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**GRADIENT ESTIMATES AND ENTROPY FORMULAE OF
POROUS MEDIUM AND FAST DIFFUSION EQUATIONS FOR
THE WITTEN LAPLACIAN**

GUANGYUE HUANG AND HAIZHONG LI

GRADIENT ESTIMATES AND ENTROPY FORMULAE OF POROUS MEDIUM AND FAST DIFFUSION EQUATIONS FOR THE WITTEN LAPLACIAN

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We study gradient estimates for the positive solutions of the porous medium equations and the fast diffusion equations

$$u_t = \Delta_\phi(u^p)$$

associated with the Witten Laplacian on Riemannian manifolds. Under the assumption that the m -dimensional Bakry–Emery Ricci curvature is bounded from below, we obtain some gradient estimates which generalize some previous results of Lu et. al. and Huang et. al. As applications, several parabolic Harnack inequalities are obtained. Moreover, inspired by X.-D. Li's work, we also extend the entropy formulae introduced by Lu et. al. to the porous medium equations and the fast diffusion equations associated with the Witten Laplacian. We prove some monotonicity theorems for such entropy on compact Riemannian manifolds with nonnegative m -dimensional Bakry–Emery Ricci curvature.

1. Introduction

Let (M^n, g) be an n -dimensional complete Riemannian manifold. P. Li and Yau [1986] considered positive solutions of the heat equation

$$(1-1) \quad u_t = \Delta u$$

and proved the following gradient estimates.

Theorem A [Li and Yau 1986]. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, where $\text{Ric}(B_p(2R))$ denotes the Ricci curvature on the geodesic ball $B_p(2R)$ with radius $2R$ and K is a nonnegative constant. Let u be a*

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positive solution of (1-1) on $B_p(2R) \times [0, T]$. Then, on $B_p(R)$, we have

$$(1-2) \quad \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C(n)\alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha-1} + \sqrt{K}R \right) + \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t},$$

where $\alpha > 1$ is a constant and $C(n)$ is a constant depending only on n . Moreover, taking $R \rightarrow \infty$, (1-2) yields the following estimate on (M^n, g) :

$$(1-3) \quad \frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t}.$$

J. F. Li and X. J. Xu [2011] obtained new Li–Yau-type gradient estimates for positive solutions of the heat equation (1-1) on complete Riemannian manifolds. For related research and some improvements on Li–Yau-type gradient estimates of (1-1), see [Yau 1994; 1995; Bakry and Qian 1999; Hamilton 1993; Li 2005; Davies 1989]. The equation

$$(1-4) \quad u_t = \Delta(u^p)$$

with $p > 1$ is called the porous medium equation, which is a nonlinear extension of the classical heat equation. For various values of $p > 1$, it has appeared in different applications to model diffusive phenomena (see [Vázquez 2007; Aronson and Bénilan 1979; Lu et al. 2009] and the references therein). Equation (1-4) with $p \in (0, 1)$ is called the fast diffusion equation, which appears in plasma physics and in geometric flows. However, there are remarkable differences between the porous medium equations and the fast diffusion equation; see [Vázquez 2006; Daskalopoulos and Kenig 2007]. For the study of gradient estimates of (1-4), see [Huang et al. 2013; Aronson and Bénilan 1979; Vázquez 2007; Xu 2012].

Lu, Ni, Vázquez, and Villani studied gradient estimates of (1-4) and proved the following results.

Theorem B [Lu et al. 2009, Theorem 3.3]. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, where $\text{Ric}(B_p(2R))$ denotes the Ricci curvature on the geodesic ball $B_p(2R)$ with radius $2R$ and K is a nonnegative constant. Let u be a positive solution to (1-4) with $p > 1$. Let $v = (p/(p-1))u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then, for any $\alpha > 1$, on $B_p(R)$, we have*

$$(1-5) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{C(n)Ma\alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha-1} \frac{ap^2}{p-1} + (1 + \sqrt{K}R) \right) + \frac{\alpha^2}{\alpha-1} aMK + \frac{a\alpha^2}{t},$$

where $a = n(p-1)/(n(p-1) + 2)$. Moreover, taking $R \rightarrow \infty$, (1-5) yields the following estimate on (M^n, g) :

$$(1-6) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha-1} aMK + \frac{a\alpha^2}{t}.$$

Now we rewrite the inequality (1-6) as

$$(1-7) \quad |\nabla v|^2 - \alpha v_t \leq \frac{\alpha^2}{\alpha-1} aMK v + \frac{a\alpha^2 v}{t}.$$

Since $(p-1)v = pu^{p-1}$, we have $(p-1)v \rightarrow 1$ as $p \rightarrow 1$. As $p \rightarrow 1$, we have $M \rightarrow 1$,

$$|\nabla v|^2 \rightarrow \frac{|\nabla u|^2}{u^2}, \quad v_t \rightarrow \frac{u_t}{u}, \quad av \rightarrow \frac{n}{2}.$$

Consequently, (1-7) becomes Li and Yau's inequality (1-3). Therefore, for a complete noncompact Riemannian manifold (M^n, g) , estimate (1-6) in the result of Lu, Ni, Vázquez and Villani reduces to estimate (1-3) when $p \rightarrow 1$.

Let $\phi \in C^2(M^n)$. The Witten Laplacian associated with ϕ is defined by

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla,$$

which is symmetric with respect to the $L^2(M^n)$ inner product under the weighted measure

$$d\mu = e^{-\phi} dv,$$

that is,

$$\int_{M^n} u \Delta_\phi v d\mu = - \int_{M^n} \nabla u \nabla v d\mu = \int_{M^n} v \Delta_\phi u d\mu \quad \text{for all } u, v \in C_0^\infty(M^n).$$

Following [Bakry and Émery 1985; Bakry 1994; Li 2005; Wei and Wylie 2009], we introduce the m -dimensional Bakry–Emery Ricci curvature associated with the Witten Laplacian by

$$\text{Ric}_\phi^m = \text{Ric} + \nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi,$$

where $m \geq n$ is a constant and $m = n$ if and only if ϕ is a constant. Define

$$\text{Ric}_\phi = \text{Ric} + \nabla^2\phi.$$

Then Ric_ϕ can be seen as the ∞ -dimensional Bakry–Emery Ricci curvature. In this paper, we study the following equation associated with the Witten Laplacian:

$$(1-8) \quad u_t = \Delta_\phi(u^p)$$

with $p > 0$ and $p \neq 1$. For $p > 1$ and $p \in (0, 1)$, we derive an analogue of the estimates of Lu, Ni, Vázquez, and Villani and a Davies-type estimate. Moreover,

for $p > 1$, we obtain a Hamilton-type estimate and an analogue of the estimates of Li and Xu. In particular, our results generalize the ones in [Huang et al. 2013].

First we consider gradient estimates of (1-8) under the assumption that the m -dimensional Bakry–Emery Ricci curvature is bounded from below, and obtain the following results. We set once and for all

$$(1-9) \quad \tilde{a} = \frac{m(p-1)}{m(p-1)+2}.$$

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with*

$$\text{Ric}_\phi^m(B_p(2R)) \geq -K,$$

where $\text{Ric}_\phi^m(B_p(2R))$ denotes the m -dimensional Bakry–Emery Ricci curvature on the geodesic ball $B_p(2R)$ with radius $2R$, and K is a nonnegative constant. Let u be a positive solution to the porous medium equation (1-8) with $p > 1$. Let $v = (p/(p-1))u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then, for any $\alpha > 1$, on $B_p(R)$, we have

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} &\leq \tilde{a} \alpha^2 M \frac{C(m)}{R^2} \left(\frac{\alpha^2}{\alpha-1} \frac{\tilde{a} p^2}{p-1} + 1 + \sqrt{K} R \coth(\sqrt{K} R) \right) \\ &\quad + \frac{\alpha^2}{(\alpha-1)} \tilde{a} M K + \frac{\tilde{a} \alpha^2}{t}. \end{aligned}$$

Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(1-10) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha-1} \tilde{a} M K + \frac{\tilde{a} \alpha^2}{t}.$$

Corollary 1.2. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}_\phi^m \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p > 1$. Set*

$$v = \frac{p}{p-1} u^{p-1}, \quad M = (p-1) \sup_{M^n \times [0, T]} v, \quad \tilde{M} = \inf_{M^n \times [0, T]} v.$$

Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, we have

$$v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\tilde{a}\alpha} \exp \left(\frac{\alpha \text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} + \frac{\alpha}{\alpha-1} \tilde{a} M K (t_2 - t_1) \right),$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Theorem 1.3. *Let (M^n, g) and K be as in Theorem 1.1. Let u be a positive solution to the fast diffusion equation (1-8) with $p \in (1 - 2/m, 1)$. Set*

$$v = \frac{p}{p-1} u^{p-1}, \quad M = (1-p) \max_{B_p(2R) \times [0, T]} (-v).$$

