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HARER STABILITY AND ORBIFOLD COHOMOLOGY

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In this paper we review the combinatorics of the twisted sectors of $\mathcal{M}_{g,n}$, and we exhibit a formula for the age of each of them in terms of the combinatorial data. Then we show that orbifold cohomology of $\mathcal{M}_{g,n}$ when $g \rightarrow \infty$ reduces to its ordinary cohomology. We do this by showing that the twisted sector of minimum age is always the hyperelliptic twisted sector with all markings in the Weierstrass points; the age of the latter moduli space is just half its codimension in $\mathcal{M}_{g,n}$.

1. Introduction

In recent years there have been lots of new results on geometrical and topological properties of the moduli space $\mathcal{M}_{g,n}$ parametrizing smooth curves of genus g with n distinct marked points on it. When $2g - 2 + n > 0$, this moduli space is a smooth Deligne–Mumford stack, or an orbifold, and its coarse moduli space is a quasiprojective variety of dimension $3g - 3 + n$. When $n > 2g + 2$, every marked curve is rigid, therefore the moduli space is actually a smooth quasiprojective variety.

A celebrated result states that there are isomorphisms

$$(1) \quad H^k(\mathcal{M}_{g,n}, \mathbb{Q}) \cong H^k(\mathcal{M}_{g+1,n}, \mathbb{Q}) \quad \text{when } 3k + 2 \leq 2g.$$

These isomorphisms were introduced in [Harer 1985], but the ranges of their validity have been gradually improved over time by the efforts of different authors. This allows the definition of the *stable cohomology*, denoted $H^*(\mathcal{M}_{\infty,n}, \mathbb{Q})$. The tautological classes κ and ψ are preserved by the above isomorphisms when g is sufficiently large.

A recent result, whose proof was completed in [Madsen and Weiss 2007], asserts that the resulting maps

$$(2) \quad \mathbb{Q}[\kappa_1, \kappa_2, \dots] \otimes \mathbb{Q}[\psi_1, \dots, \psi_n] \rightarrow H^*(\mathcal{M}_{\infty,n}).$$

are also isomorphisms. (More precisely, the paper cited shows the result in the case $n = 0$; the extension to $n > 0$ follows from [Looijenga 1996, Proposition 2.1].)

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We refer the reader to [Kirwan 2002; Wahl 2012] for a survey of these topological results.

In the latest years, building on earlier results in topology [Kawasaki 1979] and theoretical physics [Dixon et al. 1985; 1986], it has become clearer that when studying the geometry and topology of orbifolds, one should include in the study the *twisted sectors* of the orbifold itself. We refer to [Adem et al. 2007] for an introduction to this emerging new subject. In particular, the cohomology theory of an orbifold is enriched by the so-called *orbifold cohomology*, introduced by Chen and Ruan in [2004]. As a graded vector space, the orbifold cohomology is the direct sum of the cohomology of the original orbifold and of the cohomology of the twisted sectors; the degree of the cohomology classes of each twisted sector is shifted in orbifold cohomology by (twice) a rational number called *age*. This number is not of topological nature, in fact it depends on the complex structure. Its geometric significance appears in [Jarvis et al. 2007] as the virtual rank of an element in the rational K -theory of the twisted sector also known as “half of the normal bundle”, this element plays a key role in orbifold intersection theory.

In this note, we introduce the twisted sectors of $\mathcal{M}_{g,n}$ in the combinatorial description of [Pagani 2012; Pagani and Tommasi 2013], we write a closed formula for the age of the twisted sectors of $\mathcal{M}_{g,n}$. (The two papers just cited contain the special cases of this formula for $\mathcal{M}_{2,n}$ and \mathcal{M}_g , respectively.) Our main result is [Theorem 1](#), which states that for fixed (g, n) , the twisted sector of minimum age is the hyperelliptic twisted sector with marked Weierstrass points. It is a well-known and classical fact, which we review in [Proposition 1](#), that the twisted sectors of $\mathcal{M}_{g,n}$ have codimension higher than $g - 2 + n$, with equality only for the hyperelliptic locus. Our novel contribution here is that the virtual rank of “half of the normal bundle” (see above) is strictly greater than $\frac{1}{2}(g - 2 + n)$, with equality only for the hyperelliptic twisted sector. This inequality might have further geometric consequences, besides the implications in orbifold cohomology investigated in this note. (The study of the age of the twisted sectors of various types of moduli spaces of curves, has also recently played a significant role in the investigation of the singularities of the coarse moduli space.)

