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REFLEXIVE OPERATOR ALGEBRAS ON BANACH SPACES

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In this paper we study the reflexivity of a unital strongly closed algebra of operators with complemented invariant subspace lattice on a Banach space. We prove that if such an algebra contains a complete Boolean algebra of projections of finite uniform multiplicity and with the direct sum property, then it is reflexive, i.e., it contains every operator that leaves invariant every closed subspace in the invariant subspace lattice of the algebra. In particular, such algebras coincide with their bicommutant.

1. Introduction

Let $A \subset B(X)$ denote a strongly closed algebra of operators on the Banach space X. Suppose that A has the property that each of its invariant subspaces has an invariant complement. If A contains a complete Boolean algebra of projections of finite uniform multiplicity and with the direct sum property as defined below, we prove that A is reflexive in the sense that it contains all the operators which leave its closed invariant subspaces invariant (Theorem 15). In particular such an algebra is equal to its bicommutant A'' (Corollary 22). The problem of whether a strongly closed algebra of operators with complemented invariant subspace lattice is reflexive started to be studied in the sixties. This problem is a generalization of the invariant subspace problem in operator theory. Arveson [1967] introduced a technique for studying the particular case of transitive algebras on Hilbert spaces, namely the strongly closed algebras of operators on Hilbert spaces that have no nontrivial closed invariant subspaces. He proved that every transitive algebra that contains a maximal abelian von Neumann algebra coincides with the full algebra $B(X)$ if X is a complex Hilbert space. Douglas and Pearcy [1972] extended the result of Arveson to the case of transitive operator algebras containing an abelian von Neumann algebra of finite multiplicity. Hoover [1973] extended the result of Douglas and Pearcy to the case of reductive operator algebras on Hilbert spaces that contain

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abelian von Neumann algebras of finite multiplicity. Hoover proved that every reductive operator algebra (that is a strongly closed subalgebra for which every closed invariant subspace is reducing) which contains an abelian von Neumann algebra of finite multiplicity is self-adjoint. The transitive algebra result of Douglas and Pearcy was generalized in [Önder and Orhon 1989] to the case of transitive algebras on Banach spaces that contain a n-fold direct sum of a cyclic complete Boolean algebra of projections. The case of operator algebras on Banach spaces with complemented invariant subspace lattice was considered by Rosenthal and Sourour [1977]. They proved that every strongly closed algebra of operators with complemented invariant subspace lattice containing a complete Boolean algebra of projections of uniform multiplicity one is reflexive.

In this paper we build upon the techniques introduced by Arveson and developed in [Douglas and Pearcy 1972; Radjavi and Rosenthal 1973] for invariant subspaces of operator algebras as well as Bade's multiplicity theory of Boolean algebras of projections [Bade 1955; 1959]. We also use the results of [Foguel 1959] and [Tzafriri 1967] about the commutant of Boolean algebras of projections of finite multiplicity.

2. Notation and preliminary results

2.1. ***Invariant subspaces of operator algebras.*** Let X be a complex Banach space and $B(X)$ the algebra of all bounded linear operators on X. We will denote by $X^{(n)}$ the direct sum of n copies of X and, if $S \subset B(X)$, we set

$$S^{(n)} = \big\{a \oplus a \oplus \cdots \oplus a \in B(X^{(n)}) \,\big|\, a \in S\big\}.$$

If $S \subset B(Y)$, where Y is a Banach space, we denote by $\operatorname{Lat} S$ the collection of all closed linear subspaces of Y that are invariant under every element of S. If L is a collection of closed linear subspaces of Y, we denote by $\operatorname{alg} L$ the (strongly closed) algebra of operators on Y that leave every element of L invariant. An algebra $A \subset B(X)$ is called *reflexive* if $\operatorname{algLat} A = A$.

In what follows all the subalgebras $A \subset B(X)$ will be assumed to be strongly closed and containing the identity operator $I \in B(X)$.

Remark 1. Let $A \subset B(X)$ be a strongly closed algebra with $I \in A$ and $b \in B(X)$. If $\operatorname{Lat} A^{(n)} \subset \operatorname{Lat} b^{(n)}$ for every $n \in \mathbb{N}$, then $b \in A$.

Proof. Indeed, then for every finite set of elements $\{x_1, x_2, \ldots, x_n\} \subset X$ we have that $K = \overline{\{ax_1 \oplus ax_2 \oplus \cdots \oplus ax_n \mid a \in A\}} \in \operatorname{Lat} A^{(n)}$ and therefore $K \in \operatorname{Lat} b^{(n)}$. This means that $b \in A$, since A is strongly closed. □

Proposition 2. *Let $A \subset B(X)$ be a strongly closed algebra with complemented invariant subspace lattice and with $I \in A$. Let $q \in B(X)$ be a projection.*

(i) *If $q \in A$, the algebra $qAq \subset B(qX)$ has complemented invariant subspace lattice and* $\mathrm{algLat}(qAq) = q(\mathrm{algLat}\, A)q$.