Then, for any $0 < \alpha < 1$, we have on $B_p(R)$

$$(1-11) \quad -\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq \frac{(-\tilde{a})\alpha^2 M C(m)}{A(\varepsilon_1, \varepsilon_2)} \frac{C(m)}{R^2} \left(\frac{(-\tilde{a})\alpha^2 p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + 1 + \sqrt{K} R \coth(\sqrt{K} R) \right) + \frac{(-\tilde{a})\alpha^2 MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})} A(\varepsilon_1, \varepsilon_2)} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)t},$$

where $\varepsilon_1, \varepsilon_2 \in (0, 1)$ are positive constants satisfying

$$A(\varepsilon_1, \varepsilon_2) := [1 - \tilde{a}(1-\alpha)] - \frac{(1 + \varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1 - \varepsilon_1)(1-\alpha-\tilde{a})} > 0.$$

Taking $R \rightarrow \infty$ and $\alpha \rightarrow 1$, we thus obtain the following estimate on (M^n, g) with $\text{Ric}_\phi^m \geq 0$:

$$(1-12) \quad -\frac{|\nabla v|^2}{v} + \frac{v_t}{v} \leq -\frac{\tilde{a}}{t}.$$

Corollary 1.4. Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}_\phi^m \geq 0$. Let u be a positive solution to (1-8) with $p \in (1 - 2/m, 1)$. Set

$$v = \frac{P}{p-1} u^{p-1}, \quad M = (1-p) \sup_{M^n \times [0, T]} (-v), \quad \tilde{M} = \inf_{M^n \times [0, T]} (-v).$$

Then, for any $x_1, x_2 \in M^n$ and $0 < t_1 < t_2 < T$, we have

$$(1-13) \quad -v(x_2, t_2) \leq -v(x_1, t_1) \left(\frac{t_2}{t_1} \right)^{-\tilde{a}} \exp \frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)},$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 1.5. Clearly, our estimate (1-10) reduces to (1-6) (see [Lu et al. 2009]) by letting $m = n$. Moreover, for $p \in (0, 1)$, [Lu et al. 2009, Theorem 4.1] can be obtained from our Theorem 1.3 by taking $m = n$.

Theorem 1.6. Let (M^n, g) and K be as in Theorem 1.1. Let u be a positive solution to the fast diffusion equation (1-8) with $p \in (1 - 2/m, 1)$. Set

$$v = \frac{P}{p-1} u^{p-1}, \quad M = (1-p) \max_{B_p(2R) \times [0, T]} (-v).$$

Then, for any $0 < \alpha < 1$, we have on $B_p(R)$

$$\begin{aligned}
-\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq & \left\{ C(\tilde{a}, \alpha) \frac{p}{\sqrt{1-p}} \sqrt{M} \frac{C}{R} \right. \\
& + \left[\left(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}) \right) MK + \frac{1-\alpha-\tilde{a}}{t} \right. \\
& \left. \left. + (1-p)(1-\alpha-\tilde{a}) M \frac{C(m)}{R^2} (1 + \sqrt{K} R \coth(\sqrt{K} R)) \right] \right\}^{\frac{1}{2}}.
\end{aligned}$$

Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(1-14) \quad -\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq \left(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}) \right) MK + \frac{1-\alpha-\tilde{a}}{t}.$$

Corollary 1.7. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}_\phi^m \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p \in (1 - 2/m, 1)$. Let*

$$v = \frac{p}{p-1} u^{p-1}, \quad M = (1-p) \sup_{M^n \times [0, T]} (-v), \quad \tilde{M} = \inf_{M^n \times [0, T]} (-v).$$

Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $0 < \alpha < 1$, we have

$$\begin{aligned}
-v(x_2, t_2) \leq & -v(x_1, t_1) \left(\frac{t_2}{t_1} \right)^{(1-\alpha-\tilde{a})/\alpha} \\
& \times \exp \left(\frac{\alpha \text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} + \left(\frac{\alpha}{2(1-\alpha)} + \frac{2(1-\tilde{a})}{\alpha} \right) MK (t_2 - t_1) \right)
\end{aligned}$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 1.8. For complete Riemannian manifolds with $p \in (0, 1)$, Corollary 4.2 of [Lu et al. 2009] shows that, if $\text{Ric} \geq 0$, then

$$(1-15) \quad -\frac{|\nabla v|^2}{v} + \frac{v_t}{v} \leq -\frac{a}{t};$$

while if $\text{Ric} \geq -K$ and $0 < \alpha < 1$, then, for any $\varepsilon > 0$ satisfying

$$C(a, \alpha, \varepsilon) := 1 + (-a)(1-\alpha) - \frac{(1-\alpha)(1-a)^2}{(1-\alpha) - a - (1-\alpha)\varepsilon^2} > 0,$$

we have

$$(1-16) \quad -\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq \frac{(-a)\alpha^2}{C(a, \alpha, \varepsilon)} \left(\frac{1}{t} + \frac{\sqrt{C(a, \alpha, \varepsilon)}}{(1-\alpha)\varepsilon} MK \right).$$

Obviously, our estimate (1-14) reduces to (1-15) when $m = n$ and $\alpha \rightarrow 1$. Moreover, (1-14) is independent of ε .

Denote by R the scalar curvature of the metric g . Perelman [2002] introduced the \mathcal{W} -entropy functional as

$$(1-17) \quad \mathcal{W}(g, f, \tau) = \int_{M^n} (\tau(R + |\nabla f|^2) + f - n) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv,$$

where τ is a positive scale parameter and $f \in C^\infty(M^n)$ satisfies

$$\int_{M^n} \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv = 1.$$

By [Perelman 2002], we know that the \mathcal{W} -entropy is monotone increasing under the Ricci flow, and its critical points are given by gradient shrinking solitons. Ni [2004a; 2004b] considered the \mathcal{W} -entropy for the linear heat equation

$$(1-18) \quad u_\tau = \Delta u$$

on complete Riemannian manifolds. More precisely, Ni [2004a] introduced the \mathcal{W} -entropy associated with (1-18) by

$$(1-19) \quad \mathcal{W}(g, f, \tau) = \int_{M^n} [\tau|\nabla f|^2 + f - n] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv,$$

where $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$ is a positive solution to (1-18) and $\int_{M^n} u dv = 1$, and proved that

$$(1-20) \quad \frac{d}{d\tau} \mathcal{W}(g, f, \tau) = -2 \int_{M^n} \tau \left(\left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + \text{Ric}(\nabla f, \nabla f) \right) u dv.$$

Thus, if the Ricci curvature is nonnegative, the \mathcal{W} -entropy defined by (1-19) is non-increasing on complete Riemannian manifolds. For research on the monotonicity of \mathcal{W} -entropy for other geometric heat flows on Riemannian manifolds, see [Kotschwar and Ni 2009; Ecker 2007; Ni 2004a; 2004b; Lu et al. 2009]. X.-D. Li [2011; 2012; 2013] studied the \mathcal{W}_m -entropy associated with the Witten Laplacian to the linear heat equation

$$(1-21) \quad u_\tau = \Delta_\phi u$$

on complete Riemannian manifolds satisfying the μ -bounded geometry condition. More precisely, [Li 2012] introduced the \mathcal{W}_m -entropy associated with (1-21) by

$$(1-22) \quad \mathcal{W}_m(g, f, \tau) = \int_{M^n} [\tau|\nabla f|^2 + f - m] \frac{e^{-f}}{(4\pi\tau)^{m/2}} d\mu,$$

where $u = \frac{e^{-f}}{(4\pi\tau)^{m/2}}$ is a positive solution to (1-21), and proved that if there exist

two constants $m > n$ and $K \geq 0$ such that $\text{Ric}_\phi^m \geq -K$, then

$$(1-23) \quad \frac{d}{d\tau} \mathfrak{W}_m(g, f, \tau) = -2 \int_{M^n} \tau \left(\left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + \text{Ric}_\phi^m(\nabla f, \nabla f) \right) u \, d\mu \\ - \frac{2}{m-n} \int_{M^n} \tau \left(\nabla \phi \nabla f + \frac{m-n}{2\tau} \right)^2 u \, d\mu.$$

Thus, if $\text{Ric}_\phi^m \geq 0$, then $\mathfrak{W}_m(g, f, \tau)$ is nonincreasing along the heat equation (1-21). For the study of the Witten Laplacian associated with the m -dimensional Bakry–Emery Ricci curvature on complete Riemannian manifolds, see [Wei and Wylie 2009; Wang 2004; 1997; Qian 1998; 1997; Ni 2002; Li 2005; Fang et al. 2009; Bakry and Qian 2005; Bakry 1994; Bakry and Émery 1985]. Let u be a positive solution to (1-4), and let $v = (p/(p-1))u^{p-1}$. Lu et. al. [2009] introduced

$$\mathcal{N}_p(g, u, t) = -t^a \int_{M^n} uv \, dv$$

and

$$(1-24) \quad \mathfrak{W}_p(g, u, t) = \frac{d}{dt} [t \mathcal{N}_p(g, u, t)] = t^{a+1} \int_{M^n} \left(p \frac{|\nabla v|^2}{v} - \frac{a+1}{t} \right) uv \, dv,$$

where $a = \frac{n(p-1)}{n(p-1)+2}$. They proved that if M^n is compact,

$$(1-25) \quad \frac{d}{dt} \mathfrak{W}_p(g, u, t) \\ = -2(p-1)t^{a+1} \int_{M^n} \left(\left| \nabla^2 v + \frac{g}{[n(p-1)+2]t} \right|^2 + \text{Ric}(\nabla v, \nabla v) \right) uv \, dv \\ - 2t^{a+1} \int_{M^n} \left((p-1)\Delta v + \frac{a}{t} \right)^2 uv \, dv.$$