Combining [Theorem 1](#) with Harer stability, we obtain that the orbifold cohomology of $\mathcal{M}_{g,n}$ stabilizes. Combining further our main result with the theorem of Madsen–Weiss, we can explicitly compute the orbifold cohomology of $\mathcal{M}_{g,n}$ in low degrees. Indeed, from [Theorem 1](#), we deduce

$$(3) \quad H_{\text{orb}}^k(\mathcal{M}_{g,n}, \mathbb{Q}) = H^k(\mathcal{M}_{g,n}, \mathbb{Q}) \quad \text{if } k < g - 2 + n \text{ or } n > 2g + 2.$$

(There are no twisted sectors of $\mathcal{M}_{g,n}$ if and only if $n > 2g + 2$).

The stabilization of orbifold cohomology was conjectured by Fantechi in the discussion following her talk [Fantechi 2009] at MSRI. We acknowledge her for

the insight in this topic. We also thank Stefano Maggiolo for having significantly improved the computer program that plays a role at the end of the proof of our main result. The author was supported by DFG project Hu 337/6-2.

2. The twisted sectors of $\mathcal{M}_{g,n}$ and their age

In this section we review the combinatorics of the twisted sectors of $\mathcal{M}_{g,n}$. This description of the twisted sectors of $\mathcal{M}_{g,n}$ was obtained in [Pagani 2012] for $n \geq 1$ or $g = 2$, and in [Pagani and Tommasi 2013] for the remaining cases $\mathcal{M}_{g,0}$, $g \geq 3$.

Let us fix (g, n) with $2g - 2 + n > 0$. A (g, n) -admissible datum consists of non-negative integers $(g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1})$ such that $N \geq 2$ and

$$(4) \quad 2g - 2 = N(2g' - 2) + \sum_{i=1}^{N-1} (N - \gcd(i, N)) d_i,$$

$$(5) \quad \sum_{i=1}^{N-1} i d_i \equiv 0 \pmod{N},$$

$$(6) \quad \sum_{i=1}^{N-1} a_i = n, \quad a_i \leq d_i, \quad a_i = 0 \text{ if } \gcd(i, N) \neq 1,$$

$$(7) \quad n = g' = 0 \implies \text{the g.c.d. of } N \text{ and of the } i\text{'s such that } d_i \neq 0 \text{ is } 1.$$

Each (g, n) -admissible datum corresponds to $\binom{n}{a_1, \dots, a_{N-1}}$ twisted sectors of $\mathcal{M}_{g,n}$ that are related each to the other by an (a_1, \dots, a_{N-1}) -permutation of the n marked points. Since we will only investigate properties of the twisted sectors of $\mathcal{M}_{g,n}$ that do not depend on this permutation, from now on we shall slightly abuse the notation and identify each twisted sector Y of $\mathcal{M}_{g,n}$ with its (g, n) -admissible datum

$$Y \sim (g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1}).$$

These facts follow from [Pagani 2012, Proposition 2.13] for $n \geq 1$ and from [Pagani and Tommasi 2013, Corollary 2.16, Theorem 2.19] in the case $n = 0$.

We observe that, from condition (4), there are no (g, n) -admissible data when $n > 2g + 2$; in particular, this is the case when g equals 0.

For completeness, we briefly recall our description of the twisted sectors of $\mathcal{M}_{g,n}$, from which the above correspondence follows. For more details, we refer to [Pagani 2012, Section 2.b] for the case $n \geq 1$ and to [Pagani and Tommasi 2013, Section 2.b] for the case $n = 0$.

Construction 1. A twisted sector of $\mathcal{M}_{g,n}$ parametrizes connected cyclic covers of order N of curves of genus g' with total space a curve of genus g , where the n marked points are chosen among the points of total ramification. The branch divisor of the cyclic cover splits into $N - 1$ divisors, some of which are possibly

empty. Indeed to any point p in the branch divisor D , let q be any point in the fiber of p under the cyclic cover map; we define H_p as the stabilizer of the action of $\mathbb{Z}/N\mathbb{Z}$ at q , and ψ_p as the character of the action of H_p on the cotangent space in q . Then, for $0 < i < N$, we define D_i as the subset of D of those p such that H_p equals the subgroup generated by i in $\mathbb{Z}/N\mathbb{Z}$, and such that $\psi_p(i)$ equals ω_N , a fixed generator for μ_N : the group of N -th roots of 1. In addition to (g, g', N) , the admissible datum consists of $d_i := |D_i|$ and of a_i , the number of chosen marked points in the preimage of D_i under the cyclic cover map.

