots, \hat{a}_{N-1})$, we have $(A_\varnothing, B_\varnothing) \in T$.

For any $(A, B) \in T$ we define (A^-, B^-) as follows: A^- consists of $x_0 < x_1 - 1 < x_2 - 2 < \cdots < x_p - p$ where $x_0 < x_1 < \cdots < x_p$ are the elements of A; B^- consists of $y_1 - 1 < y_2 - 2 < \cdots < y_q - q$ where $y_1 < y_2 < \cdots < y_q$ are the elements of B.

We can enumerate the elements of T as in [Lusztig 1984b, 11.5]. Let J be the set of all $c \in \mathbb{N}$ such that c appears exactly once in the sequence

$$(\hat{a}_0, \hat{a}_1, \hat{a}_2, \dots, \hat{a}_N) = (a_0, a_1+1, a_2+1, a_3+2, a_4+2, \dots, a_{N-1}+N/2, a_N+N/2).$$

A nonempty subset I of J is said to be an interval if it is of the form $\{i, i+1, i+2, \dots, j\}$ with $i-1 \notin J$, $j+1 \notin J$ and with $i \neq 0$. Let $\mathcal{I}$ be the set of intervals of J. For any $s \in \{1, 3, \dots, 2M-1\}$, the set $I_s := \{\hat{a}_{i_s}, \hat{a}_{i_s+1}, \hat{a}_{i_s+2}, \dots, \hat{a}_{i_s+2m_s+1}\}$ is either a single interval or a union of intervals $I_s^1 \sqcup I_s^2 \sqcup \dots \sqcup I_s^{t_s}$ ($t_s \geq 2$) where $\hat{a}_{i_s} \in I_s^1$, $\hat{a}_{i_s+2m_s+1} \in I_s^{t_s}$, $|I_s^1|$, $|I_s^{t_s}|$ are odd, $|I_s^h|$ are even for $h \in [2, t_s-1]$ and any element in I_s^e is $<$ than any element in $I_s^{e'}$ for $e < e'$. Let $\mathcal{I}_s$ be the set of all $I \in \mathcal{I}$ such that $I \subset I_s$. Let H be the set of all $c \in J$ such that c does not belong to any interval. For any subset $\alpha \subset \mathcal{I}$ we consider

$$A_\alpha = \bigcup_{I \in \mathcal{I}-\alpha} (I \cap A_\varnothing) \cup \bigcup_{I \in \alpha} (I \cap B_\varnothing) \cup (H \cap A_\varnothing) \cup (A_\varnothing \cap B_\varnothing),$$
$$B_\alpha = \bigcup_{I \in \mathcal{I}-\alpha} (I \cap B_\varnothing) \cup \bigcup_{I \in \alpha} (I \cap A_\varnothing) \cup (H \cap B_\varnothing) \cup (A_\varnothing \cap B_\varnothing).$$

Then $(A_\alpha, B_\alpha) \in T$ and the map $\alpha \mapsto (A_\alpha, B_\alpha)$ is a bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$. (Note that if $\alpha = \varnothing$ then (A_α, B_α) agrees with the earlier definition of $(A_\varnothing, B_\varnothing)$.)

Let $T' = \{(A, B) \in T; |A| = |A_\varnothing|, |B| = |B_\varnothing|\}$, $T_1 = \{(A, B) \in T'; A^- \dot{\cup} B^- = A_\varnothing^- \dot{\cup} B_\varnothing^-\}$. Let $\mathcal{P}(\mathcal{I})'$ and $\mathcal{P}(\mathcal{I})_1$ be the subsets of $\mathcal{P}(\mathcal{I})$ corresponding to T' and T_1 under the bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$.

Now let X be a subset of $\{1, 3, \dots, 2M-1\}$. Let $\alpha_X = \bigcup_{s \in X} \mathcal{I}_s \in \mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $$A_{\alpha_X}^- = \mathfrak{A}_{\mathfrak{a}_X}, \quad B_{\alpha_X}^- = \mathfrak{B}_{\mathfrak{a}_X}$$

(in the notation of 1.6). In particular we have $(A_{\alpha_X}, B_{\alpha_X}) \in T_1$. Thus $|T_1| \geq 2^M$. Using Lemma 1.2 we see that

(b) $$|T_1| = 2^M \quad \textit{and} \quad T_1 = \big\{(A_{\alpha_X}, B_{\alpha_X});\, X \subset \{1, 3, \dots, 2M-1\}\big\}.$$

Using (a) and (b) we deduce:

(c) *The map $T_1 \to \mathfrak{T}_1$ given by $(A, B) \mapsto (A^-, B^-)$ is a bijection.*

2. Combinatorics (continued)

2.1. Let $N \in \mathbb{N}$. Let

$$a := (a_0, a_1, a_2, \dots, a_N) \in \mathbb{N}^{N+1}$$

be such that $a_0 \le a_1 \le a_2 \le \cdots \le a_N,\ a_0 < a_2 < a_4 < \cdots,\ a_1 < a_3 < a_5 < \cdots$ and such that the set $\mathcal{J} := \{i \in [0, N];\ a_i \text{ appears exactly once in } a\}$ is nonempty. Now $\mathcal{J}$ consists of $\mu+1$ elements $i_0 < i_1 < \cdots < i_\mu$, where $\mu \in \mathbb{N}$, $\mu = N \bmod 2$. We have $i_s = s \bmod 2$ for $s \in [0, \mu]$. Hence for any $s \in [0, \mu-1]$ we have $i_{s+1} = i_s + 2m_s + 1$ for some $m_s \in \mathbb{N}$. Let $\mathscr{E}$ be the set of $b := (b_0, b_1, b_2, \dots, b_N) \in \mathbb{N}^{N+1}$ such that $b_0 < b_2 < b_4 < \cdots,\ b_1 < b_3 < b_5 < \cdots$ and such that $[b] = [a]$ (we denote by $[b], [a]$ the multisets $\{b_0, b_1, \dots, b_N\}, \{a_0, a_1, \dots, a_N\}$). We have $a \in \mathscr{E}$. For $b \in \mathscr{E}$ we set

$$\mathring{b} = (\mathring{b}_0, \mathring{b}_1, \mathring{b}_2, \dots, \mathring{b}_N) = (b_0, b_1, b_2+1, b_3+1, b_4+2, b_5+2, \dots) \in \mathbb{N}^{N+1}.$$