(ii) *If $q \in A'$, where A' denotes the commutant of A, the strong operator closure $\overline{qAq}^{\mathrm{so}} \subset B(qX)$ is an algebra with complemented invariant subspace lattice.*

Proof. We prove first (i). Clearly, qAq is a strongly closed subalgebra of $B(qX)$ whose unit is q. Let $L \subset qX$, $L \in \mathrm{Lat}(qAq)$. We define $\tilde{L} = \overline{AqL}$, the closure being taken in X. Then, obviously, $\tilde{L} \in \mathrm{Lat}\, A$ and, therefore $\tilde{L}$ has a complement $\tilde{L}^c$ in $\mathrm{Lat}\, A$. Since $q \in A$, we have $q\tilde{L} \subset \tilde{L}$, also $q\tilde{L}^c \subset \tilde{L}^c$ and $q\tilde{L}^c \in \mathrm{Lat}(qAq)$. Moreover, it is immediate that $q\tilde{L}$ and $q\tilde{L}^c$ are closed linear subspaces of qX such that

$$q\tilde{L} \oplus q\tilde{L}^c = qX.$$

On the other hand we have $L \subset q\tilde{L} = q\overline{AqL} \subset \overline{qAqL} \subset \bar{L} = L$. Hence

$$q\tilde{L} = L.$$

It follows that L is complemented in $\mathrm{Lat}(qAq)$ and so qAq has complemented invariant subspace lattice. Let now $b \in \mathrm{algLat}\, A$ and $L \in \mathrm{Lat}(qAq)$. By the above argument, there exists $\widetilde{L} \in \mathrm{Lat}\, A$ such that $L = q\tilde{L}$. Hence $b\widetilde{L} \subset \tilde{L}$. Therefore, since $q\tilde{L} = L \subset \tilde{L}$ it follows that $qbqL \subset L$, so $qbq \in \mathrm{algLat}(qAq)$. Conversely, let $c \in \mathrm{algLat}(qAq)$ and let $\tilde{c} \in B(X)$ be the extension of c to X that equals 0 on $(I - q)X$. Then, it is straightforward to show that $\tilde{c} \in \mathrm{algLat}\, A$ and $c = q\tilde{c}q$ and so the proof is completed.

To establish (ii), let $K \in \mathrm{Lat}(qAq)$. Since $q \in A'$, it follows that $K \in \mathrm{Lat}\, A$ and therefore K has a complement $K^c \in \mathrm{Lat}\, A$. Then, clearly $K^c \cap qX \in \mathrm{Lat}(qAq)$ and $K + K^c \cap qX = qX$. □

We will also need the following:

Lemma 3. *Let $A \subset B(X)$ be an algebra with complemented invariant subspace lattice and let $K \in \mathrm{Lat}\, A$. If $p \in A'$ is the projection on K and $t_1, t_2, \dots, t_n \in (pAp)'$, for some $n \in \mathbb{N}$, then the subspace*

$$\Gamma_{\{t_1,t_2,\dots,t_n;\,p\}} = \left\{ x \oplus t_1x \oplus t_2x \oplus \cdots \oplus t_nx \;\middle|\; x \in pX \right\} \in \mathrm{Lat}\, A^{(n+1)}$$

is complemented in $\mathrm{Lat}\, A^{(n+1)}$.

Proof. Since A has complemented invariant subspace lattice and $pX = K \in \mathrm{Lat}\, A$, it follows that the subspace $(1 - p)X = (pX)^c = K^c$ belongs to $\mathrm{Lat}\, A$. It is then clear that $(pX)^c \oplus X^{(n)}$ is a complement of $\Gamma_{\{t_1,t_2,\dots,t_n;\,p\}}$ in $\mathrm{Lat}\, A^{(n+1)}$. □

Remark 4. Let $I \in A \subset B(X)$ be a strongly closed subalgebra with complemented invariant subspace lattice. If A is reflexive, then $A'' = A$ where A'' denotes the bicommutant of A.

Proof. If $a \in A''$, then, in particular, a commutes with every projection on an invariant subspace of A. Therefore $a \in \operatorname{algLat} A = A$. □

The following concept is defined in [Radjavi and Rosenthal 1973, §8.2], for instance.

Definition 5. Let $A \subset B(X)$ be a subalgebra. A linear operator T defined on a not necessarily closed linear subspace $P \subset X$ is called a graph transformation for A if there exist finitely many linear operators $T_1, T_2, \ldots, T_l$, all defined on P, such that

$$\{x \oplus Tx \oplus T_1x \oplus T_2x \oplus \cdots \oplus T_lx \mid x \in P\} \in \operatorname{Lat} A^{(l+2)}.$$

Remark 6. Let $K \in \operatorname{Lat} A^{(n)}$, $n \in \mathbb{N}$. Define

$$K_0 = \{\boldsymbol{x} \in X^{(n-1)} \mid 0 \oplus \boldsymbol{x} \in K\} \in \operatorname{Lat} A^{(n-1)}.$$

Then, if K_0 is complemented in $\operatorname{Lat} A^{(n-1)}$ with complement K_0^c it follows that there exist graph transformations for A: $T_1, T_2, \ldots, T_{n-1}$, defined on a linear subspace, $P \subset X$, such that

$$(X \oplus K_0^c) \cap K = \{x \oplus T_1x \oplus T_2x \oplus \cdots \oplus T_{n-1}x \mid x \in P\}.$$

Proof. Straightforward. □

2.2. *Boolean algebras of projections in Banach spaces and spectral operators.* Let $\mathcal{B}$ be a complete Boolean algebra of projections in a (complex) Banach space X (as defined for instance in [Bade 1955] or in [Dunford and Schwartz 1988, Chapter XVII]). It is known [Stone 1949] that there exists an extremally disconnected compact Hausdorff topological space Ω (that is a compact Hausdorff space in which the closure of every open set in it is open), such that $\mathcal{B}$ is equivalent as a Boolean algebra with the Boolean algebra of open and closed subsets of Ω. We will denote by Σ the collection of Borel sets of Ω. Such a compact Hausdorff space is called a Stonean space.

The following remark collects some results about the complete Boolean algebras of projections in Banach spaces that will be used in this paper.