Given a (g, n) -admissible datum, we can construct a moduli space of cyclic covers as in the paragraph above. Condition (4) is the Riemann–Hurwitz formula, condition (5) is a compatibility condition that guarantees the existence of a (not necessarily connected) cyclic cover with the data d_i and N , condition (6) corresponds to the fact that the marked points must be points of total ramification for the cover. Now if $n \geq 1$, it is easy to see that the total space of the cover is forced to be connected and that the moduli space parametrizing such covers is also connected. If instead $n = 0$, it is shown in [Pagani and Tommasi 2013, Theorem 2.19] that there is always one connected component of the moduli space that parametrizes *connected* cyclic covers. This component may possibly be empty only when $g' = 0$, condition (7) rules out precisely these cases.

Let us fix a twisted sector $(g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1})$. Since Y admits a finite map to $\mathcal{M}_{g', \sum d_i}$, its dimension is $3g' - 3 + \sum d_i$, its *codimension in $\mathcal{M}_{g,n}$* is

$$(8) \quad \text{codim}(Y) := 3g - 3g' - \sum_{i=1}^{N-1} d_i + n,$$

and its *twin* is $(g', N; d_{N-1}, \dots, d_1, a_{N-1}, \dots, a_1)$. If (g, n) is fixed and n is at most $2g + 2$, the *hyperelliptic twisted sector with n marked Weierstrass points* is $(g' = 0, N = 2; d_1 = 2g + 2, a_1 = n)$. In short, we will also call it simply the *hyperelliptic twisted sector*; from (8) it has codimension $g - 2 + n$.

The next result is classical, but we review it for completeness.

Proposition 1. *The codimension of any twisted sector Y of $\mathcal{M}_{g,n}$ satisfies*

$$\text{codim}(Y) \geq g - 2 + n,$$

with equality if and only if Y is the hyperelliptic twisted sector with n marked Weierstrass points.

Proof. Using (8), our statement is reduced to proving the inequality

$$(9) \quad \sum d_i \leq 2g - 3g' + 2.$$

Using (4), we have

$$(10) \quad \frac{N}{2} \sum d_i \leq \sum d_i(N - \gcd(i, N)) = 2g - 2 - N(2g' - 2);$$

therefore, it is enough to show that

$$\frac{2}{N}(2g - 2 - N(2g' - 2)) \leq 2g - 3g' + 2.$$

Or, rearranging the terms, that

$$(11) \quad (2N - 4)(g - 1) + Ng' \geq 0.$$

This is clearly always true. Equality holds if and only if $g' = 0$ and $N = 2$. \square

Every twisted sector Y is assigned a rational number, first defined by Chen and Ruan in [2004], which is called the *degree shifting number*, *age*, or *fermionic shift*. Orbifold cohomology is then the direct sum of the ordinary cohomology and of the cohomology of all the twisted sectors, where the latter is shifted in degree by twice the age. For completeness, we briefly review the Chen–Ruan definition of the degree shifting number, building on Construction 1.

Construction 2. Let $f: Y \rightarrow \mathcal{M}_{g,n}$ be the natural map from the twisted sector to the moduli stack of curves. The group μ_N of N -th roots of 1 acts on $f^*(T_{\mathcal{M}_{g,n}})$, the action can be diagonalized, and each eigenvalue at a point of Y has the form $\lambda_k = e^{2\pi i \alpha_k}$, where the $\alpha_k \in [0, 1) \cap \mathbb{Q}$ are the “logarithms” of the eigenvalues. It is not difficult to see that the function $\sum_k \alpha_k$ is well-defined and constant on Y , thus the *age* of Y is defined as

$$(12) \quad a(Y) := \sum_k \alpha_k \in \mathbb{Q}.$$

Moreover, by the very definition of twisted sector, the action of μ_N on T_Y is trivial, thus in the definition (12) it is equivalent to sum the “logarithms” of the eigenvalues of the normal bundle $N_Y \mathcal{M}_{g,n}$, where the latter is defined by the exact sequence of vector bundles

$$0 \rightarrow T_Y \rightarrow f^*(T_{\mathcal{M}_{g,n}}) \rightarrow N_Y \mathcal{M}_{g,n} \rightarrow 0.$$

The age of a twisted sector can be interpreted as the virtual rank of an element in the rational K -theory of Y that plays an important role in orbifold intersection theory, see [Jarvis et al. 2007, Definition 1.3, Sections 1.3 and 4].