In particular, if the Ricci curvature is nonnegative, the entropy defined in (1-24) is nonincreasing on compact Riemannian manifolds when $p > 1$. For $p < 1$, using the Cauchy–Schwarz inequality, they proved from (1-25) that

$$(1-26) \quad \frac{d}{dt} \mathfrak{W}_p(g, u, t) \\ \leq -2t^{a+1} \int_{M^n} \left[\frac{n(p-1)+1}{n(p-1)} \left((p-1)\Delta v + \frac{a}{t} \right)^2 + (p-1) \text{Ric}(\nabla v, \nabla v) \right] uv \, dv.$$

Clearly, if the Ricci curvature is nonnegative and $p \in (1 - 1/n, 1)$, then (1-26) shows that $(d/dt)\mathfrak{W}_p(g, u, t) \leq 0$ and the entropy defined in (1-24) is nonincreasing on compact Riemannian manifolds.

Inspired by [Li 2012], in this paper we also study the $\mathcal{W}_{p,m}$ -entropy for (1-8) associated with the Witten Laplacian on compact Riemannian manifolds with $p > 0$ and $p \neq 1$. First we define

$$(1-27) \quad \mathcal{N}_{p,m}(g, u, t) = -t^{\tilde{a}} \int_{M^n} uv \, d\mu,$$

where the $\mathcal{W}_{p,m}$ -entropy is defined by

$$(1-28) \quad \mathcal{W}_{p,m}(g, u, t) = \frac{d}{dt} [t \mathcal{N}_{p,m}(g, u, t)],$$

When the m -dimensional Bakry–Emery Ricci curvature is bounded from below, we prove the following.

Theorem 1.9. *Let (M^n, g) be a compact Riemannian manifold. If u is a positive solution to the porous medium equation (1-8) with $p > 1$, then*

$$(1-29) \quad \frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) = -t^{\tilde{a}} \int_{M^n} \left((p-1) \Delta_\phi v + \frac{\tilde{a}}{t} \right) uv \, d\mu,$$

where $v = (p/(p-1))u^{p-1}$. In particular, if $\text{Ric}_\phi^m \geq 0$, then $\mathcal{N}_{p,m}(g, u, t)$ is nonincreasing in t . Moreover,

$$(1-30) \quad \mathcal{W}_{p,m}(g, u, t) = t^{\tilde{a}+1} \int_{M^n} \left(p \frac{|\nabla v|^2}{v} - \frac{\tilde{a}+1}{t} \right) uv \, d\mu$$

and

$$(1-31) \quad \begin{aligned} & \frac{d}{dt} \mathcal{W}_{p,m}(g, u, t) \\ &= -2(p-1)t^{\tilde{a}+1} \int_{M^n} \left(\left| \nabla^2 v + \frac{g}{[m(p-1)+2]t} \right|^2 + \text{Ric}_\phi^m(\nabla v, \nabla v) \right. \\ & \quad \left. + \frac{1}{m-n} \left| \nabla \phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \right) uv \, d\mu \\ & \quad - 2t^{\tilde{a}+1} \int_{M^n} \left| (p-1) \Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 uv \, d\mu. \end{aligned}$$

In particular, if $\text{Ric}_\phi^m \geq 0$, then $\mathcal{W}_{p,m}(g, u, t)$ is nonincreasing in t .

Theorem 1.10. *Let (M^n, g) be a compact Riemannian manifold. If u is a positive solution to the fast diffusion equation (1-8) with $p \in (0, 1)$, then*

$$(1-32) \quad \frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) = -t^{\tilde{a}} \int_{M^n} \left((p-1) \Delta_\phi v + \frac{\tilde{a}}{t} \right) uv \, d\mu,$$

where $v = \frac{p}{p-1}u^{p-1}$. In particular, if $\text{Ric}_\phi^m \geq 0$ and $p \in (1 - 2/m, 1)$, then

$\mathcal{N}_{p,m}(g, u, t)$ is nonincreasing in t . Moreover, we have

$$(1-33) \quad \mathcal{W}_{p,m}(g, u, t) = t^{\tilde{a}+1} \int_{M^n} \left(p \frac{|\nabla v|^2}{v} - \frac{\tilde{a}+1}{t} \right) uv \, d\mu,$$

and, for any positive constant $\varepsilon \geq m - n$ and $1 - \frac{1}{n+\varepsilon} \leq p \leq 1 - \frac{m-n}{m\varepsilon}$,

$$(1-34) \quad \begin{aligned} & \frac{d}{dt} \mathcal{W}_{p,m}(g, u, t) \\ & \leq 2t^{\tilde{a}+1} \int_{M^n} \left((1-p) \operatorname{Ric}_\phi^m(\nabla v, \nabla v) + \left(\frac{1-n(1-p)}{n(1-p)} - \frac{\varepsilon}{n} \right) \left| (p-1) \Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 \right. \\ & \quad \left. + \left(\frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon} \right) \left| \nabla \phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \right) uv \, d\mu. \end{aligned}$$

In particular, if $\operatorname{Ric}_\phi^m \geq 0$, then $\mathcal{W}_{p,m}(g, u, t)$ is nonincreasing in t .

Remark 1.11. If $m = n$, we see that ϕ is a constant. Then (1-31) becomes [Lu et al. 2009, (5.6)]. By letting $m = n$ and $\varepsilon \rightarrow 0$, (1-34) becomes (1-26), which is [Lu et al. 2009, Corollary 5.10].

Remark 1.12. After we submitted our paper, the referee pointed out to us [Li and Li 2013; Wang and Chen 2013; Wang et al. 2013], in which some related problems are studied. Specifically, S. Li and X.-D. Li [2013] derived the W -entropy formula for the Witten Laplacian on manifolds with time dependent metrics and potentials. Wang and Chen [2013] obtained Aronson–Bénilan-type estimates for the porous medium equations associated with the Witten Laplacian. Wang, Yang, and Chen [Wang et al. 2013] studied the weighted p -Laplacian heat equation and proved an optimal gradient estimate and the W -entropy monotonicity formula, which generalized the results of [Kotschwar and Ni 2009]. We note that the first version of this paper was posted on arXiv (1203.5482) on March 25 of 2012.

2. Proofs of Theorems 1.1 and 1.3

Let $v = (p/(p-1))u^{p-1}$. By virtue of (1-8), we have $v_t = (p-1)v\Delta_\phi v + |\nabla v|^2$, which is equivalent to

$$(2-1) \quad \frac{v_t}{v} = (p-1)\Delta_\phi v + \frac{|\nabla v|^2}{v}.$$

As in [Lu et al. 2009], we introduce the differential operator

$$(2-2) \quad \mathcal{L} = \partial_t - (p-1)v\Delta_\phi.$$

Lemma 2.1. Let $F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \varphi$, where $\alpha = \alpha(t)$ and $\varphi = \varphi(t)$ are functions of t . Set

$$L_0(F) = -\frac{1}{\tilde{a}}[(p-1)\Delta_\phi v]^2 - 2(p-1)\text{Ric}_\phi^m(\nabla v, \nabla v) + 2p\nabla v \nabla F + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha' \frac{v_t}{v} - \varphi'.$$

(1) If $p > 1$, then $\mathcal{L}(F) \leq L_0(F)$.

(2) If $p \in (0, 1)$, then $\mathcal{L}(F) \geq L_0(F)$.