Let $[\mathring{b}]$ be the multiset $\{\mathring{b}_0, \mathring{b}_1, \mathring{b}_2, \dots, \mathring{b}_N\}$. For any $s \in [0, \mu-1] \in 2\mathbb{N}$ we define $a^{\{s\}} = (a_0^{\{s\}}, a_1^{\{s\}}, a_2^{\{s\}}, \dots, a_N^{\{s\}}) \in \mathscr{E}$ by

$$(a_{i_s}^{\{s\}}, a_{i_s+1}^{\{s\}}, a_{i_s+2}^{\{s\}}, a_{i_s+3}^{\{s\}}, \dots, a_{i_s+2m_s}^{\{s\}}, a_{i_s+2m_s+1}^{\{s\}}) \\ = (a_{i_s+1}, a_{i_s}, a_{i_s+3}, a_{i_s+2}, \dots, a_{i_s+2m_s+1}, a_{i_s+2m_s})$$

and $a_i^{\{s\}} = a_i$ if $i \in [0, N] - [i_s, i_{s+1}]$. More generally, for a subset X of $[0, \mu-1] \cap 2\mathbb{N}$ we define $a^X = (a_0^X, a_1^X, a_2^X, \dots, a_N^X) \in \mathscr{E}$ by $a_i^X = a_i^{\{s\}}$ if $s \in X$, $i \in [i_s, i_{s+1}]$, and $a_i^X = a_i$ for all other $i \in [0, N]$. Note that $[\mathring{a}^X] = [\mathring{a}]$. Conversely:

Lemma 2.2. *Let $b \in \mathscr{E}$ be such that $[\mathring{b}] = [\mathring{a}]$. Then there exists $X \subset [0, \mu-1] \cap 2\mathbb{N}$ such that $b = a^X$.*

The proof is given in 2.3–2.5.

2.3. We argue by induction on μ. By the argument in 1.3 we have

(a) $$b_i = a_i \quad \text{for } i < i_0,$$

(b) $$b_i = a_i \quad \text{for } i > i_\mu.$$

If $\mu = 0$ we see that $b = a$ and there is nothing further to prove. In the rest of the proof we assume that $\mu \ge 1$.

2.4. From 2.3 we see that $(a_{i_0}, a_{i_0+1}, \dots, a_N) = (a_{i_0} < x_1 = x_1 < x_2 = x_2 < \cdots < x_p = x_p < a_{i_1} < \dots)$ (for some p) has the same entries as $(b_{i_0}, b_{i_0+1}, \dots, b_N)$ (in some order). Hence the pair $(b_{i_0}, b_{i_0+2}, b_{i_0+4}, \dots), (b_{i_0+1}, b_{i_0+3}, b_{i_0+5}, \dots)$ must have one of the following four forms.

$$(a_{i_0}, x_1, x_2, \dots, x_p, \dots), (x_1, x_2, \dots, x_p, a_{i_1}, \dots),$$

$$(x_1, x_2, \dots, x_p, a_{i_1}, \dots), (a_{i_0}, x_1, x_2, \dots, x_p, \dots),$$
$$(a_{i_0}, x_1, x_2, \dots, x_p, a_{i_1}, \dots), (x_1, x_2, \dots, x_p, \dots),$$
$$(x_1, x_2, \dots, x_p, \dots), (a_{i_0}, x_1, x_2, \dots, x_p, a_{i_1}, \dots).$$

Hence $(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_N)$ must have one of the following four forms.

(I) $(a_{i_0}, x_1, x_1, x_2, x_2, \dots, x_p, x_p, a_{i_1}, \dots)$,

(II) $(x_1, a_{i_0}, x_2, x_1, x_3, x_2, \dots, x_p, x_{p-1}, a_{i_1}, x_p, \dots)$,

(III) $(a_{i_0}, x_1, x_1, x_2, x_2, \dots, x_p, x_p, z, a_{i_1}, \dots)$,

(IV) $(x_1, a_{i_0}, x_2, x_1, x_3, x_2, \dots, x_p, x_{p-1}, z', x_p, z'', a_{i_1}, \dots)$,

where $a_{i_1} < z$, $a_{i_1} < z' \le z''$ and all entries in $\dots$ are $> a_{i_1}$. Correspondingly, $(\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots, \mathring{b}_N)$ must have one of the following four forms.