The age of a twisted sector of $\mathcal{M}_{g,n}$ can explicitly be determined in terms of its admissible datum. From [Pagani and Tommasi 2013, Proposition 5.6] and [Pagani

2012, Lemma 4.6], we have the following formula for the age:

$$(13) \quad a(Y) = \frac{(3g' - 3)(N - 1)}{2} + \frac{1}{N} \sum_{\gcd(i, N)=1} a_i \sum_{k=1}^{N-1} k \sigma(k, i) + \frac{1}{N} \sum_{i=1}^{N-1} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k, i) \right),$$

where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of $x \in \mathbb{Q}^+$, and

$$\sigma(k, i) := \begin{cases} 0 & \text{if } ki + \gcd(i, N) \equiv 0 \pmod{N}, \\ 1 & \text{if } ki + \gcd(i, N) \not\equiv 0 \pmod{N}. \end{cases}$$

Using only (13), it is an easy exercise to check that, if Y and Y' are twins, then

$$(14) \quad a(Y) + a(Y') = \text{codim}(Y) = \text{codim}(Y').$$

For example, when a twisted sector Y is twin to itself (this happens always, for example, when $N = 2$), its age is half its codimension.

3. The twisted sectors of minimum age

Using only the combinatorial description of the previous section, and in analogy with Proposition 1, we can prove the main result of this note. From now on, we assume $2g - 2 + n > 0$.

Theorem 1. *The age of any twisted sector Y satisfies $2a(Y) \geq g - 2 + n$, with equality if and only if Y is the hyperelliptic twisted sector with n marked Weierstrass points.*

The marked hyperelliptic twisted sector is, using the terminology established in the previous section, twin to itself. Therefore its age is half its codimension: $\frac{1}{2}(g - 2 + n)$.

This implies the following corollary, relevant for orbifold cohomology:

$$(15) \quad H^k(\mathcal{M}_{g,n}, \mathbb{Q}) = H_{\text{orb}}^k(\mathcal{M}_{g,n}, \mathbb{Q}) \quad \text{if } k < g - 2 + n \text{ or } n > 2g + 2.$$

There are no twisted sectors of $\mathcal{M}_{g,n}$ if and only if $n > 2g + 2$; otherwise our bound on the cohomological degree k is sharp.

Using the stability results for ordinary cohomology, we deduce:

Corollary 1. *The isomorphisms (1) are, in fact, isomorphisms*

$$H_{\text{orb}}^k(\mathcal{M}_{g,n}, \mathbb{Q}) \cong H_{\text{orb}}^k(\mathcal{M}_{g+1,n}, \mathbb{Q})$$

when $k \leq \min(g - 3 + n, 2g/3 - 2/3)$.

In particular, we can interpret this by saying that orbifold cohomology of $\mathcal{M}_{g,n}$ “trivially stabilizes” when $g \rightarrow \infty$, and the stable orbifold cohomology of $\mathcal{M}_{g,n}$ coincides with its ordinary stable cohomology. The only pairs (g, n) for which $2g/3 - 2/3 > g - 3 + n$ occur are

$$(1, 1), (2, 0), (3, 0), (4, 0).$$

In these special cases, the ranges for k in [Corollary 1](#) are optimal, whereas in all other cases our ranges coincide with the ranges of stability for ordinary cohomology: $k < 2g/3 - 2/3$. The latter ranges are known to be optimal when $g \equiv 2 \pmod{3}$. More details on the sharpness of the ranges for cohomological stability can be found in [\[Wahl 2012, p. 2\]](#).

Combining [\(15\)](#) with the isomorphisms [\(2\)](#), we see how the orbifold cohomology of $\mathcal{M}_{g,n}$ is explicitly computable in low degrees.

We now move to the proof of [Theorem 1](#). Thanks to [Proposition 1](#) and to [\(14\)](#), what we have to prove is in fact

$$(16) \quad \begin{aligned} |a(Y) - a(Y')| &\leq \text{codim}(Y) - (g - 2 + n) \\ &= 2g + 2 - 3g' - \sum d_i, \end{aligned}$$

with equality only when Y is the hyperelliptic twisted sector with n marked Weierstrass points.