Proof. We only give the proof for the case where $p > 1$; the other case is similar. By a direct calculation, we have

$$(2-3) \quad \mathcal{L}\left(\frac{f}{g}\right) = \frac{1}{g}\mathcal{L}(f) - \frac{f}{g^2}\mathcal{L}(g) + 2(p-1)v\nabla\frac{f}{g}\nabla\log g \quad \text{for all } f, g \in C^\infty(M).$$

Using (2-1), we obtain

$$(2-4) \quad \mathcal{L}(v_t) = (p-1)v_t\Delta_\phi v + 2\nabla v \nabla v_t.$$

It is well known that, for the m -dimensional Bakry–Emery Ricci curvature, we have the following Bochner formula (for the elementary proof, see [Ledoux 2000; Li 2005]):

$$(2-5) \quad \begin{aligned} \frac{1}{2}\Delta_\phi(|\nabla w|^2) &= |\nabla^2 w|^2 + \nabla w \nabla \Delta_\phi w + \text{Ric}_\phi(\nabla w, \nabla w) \\ &\geq \frac{1}{n}|\Delta w|^2 + \nabla w \nabla \Delta_\phi w + \text{Ric}_\phi(\nabla w, \nabla w) \\ &\geq \frac{1}{m}|\Delta_\phi w|^2 + \nabla w \nabla \Delta_\phi w + \text{Ric}_\phi^m(\nabla w, \nabla w). \end{aligned}$$

It follows from $p > 1$ that

$$\begin{aligned} \mathcal{L}(|\nabla v|^2) &\leq 2\nabla v \nabla v_t - 2(p-1)v\left(\frac{1}{m}|\Delta_\phi v|^2 + \nabla v \nabla \Delta_\phi v + \text{Ric}_\phi^m(\nabla v, \nabla v)\right) \\ &= 2\nabla v \nabla[(p-1)v\Delta_\phi v + |\nabla v|^2] \\ &\quad - 2(p-1)v\left(\frac{1}{m}|\Delta_\phi v|^2 + \nabla v \nabla \Delta_\phi v + \text{Ric}_\phi^m(\nabla v, \nabla v)\right) \\ &= 2(p-1)|\nabla v|^2\Delta_\phi v + 2\nabla v \nabla(|\nabla v|^2) \\ &\quad - \frac{2(p-1)}{m}v(\Delta_\phi v)^2 - 2(p-1)v\text{Ric}_\phi^m(\nabla v, \nabla v). \end{aligned}$$

Applying this and (2-4) to (2-3) yields

$$(2-6) \quad \mathcal{L}\left(\frac{v_t}{v}\right) = (p-1)\frac{v_t}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla v_t - \frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2(p-1)v\nabla\frac{v_t}{v}\nabla\log v$$

and

$$\begin{aligned} \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) &\leq 2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) \\ &\quad - \frac{2(p-1)}{m}(\Delta_\phi v)^2 - 2(p-1)\text{Ric}_\phi^m(\nabla v, \nabla v) - \frac{|\nabla v|^4}{v^2} + 2(p-1)v\nabla\frac{|\nabla v|^2}{v}\nabla\log v, \end{aligned}$$

and hence

$$\begin{aligned} (2-7) \quad \mathcal{L}(F) &= \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) - \alpha\mathcal{L}\left(\frac{v_t}{v}\right) - \alpha'\frac{v_t}{v} - \varphi' \\ &\leq 2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \frac{2(p-1)}{m}(\Delta_\phi v)^2 \\ &\quad - 2(p-1)\text{Ric}_\phi^m(\nabla v, \nabla v) - \frac{|\nabla v|^4}{v^2} + 2(p-1)v\nabla\frac{|\nabla v|^2}{v}\nabla\log v \\ &\quad - \alpha(p-1)\frac{v_t}{v}\Delta_\phi v - \alpha\frac{2}{v}\nabla v\nabla v_t + \alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v} \\ &\quad - 2\alpha(p-1)v\nabla\frac{v_t}{v}\nabla\log v - \alpha'\frac{v_t}{v} - \varphi'. \end{aligned}$$

Noticing that

$$2(p-1)v\nabla\frac{|\nabla v|^2}{v}\nabla\log v - 2\alpha(p-1)v\nabla\frac{v_t}{v}\nabla\log v = 2(p-1)\nabla v\nabla F$$

and

$$\frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \alpha\frac{2}{v}\nabla v\nabla v_t = \frac{2}{v}\nabla v\nabla[(F + \varphi)v] = 2(F + \varphi)\frac{|\nabla v|^2}{v} + 2\nabla v\nabla F,$$

we obtain

$$\begin{aligned} (2-8) \quad 2(p-1)v\left(\nabla\frac{|\nabla v|^2}{v} - \alpha\nabla\frac{v_t}{v}\right)\nabla\log v + \frac{2}{v}\nabla v\nabla(|\nabla v|^2) - \alpha\frac{2}{v}\nabla v\nabla v_t \\ = 2p\nabla v\nabla F + 2(F + \varphi)\frac{|\nabla v|^2}{v} \\ = 2p\nabla v\nabla F + 2\left(\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}\right)\frac{|\nabla v|^2}{v}. \end{aligned}$$

On the other hand, using (2-1) again, we have

$$\begin{aligned}
(2-9) \quad & 2(p-1) \frac{|\nabla v|^2}{v} \Delta_\phi v - \frac{|\nabla v|^4}{v^2} - \alpha(p-1) \frac{v_t}{v} \Delta_\phi v + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
& = 2 \frac{|\nabla v|^2}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
& = (2\alpha + 2) \frac{v_t}{v} \frac{|\nabla v|^2}{v} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left(\frac{v_t}{v} \right)^2.
\end{aligned}$$

Combining (2-8) with (2-9) gives

$$\begin{aligned}
(2-10) \quad & 2(p-1)v \nabla \frac{|\nabla v|^2}{v} \nabla \log v - 2\alpha(p-1)v \nabla \frac{v_t}{v} \nabla \log v + \frac{2}{v} \nabla v \nabla (|\nabla v|^2) \\
& - \alpha \frac{2}{v} \nabla v \nabla v_t + 2(p-1) \frac{|\nabla v|^2}{v} \Delta_\phi v - \frac{|\nabla v|^4}{v^2} - \alpha(p-1) \frac{v_t}{v} \Delta_\phi v + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
& = 2p \nabla v \nabla F - \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\
& = 2p \nabla v \nabla F - [(p-1)\Delta_\phi v]^2 + (1-\alpha) \left(\frac{v_t}{v} \right)^2.
\end{aligned}$$

Putting (2-10) into (2-7) yields

$$\begin{aligned}
\mathcal{L}(F) & \leq -\frac{2(p-1)}{m} (\Delta_\phi v)^2 - 2(p-1) \operatorname{Ric}_\phi^m(\nabla v, \nabla v) + 2p \nabla v \nabla F \\
& \quad - [(p-1)\Delta_\phi v]^2 + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \varphi' \\
& = -\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 - 2(p-1) \operatorname{Ric}_\phi^m(\nabla v, \nabla v) + 2p \nabla v \nabla F \\
& \quad + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \varphi',
\end{aligned}$$

which completes the proof of (1) in Lemma 2.1. \square

Proof of Theorem 1.1. Let ξ be a cut-off function such that $\xi(r) = 1$ for $r \leq 1$, $\xi(r) = 0$ for $r \geq 2$, $0 \leq \xi(r) \leq 1$, and

$$0 \geq \xi'(r) \geq -c_1 \xi^{1/2}(r), \quad \xi''(r) \geq -c_2,$$

for positive constants c_1 and c_2 . With $\rho(x)$ the distance between x and p in M^n , let

$$\psi(x) = \xi \left(\frac{\rho(x)}{R} \right).$$

Making use of an argument of Calabi [1958] (see also [Cheng and Yau 1975]), we can assume without loss of generality that the function ψ is smooth in $B_p(2R)$. Then we have

$$(2-11) \quad \frac{|\nabla \psi|^2}{\psi} \leq \frac{C}{R^2}.$$

By the comparison theorem with respect to the Witten Laplacian (see [Li 2005, p. 1324])

$$\Delta_\phi \rho \geq \sqrt{(m-1)K} \coth\left(\sqrt{\frac{K}{m-1}}\rho\right),$$

we have

$$(2-12) \quad \Delta_\phi \psi = \frac{\xi' \Delta_\phi \rho}{R} + \frac{\xi'' |\nabla \rho|^2}{R^2} \geq -\frac{C(m)}{R^2} (1 + \sqrt{K} R \coth(\sqrt{K} R)).$$

Define $\tilde{F} = |\nabla v|^2/v - \alpha v_t/v$, where $\alpha > 1$ is a constant. Under the assumption that $\text{Ric}_\phi^m \geq -K$, Lemma 2.1(1) shows that

$$(2-13) \quad \begin{aligned} \mathcal{L}(\tilde{F}) &\leq -\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2(p-1)K |\nabla v|^2 + 2p \nabla v \nabla \tilde{F} \\ &\leq -\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{v} + 2p \nabla v \nabla \tilde{F}. \end{aligned}$$

Set $G = t\psi \tilde{F}$. Next we will apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then, at the point (x_0, s) , we have

$$\mathcal{L}(G) \geq 0, \quad \nabla \tilde{F} = -\frac{\tilde{F}}{\psi} \nabla \psi,$$

and, by use of (2-13), we have

$$(2-14) \quad \begin{aligned} 0 &\leq \mathcal{L}(G) \\ &= s\psi \mathcal{L}(\tilde{F}) - s(p-1)v \tilde{F} \Delta_\phi \psi - 2s(p-1)v \nabla \tilde{F} \nabla \psi + \psi \tilde{F} \\ &= s\psi \mathcal{L}(\tilde{F}) - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} \\ &\leq s\psi \left(-\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{v} + 2p \nabla v \nabla \tilde{F} \right) \\ &\quad - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} \\ &\leq -\frac{s\psi}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2s\psi MK \frac{|\nabla v|^2}{v} + 2\frac{p}{\sqrt{p-1}} \sqrt{MG} \frac{|\nabla v|}{\sqrt{v}} \frac{|\nabla \psi|}{\psi} \\ &\quad - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s}. \end{aligned}$$