(I) $(a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \dots, x_p + h + p - 1, x_p + h + p, a_{i_1} + h + p, \dots)$,

(II) $(x_1+h, a_{i_0}+h, x_2+h+1, x_1+h+1, x_3+h+2, x_2+h+2, \dots, x_p+h+p-1, x_{p-1} + h + p - 1, a_{i_1} + h + p, x_p + h + p, \dots)$,

(III) $(a_{i_0}+h, x_1+h, x_1+h+1, x_2+h+1, x_2+h+2, \dots, x_p+h+p-1, x_p+h+p, z + p, a_{i_1} + h + p + 1, \dots)$,

(IV) $(x_1+h, a_{i_0}+h, x_2+h+1, x_1+h+1, x_3+h+2, x_2+h+2, \dots, x_p+h+p-1, x_{p-1}+h+p-1, z'+h+p, x_p+h+p, z''+h+p+1, a_{i_1}+h+p+1, \dots)$,

where $h = i_0/2$ and in cases (III) and (IV) $a_{i_1} + h + p$ is not an entry of $(\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots)$.

Since $(\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots)$ is given by (I) we see that $a_{i_1} + h + p$ is an entry of $(\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots)$. Using 2.3 we see that

$$\{\mathring{a}_{i_0}, \mathring{a}_{i_0+1}, \mathring{a}_{i_0+2}, \dots\} = \{\mathring{b}_{i_0}, \mathring{b}_{i_0+1}, \mathring{b}_{i_0+2}, \dots\}$$

as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus we have *either*

(a) $$(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_{i_1}) = (a_{i_0}, a_{i_0+1}, a_{i_0+2}, \dots, a_{i_1})$$

or

(b) $$(b_{i_0}, b_{i_0+1}, b_{i_0+2}, \dots, b_{i_1}) = (a_{i_0+1}, a_{i_0}, a_{i_0+3}, a_{i_0+2}, \dots, a_{i_1}, a_{i_1-1}).$$

From 2.3 and (a)+(b) we see that if $\mu = 1$ then Lemma 2.2 holds. Thus in the rest of the proof we can assume that $\mu \ge 2$.

2.5. Let $a' = (a_{i_1+1}, a_{i_1+2}, \dots, a_N)$, $b' = (b_{i_1+1}, b_{i_1+2}, \dots, b_N)$,

$$\mathring{a}' = (a_{i_1+1}, a_{i_1+2}, a_{i_1+3}+1, a_{i_1+4}+1, a_{i_1+5}+2, a_{i_1+6}+2, \dots),$$
$$\mathring{b}' = (b_{i_1+1}, b_{i_1+2}, b_{i_1+3}+1, b_{i_1+4}+1, b_{i_1+5}+2, b_{i_1+6}+2, \dots).$$

From $[\mathring{b}] = [\mathring{a}]$, (a) in 2.3 and (a)+(b) in 2.4 we see that the multiset formed by the entries of $\mathring{a}'$ coincides with the multiset formed by the entries of $\mathring{b}'$. Using the induction hypothesis we see that there exists $X' \subset [2, \mu - 1] \cap 2\mathbb{N}$ such that $b' = a'^{X'}$ where $a'^{X'}$ is defined in terms of a', X' in the same way as a^X (see 2.1) was defined in terms of a, X. We set $X = X'$ if we are in case (a) of 2.4 and $X = \{0\} \cup X'$ if we are in case (b). Then we have $b = a^X$ (see again (a) and (b) of 2.4), as required. This completes the proof of Lemma 2.2.

2.6. We shall use the notation of 2.1. Let $\mathfrak{T}$ be the set of all unordered pairs $(\mathfrak{A}, \mathfrak{B})$ where $\mathfrak{A}, \mathfrak{B}$ are subsets of $\{0, 1, 2, \dots\}$ and $\mathfrak{A} \dot\cup \mathfrak{B} = (a_0, a_1, a_2, \dots, a_N)$ as multisets. For example, setting

$$\mathfrak{A}_\varnothing = \{a_i; i \in [0, N] \cap 2\mathbb{N}\} \quad \text{and} \quad \mathfrak{B}_\varnothing = \{a_i; i \in [0, N] \cap (2\mathbb{N}+1)\},$$

we have $(\mathfrak{A}_\varnothing, \mathfrak{B}_\varnothing) \in \mathfrak{T}$. For any subset $\mathfrak{a}$ of $\mathcal{J}$ we consider

$$\mathfrak{A}_\mathfrak{a} = ((\mathcal{J} - \mathfrak{a}) \cap \mathfrak{A}_\varnothing) \cup (\mathfrak{a} \cap \mathfrak{B}_\varnothing) \cup (\mathfrak{A}_\varnothing \cap \mathfrak{B}_\varnothing),$$
$$\mathfrak{B}_\mathfrak{a} = ((\mathcal{J} - \mathfrak{a}) \cap \mathfrak{B}_\varnothing) \cup (\mathfrak{a} \cap \mathfrak{A}_\varnothing) \cup (\mathfrak{A}_\varnothing \cap \mathfrak{B}_\varnothing).$$

Then $(\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a}) = (\mathfrak{A}_{\mathcal{J}-\mathfrak{a}}, \mathfrak{A}_{\mathcal{J}-\mathfrak{a}}) \in \mathfrak{T}$ and the map $\mathfrak{a} \mapsto (\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a})$ induces a bijection $\bar{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathfrak{T}$. (Note that if $\mathfrak{a} = \varnothing$ then $(\mathfrak{A}_\mathfrak{a}, \mathfrak{B}_\mathfrak{a})$ agrees with the earlier definition of $(\mathfrak{A}_\varnothing, \mathfrak{B}_\varnothing)$.)