We introduce some notation. Let

$$\Sigma := \{d \in \mathbb{N} \mid d \text{ divides } N, d \neq N\}$$

be the set of proper divisors of N , and let

$$(17) \quad a(Y)_{\text{mark}} := \frac{1}{N} \sum_{\gcd(i,N)=1} a_i \sum_{k=1}^{N-1} k \sigma(k, i),$$

$$(18) \quad a(Y)_{\sigma} := \frac{1}{N} \sum_{\gcd(i,N)=\sigma} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k, i) \right).$$

We can rewrite formula [\(13\)](#) for the age of a twisted sector Y as:

$$a(Y) = \frac{(3g' - 3)(N - 1)}{2} + a(Y)_{\text{mark}} + \sum_{\sigma \in \Sigma} a(Y)_{\sigma}.$$

The term $a(Y)_{\text{mark}}$ is the contribution to the age of Y coming from the marked points, and as such it is zero when $n = 0$. Of course now we have the estimate

$$(19) \quad |a(Y) - a(Y')| \leq |a(Y)_{\text{mark}} - a(Y')_{\text{mark}}| + \sum_{\sigma \in \Sigma} |a(Y)_{\sigma} - a(Y')_{\sigma}|.$$

We can give estimates for each term in the right hand side of [\(19\)](#).

Lemma 1. *The following inequalities hold:*

$$(20) \quad |a(Y)_{\text{mark}} - a(Y')_{\text{mark}}| \leq \frac{N-2}{N} \sum_{\gcd(i,N)=1} a_i,$$

$$(21) \quad |a(Y)_{\sigma} - a(Y')_{\sigma}| \leq \frac{(N-2\sigma)(N/\sigma+5)}{6N} \sum_{\gcd(i,N)=\sigma} d_i.$$

Proof. Let us begin with the contribution coming from the marked points. The left hand side of (20) is equal to

$$(22) \quad \frac{1}{N} \left| \sum a_i (\lambda(i) - \lambda(N-i)) \right|,$$

where $\lambda(s)$ is the multiplicative inverse of s modulo N . The maximum of the absolute value of

$$\lambda(i) - \lambda(N-i) = 2\lambda(i) - N$$

is obtained when i is either 1 or $N-1$.

As for the second inequality, we separate the two summands in the right hand side of (18). For the first term, consider the function of i

$$g_N^{\sigma}(i) := \left| \sum_{k=1}^{N-1} k \left(\left\{ \frac{ik}{N} \right\} - \left\{ \frac{(N-i)k}{N} \right\} \right) \right|.$$

Its maximum among the values of i such that $\gcd(i, N) = \sigma$ is obtained for $i = \sigma$ or for $i = N - \sigma$. From this, we obtain

$$(23) \quad \left| \sum_{k=1}^{N-1} k \left\{ \frac{ik}{N} \right\} - \left\{ \frac{(N-i)k}{N} \right\} \right| \leq g_N^{\sigma}(\sigma) = \frac{1}{6} \left(\frac{N}{\sigma} - 1 \right) (N - 2\sigma).$$

The second term is treated similarly to the contribution coming from the marked points. The maximum of the absolute value of

$$\sum_k (\sigma(k, i) - \sigma(k, N-i)) = 2i - N$$

is obtained when i is either σ or $N - \sigma$. Combining this fact with (23), we get the desired inequality. □

Proof of Theorem 1. As we have already observed, it suffices to prove (16). By using the Riemann–Hurwitz formula (4) to eliminate the variable g , the right hand side of (16) can be rearranged to

$$(2N-3)g' - 2(N-2) + \sum_{\sigma \in \Sigma} (N-\sigma-1) \sum_{\gcd(i,N)=\sigma} d_i.$$

Now let us define for convenience the function

$$f_N(\sigma) := (N - \sigma - 1) - \frac{(N - 2\sigma)\left(\frac{N}{\sigma} + 5\right)}{6N} = \frac{\left(6 - \frac{1}{\sigma}\right)N^2 - (6\sigma + 9)N + 10\sigma}{6N}$$

for any integer $N \geq 2$ and any σ a real number between 1 and $N/2$. By using (20) and (21), in order to prove (16) it is enough to prove

$$(24) \quad -\frac{N-2}{N} \sum_{\gcd(i,N)=1} a_i + \sum_{\sigma \in \Sigma} f_N(\sigma) \sum_{\gcd(i,N)=\sigma} d_i \geq (2-2g')(N-2) - g',$$

with equality only in the case of the hyperelliptic twisted sector. Note that $a_i \leq d_i$ from inequality (6).