Remark 7. (i) If $\mathcal{B}$ is a complete Boolean algebra of projections, then there is a regular countably additive spectral measure E in X defined on the family of Borel sets in Ω such that the mapping

$$S(f) = \int_\Omega f(w)\, E(dw)$$

is a continuous isomorphism of the algebra $C(\Omega)$ of continuous functions on Ω onto the uniformly closed algebra of operators, B, generated by $\mathcal{B}$.

(ii) The algebra B coincides with the strongly closed algebra generated by $\mathcal{B}$ and consists of spectral operators of scalar type.

(iii) The range of E is precisely the Boolean algebra $\mathcal{B}$.

(iv) $\mathcal{B}$ is norm bounded.

Proof. (i) and (iii) follow from [Dunford and Schwartz 1988, Lemma XVII.3.9]. Point (ii) is Corollary XVII.3.17 of the same reference, and (iv) follows from [Bade 1955, Theorem 2.2]. □

Let $\mathcal{B} \subset B(X)$ be a complete Boolean algebra of projections that contains the identity projection $I \in B(X)$. We say that $I \in \mathcal{B}$ has multiplicity k, $k \in \mathbb{N}$, if there are $x_1, x_2, \ldots, x_k \in X$ such that $\overline{\text{lin}}\,\{ex_i \mid e \in \mathcal{B}, 1 \le i \le k\} = X$ and no subset of X of cardinality less than k has this property [Bade 1959, Definition 3.2]. The Boolean algebra $\mathcal{B}$ is said to be of uniform multiplicity k if every projection $e \in \mathcal{B}$, $e \neq 0$ has multiplicity k. For each i, $1 \le i \le k$, define $\mathfrak{M}(x_i) = \overline{\text{lin}}\{ex_i \mid e \in \mathcal{B}\}$. Here, $\overline{\text{lin}}\{ex_i \mid e \in \mathcal{B}\}$ denotes the closed linear subspace of X spanned by $\{ex_i \mid e \in \mathcal{B}\}$.

The next remark collects some known results from [Bade 1959] (see also [Dunford and Schwartz 1988]).

Remark 8. Let $\mathcal{B}$ be a complete Boolean algebra of finite uniform multiplicity n, $n \in \mathbb{N}$, and let $\{x_1, \ldots, x_n\}$ be a set of vectors such that

$$\overline{\text{lin}}\{ex_i \mid e \in \mathcal{B}, 1 \le i \le n\} = X.$$

(i) There are $x_i^* \in X^*$, $i = 1, 2, \ldots, n$, where X^* is the dual Banach space of X, such that each of the measures $\mu_i(\delta) = x_i^* E(\delta) x_i$, $i \in \{1, 2, \ldots, n\}$, $\delta \in \Sigma$ vanishes on sets of first category of Ω and $\mu_i(\sigma) \neq 0$ if σ has nonempty interior. The measures μ_i are equivalent and $x_i^*(\mathfrak{M}(x_j)) = \{0\}$ for $i \neq j$.

(ii) There exists a continuous injective linear map V of X onto a dense linear subspace $L \subset \sum_{i=1}^n L^1(\Omega, \Sigma, \mu_i)$ such that if $V(x) = \boldsymbol{f} = \sum f_i$, then:

(a)
$$x_i^* E(\delta) x = \int_\delta f_i(\omega)\, \mu_i(d\omega),$$

for $\delta \in \Sigma$. In particular, $V(x_i) = 0 \oplus \cdots \oplus \chi_\Omega \oplus \cdots \oplus 0$, where $\chi_\Omega = 1$ is in the i-th place in the direct sum.

(b)
$$x = \lim_{m \to \infty} \sum_{i=1}^{n} S(f_i \chi_{\delta_m}) x_i,$$

where χ_{δ_m} is the characteristic function of

$$\delta_m = \big\{\omega \,\big|\, |f_i(\omega)| \le m,\ i = 1, 2, \ldots, n\big\}.$$

(iii) The linear space L is a Banach space when endowed with the norm

$$\|\boldsymbol{f}\|_0 = \max_{1 \le i \le n} \|f_i\|_1 + \|V^{-1}(\boldsymbol{f})\|,$$

and V is a Banach space isomorphism between X and $(L, \|\cdot\|_0)$.

Proof. Points (i) and (ii) follow from [Bade 1959, Lemma 5.1 and Theorem 5.2] (see also [Dunford and Schwartz 1988, Theorem XVIII.3.19]). The proof of (iii) is immediate. □

A function f is called E-essentially bounded if

$$\inf_{E(\delta)=1} \sup_{\omega\in\delta} |f(\omega)|$$

is finite [Dunford and Schwartz 1988, Definition 7].

Denote by $EB(\Omega, \Sigma)$ the set of all E-essentially bounded Σ-measurable functions.