Let $\mathfrak{T}'$ be the set of all $(\mathfrak{A}, \mathfrak{B}) \in \mathfrak{T}$ such that $|\mathfrak{A}| = |\mathfrak{A}_\varnothing|$ and $|\mathfrak{B}| = |\mathfrak{B}_\varnothing|$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \dots, \{a_{i_{\mu-2}}, a_{i_{\mu-1}}\} \quad \text{if } N \text{ is odd},$$
$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \dots, \{a_{i_{\mu-1}}, a_{i_\mu}\} \quad \text{if } N \text{ is even}.$$

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \dots, \{a_{i_{\mu-1}}, a_{i_\mu}\} \quad \text{if } N \text{ is odd},$$
$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \dots, \{a_{i_{\mu-2}}, a_{i_{\mu-1}}\} \quad \text{if } N \text{ is even}.$$

Let $\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$ be the images of $\mathcal{P}(\mathcal{J})_0$ and $\mathcal{P}(\mathcal{J})_1$ under the obvious map $\mathcal{P}(\mathcal{J}) \to \bar{\mathcal{P}}(\mathcal{J})$. Then:

(a) $\bar{\mathcal{P}}(\mathcal{J})_0$ *and* $\bar{\mathcal{P}}(\mathcal{J})_1$ *are opposed Lagrangian subspaces of the symplectic vector space* $\bar{\mathcal{P}}_{\mathrm{ev}}(\mathcal{J})$, (,) (see 0.7); *hence* (,) *defines an identification* $\bar{\mathcal{P}}(\mathcal{J})_1 = \bar{\mathcal{P}}(\mathcal{J})_0^*$, where $\bar{\mathcal{P}}(\mathcal{J})_0^*$ is the vector space dual to $\bar{\mathcal{P}}(\mathcal{J})_0$.

Let $\mathfrak{T}_0$ and $\mathfrak{T}_1$ be the subsets of $\mathfrak{T}$ corresponding to $\bar{\mathcal{P}}(\mathcal{J})_0$ and $\bar{\mathcal{P}}(\mathcal{J})_1$ under the bijection $\bar{\mathcal{P}}(\mathcal{J}) \leftrightarrow \mathfrak{T}$. Note that $\mathfrak{T}_0 \subset \mathfrak{T}'$, $\mathfrak{T}_1 \subset \mathfrak{T}'$, $|\mathfrak{T}_0| = |\mathfrak{T}_1| = 2^{\lfloor \mu/2 \rfloor}$.

For any $X \subset [0, \mu - 1] \cap 2\mathbb{N}$ we set $\mathfrak{a}_X = \bigcup_{s \in X} \{a_{i_s}, a_{i_{s+1}}\} \in \mathcal{P}(\mathcal{J})$. Then $(\mathfrak{A}_{\mathfrak{a}_X}, \mathfrak{B}_{\mathfrak{a}_X})$ is related to a^X in 2.1 as follows:

$$\mathfrak{A}_{\mathfrak{a}_X} = \{a_i^X; i \in [0, N] \cap 2\mathbb{N}\}, \mathfrak{B}_{\mathfrak{a}_X} = \{a_i^X; i \in [0, N] \cap (2\mathbb{N} + 1)\}.$$

2.7. We shall use the notation of 2.1. Let T be the set of all unordered pairs (A, B) where A is a subset of $\{0, 1, 2, \dots\}$, B is a subset of $\{1, 2, 3, \dots\}$, A contains no consecutive integers, B contains no consecutive integers, and $A \dot{\cup} B = (\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \dots, \mathring{a}_N)$ as multisets. For example, setting

$$A_\varnothing = \{\mathring{a}_i; i \in [0, N] \cap 2\mathbb{N}\} \quad \text{and} \quad B_\varnothing = (\mathring{a}_i; i \in [0, N] \cap (2\mathbb{N} + 1)\},$$

we have $(A_\varnothing, B_\varnothing) \in T$.

For any $(A, B) \in T$ we define (A^-, B^-) as follows: A^- consists of $x_1 < x_2 - 1 < x_3 - 2 < \dots < x_p - p + 1$ where $x_1 < x_2 < \dots < x_p$ are the elements of A; B^- consists of $y_1 < y_2 - 1 < \dots < y_q - q + 1$ where $y_1 < y_2 < \dots < y_q$ are the elements of B.

We can enumerate the elements of T as in [Lusztig 1984b, 11.5]. Let J be the set of all $c \in \mathbb{N}$ such that c appears exactly once in the sequence

$$(\mathring{a}_0, \mathring{a}_1, \mathring{a}_2, \dots, \mathring{a}_N) = (a_0, a_1, a_2 + 1, a_3 + 1, a_4 + 2, a_5 + 2, \dots).$$

A nonempty subset I of J is said to be an interval if it is of the form $\{i, i + 1, i + 2, \dots, j\}$ with $i - 1 \notin J$, $j + 1 \notin J$. Let $\mathcal{I}$ be the set of intervals of J. For any $s \in [0, \mu - 1] \cap 2\mathbb{N}$, the set $I_s := \{\mathring{a}_{i_s}, \mathring{a}_{i_s+1}, \mathring{a}_{i_s+2}, \dots, \mathring{a}_{i_s+2m_s+1}\}$ is either a single interval or a union of intervals $I_s^1 \sqcup I_s^2 \sqcup \dots \sqcup I_s^{t_s}$ $(t_s \geq 2)$ where $\mathring{a}_{i_s} \in I_s^1$, $\mathring{a}_{i_s+2m_s+1} \in I_s^{t_s}$, $|I_s^1|$, $|I_s^{t_s}|$ are odd, $|I_s^h|$ are even for $h \in [2, t_s - 1]$ and any element in I_s^e is $<$ than any element in $I_s^{e'}$ for $e < e'$. Let $\mathcal{I}_s$ be the set of all $I \in \mathcal{I}$ such that $I \subset I_s$. For any subset $\alpha \subset \mathcal{I}$ we consider

$$A_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap A_\varnothing) \cup \bigcup_{I \in \alpha} (I \cap B_\varnothing) \cup (A_\varnothing \cap B_\varnothing),$$
$$B_\alpha = \bigcup_{I \in \mathcal{I} - \alpha} (I \cap B_\varnothing) \cup \bigcup_{I \in \alpha} (I \cap A_\varnothing) \cup (A_\varnothing \cap B_\varnothing).$$

Then $(A_\alpha, B_\alpha) \in T$ and the map $\alpha \mapsto (A_\alpha, B_\alpha)$ is a bijection $\bar{\mathcal{P}}(\mathcal{I}) \leftrightarrow T$. (Note that if $\alpha = \varnothing$ then (A_α, B_α) agrees with the earlier definition of $(A_\varnothing, B_\varnothing)$.)