The left hand side of (24) is always nonnegative for any integer $N \geq 2$, because

$$\tilde{f}_N(1) := f_N(1) - \frac{N-2}{N} = \frac{(N-2)(5N-11)}{6N} \geq 0.$$

Therefore when $g' > 0$, the strict inequality (24) holds evidently, as the right hand side is strictly smaller than 0. Thus all we have to prove is (24) when $g' = 0$, a case in which we always have that $\sum d_i \geq 3$ (this follows from condition (4) with $g' = 0$ and $g > 0$).

We start with the case $g' = 0$ and $\sum d_i \geq 4$. The function f_N is concave (just look at its second derivative with respect to σ), thus it has its minimum either in 1 or in $N/2$, and we have $f_N(N/2) = (N-2)/2$. The following two inequalities hold in this case:

$$(25) \quad \tilde{f}_N(1) \sum d_i \leq 2(N-2),$$

$$(26) \quad f_N(N/2) \sum d_i \leq 2(N-2),$$

and they suffice to prove (24). If (24) is an equality, then either (25) or (26) must be an equality. If $N = 2$, we are precisely in the case of the hyperelliptic twisted sector. If $N > 2$, the inequality (25) is strict, so (26) must be an equality and therefore $\sum d_i = 4$. So if (24) is an equality, with $g' = 0$, $N > 2$ and $\sum d_i = 4$, then $d_{N/2} = 4$, but this implies $g = 1$ by (4), hence $n \geq 1$, and this case does not exist because of (6).

So we are left with the case $g' = 0$ and $\sum d_i = 3$. A large number of twisted sectors still falls into this last category, but not the hyperelliptic twisted sector. We set the three nonzero d_i 's to 1, and denote them by $d_{\sigma_1} = d_{\sigma_2} = d_{\sigma_3} = 1$. Then it suffices to prove the strict inequality

$$(27) \quad \left(6 - \sum_{i=1}^3 \frac{1}{\sigma_i}\right)N^2 - \left(3 + 6 \sum_{i=1}^3 \sigma_i\right)N + 10 \sum_{i=1}^3 \sigma_i > 6n(N-2).$$

If N is fixed, there are only finitely many possibilities for the variables involved in (27). The constraints are

$$(28) \quad \begin{cases} \sigma_1 + \sigma_2 + \sigma_3 < N, \\ a\sigma_1 + b\sigma_2 + c\sigma_3 = N \quad \text{for some } a, b, c \in \mathbb{N}^+, \\ n \leq |\{i \mid \sigma_i = 1\}| \leq 3, \\ \sigma_i \text{ divides } N, \sigma_i \neq N, \end{cases}$$

where all the quantities involved are integers. The first is a consequence of Riemann–Hurwitz (4) (assuming $g > 1$), the second follows from (5) and the third from (6). From now on, we aim at proving (27) for N greater than a certain explicit constant. We will repeatedly use that the left hand side of (27), for fixed n, N , is a concave function in the domain of definition (28). We can also assume for convenience that $\sigma_1 \leq \sigma_2 \leq \sigma_3$.

- If $n = 3$, then from (28) we deduce $\sigma_1 = \sigma_2 = \sigma_3 = 1$. The inequality (27) is satisfied when $N > 11$.
- If $n = 2$, from (28), we have that $\sigma_1 = \sigma_2 = 1$. It is enough to check (27) for the extreme values $\sigma_3 = 1$ and $\sigma_3 = N/2$. The first follows from the case $n = 3$, by checking the case of $\sigma_3 = N/2$ we see that (27) is valid when $N > 22$.
- If $n = 1$, from (28) $\sigma_1 = 1$, so we have $1 \leq \sigma_2 \leq \sigma_3 \leq N/2$ and $\sigma_2 + \sigma_3 < N - 1$. It is enough to check the extremal values. The case when $\sigma_2 = 1$ follows from the case $n = 2$. From the second point in (28), if $\sigma_3 = N/2$, then σ_2 is either 1 or 2; in the latter case (27) is valid when $N > 14$. Finally, when $\sigma_2 = \sigma_3 = N/3$, (27) is always valid.
- If $n = 0$, we can assume $\sigma_i \geq 2$, since the other cases fall in the above paragraph. Moreover, there are six extremal cases that fulfill the first and the last of (28):

$$(2, 2, 2), \left(2, 2, \frac{N}{2}\right), \left(2, \frac{N}{3}, \frac{N}{2}\right), \left(\frac{N}{7}, \frac{N}{3}, \frac{N}{2}\right), \left(\frac{N}{5}, \frac{N}{4}, \frac{N}{2}\right), \left(\frac{N}{4}, \frac{N}{3}, \frac{N}{3}\right).$$