Lemma 9. *With the notations in Remark 8, if $\varphi \in EB(\Omega, \Sigma)$, then the operator $M_\varphi(\boldsymbol{f}) = \varphi \boldsymbol{f}$ is a well defined, bounded operator on $(L, \|\cdot\|_0)$ and $M_\varphi = VS(\varphi)V^{-1}$. Here $\varphi\boldsymbol{f} = \varphi f_1 \oplus \varphi f_2 \oplus \cdots \oplus \varphi f_n$. Thus*

$$VBV^{-1} = \{M_\varphi \mid \varphi \in EB(\Omega, \Sigma)\}.$$

Proof. Let $\boldsymbol{f} \in L$ and $x = S(\varphi)V^{-1}(\boldsymbol{f})$. Then, according to point (a) in Remark 8 (ii), if $\boldsymbol{g} = V(x)$, we have $x_i^* S(\chi_\delta)x = \int_\delta g_i(w)\,\mu_i(dw)$ for every Borel set $\delta \in \Sigma$. On the other hand,

$$x_i^* S(\chi_\delta)x = x_i^* S(\chi_\delta)S(\varphi)V^{-1}(\boldsymbol{f}) = x_i^* S(\chi_\delta\varphi)V^{-1}(\boldsymbol{f}) = \int_\delta \varphi(w) f_i(w)\,\mu_i(dw).$$

Hence $g_i = \varphi f_i$ μ_i-a.e., so $\boldsymbol{g} = \varphi\boldsymbol{f}$ a.e. and the proof is completed. □

In [Dieudonné 1956] is presented an example of a Boolean algebra of projections, $\mathcal{B}$, such that every nonzero projection $e \in \mathcal{B}$ has multiplicity 2. However, for no choice of $x_1, x_2 \in X$ or $e \in \mathcal{B}$, $e \neq 0$ is eX the algebraic sum of $\mathfrak{M}(ex_1)$ and $\mathfrak{M}(ex_2)$. In the rest of this paper we will consider only Boolean algebras of finite uniform multiplicity with the direct sum property:

Definition 10. We say that the complete Boolean algebra $\mathcal{B}$ of uniform multiplicity k has the direct sum property if X is the algebraic (and therefore, Banach) direct sum of $\mathfrak{M}(x_i)$, $1 \le i \le k$.

A particular case of a Boolean algebra of uniform multiplicity k with the direct sum property is the k-fold direct sum of k copies of a cyclic Boolean algebra of projections. Other examples are presented in [Foguel 1959].

Lemma 11. *Suppose that $\mathcal{B}$ is a complete Boolean algebra of projections of uniform multiplicity k with the direct sum property. Then, for every $\epsilon > 0$ there exist $e \in \mathcal{B}$, $e = E(\rho)$, and $\rho \in \Sigma$ with $\mu_l(\rho^c) < \epsilon$ for every $1 \le l \le k$ (where ρ^c is the complement of ρ) such that for every $\{\varphi_{ij} \mid 1 \le i, j \le k\} \subset EB(\Omega, \Sigma)$, the matrix $[\varphi_{ij}\chi_\rho]$ is a bounded linear operator on $(L, \|\cdot\|_0)$ and $[\varphi_{ij}\chi_\rho]$ belongs to the commutant $\mathcal{B}'$ of $\mathcal{B}$.*

Proof. Since the measures μ_l, $1 \le l \le k$ are equivalent, let $h_{ml} = d\mu_m/d\mu_l$, $1 \le m, l \le k$ be the corresponding Radon Nikodym derivative. Let $\epsilon > 0$ be arbitrary. Fix $1 \le m, l \le k$. Then, since $\bigcup_{n=1}^{\infty} \{1/n \le h_{ml} \le n\} = \Omega$, there is a $n \in \mathbb{N}$ such that $\mu_l(\{1/n \le h_{ml} \le n\}^c) < \epsilon/k^2$. Therefore there is a $n \in \mathbb{N}$ such that $\mu_l(\{1/n \le h_{ml} \le n\}^c) < \epsilon$ for every $1 \le m, l \le k$. Let $\rho = \{1/n \le h_{ml} \le n\} \in \Sigma$. It is easy to see that for every Borel subset $\sigma \subset \rho$ we have $\mu_i(\sigma)/n \le \mu_j(\sigma) \le n\mu_i(\sigma)$ for all $1 \le i, j \le k$. Hence all the spaces $M_{\chi_\rho} L^1(\mu_i) = \chi_\rho L^1(\mu_i)$, $1 \le i \le k$, are equal as sets and mutually isomorphic as Banach spaces. Then, clearly,

$$\chi_\rho L = \chi_\rho L^1(\mu_1) \oplus \chi_\rho L^1(\mu_2) \oplus \cdots \oplus \chi_\rho L^1(\mu_k).$$

Since $\mathcal{B}$ has the direct sum property, we also have

$$E(\rho)X = E(\rho)\mathfrak{M}(x_1) \oplus \cdots \oplus E(\rho)\mathfrak{M}(x_k)$$

and the lemma follows. □

For the definition and basic facts about spectral operators on Banach spaces we refer to [Dunford and Schwartz 1988]. We will need the following result, which follows from [Tzafriri 1967, Theorem 2] and [Foguel 1959, Lemma 2.1 and Theorem 2.3].

Remark 12. Let $T \in B(X)$ and let $\mathcal{B}$ be a complete Boolean algebra of projections in X, of uniform multiplicity k, $k \in \mathbb{N}$. If T commutes with the strongly closed algebra B generated by $\mathcal{B}$, then there exists an increasing sequence of projections $\{e_m = E(\chi_{\delta_m}) \mid m \in \mathbb{N}\} \subset \mathcal{B}$ such that $\{e_m\}$ converges strongly to the identity $I \in B(X)$ and Te_m is a spectral operator of finite type for every m. Moreover, if $T \in B'$ is a spectral operator then T is the sum of a spectral operator R of scalar type in B' and a nilpotent operator Q of order k, $Q \in B'$.

Next we will study the dense linear subspaces of X that are invariant under every element of B, where B is the strongly closed algebra generated by $\mathcal{B}$, the complete Boolean algebra of projections of uniform multiplicity k with the direct sum property. The following lemma is an extension to the case of Banach spaces and an improvement on [Douglas and Pearcy 1972, Lemma 3.3]. Using Remark 8 and Lemma 9, we will identify X with L and B with $\{VS(\varphi)V^{-1} \mid S(\varphi) \in B\}$.