Let

$$T' = \{(A, B) \in T; |A| = |A_\varnothing|, |B| = |B_\varnothing|\},$$
$$T_1 = \{(A, B) \in T'; A^- \dot{\cup} B^- = A_\varnothing^- \dot{\cup} B_\varnothing^-\}.$$

Let $\bar{\mathcal{P}}(\mathcal{I})'$ and $\bar{\mathcal{P}}(\mathcal{I})_1$ be the subsets of $\bar{\mathcal{P}}(\mathcal{I})$ corresponding to T' and T_1 under the bijection $\bar{\mathcal{P}}(\mathcal{I}) \leftrightarrow T$.

Now let X be a subset of $[0, \mu-1] \cap 2\mathbb{N}$. Let $\alpha_X = \bigcup_{s \in X} \mathcal{I}_s \in \mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $$A^-_{\alpha_X} = \mathfrak{A}_{\mathfrak{a}_X}, \quad B^-_{\alpha_X} = \mathfrak{B}_{\mathfrak{a}_X}$$

(in the notation of 2.6). In particular we have $(A_{\alpha_X}, B_{\alpha_X}) \in T_1$. Thus $|T_1| \geq 2^{\lfloor \mu/2 \rfloor}$. Using Lemma 2.2 we see that

(b) $$|T_1| = 2^{\lfloor \mu/2 \rfloor} \quad \textit{and} \quad T_1 = \big\{(A_{\alpha_X}, B_{\alpha_X});\ X \subset [0, \mu-1] \cap 2\mathbb{N}\big\}.$$

Using (a) and (b) we deduce:

(c) *The map $T_1 \to \mathfrak{T}_1$ given by $(A, B) \mapsto (A^-, B^-)$ is a bijection.*

3. Proof of Theorem 0.4 and of Corollary 0.5

3.1. If G is simple adjoint of type A_n, $n \geq 1$, then Theorem 0.4 and Corollary 0.5 are obvious: we have $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.2. Assume that $G = Sp_{2n}(\Bbbk)$ where $n \geq 2$. Let N be a sufficiently large even integer. Now $u : \Bbbk^{2n} \to \Bbbk^{2n}$ has i_e Jordan blocks of size e ($e = 1, 2, 3, \dots$). Here $i_1, i_3, i_5, \dots$ are even. Let $\Delta = \{e \in \{2, 4, 6, \dots\};\ i_e \geq 1\}$. Then $A(u)$ can be identified in the standard way with $\mathcal{P}(\Delta)$. Hence the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the F_2-vector space dual to $\mathcal{P}(\Delta)$) may be also canonically identified with $\mathcal{P}(\Delta)$ itself (so that the basis given by the one-element subsets of Δ is self-dual).

To the partition $1i_1 + 2i_2 + 3i_3 + \cdots$ of $2n$ we associate a pair (A, B) as in [Lusztig 1984b, 11.6] (with $N, 2m$ replaced by $2n, N$). We have $A = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \dots, \hat{a}_N)$, $B = (\hat{a}_1, \hat{a}_3, \dots, \hat{a}_{N-1})$, where $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2 \leq \cdots \leq \hat{a}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 1.1. (Here we use that C is special.) Now the definitions and results in Section 1 are applicable. As in [Lusztig 1984a, 4.5] the family $\mathcal{F}$ is in canonical bijection with $\mathfrak{T}'$ in 1.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \dots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of Δ in increasing order $e_1 < e_2 < \cdots < e_{f'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [Lusztig 1984b, 11.6]. Using this bijection we see that $A(u)$ and $\hat{A}(u)$ are identified with the F_2-vector space $\mathcal{P}(\mathcal{I})$ with basis given by the one-element subsets of $\mathcal{I}$. Let $\pi : \mathcal{P}(\mathcal{I}) \to \mathcal{P}(\mathcal{I})_1^*$ (the dual of $\mathcal{P}(\mathcal{I})_1$ in 1.7) be the (surjective) F_2-linear map which to $X \subset \mathcal{I}$ associates the linear form $L \mapsto |X \cap L| \bmod 2$ on $\mathcal{P}(\mathcal{I})_1$. We will show that

(a) $$\ker \pi = \mathcal{K}(u), \quad \text{with } \mathcal{K}(u) \text{ as in 0.1.}$$

We identify $\mathrm{Irr}_C W$ with T' (see 1.7) via the restriction of the bijection in [Lusztig 1984b, (12.2.4)] (we also use the description of the Springer correspondence in [Lusztig 1984b, 12.3]). Under this identification the subset $\mathrm{Irr}^*_C W$ of $\mathrm{Irr}_C W$ becomes the subset T_1 (see 1.7) of T'. Via the identification $\mathcal{P}(\mathcal{I})' \leftrightarrow T'$ in 1.7 and $\hat{A}(u) \leftrightarrow \mathcal{P}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from T' to $\hat{A}(u)$ becomes the obvious imbedding $\mathcal{P}(\mathcal{I})' \to \mathcal{P}(\mathcal{I})$ (we use again [Lusztig 1984b, 12.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})_1$ we have $|X \cap L| = 0 \bmod 2$. Thus, (a) holds.