We check that (27) for the extremal cases is satisfied when $N > 36$ (the inequality is sharp in the case of the fourth triple).

To conclude the proof, we have to check that (16) holds in the cases when $g' = 0$, $\sum d_i = 3$ and $N < 37$, which imply $g \leq 17$. These cases are only finitely many, and can be handled with the help of a computer program.¹ \square

Let us conclude with some remarks.

¹The source code of a C++ program that lists all twisted sectors of \mathcal{M}_g , each one with its age, is available at <http://pcwww.liv.ac.uk/~pagani/twisted.cpp>.

Remark 1. We list the number of twisted sectors of \mathcal{M}_g for $1 \leq g \leq 17$, to give an idea of its rapid growth:

(7, 17, 47, 72, 76, 203, 196, 225, 415, 537, 427, 1040, 811, 779, 1750, 1860, 1371).

Then the number of twisted sectors of \mathcal{M}_g with $g' = 0$:

(7, 16, 43, 65, 64, 193, 163, 207, 372, 485, 359, 983, 657, 866, 1592, 1636, 1115).

And, finally, the number of twisted sectors of \mathcal{M}_g with $g' = 0$ and $\sum d_i = 3$:

(6, 12, 32, 38, 42, 108, 76, 100, 184, 190, 150, 352, 162, 286, 544, 382, 196).

These final 2860 twisted sectors are those for which we performed the computer assisted calculation mentioned in the last paragraph of the proof of [Theorem 1](#).

Remark 2. One can also ask what are the twisted sectors of small age in $\mathcal{M}_{g,n}$, after the marked hyperelliptic one. Here we list, for fixed (g, n) , the first twisted sectors in order of increasing age: marked hyperelliptic, marked bielliptic, \dots , (marked) double covers of curves of genus $\lceil g/2 \rceil$. We remark that the ranges of existence of those twisted sectors, in terms of g and n , are, respectively,

$$n \leq 2g + 2, \quad n \leq 2g - 2, \quad n \leq 2g - 6, \quad \dots, \quad n \leq 1 + (-1)^g;$$

their ages are, respectively,

$$\frac{g - 2 + n}{2}, \quad \frac{g - 1 + n}{2}, \quad \frac{g + n}{2}, \quad \dots, \quad \frac{3\lceil g/2 \rceil - 1 - (-1)^g + n}{2}.$$

After all these, there is one marked trigonal cyclic twisted sector (when $n \leq g + 2$). Then the picture becomes more complicated, and we do not know the answer. For example, we have empirically observed that the minimum age among twisted sectors of codimension k can be bigger than the minimum age among twisted sectors of codimension $k + 1$.

The validity of the statements that we made in this remark require a long combinatorial proof along the lines of the proof of [Theorem 1](#), which we do not include in this note as it is not really relevant to our scope.

Remark 3. Condition (7) has not been used in any of the steps of the proof of [Theorem 1](#), which could then have been stated slightly more generally for the twisted sectors of the moduli spaces of not necessarily connected smooth curves of genus g .

Remark 4. There is no such thing as a stable cohomology in low degrees for $\bar{\mathcal{M}}_{g,n}$; it is a classical fact for example that even the second Betti number (which equals the dimension of the Picard group in this case) grows exponentially in g . It still makes sense to ask for the twisted sector of minimum age of $\bar{\mathcal{M}}_{g,n}$, but the answer

is much easier. To fix the ideas, we give the answer when $g + n > 3$ and $g > 0$: the cases in which the generalized hyperelliptic locus has codimension > 1 . Then the unique twisted sector of minimum age is the codimension-1 locus, consisting generically of a smooth elliptic curve glued at the origin to a smooth curve of genus $g - 1$ carrying all the marked points, and with the automorphism induced by the pair (elliptic involution on the elliptic curve, identity). Its age is $\frac{1}{2}$: half its codimension.

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