Lemma 13. *Let $k \in \mathbb{N}$ and B the strongly closed algebra generated by the Boolean algebra of projections of uniform multiplicity k, $\mathcal{B} \subset B(X)$ and with the direct sum property. With the above notations, suppose that $\mathcal{D} \subset X$ is a dense linear subspace which is invariant under all operators in B. Then, for every $\epsilon > 0$, there exists an open and closed set $\lambda_\epsilon \subset \Omega$ such that*

(i) $\mu_i(\lambda_\epsilon^c) < \epsilon$, $i = 1, 2, \ldots, k$, *where λ_ϵ^c is the complement of λ_ϵ in Ω, and*

(ii) $\chi_{\lambda_\epsilon} \boldsymbol{e}_j \in \mathfrak{D}$ *for all* $j \in \{1, 2, \dots, k\}$, *where* $\{\boldsymbol{e}_j \mid j = 1, 2, \dots, k\}$ *is the standard basis of* $\mathbb{C}^{(k)}$.

Proof. If $\boldsymbol{z} = (z^1, z^2, \dots, z^k) \in \mathbb{C}^{(k)}$, consider the norm

$$\|\boldsymbol{z}\| = \max\left\{|z^p| \;\middle|\; 1 \le p \le k\right\}.$$

It is easy to see that there exists $\alpha > 0$ such that if the set $\{h_1, h_2, \dots, h_k\} \subset \mathbb{C}^{(k)}$ satisfies $\|h_i - \boldsymbol{e}_i\| < \alpha$, $i = 1, 2, \dots, k$, then the set $\{h_1, h_2, \dots, h_k\}$ is linearly independent. Let now $\epsilon > 0$ be arbitrary. We can choose $\alpha < \epsilon^2/2$. Let $\rho \in \Sigma$, $\mu_l(\rho) < \epsilon/2$, $1 \le l \le k$, be as in Lemma 11. Since Ω is extremally disconnected, we can assume that ρ is an open and closed set. For every j, $1 \le j \le k$ let $\boldsymbol{g}_j(w) = \boldsymbol{e}_j$, if $w \in \rho$ and $\boldsymbol{g}_j(w) = 0$ if $\omega \in \rho^c$. Since by point (a) of Remark 8 (ii) we have that $\boldsymbol{g}_j \in \chi_\rho L$ for every j, $1 \le j \le k$ and $\chi_\rho \mathfrak{D}$ is dense in $\chi_\rho L$, it follows that there exists a set of elements $\{\boldsymbol{l}_i \mid 1 \le i \le k\} \subset \chi_\rho \mathfrak{D}$, $\boldsymbol{l}_i = l_i^1 \oplus l_i^2 \oplus \cdots \oplus l_i^k$ such that

$$\|\boldsymbol{l}_i - \boldsymbol{g}_i\|_0 = \max_{1 \le p \le k} \left\{\|l_i^p - g_i^p\|_0 = \|l_i^p - g_i^p\|_1 + \|T^{-1}(l_i^p - g_i)^p\|\right\} < \alpha < \epsilon^2.$$

Let $\delta_\epsilon = \bigcap_{i=1}^{k} \left\{\omega \in \rho \;\middle|\; |l_i^p(w) - g_i^p(w)| \ge \epsilon \text{ and } 1 \le p \le k\right\}$. Then we have

$$\epsilon^2/2 > \alpha > \max\left\{\|l_i^p - g_i^p\|_1 \;\middle|\; 1 \le i, p \le k\right\} \ge \epsilon \mu_m(\delta_\epsilon) \quad \text{for } 1 \le m \le k.$$

Hence $\mu_m(\delta_\epsilon) < \epsilon/2$ for $m = 1, 2, \dots, k$. Assuming that $\epsilon < 2$, it follows that $\mu_m(\delta_\epsilon^c) \ne 0$ and since Ω is a Stonean space, and μ_m a normal measure, $\mu_m(\delta_\epsilon^c) = \mu_m((\delta_\epsilon^c)^\circ)$ where $(\delta_\epsilon^c)^\circ$ is the interior of δ_ϵ^c. The same argument as the preceding one shows that there exists an open and closed subset $\sigma_\epsilon \subset (\delta_\epsilon^c)^\circ$ with $\mu_m(\sigma_\epsilon^c) < \epsilon/2$. Let $\lambda_\epsilon = \rho \cap \sigma_\epsilon$. Then, $\mu_m(\lambda_\epsilon) < \epsilon$ for all $1 \le m \le k$. It follows that all the components of the vectors $l_i^\epsilon = l_i \chi_{\lambda_\epsilon} \in L$ are in $EB(\Omega, \Sigma)$. Let M be the matrix whose i-th column is l_i^ϵ. Then, using Lemma 11, it follows that M is a bounded linear operator that commutes with every element in B, so $M \in B^{'}$. The choice of α implies that $M(w)$ is nonsingular for every $\omega \in \lambda_\epsilon$. Consider the matrix N defined as follows:

$$N(w) = \begin{cases} M(w)^{-1} & \text{if } w \in \lambda_\epsilon, \\ \quad 0 & \text{if } w \in \lambda_\epsilon^c. \end{cases}$$

By restricting N to an open and closed subset of λ_ϵ, if necessary, we can apply Lemma 11 again and get $N \in B'$. It follows that the columns of the product MN are linear combinations of vectors in $\mathfrak{D}$ with coefficients in B. Since $\mathfrak{D}$ is invariant under B we have that these columns belong to $\mathfrak{D}$. Since $M(w)N(w) = I$ for $w \in \lambda_\epsilon$ the proof is completed. □

We will use next the following results about spectral operators and their resolutions of the identity from [Dunford and Schwartz 1988].