Applying

$$[(p-1)\Delta_\phi v]^2 = \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \left(\frac{\alpha-1}{\alpha}\right)^2 \frac{|\nabla v|^4}{v^2}$$

to (2-14), we obtain

$$(2-15) \quad 0 \leq -\frac{1}{\tilde{a}s\alpha^2}G^2 - \frac{2(\alpha-1)\psi}{\tilde{a}\alpha^2}G \frac{|\nabla v|^2}{v} - \frac{s\psi^2}{\tilde{a}} \left(\frac{\alpha-1}{\alpha}\right)^2 \frac{|\nabla v|^4}{v^2} \\ + 2s\psi^2 MK \frac{|\nabla v|^2}{v} + 2\frac{p}{\sqrt{p-1}}\sqrt{M\psi}G \frac{|\nabla v|}{\sqrt{v}} \frac{|\nabla\psi|}{\sqrt{\psi}} \\ - (p-1)v(\Delta_\phi\psi)G + 2(p-1)v \frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s}.$$

Since $-Ax^2 + Bx \leq \frac{B^2}{4A}$, we have

$$-\frac{s\psi^2}{\tilde{a}} \left(\frac{\alpha-1}{\alpha}\right)^2 \frac{|\nabla v|^4}{v^2} + 2s\psi^2 MK \frac{|\nabla v|^2}{v} \leq \frac{\tilde{a}\alpha^2 s\psi^2 M^2 K^2}{(\alpha-1)^2}$$

and

$$-\frac{2(\alpha-1)\psi}{\tilde{a}\alpha^2}G \frac{|\nabla v|^2}{v} + \frac{2p}{\sqrt{p-1}}\sqrt{M\psi}G \frac{|\nabla v|}{\sqrt{v}} \frac{|\nabla\psi|}{\sqrt{\psi}} \leq \frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{|\nabla\psi|^2}{\psi}G.$$

We now set

$$(2-16) \quad P(K, R) = 1 + \sqrt{K}R \coth(\sqrt{K}R).$$

From (2-15) we obtain

$$0 \leq -\frac{1}{\tilde{a}s\alpha^2}G^2 + \frac{\tilde{a}\alpha^2 s\psi^2 M^2 K^2}{(\alpha-1)^2} + \frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{|\nabla\psi|^2}{\psi}G \\ - (p-1)v(L\psi)G + 2(p-1)v \frac{|\nabla\psi|^2}{\psi}G + \frac{\psi G}{s} \\ \leq -\frac{1}{\tilde{a}s\alpha^2}G^2 + \left(\frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) + \frac{\psi}{s} \right) G \\ + \frac{\tilde{a}\alpha^2 s\psi^2 M^2 K^2}{(\alpha-1)^2}.$$

Solving this quadratic inequality for G yields

$$G \leq \frac{\tilde{a}s\alpha^2}{2} \left\{ \frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) + \frac{\psi}{s} \right. \\ \left. + \left[\left(\frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) + \frac{\psi}{s} \right)^2 + \frac{4\psi^2 M^2 K^2}{(\alpha-1)^2} \right]^{\frac{1}{2}} \right\} \\ \leq \tilde{a}s\alpha^2 \left\{ \frac{\tilde{a}\alpha^2 p^2 M}{2(p-1)(\alpha-1)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) + \frac{\psi}{s} + \frac{\psi MK}{\alpha-1} \right\}.$$

Hence we have

$$G(x, T) \leq G(x_0, s) \\ \leq \tilde{a}T\alpha^2 \frac{C(m)}{R^2} \left(\frac{\alpha^2}{(p-1)(\alpha-1)} \tilde{a}p^2 + P(K, R) \right) M + \frac{\alpha^2}{\alpha-1} \tilde{a}TMK + \tilde{a}\alpha^2.$$

This implies that, for all $x \in B_p(R)$,

$$(2-17) \quad F(x, T) \leq \tilde{a}\alpha^2 M \frac{C(m)}{R^2} \left(\frac{\alpha^2}{\alpha-1} \frac{\tilde{a}p^2}{p-1} + P(K, R) \right) + \frac{\alpha^2}{\alpha-1} \tilde{a}MK + \frac{\tilde{a}\alpha^2}{T}.$$

Since T is arbitrary, we complete the proof of Theorem 1.1. \square

Proof of Corollary 1.2. Along the lines of Li and Yau, we will establish a Harnack inequality from a general estimate

$$(2-18) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0.$$

Rewrite (2-18) as

$$-\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right).$$

Let $f = \log v$. Then we have

$$-f_t = -\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right) \leq \frac{1}{\alpha(t)} (\varphi(t) - \tilde{M}|\nabla f|^2).$$

Let γ be a shortest geodesic joining x_1 and x_2 , and set $\gamma : [t_1, t_2] \rightarrow M^n$, $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$. Define a curve ζ in $M^n \times (0, \infty)$, $\zeta : [t_1, t_2] \rightarrow M^n \times (0, \infty)$ by $\zeta(t) = (\gamma(t), t)$. Then $\zeta(t_1) = (x_1, t_1)$ and $\zeta(t_2) = (x_2, t_2)$. Set $\rho = d(x_1, x_2)$. Then $|\dot{\gamma}| = \rho/(t_2 - t_1)$ and

$$(2-19) \quad f(x_1, t_1) - f(x_2, t_2) = \int_{t_2}^{t_1} \frac{d}{dt} f(\zeta(t)) dt = \int_{t_2}^{t_1} (\langle \dot{\gamma}, \nabla f \rangle + f_t) dt \\ = \int_{t_1}^{t_2} (-\langle \dot{\gamma}, \nabla f \rangle + (-f_t)) dt \\ \leq \int_{t_1}^{t_2} \left(|\dot{\gamma}| |\nabla f| + \frac{1}{\alpha(t)} (\varphi(t) - \tilde{M}|\nabla f|^2) \right) dt \\ = \int_{t_1}^{t_2} \left(-\frac{\tilde{M}}{\alpha(t)} |\nabla f|^2 + |\dot{\gamma}| |\nabla f| \right) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt \\ \leq \frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt,$$

where in the last inequality we used $-Ax^2 + Bx \leq \frac{B^2}{4A}$ and $|\dot{\gamma}| = \rho/(t_2 - t_1)$.

Let $\alpha > 1$ be a constant and set $\varphi = \frac{\alpha^2}{(\alpha-1)} \tilde{a}MK + \frac{\tilde{a}\alpha^2}{t}$. We have from (2-19)

$$(2-20) \quad f(x_1, t_1) - f(x_2, t_2) \leq \int_{t_1}^{t_2} \left(\frac{\alpha\rho^2}{4\tilde{M}(t_2-t_1)^2} + \frac{\alpha}{\alpha-1} \tilde{a}MK + \frac{\tilde{a}\alpha}{t} \right) dt \\ = \frac{\alpha\rho^2}{4\tilde{M}(t_2-t_1)} + \frac{\alpha}{\alpha-1} \tilde{a}MK(t_2-t_1) + \tilde{a}\alpha \log \frac{t_2}{t_1}.$$

Therefore, we arrive at

$$v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\tilde{a}\alpha} \exp \left(\frac{\alpha\rho^2}{4\tilde{M}(t_2-t_1)} + \frac{\alpha}{\alpha-1} \tilde{a}MK(t_2-t_1) \right). \quad \square$$

Proof of Theorem 1.3. When $p \in (0, 1)$, we have $v < 0$, and from Lemma 2.1(2)

$$\mathcal{L}(-\tilde{F}) \leq \frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2(p-1) \text{Ric}_\phi^m(\nabla v, \nabla v) + 2p \nabla v \nabla(-\tilde{F}) - (1-\alpha) \left(\frac{v_t}{v} \right)^2,$$

which implies

$$(2-21) \quad \mathcal{L}(-\tilde{F}) \leq \frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{-v} + 2p \nabla v \nabla(-\tilde{F}) - (1-\alpha) \left(\frac{v_t}{v} \right)^2.$$

Define $G = t\psi(-\tilde{F})$. We'll apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then, at the point (x_0, s) , we have

$$\mathcal{L}(G) \geq 0, \quad \nabla(-\tilde{F}) = -\frac{\tilde{F}}{\psi} \nabla\psi$$

and, by use of (2-21), we have

$$(2-22) \quad 0 \leq \mathcal{L}(G) = s\psi \mathcal{L}(-\tilde{F}) - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} \\ \leq s\psi \left(\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{-v} + 2p \nabla v \nabla(-\tilde{F}) \right) \\ - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} - (1-\alpha)s\psi \left(\frac{v_t}{v} \right)^2 \\ \leq \frac{s\psi}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2s\varphi MK \frac{|\nabla v|^2}{-v} \\ + 2 \frac{p}{\sqrt{1-p}} \sqrt{MG} \frac{|\nabla v|}{\sqrt{-v}} \frac{|\nabla \psi|}{\psi} - (p-1)v \frac{\Delta_\phi \psi}{\psi} G \\ + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} - (1-\alpha)s\psi \left(\frac{v_t}{v} \right)^2.$$