Using (a) we have canonically $\bar{A}(u) = \mathcal{P}(\mathcal{I})^*_1$ via π. We define an F_2-linear map $\mathcal{P}(\mathcal{I})_1 \to \bar{\mathcal{P}}(\mathcal{J})_1$ (see 1.6) by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in \{1, 3, \dots, 2M-1\}$ (I_s as in 1.7). This is an isomorphism; it corresponds to the bijection 1.7(c) under the identification $T_1 \leftrightarrow \mathcal{P}(\mathcal{I})_1$ in 1.7 and the identification $\mathfrak{T}_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{J})_1$ in 1.6. Hence we can identify $\mathcal{P}(\mathcal{I})^*_1$ with $\bar{\mathcal{P}}(\mathcal{J})^*_1$ and with $\bar{\mathcal{P}}(\mathcal{J})_0$ (see 1.6(a)). We obtain an identification $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{J})_0$.

By [Lusztig 1984a, 4.5] we have $\boldsymbol{X}_{\mathcal{F}} = \bar{\mathcal{P}}(\mathcal{J})$. Using 1.6(a) we see that $\bar{\mathcal{P}}(\mathcal{J}) = M(\bar{\mathcal{P}}(\mathcal{J})_0) = M(\bar{A}(u))$ canonically so that Theorem 0.4 holds in our case. From the arguments above we see that in our case Corollary 0.5 follows from 1.7(c).

3.3. Assume that $G = \mathrm{SO}_n(\Bbbk)$ where $n \geq 7$. Let N be a sufficiently large integer such that $N = n \bmod 2$. Now $u : \Bbbk^n \to \Bbbk^n$ has i_e Jordan blocks of size e ($e = 1, 2, 3, \dots$). Here $i_2, i_4, i_6, \dots$ are even. Let $\Delta = \{e \in \{1, 3, 5, \dots\}; i_e \geq 1\}$. If $\Delta = \varnothing$ then $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$ and $\mathcal{G}_{\mathcal{F}} = \{1\}$ so that the result is trivial.

In the remainder of this subsection we assume that $\Delta \neq \varnothing$. Then $A(u)$ can be identified in the standard way with the F_2-subspace $\mathcal{P}_{\mathrm{ev}}(\Delta)$ of $\mathcal{P}(\Delta)$ and the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the F_2-vector space dual to $A(u)$) becomes $\bar{\mathcal{P}}(\Delta)$; the obvious pairing $A(u) \times \hat{A}(u) \to F_2$ is induced by the inner product $L, L' \mapsto |L \cap L'| \bmod 2$ on $\mathcal{P}(\Delta)$.

To the partition $1i_1 + 2i_2 + 3i_3 + \cdots$ of n we associate a pair (A, B) as in [Lusztig 1984b, 11.7] (with N, M replaced by n, N). We have $A = \{\mathring{a}_i; i \in [0, N] \cap 2\mathbb{N}\}$, $B = (\mathring{a}_i; i \in [0, N] \cap (2\mathbb{N}+1)\}$ where $\mathring{a}_0 \leq \mathring{a}_1 \leq \mathring{a}_2 \leq \cdots \leq \mathring{a}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 2.1. (Here we use that C is special.) Now the definitions and results in §2 are applicable. As in [Lusztig 1984a, 4.5] (if N is even) or [Lusztig 1984a, 4.6] (if N is odd) the family $\mathcal{F}$ is in canonical bijection with $\mathfrak{T}'$ in 2.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \dots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of Δ in increasing order $e_1 < e_2 < \cdots < e_{f'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [Lusztig 1984b, 11.7]. Using this bijection we see that $A(u)$ is identified with $\mathcal{P}_{\mathrm{ev}}(\mathcal{I})$ and $\hat{A}(u)$ is identified with $\bar{\mathcal{P}}(\mathcal{I})$. For any $X \in \mathcal{P}_{\mathrm{ev}}(\mathcal{I})$, the assignment $L \mapsto |X \cap L| \bmod 2$ can be viewed as an element of $\bar{\mathcal{P}}(\mathcal{I})_1^*$ (the dual space of $\bar{\mathcal{P}}(\mathcal{I})_1$ in 2.7 which by 2.7(b) is an F_2-vector space of dimension $2^{\lfloor \mu/2 \rfloor}$). This induces a (surjective) F_2-linear map $\pi : \mathcal{P}_{\mathrm{ev}}(\mathcal{I}) \to \bar{\mathcal{P}}(\mathcal{I})_1^*$. We will show that

(a) $\qquad \ker \pi = \mathcal{K}(u), \quad$ with $\mathcal{K}(u)$ as in 0.1.

We identify $\mathrm{Irr}_C W$ with T' (see 2.7) via the restriction of the bijection in [Lusztig 1984b, (13.2.5)] if N is odd or [ibid., (13.2.6)] if N is even (we also use the description of the Springer correspondence in [Lusztig 1984b, 13.3]). Under this identification the subset $\mathrm{Irr}_C^* W$ of $\mathrm{Irr}_C W$ becomes the subset T_1 (see 2.7) of T'. Via the identification $\bar{\mathcal{P}}(\mathcal{I})' \leftrightarrow T'$ in 2.7 and $\hat{A}(u) \leftrightarrow \bar{\mathcal{P}}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from T' to $\hat{A}(u)$ becomes the obvious imbedding $\bar{\mathcal{P}}(\mathcal{I})_0 \to \bar{\mathcal{P}}(\mathcal{I})$ (we use again [ibid., 13.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in \mathcal{P}_{\mathrm{ev}}(\mathcal{I})$ such that for any $L \in \mathcal{P}(\mathcal{I})$ representing a vector in $\bar{\mathcal{P}}(\mathcal{I})_1$ we have $|X \cap L| = 0 \bmod 2$. Thus, (a) holds.