Remark 14. If the operator M commutes with the spectral operator T, then M commutes with every resolution of the identity of T.

Proof. This is [Dunford and Schwartz 1988, Corollary XV.3.7]. □

3. Algebras with complemented invariant subspace lattices

In this section we will prove our main result:

Theorem 15. *Let B be the strongly closed subalgebra of $B(X)$ generated by a complete Boolean algebra of projections $\mathcal{B} \subset B(X)$ of finite uniform multiplicity, k, with the direct sum property. If $A \subset B(X)$ is a strongly closed algebra with complemented invariant subspace lattice that contains B, then A is reflexive.*

The proof of this theorem will be given after a series of auxiliary results. In the rest of this section $\mathcal{B}$ and B will be as in Theorem 15. We will identify X with $(L, \|\cdot\|_0)$ as in Remark 8.

Proposition 16. *Let B as in Theorem 15 and let T be a densely defined closed operator on X which commutes with B. There exists an increasing sequence of projections $\{q_p\}_{p=1}^{\infty} \subset \mathcal{B}$ that converges strongly to I such that Tq_p is a spectral operator of finite type for every $p \in \mathbb{N}$.*

Proof. Let $\mathcal{D} \subset X$ be the (dense) domain of T. Since T commutes with B it follows that $\mathcal{D}$ is invariant under B. By Lemma 13 it follows that for every $p \in \mathbb{N}$ there is an open and closed subset $\sigma_p \subset \Omega$ such that $\chi_{\sigma_p} \oplus \chi_{\sigma_p} \oplus \cdots \oplus \chi_{\sigma_p} \in \mathcal{D}$ and $\mu_l(\sigma_p^c) < 1/2p$ for every $1 \leq l \leq k$. Define $r_p = S(\chi_{\sigma_p}) \in \mathcal{B}$. Obviously, we can take $r_p \leq r_{p+1}$ (in the sense that $r_pX \subset r_{p+1}X$) for every $p \in \mathbb{N}$. Therefore Tr_p $(p \in \mathbb{N})$ is a bounded operator and $r_p \nearrow I$. On the other hand, by Remark 12, since $Tr_p \in B'$, for every $p \in \mathbb{N}$, there exists a Borel set $\delta_p \in \Sigma$ such that, for all $1 \leq l \leq p$, we have $\mu_l(\delta_p^c) < 1/2p$. Furthermore, if $q_p = S(\chi_{\delta_p \cap \sigma_p})$, then Tq_p is a spectral operator of finite type. Clearly $\{q_p\}$ is an increasing sequence of projections in $\mathcal{B}$ that converges strongly to I and the proof is completed. □

Proposition 17. *Assume that B is as in the statement of Theorem 15. Let T be a densely defined graph transformation for $B \subset B(X)$. Then there exists an increasing sequence of projections $\{q_p\}_{p=1}^{\infty} \subset \mathcal{B}$ that converges strongly to I such that Tq_p is a spectral operator of finite type for every $p \in \mathbb{N}$. In particular every such transformation is closable and its closure commutes with B.*

Proof. Let T be a densely defined graph transformation for B with domain $\mathcal{D}_T$. Since T is a graph transformation for B, there exists $l \in \mathbb{N}$ and operators $T_1, T_2, \ldots, T_{l-2}$ such that the subspace

$$Z = \left\{ x \oplus Tx \oplus T_1x \oplus T_2x \oplus \cdots \oplus T_{l-2}x \;\middle|\; x \in \mathcal{D}_T \right\}$$

belongs to Lat $B^{(l)}$. Define $\Delta_{l-1} = \{x \oplus x \oplus \cdots \oplus \mid x \in X\} \subset X^{(l-1)}$. Then it can be easily seen that the subspace

$$\Delta_{l-1}^c = \left\{ x_1 \oplus x_2 \oplus \cdots \oplus x_{l-1} \;\middle|\; x_i \in X \text{ with } \sum_{i=1}^{l-1} x_i = 0 \right\}$$

is a Banach subspace complement of Δ_{l-1} which is invariant under every element of $B^{(l-1)}$. The operator $\widetilde{T}$ defined by

$$\widetilde{T}(x \oplus x \oplus \cdots \oplus x) = Tx \oplus T_1 x \oplus \cdots \oplus T_{l-2} x \quad \text{if } x \in \mathfrak{D}_T$$

and

$$\widetilde{T}(x_1 \oplus x_2 \oplus \cdots \oplus x_{l-1}) = 0 \quad \text{if } x_1 \oplus x_2 \oplus \cdots \oplus x_{l-1} \in \Delta_{l-1}^c$$

is a closed, densely defined operator which commutes with $B^{(l-1)}$. An application of Proposition 16 with k replaced by $k(l-1)$ completes the proof. □

Remark 18. Let $A \subset B(X)$ be a strongly closed algebra with complemented invariant subspace lattice and $I \in A$. Then, if $Q \in A'$ is such that $Q^2 = 0$ it follows that $Q \in (\operatorname{algLat} A)'$.