Applying the equalities

$$[(p-1)\Delta_\phi v]^2 = \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \left(\frac{\alpha-1}{\alpha}\right)^2 \frac{|\nabla v|^4}{v^2}$$

and

$$\left(\frac{v_t}{v}\right)^2 = \frac{1}{\alpha^2} \left(-\tilde{F} + \frac{|\nabla v|^2}{v}\right)^2 = \frac{1}{\alpha^2} (-\tilde{F})^2 + \frac{2}{\alpha^2} (-\tilde{F}) \frac{|\nabla v|^2}{v} + \frac{1}{\alpha^2} \frac{|\nabla v|^4}{v^2}$$

to (2-22), we obtain

$$(2-23) \quad 0 \leq \frac{1}{\tilde{a}s\alpha^2} \left((1-\tilde{a}(1-\alpha))G^2 - 2(1-\tilde{a})(1-\alpha)s\psi G \frac{|\nabla v|^2}{-v} + s^2\psi^2(1-\alpha)(1-\alpha-\tilde{a}) \frac{|\nabla v|^4}{v^2} \right) \\ + 2s\psi^2 MK \frac{|\nabla v|^2}{-v} + 2 \frac{p}{\sqrt{1-p}} \sqrt{M\psi} G \frac{|\nabla v|}{\sqrt{-v}} \frac{|\nabla\psi|}{\sqrt{\psi}} \\ - (p-1)v(\Delta_\phi\psi)G + 2(p-1)v \frac{|\nabla\psi|^2}{\psi} G + \frac{\psi G}{s}.$$

Next we employ a method similar to that in [Lu et al. 2009, Theorem 4.1]. Since $p \in (1-2/m, 1)$, we have $\tilde{a} < 0$. Thus we have, for any positive constants $\varepsilon_1, \varepsilon_2$,

$$2s\psi^2 MK \frac{|\nabla v|^2}{-v} \leq -\varepsilon_1 \frac{s^2\psi^2}{\tilde{a}s\alpha^2} (1-\alpha)(1-\alpha-\tilde{a}) \frac{|\nabla v|^4}{v^2} - \frac{1}{\varepsilon_1} \frac{\tilde{a}s\alpha^2(p-1)^2\psi^2 M^2 K^2}{(1-\alpha)(1-\alpha-\tilde{a})},$$

and

$$2 \frac{p}{\sqrt{1-p}} \sqrt{M\psi} G \frac{|\nabla v|}{\sqrt{-v}} \frac{|\nabla\psi|}{\sqrt{\psi}} \\ \leq -\varepsilon_2 \frac{2}{\tilde{a}s\alpha^2} (1-\tilde{a})(1-\alpha)s\psi G \frac{|\nabla v|^2}{-v} - \frac{\tilde{a}\alpha^2 p^2 M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} \frac{|\nabla\psi|^2}{\psi} G.$$

Hence we get from (2-23) that

$$0 \leq -\frac{1}{\tilde{a}s\alpha^2} \left(-(1-\tilde{a}(1-\alpha))G^2 + 2(1+\varepsilon_2)(1-\tilde{a})(1-\alpha)s\psi G \frac{|\nabla v|^2}{-v} - (1-\varepsilon_1)s^2\psi^2(1-\alpha)(1-\alpha-\tilde{a}) \frac{|\nabla v|^4}{v^2} \right) \\ - \frac{1}{\varepsilon_1} \frac{as\alpha^2\psi^2 M^2 K^2}{(1-\alpha)(1-\alpha-\tilde{a})} - \frac{\tilde{a}\alpha^2 p^2 M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} \frac{|\nabla\psi|^2}{\psi} G \\ - (p-1)v(\Delta_\phi\psi)G + 2(p-1)v \frac{|\nabla\psi|^2}{\psi} G + \frac{\psi G}{s},$$

which can be rewritten as

$$(2-24) \quad 0 \leq \frac{1}{\tilde{a}s\alpha^2} \left(1 - \tilde{a}(1-\alpha) - \frac{(1+\varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1-\varepsilon_1)(1-\alpha-\tilde{a})} \right) G^2 \\ - \frac{1}{\varepsilon_1} \frac{\tilde{a}s\alpha^2\psi^2 M^2 K^2}{(1-\alpha)(1-\alpha-\tilde{a})} - \frac{\tilde{a}\alpha^2 p^2 M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} \frac{|\nabla\psi|^2}{\psi} G \\ - (p-1)v(\Delta_\phi\psi)G + 2(p-1)v \frac{|\nabla\psi|^2}{\psi} G + \frac{\psi G}{s}.$$

Taking $\varepsilon_1, \varepsilon_2$ such that

$$(2-25) \quad 1 - \tilde{a}(1-\alpha) - \frac{(1+\varepsilon_2)^2(1-\tilde{a})^2(1-\alpha)}{(1-\varepsilon_1)(1-\alpha-\tilde{a})} =: A(\varepsilon_1, \varepsilon_2) > 0,$$

we obtain from (2-24), with $P(K, R)$ as in (2-16),

$$0 \leq -\frac{1}{(-\tilde{a})s\alpha^2} A(\varepsilon_1, \varepsilon_2) G^2 \\ + \left(\frac{(-\tilde{a})\alpha^2 p^2 M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) + \frac{\psi}{s} \right) G \\ + \frac{(-\tilde{a})s\alpha^2\psi^2 M^2 K^2}{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})}.$$

Solving this quadratic inequality for G yields

$$(2-26) \quad G \leq \frac{(-\tilde{a})s\alpha^2}{A(\varepsilon_1, \varepsilon_2)} \left(\frac{(-\tilde{a})\alpha^2 p^2 M}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} \frac{C}{R^2} + M \frac{C(m)}{R^2} P(K, R) \right. \\ \left. + \frac{\psi}{s} + \frac{\psi MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})}} \sqrt{A(\varepsilon_1, \varepsilon_2)} \right).$$

Hence we have

$$(2-27) \quad G(x, T) \leq G(x_0, s) \\ \leq \frac{(-\tilde{a})T\alpha^2 M}{A(\varepsilon_1, \varepsilon_2)} \frac{C(m)}{R^2} \left(\frac{(-\tilde{a})\alpha^2 p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + P(K, R) \right) \\ + \frac{(-\tilde{a})T\alpha^2 MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})} A(\varepsilon_1, \varepsilon_2)} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)},$$

and, for $x \in B_p(R)$,

$$-F(x, t) \leq \frac{(-\tilde{a})\alpha^2 M}{A(\varepsilon_1, \varepsilon_2)} \frac{C(m)}{R^2} \left(\frac{(-\tilde{a})\alpha^2 p^2}{2\varepsilon_2(1-\tilde{a})(1-\alpha)(1-p)} + P(K, R) \right) \\ + \frac{(-\tilde{a})\alpha^2 MK}{\sqrt{\varepsilon_1(1-\alpha)(1-\alpha-\tilde{a})} A(\varepsilon_1, \varepsilon_2)} + \frac{(-\tilde{a})\alpha^2}{A(\varepsilon_1, \varepsilon_2)t}.$$

This completes the proof of Theorem 1.3. \square

Proof of Corollary 1.4. Choosing $f = \log(-v)$ and $\varphi(t) = -\frac{\tilde{a}}{t}$, we get from (2-19)

$$f(x_2, t_2) - f(x_1, t_1) \leq \int_{t_1}^{t_2} \left(\frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} - \frac{\tilde{a}}{t} \right) dt = \frac{\rho^2}{4\tilde{M}(t_2 - t_1)} - \tilde{a} \log \frac{t_2}{t_1}. \quad \square$$

3. Proof of Theorem 1.6

Proof. Define $\bar{F} = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v}$, where $\alpha \in (0, 1)$ is constant. Lemma 2.1(2) shows that

$$\begin{aligned} (3-1) \quad \mathcal{L}(-\bar{F}) &\leq \frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{-v} + 2p \nabla v \nabla(-\bar{F}) - (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\ &= \frac{1}{\tilde{a}\alpha^2} \left(-\bar{F} - (1-\alpha) \frac{|\nabla v|^2}{-v} \right)^2 + 2MK \frac{|\nabla v|^2}{-v} + 2p \nabla v \nabla(-\bar{F}) \\ &\quad - \frac{1-\alpha}{\alpha^2} \left(-\bar{F} - \frac{|\nabla v|^2}{-v} \right)^2. \end{aligned}$$