Using (a) we have canonically $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{I})_1^*$ via π. We have an F_2-linear map $\bar{\mathcal{P}}(\mathcal{I})_1 \to \bar{\mathcal{P}}(\mathcal{J})_0$ (see 2.6) induced by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in [0, \mu-1] \cap 2\mathbb{N}$ (I_s as in 2.7). This is an isomorphism; it corresponds to the bijection 2.7(c) under the identification $T_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{I})_1$ in 2.7 and the identification $\mathfrak{T}_1 \leftrightarrow \bar{\mathcal{P}}(\mathcal{J})_0$ in 2.6. Hence we can identify $\bar{\mathcal{P}}(\mathcal{I})_1^*$ with $\bar{\mathcal{P}}(\mathcal{J})_0^*$ and with $\bar{\mathcal{P}}(\mathcal{J})_1$ (see 2.6(a)). We obtain an identification $\bar{A}(u) = \bar{\mathcal{P}}(\mathcal{J})_1$.

By [Lusztig 1984a, 4.6] we have $X_{\mathcal{F}} = \bar{\mathcal{P}}_{\mathrm{ev}}(\mathcal{J})$. Using 2.6(a) we see that $\bar{\mathcal{P}}(\mathcal{J}) = M(\bar{\mathcal{P}}(\mathcal{J})_1) = M(\bar{A}(u))$ canonically so that Theorem 0.4 holds in our case. From the arguments above we see that in our case Corollary 0.5 follows from 2.7(c).

3.4. In 3.5–3.9 we consider the case where G is simple adjoint of exceptional type. In each case we list the elements of the set $\mathrm{Irr}_C W$ for each special unipotent class C of G; an element e of $\mathrm{Irr}_C W - \mathrm{Irr}_C^* W$ is listed as $[e]$. (The notation for the various C is as in [Spaltenstein 1985]; the notation for the objects of $\mathrm{Irr} W$ is as in [Spaltenstein 1985] (for type E_n) and as in [Lusztig 1984a, 4.10] for type F_4.) In each case the structure of $A(u)$, $\bar{A}(u)$ (for $u \in C$) is indicated; here S_n denotes the symmetric group in n letters. The order in which we list the objects in $\mathrm{Irr}_C W$ corresponds to the following order of the irreducible representations of $A(u) = S_n$:

$$
\begin{aligned}
&1, \epsilon \ (n=2),\\
&1, r, \epsilon \ (n=3,\ G \neq G_2),\\
&1, r \ (n=3,\ G = G_2),\\
&1, \lambda^1, \lambda^2, \sigma \ (n=4),\\
&1, \nu, \lambda^1, \nu', \lambda^2, \lambda^3 \ (n=5),
\end{aligned}
$$

in the notation of [Lusztig 1984a, 4.3]. Now Theorem 0.4 and Corollary 0.5 follow in our case from the tables in 3.5–3.9 and the definitions in [Lusztig 1984a, 4.8–4.13]. (In those tables S_n is the symmetric group in n letters.)

3.5. Assume that G is of type E_8.

$\mathrm{Irr}_{E_8} W = \{1_0\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_8(a_1)} W = \{8_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_8(a_2)} W = \{35_2\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_7A_1} W = \{112_3, 28_8\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_8} W = \{210_4, 160_7\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_7(a_1)A_1} W = \{560_5, [50_8]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_7(a_1)} W = \{567_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_8(a_1)} W = \{700_6, 300_8\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_7(a_2)A_1} W = \{1400_7, 1008_9, 56_{19}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{A_8} W = \{1400_8, 1575_{10}, 350_{14}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{D_7(a_1)} W = \{3240_9, [1050_{10}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_8(a_3)} W = \{2240_{10}, [175_{12}], 840_{13}\}$; $A(u) = S_3$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_6A_1} W = \{2268_{10}, 1296_{13}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_6(a_1)A_1} W = \{4096_{11}, 4096_{12}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_6} W = \{525_{12}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_7(a_2)} W = \{4200_{12}, 3360_{13}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_6(a_1)} W = \{2800_{13}, 2100_{16}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_5A_2} W = \{4536_{13}, [840_{14}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_6(a_1)A_1} W = \{6075_{14}, [700_{16}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_6A_1} W = \{2835_{14}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_6} W = \{4200_{15}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_6(a_1)} W = \{5600_{15}, 2400_{17}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{2A_4} W = \{4480_{16}, 4536_{18}, 5670_{18}, 1400_{20}, 1680_{22}, 70_{32}\}$; $A(u) = S_5$, $\bar{A}(u) = S_5$.
$\mathrm{Irr}_{D_5} W = \{2100_{20}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{(A_5A_1)''} W = \{5600_{21}, 2400_{23}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_4A_2} W = \{4200_{15}, [168_{24}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4A_2A_1} W = \{2835_{22}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4A_2} W = \{4536_{23}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_5(a_1)} W = \{2800_{25}, 2100_{28}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_42A_1} W = \{4200_{24}, 3360_{25}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_4} W = \{525_{36}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4A_1} W = \{4096_{26}, 4096_{27}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_4} W = \{2268_{30}, 1296_{33}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_4(a_1)A_2} = \{2240_{28}, 840_{31}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_3A_2} W = \{3240_{31}, [972_{32}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_4(a_1)A_1} W = \{1400_{32}, 1575_{34}, 350_{38}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{D_4(a_1)} W = \{1400_{37}, 1008_{39}, 56_{49}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.