Proof. The proof of [Feintuch and Rosenthal 1973, Lemma 3] for the particular case of Hilbert spaces can be extended to the case of Banach spaces. Indeed, let $Q \in A'$ be such that $Q^2 = 0$. Then, if $Y = \ker Q$ is the null space of Q, Y is in Lat A and since A has a complemented invariant subspace lattice, Y has a complement, Y^c in Lat A. Therefore Q can be written as a matrix

$$Q = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix},$$

and every $a \in A$ can be written as the matrix

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

Moreover, every $b \in \operatorname{algLat} A$, can be written as a matrix

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}.$$

Since $aQ = Qa$ it follows that $ca_2 = a_1 c$. Hence the subspace $\{cx \oplus x \mid x \in Y^c\}$ belongs to Lat A and is therefore invariant for algLat A. It follows that $cb_2 = b_1 c$, so $Qb = bQ$. □

Part (i) of the next result is a generalization of Remark 18.

Proposition 19. *Let $A \subset B(X)$ be an algebra with complemented invariant subspace lattice.*

(i) *If $Q \in A'$ is a nilpotent operator*, *then $Q \in (\operatorname{algLat} A)'$.*

(ii) *If $T = R + Q$ is a spectral operator of finite type* (*where R is spectral of scalar type and Q is nilpotent*) *and $T \in A'$*, *then $R \in (\operatorname{algLat} A)'$ and $N \in (\operatorname{algLat} A)'$.*

Proof. We will prove point (i) of this proposition by induction. By Remark 18, if $Q \in A'$ and $Q^2 = 0$, then $Q \in (\operatorname{algLat} A)'$. Suppose that for every operator $Q \in A'$ with $Q^n = 0$ it follows that $Q \in (\operatorname{algLat} A)'$ and let $Q \in A'$ with $Q^{n+1} = 0$. Let p_0 denote a projection on $\ker Q$ such that $p_0 \in A'$. Since $Qp_0 = 0$ it follows that

$$(1 - p_0)Q = (1 - p_0)Q(1 - p_0)$$

and therefore

$$(1 - p_0)Q^k = ((1 - p_0)Q(1 - p_0))^k, \quad k \in \mathbb{N}.$$

Since $Q^{n+1} = 0$ we have $Q^n(X) \subset \ker Q$ and therefore

$$0 = (1 - p_0)Q^n = ((1 - p_0)Q(1 - p_0))^n.$$

By hypothesis, $(1 - p_0)Q = (1 - p_0)Q(1 - p_0) \in (\operatorname{algLat} A)'$. On the other hand, since $Q \in A'$ and $p_0 \in A'$ we have $p_0 Q \in A'$. Since obviously $(p_0 Q)^2 = 0$, by Remark 18, it follows that $p_0 Q \in (\operatorname{algLat} A)'$. Therefore

$$Q = p_0 Q + (1 - p_0)Q \in (\operatorname{algLat} A)'$$

and the proof of (i) is completed.

We turn now to prove point (ii). By Remark 14, every resolution of the identity of T, $\boldsymbol{E}(\delta)$, where δ is a Borel subset of the spectrum of T, $\delta \subset \operatorname{sp}(T)$, is in A'. Therefore, since A has complemented invariant subspace lattice, it follows that $\boldsymbol{E}(\delta) \in (\operatorname{algLat} A)'$ for every Borel set $\delta \subset \operatorname{sp}(T)$. Hence $R = \int \lambda\, \boldsymbol{E}(d\lambda) \in (\operatorname{algLat} A)'$. Since $T \in A'$ and $R \in A'$ it follows that $Q \in A'$. By part (i) it follows that $Q \in (\operatorname{algLat} A)'$. $\square$

Lemma 20. *Let A be a strongly closed algebra with complemented invariant subspace lattice that contains a complete Boolean algebra of projections of finite uniform multiplicity k with the direct sum property. Then*, *if $K \in \operatorname{Lat} A^{(n)}$ for some $n \in \mathbb{N}$*, *then*, *there exists an increasing sequence of projections $\{p_m\} \subset \mathcal{B}$, $p_m \nearrow I$ such that $p_m^{(n)} K$ is complemented in $\operatorname{Lat}(p_m A p_m)^{(n)}$ for every $m \in \mathbb{N}$.*

Proof. We will prove the lemma by induction on n. For $n = 1$ the statement is obvious with $p_m = I$ for every m. Let $K \in \operatorname{Lat} A^{(n)}$. Define

$$K_0 = \{\boldsymbol{x} \in X^{(n-1)} \mid 0 \oplus \boldsymbol{x} \in K\}.$$

Obviously, $K_0 \in \operatorname{Lat} A^{(n-1)}$, so there exists an increasing sequence of projections $\{r_m\} \subset \mathcal{B}$, $r_m \nearrow I$ such that $r_m^{(n-1)} K_0$ is complemented in $\operatorname{Lat}(r_m A r_m)^{(n-1)}$. Let

$(r_m^{(n-1)}K_0)^c$ be the complement of $r_m^{(n-1)}K_0$ in $\mathrm{Lat}(r_m A r_m)^{(n-1)}$. Then,

$$\big(r_m X \oplus (r_m^{(n-1)}K_0)^c\big) \cap K \in \mathrm{Lat}\, A^{(n)}$$

and

$$r_m^{(n)}K = \big(0 \oplus r_m^{(n-1)}K_0\big) + \big(r_m X \oplus (r_m^{(n-1)}K_0)^c\big) \cap r_m^{(n)}K.$$