$\mathrm{Irr}_{2A_2} W = \{700_{42}, 300_{44}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_3} W = \{567_{46}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 2A_1} W = \{560_{47}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 A_1} W = \{210_{52}, 160_{55}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_2} W = \{112_{63}, 28_{68}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{2A_1} W = \{35_{74}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_1} W = \{8_{91}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{\varnothing} W = \{1_{120}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.6. Assume that G is adjoint of type E_7.
$\mathrm{Irr}_{E_7} W = \{1_0\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_7(a_1)} W = \{7_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_7(a_2)} W = \{27_2\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_6 A_1} W = \{56_3, 21_6\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{E_6} W = \{21_3\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_6(a_1)} W = \{120_4, 105_5\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_6(a_1)A_1} W = \{189_5, [15_7]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_6(a_1)} W = \{210_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_6} W = \{105_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_5 A_1} W = \{168_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_5} W = \{189_7\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_6(a_2)A_1} W = \{315_7, 280_9, 35_{13}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{(A_5 A_1)'} = \{405_8, 189_{10}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_5(a_1)A_1} W = \{378_9\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4 A_2} W = \{210_{10}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_5(a_1)} W = \{420_{10}, 336_{11}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_5''} W = \{105_{12}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4 A_1} W = \{512_{11}, 512_{12}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_4} W = \{105_{15}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4} W = \{420_{13}, 336_{14}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{A_3 A_2 A_1} W = \{210_{13}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_3 A_2} W = \{378_{14}, [84_{15}]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_4(a_1)A_1} W = \{405_{15}, 189_{17}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_4(a_1)} W = \{315_{16}, 280_{18}, 35_{22}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{(A_3 A_1)''} W = \{189_{20}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{2A_2} W = \{168_{21}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 3A_1} W = \{105_{21}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_3} W = \{210_{21}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

$\mathrm{Irr}_{A_2 2A_1} W = \{189_{22}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 A_1} W = \{120_{25}, 105_{26}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{3A_1''} W = \{21_{36}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2} W = \{56_{30}, 21_{33}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{2A_1} W = \{27_{37}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_1} W = \{7_{46}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{\varnothing} W = \{1_{63}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.7. Assume that G is adjoint of type E_6.
$\mathrm{Irr}_{E_6} W = \{1_0\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{E_6(a_1)} W = \{6_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_5} W = \{20_2\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_5 A_1} W = \{30_3, 15_5\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{D_5(a_1)} W = \{64_4\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4 A_1} W = \{60_5\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_4} W = \{81_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_4} W = \{24_6\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{D_4(a_1)} W = \{80_7, 90_8, 20_{10}\}$; $A(u) = S_3$, $\bar{A}(u) = S_3$.
$\mathrm{Irr}_{2A_2} W = \{24_{12}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_3} W = \{81_{10}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 2A_1} W = \{60_{11}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2 A_1} w = \{64_{13}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2} W = \{30_{15}, 15_{17}\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{2A_1} W = \{20_{20}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_1} W = \{6_{25}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{\varnothing} W = \{1_{36}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.8. Assume that G is of type F_4.
$\mathrm{Irr}_{F_4} W = \{1_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{F_4(a_1)} W = \{4_2, 2_3\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.
$\mathrm{Irr}_{F_4(a_2)} W = \{9_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{B_3} W = \{8_1\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{C_3} W = \{8_3\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{F_4(a_3)} W = \{12_1, 9_3, 6_2, 1_3\}$; $A(u) = S_4$, $\bar{A}(u) = S_4$.
$\mathrm{Irr}_{\tilde{A}_2} W = \{8_2\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_2} W = \{8_4, [1_2]\}$; $A(u) = S_2$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{A_1 \tilde{A}_1} W = \{9_4\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.
$\mathrm{Irr}_{\tilde{A}_1} W = \{4_5, 2_2\}$; $A(u) = S_2$, $\bar{A}(u) = S_2$.

$\mathrm{Irr}_{\varnothing} W = \{1_4\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.9. Assume that G is of type G_2.

$\mathrm{Irr}_{G_2} W$ is the unit representation; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

$\mathrm{Irr}_{G_2(a_1)} W$ consists of the reflection representation and the one dimensional representation on which the reflection with respect to a long (resp. short) simple coroot acts nontrivially (resp. trivially); $A(u) = S_3$, $\bar{A}(u) = S_3$.

$\mathrm{Irr}_{\varnothing} W = \{\mathrm{sgn}\}$; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$.

3.10. This completes the proof of Theorem 0.4 and that of Corollary 0.5.

We note that the definition of $\mathcal{G}_{\mathcal{F}}$ given in [Lusztig 1984a] (for type C_n, B_n) is $\bar{\mathcal{P}}(\mathcal{J})_1$ (in the setup of 3.2) and $\bar{\mathcal{P}}(\mathcal{J})_0$ (in the setup of 3.3) which is noncanonically isomorphic to $\bar{A}(u)$, unlike the definition adopted here that is, $\bar{\mathcal{P}}(\mathcal{J})_0$ (in the setup of 3.2) and $\bar{\mathcal{P}}(\mathcal{J})_1$ (in the setup of 3.3) which makes $\mathcal{G}_{\mathcal{F}}$ canonically isomorphic to $\bar{A}(u)$.

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