Since $\big(r_m X \oplus (r_m^{(n-1)}K_0)^c\big) \cap r_m^{(n)}K$ is the complement of $0 \oplus r_m^{(n-1)}K_0$ in $r_m^{(n)}K$, there exist graph transformations $T_1, T_2, \dots, T_{n-1}$ such that

$$\big(r_m X \oplus (r_m^{(n-1)}K_0)^c\big) \cap r_m^{(n)}K = \big\{x \oplus T_1 x \oplus T_2 x \oplus \cdots \oplus T_{n-1}x \;\big|\; x \in P\big\},$$

where P is a linear subspace of $r_m X$ invariant under every element of $r_m A r_m$. The closure of P in $r_m X$, $\bar{P}$, belongs to $\mathrm{Lat}(r_m A r_m)$ and hence has a complement $\bar{P}^c$ in $\mathrm{Lat}(r_m A r_m)$. For $1 \le i \le n-1$, consider the following densely defined, graph transformation on $r_m X$:

$$\widetilde{T}_i x = \begin{cases} T_i x & \text{if } x \in P, \\ 0 & \text{if } x \in \bar{P}^c. \end{cases}$$

Then $\widetilde{T}_i$ commutes with A. By Proposition 17, there exists an increasing sequence of projections $\{q_p\} \subset \mathscr{B}$, $q_p \nearrow I$ such that $\widetilde{T}_i q_p$ are bounded spectral operators of finite type. From Lemma 3 it follows that the subspace

$$\big\{q_p x \oplus \widetilde{T}_1 q_p x \oplus \widetilde{T}_2 q_p x \oplus \cdots \oplus \widetilde{T}_{n-1} q_p x \;\big|\; x \in q_p P \oplus q_p \bar{P}^c\big\}$$

is complemented in $\mathrm{Lat}(q_p A q_p)^{(n)}$. By the definition of the transformations $\widetilde{T}_i$ it follows immediately that the subspace

$$\big\{q_p x \oplus T_1 q_p x \oplus T_2 q_p x \oplus \cdots \oplus T_{n-1} q_p x \;\big|\; x \in P\big\}$$

is complemented in $\mathrm{Lat}(q_p A q_p)^{(n)}$. If we set $p_m = r_m q_m \in \mathscr{B}$ we have that $p_m \nearrow I$, $p_m^{(n)}K$ is complemented in $\mathrm{Lat}(p_m A p_m)^{(n)}$:

$$p_m^{(n)}X = \big(0 \oplus p_m^{(n-1)}K_0\big) + \big\{p_m x \oplus T_1 p_m x \oplus \cdots \oplus T_{n-1} p_m x \;\big|\; x \in P\big\} + \big((p_m P)^c \oplus p_m^{(n-1)}K_0^c\big).$$

Hence

$$p_m^{(n)}X = \big(p_m^{(n)}K\big) + \big((p_m P)^c \oplus p_m^{(n-1)}K_0^c\big),$$

and the proof of the lemma is completed. □

The following statement follows from the proof of Lemma 20.

Remark 21. If A is as in the statement of Lemma 20 and $K \in \mathrm{Lat}\, A^{(n)}$ for some $n \in \mathbb{N}$, then there exists an increasing sequence of projections $\{p_m\} \subset \mathscr{B}$, $p_m \nearrow I$ such that $p_m^{(n)}K = (0 \oplus p_m^{(n-1)}K_0) + \{p_m x \oplus T_1 p_m x \oplus \cdots \oplus T_{n-1} p_m x \mid x \in P\}$, where $K_0 = \{x \in X^{(n-1)} \mid 0 \oplus x \in K\}$ and $T_i p_m$, $1 \le i \le n-1$, $m \in \mathbb{N}$, are bounded spectral operators of finite type on the closed A-invariant subspace $p_m P$ that commute with $p_m A p_m$.

Proof of Theorem 15. Let $b \in \operatorname{algLat} A$ and $K \in \operatorname{Lat} A^{(n)}$. We will prove by induction on n that there exists an increasing sequence of projections $\{p_m\} \subset \mathcal{B}$ such that $p_m \nearrow I$ and $p_m^{(n)} K \in \operatorname{Lat}(p_m b p_m)^{(n)}$ for every $m \in \mathbb{N}$ and therefore $K \in \operatorname{Lat} b^{(n)}$; then apply Remark 1 to conclude that $b \in A$. By Remark 21, there exists an increasing sequence of projections $\{p_m\} \subset \mathcal{B}$, $p_m \nearrow I$ such that $p_m^{(n)} K = (0 \oplus p_m^{(n-1)} K_0) + \{p_m x \oplus T_1 p_m x \oplus T_2 p_m x \oplus \cdots \oplus T_{n-1} p_m x \mid x \in P\}$ where $K_0 = \{x \in X^{(n-1)} \mid 0 \oplus x \in K\}$ and $T_i p_m$, $1 \le i \le n-1$, $m \in \mathbb{N}$, are bounded spectral operators of finite type on the closed A-invariant subspace $p_m P$ that commute with $p_m A p_m$. The induction hypothesis and Proposition 2 (i) imply that $0 \oplus p_m^{(n-1)} K_0 \in \operatorname{Lat}(p_m b p_m)^{(n)}$. By Proposition 19 (ii) it follows that the bounded spectral operators of finite type $T_i p_m$, $1 \le i \le n-1$, $m \in \mathbb{N}$ commute with $p_m b p_m$. Hence $p_m^{(n)} K \in \operatorname{Lat}(p_m b p_m)^{(n)}$. Since $p_m \nearrow I$ and, by Remark 7 (iv), $\mathcal{B}$ is norm bounded, it follows that $K \in \operatorname{Lat} b^{(n)}$ and the result follows. □

Corollary 22. *Let $A \subset B(X)$ be a strongly closed algebra that contains a complete Boolean algebra of projections $\mathcal{B}$ of finite uniform multiplicity with the direct sum property. If A has complemented invariant subspace lattice, then $A = A''$ where A'' is the bicommutant of A.*

Proof. Follows from Theorem 15 and Remark 4. □

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