Let $G = t\psi(-\bar{F})$. We apply the maximum principle to G on $B_p(2R) \times [0, T]$ and assume that G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ with $G(x_0, s) > 0$ (otherwise the proof is trivial). At the point (x_0, s) , we have

$$\mathcal{L}(G) \geq 0, \quad \nabla(-\bar{F}) = -\frac{-\bar{F}}{\psi} \nabla\psi,$$

and, by use of (3-1), we get

$$\begin{aligned} 0 \leq \mathcal{L}(G) &= s\psi \mathcal{L}(-\bar{F}) - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s} \\ &\leq \frac{s\psi}{\tilde{a}\alpha^2} \left(-\bar{F} - (1-\alpha) \frac{|\nabla v|^2}{-v} \right)^2 + 2s\varphi MK \frac{|\nabla v|^2}{-v} \\ &\quad + 2 \frac{p}{\sqrt{1-p}} \sqrt{MG} \frac{|\nabla v|}{\sqrt{-v}} \frac{|\nabla \psi|}{\psi} - \frac{1-\alpha}{\alpha^2} s\psi \left(-\bar{F} - \frac{|\nabla v|^2}{-v} \right)^2 \\ &\quad - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s}. \end{aligned}$$

Let $\frac{|\nabla v|^2}{-v} = \mu(-\bar{F})$ at the point (x_0, s) . Then we have $\mu \geq 0$ and

$$\begin{aligned} (3-2) \quad 0 \leq &\frac{1}{\tilde{a}\alpha^2 s\psi} [1 - (1-\alpha)\mu]^2 G^2 + 2\mu MK G + \frac{2\sqrt{\mu}}{\sqrt{s\psi}} \frac{p}{\sqrt{1-p}} \sqrt{MG}^{3/2} \frac{|\nabla \psi|}{\psi} \\ &- \frac{1-\alpha}{\alpha^2} \frac{1}{s\psi} (1-\mu)^2 G^2 - (p-1)v \frac{\Delta_\phi \psi}{\psi} G + 2(p-1)v \frac{|\nabla \psi|^2}{\psi^2} G + \frac{G}{s}. \end{aligned}$$

Multiplying both sides of (3-2) by $s\psi/G$ yields

$$(3-3) \quad 0 \leq \frac{1}{\tilde{a}\alpha^2}[1 - (1-\alpha)\mu]^2 G + 2\mu MK s\psi + 2\sqrt{\mu s} \frac{p}{\sqrt{1-p}} \sqrt{M} \frac{|\nabla\psi|}{\sqrt{\psi G}} \\ - \frac{1-\alpha}{\alpha^2} (1-\mu)^2 G - (p-1)sv\Delta_\phi\psi + 2(p-1)sv \frac{|\nabla\psi|^2}{\psi} + \psi.$$

Introducing

$$\tilde{A} = \frac{1}{-\tilde{a}\alpha^2}[1 - (1-\alpha)\mu]^2 + \frac{1-\alpha}{\alpha^2} (1-\mu)^2, \\ \tilde{B} = \sqrt{\mu s} \frac{p}{\sqrt{1-p}} \sqrt{M} \frac{|\nabla\psi|}{\sqrt{\psi}}, \\ \tilde{C} = 2\mu MK s\psi + (1-p)s(-v) \left(-\Delta_\phi\psi + 2 \frac{|\nabla\psi|^2}{\psi} \right) + \psi,$$

we write (3-3) as

$$(3-4) \quad 0 \leq -\tilde{A}G + 2\tilde{B}G^{1/2} + \tilde{C}.$$

It is easy to see that

$$\frac{1}{\tilde{A}} = \frac{(-\tilde{a})\alpha^2}{[1 - (1-\alpha)\mu]^2 + (-\tilde{a})(1-\alpha)(1-\mu)^2} \\ = \frac{(-\tilde{a})\alpha^2}{1 + (-\tilde{a})(1-\alpha) - 2(1-\alpha)(1-\tilde{a})\mu + (1-\alpha)(1-\alpha-\tilde{a})\mu^2} \leq 1-\alpha-\tilde{a}$$

and

$$(3-5) \quad \frac{2\mu}{\tilde{A}} = \frac{2(-\tilde{a})\alpha^2\mu}{1 + (-\tilde{a})(1-\alpha) - 2(1-\alpha)(1-\tilde{a})\mu + (1-\alpha)(1-\alpha-\tilde{a})\mu^2} \\ \leq \frac{(-\tilde{a})\alpha^2}{\sqrt{[1 + (-\tilde{a})(1-\alpha)](1-\alpha)(1-\alpha-\tilde{a}) - (1-\alpha)(1-\tilde{a})}} \\ = \sqrt{[1/(1-\alpha) + (-\tilde{a})](1-\alpha-\tilde{a})} + (1-\tilde{a}) \\ \leq \frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}),$$

where the last inequality used that $\sqrt{xy} \leq \frac{1}{2}(x+y)$. Hence there exists a constant $C(\tilde{a}, \alpha)$ such that $\sqrt{\mu}/\tilde{A} \leq C(\tilde{a}, \alpha)$. Now, regarding (3-4) as a quadratic inequality in \sqrt{G} gives

$$\sqrt{G} \leq 2\tilde{B}/\tilde{A} + \sqrt{\tilde{C}/\tilde{A}},$$

and therefore

$$(3-6) \quad G^{1/2} \leq C(\tilde{a}, \alpha) \sqrt{sM} \frac{p}{\sqrt{1-p}} \frac{C}{R} + \left[\left(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}) \right) MKs + 1 - \alpha - \tilde{a} + (1-p)(1-\alpha-\tilde{a}) Ms \frac{C(m)}{R^2} P(K, R) \right]^{\frac{1}{2}}$$

Hence, for $x \in B_p(R)$, we have

$$(3-7) \quad -\frac{|\nabla v|^2}{v} + \alpha \frac{v_t}{v} \leq \left\{ C(\tilde{a}, \alpha) \frac{p}{\sqrt{1-p}} \sqrt{M} \frac{C}{R} + \left[\left(\frac{\alpha^2}{2(1-\alpha)} + 2(1-\tilde{a}) \right) MK + \frac{1-\alpha-\tilde{a}}{t} + (1-p)(1-\alpha-\tilde{a}) M \frac{C(m)}{R^2} P(K, R) \right]^{\frac{1}{2}} \right\}^2.$$

This completes the proof of Theorem 1.6. \square

On the other hand, under the assumption that $\text{Ric}_\phi^m \geq -K$ and $p > 1$, Lemma 2.1(1) shows that

$$\begin{aligned} \mathcal{L}(F) &\leq -\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2(p-1)K |\nabla v|^2 + 2p \nabla v \nabla F + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \phi' \\ &\leq -\frac{1}{\tilde{a}} [(p-1)\Delta_\phi v]^2 + 2MK \frac{|\nabla v|^2}{v} + 2p \nabla v \nabla F + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \phi'. \end{aligned}$$

Following the methods in [Huang et al. 2013], we can prove the following results.

Theorem 3.1. *Let (M^n, g) be a complete Riemannian manifold with*

$$\text{Ric}_\phi^m(B_p(2R)) \geq -K,$$

where $\text{Ric}_\phi^m(B_p(2R))$ denotes the m -dimensional Bakry–Emery Ricci curvature on the geodesic ball $B_p(2R)$ with radius $2R$, and K is a nonnegative constant. Let u be a positive solution to the porous medium equation (1-8) with $p > 1$. Set

$$v = \frac{p}{p-1} u^{p-1}, \quad M = (p-1) \max_{B_p(2R) \times [0, T]} v.$$

Then, for any $\alpha > 1$ and with \tilde{a} as in (1-9), we have on $B_p(R)$

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} &\leq \tilde{a} \alpha^2 \left\{ \frac{\sqrt{\tilde{a}} \alpha p \sqrt{M}}{\sqrt{p-1} \sqrt{\alpha-1}} \frac{C(m)}{R} + \left(\frac{1}{t} + \frac{MK}{2(\alpha-1)} + M \frac{C(m)}{R^2} P(K, R) \right)^{\frac{1}{2}} \right\}^2. \end{aligned}$$

Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(3-8) \quad \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{2(\alpha-1)} \tilde{\alpha} MK + \frac{\tilde{\alpha} \alpha^2}{t}.$$

Corollary 3.2. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric}_\phi^n \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p > 1$. Set*

$$v = \frac{P}{p-1} u^{p-1}, \quad M = (p-1) \sup_{M^n \times [0, T]} v, \quad \tilde{M} = \inf_{M^n \times [0, T]} v.$$

Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, we have

$$v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\tilde{\alpha} \alpha} \exp \left(\frac{\alpha \text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} + \frac{\alpha}{2(\alpha-1)} \tilde{\alpha} MK (t_2 - t_1) \right),$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Theorem 3.3. *Let (M^n, g) and K be as in Theorem 3.1. Let u be a positive solution to the porous medium equation (1-8) with $p > 1$. Set*

$$v = \frac{P}{p-1} u^{p-1}, \quad M = (p-1) \max_{B_p(2R) \times [0, T]} v.$$

Then, for any $\alpha > 1$, we have on $B_p(R)$

$$\begin{aligned} & \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \\ & \leq \tilde{\alpha} \alpha^2(t) M \frac{C(m)}{R^2} \left(\frac{p^2 \tilde{\alpha} \alpha^2(t)}{2(p-1)(\alpha(t) - 1)} + 3 + \sqrt{K} R \coth(\sqrt{K} R) \right) + \frac{\tilde{\alpha} \alpha^2(t)}{t}, \end{aligned}$$

where $\alpha(t) = e^{2MKt}$. Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(3-9) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{\tilde{\alpha} \alpha^2(t)}{t}.$$

Corollary 3.4. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric}_\phi^n \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p > 1$. Set*

$$v = \frac{P}{p-1} u^{p-1}, \quad M = (p-1) \sup_{M^n \times [0, T]} v, \quad \tilde{M} = \inf_{M^n \times [0, T]} v.$$

Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, we have

$$v(x_1, t_1) \leq v(x_2, t_2) \exp \left\{ \frac{e^{2MKt_2} - e^{2MKt_1}}{2MK} \left(\frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)^2} + \frac{\tilde{\alpha}}{t_1} \right) \right\},$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 3.5. Theorems 3.1 and 3.3 reduce to Theorems 1.1 and 1.2 from [Huang et al. 2013], respectively, by letting $m = n$. In particular, the estimate (3-8) improves (1-10) on complete Riemannian manifolds.

Theorem 3.6. *Let (M^n, g) and K be as in Theorem 3.1. Let u be a positive solution to the porous medium equation (1-8) with $p > 1$. Let $v = (p/(p-1))u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then, on $B_p(R)$, we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq \tilde{a} M \frac{C(m)}{R^2} \left(1 + \sqrt{K} R \coth(\sqrt{K} R) + \frac{\tilde{a} p^2}{(p-1) \tanh(MKt)} \right),$$

where $\alpha(t), \varphi(t)$ are given by

$$(3-10) \quad \begin{aligned} \varphi(t) &= \tilde{a} MK (\coth(MKt) + 1), \\ \alpha(t) &= 1 + \frac{\cosh(MKt) \sinh(MKt) - MKt}{\sinh^2(MKt)}. \end{aligned}$$

Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(3-11) \quad \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0.$$

Corollary 3.7. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric}_\phi^m \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p > 1$. Let $v = (p/(p-1))u^{p-1}$ and $M = (p-1) \sup_{M^n \times [0, T]} v$, $\tilde{M} = \inf_{M^n \times [0, T]} v$. Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, we have*

$$v(x_1, t_1) \leq v(x_2, t_2) A_1(t_1, t_2) \exp \left(\frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} (1 + A_2(t_1, t_2)) \right),$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 and

$$\begin{aligned} A_1(t_1, t_2) &= \left(\frac{\exp(2MKt_2) - 2MKt_2 - 1}{\exp(2MKt_1) - 2MKt_1 - 1} \right)^{\tilde{a}/2}, \\ A_2(t_1, t_2) &= \frac{t_2 \coth(MKt_2) - t_1 \coth(MKt_1)}{t_2 - t_1}. \end{aligned}$$

Theorem 3.8. *Let (M^n, g) and K be as in Theorem 3.1. Let u be a positive solution to the porous medium equation (1-8) with $p > 1$. Let $v = (p/(p-1))u^{p-1}$ and $M = (p-1) \max_{B_p(2R) \times [0, T]} v$. Then, on $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \\ \leq \tilde{a} \alpha^2(t) M \frac{C(m)}{R^2} \left(1 + \sqrt{K} R \coth(\sqrt{K} R) + \frac{\tilde{a} p^2 \alpha^2(t)}{(p-1) \tanh(MKt)} \right), \end{aligned}$$

where

$$(3-12) \quad \varphi(t) = \frac{\tilde{a}}{t} + \tilde{a}MK + \frac{\tilde{a}}{3}(MK)^2t \quad \text{and} \quad \alpha(t) = 1 + \frac{2}{3}MKt.$$

Taking $R \rightarrow \infty$, we thus obtain the following estimate on (M^n, g) :

$$(3-13) \quad \frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} - \varphi(t) \leq 0.$$

Corollary 3.9. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric}_\phi^n \geq -K$, where K is a nonnegative constant. Let u be a positive solution to (1-8) with $p > 1$. Set*

$$v = p/(p-1)u^{p-1}, \quad M = (p-1) \sup_{M^n \times [0, T]} v, \quad \tilde{M} = \inf_{M^n \times [0, T]} v.$$

Then, for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, we have

$$\begin{aligned} v(x_1, t_1) \leq v(x_2, t_2) & \left(\frac{t_2}{t_1}\right)^{\tilde{a}} \left(\frac{1 + \frac{2}{3}MKt_2}{1 + \frac{2}{3}MKt_1}\right)^{-\tilde{a}/4} \\ & \times \exp\left(\frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} \left(1 + \frac{1}{3}MK(t_2 + t_1)\right) + \frac{\tilde{a}}{2}MK(t_2 - t_1)\right), \end{aligned}$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 3.10. Our Theorems 3.6 and 3.8 reduce to Theorems 1.3 and 1.4 from [Huang et al. 2013], respectively, by taking $m = n$. Moreover, when t is small enough, $\alpha(t)$ and $\varphi(t)$ defined by (3-10) and (3-12) both satisfy $\alpha(t) \rightarrow 1$ and $\varphi(t) \leq 2\tilde{a}MK + \tilde{a}/t$. Hence (3-11) and (3-13) show

$$(3-14) \quad \frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} \leq 2\tilde{a}MK + \frac{\tilde{a}}{t}.$$

Clearly, for t small enough, (3-14) is better than (1-10). In this sense, (3-11) and (3-13) improve (1-10) on complete Riemannian manifolds.

4. Proofs of Theorems 1.9 and 1.10

Lemma 4.1. *If M^n is a compact Riemannian manifold and u is a positive solution to (1-8) with $p \neq 0$, then*

$$(4-1) \quad \frac{d}{dt} \int_{M^n} uv \, d\mu = (p-1) \int_{M^n} (\Delta_\phi v)uv \, d\mu = -p \int_{M^n} |\nabla v|^2 u \, d\mu.$$

Proof. From (2-1), we have $(uv)_t = vu_t + uv_t = v\Delta_\phi(u^p) + (p-1)uv\Delta_\phi v + u|\nabla v|^2$. It follows from $\nabla(u^p) = u\nabla v$ that

$$\int_{M^n} [v\Delta_\phi(u^p) + u|\nabla v|^2] \, d\mu = \int_{M^n} [-\nabla v \nabla(u^p) + u|\nabla v|^2] \, d\mu = 0.$$

Hence

$$\begin{aligned}
\frac{d}{dt} \int_{M^n} uv \, d\mu &= \int_{M^n} (uv)_t \, d\mu = \int_{M^n} [v \Delta_\phi(u^p) + (p-1)uv \Delta_\phi v + u|\nabla v|^2] \, d\mu \\
&= (p-1) \int_{M^n} (\Delta_\phi v)uv \, d\mu = p \int_{M^n} (\Delta_\phi v)u^p \, d\mu \\
&= -p \int_{M^n} \nabla v \nabla(u^p) \, d\mu = -p \int_{M^n} |\nabla v|^2 u \, d\mu. \quad \square
\end{aligned}$$

Lemma 4.2. *If M^n is a compact Riemannian manifold and u is a positive solution to (1-8) with $p \neq 0$, then*

$$\frac{d}{dt} \int_{M^n} (\Delta_\phi v)uv \, d\mu = 2 \int_{M^n} [(p-1)(\Delta_\phi v)^2 + |\nabla^2 v|^2 + \text{Ric}_\phi(\nabla v, \nabla v)]uv \, d\mu.$$

Proof. Noticing that

$$(4-2) \quad \frac{d}{dt} \int_{M^n} (\Delta_\phi v)uv \, d\mu = \int_{M^n} [(\Delta_\phi v)_t uv + (\Delta_\phi v)(uv)_t] \, d\mu,$$

a direct calculation gives

$$\begin{aligned}
(\Delta_\phi v)_t &= \Delta_\phi[(p-1)v \Delta_\phi v + |\nabla v|^2] \\
&= (p-1)[(\Delta_\phi v)^2 + 2\nabla v \nabla \Delta_\phi v + v \Delta_\phi^2 v] + \Delta_\phi |\nabla v|^2 \\
&= (p-1)(\Delta_\phi v)^2 + 2p \nabla v \nabla \Delta_\phi v + (p-1)v \Delta_\phi^2 v + 2[|\nabla^2 v|^2 + \text{Ric}_\phi(\nabla v, \nabla v)].
\end{aligned}$$

We derive from $(p-1)\nabla(uv^2) = (2p-1)uv \nabla v$ that

$$\begin{aligned}
\int_{M^n} [2p \nabla v \nabla \Delta_\phi v + (p-1)v \Delta_\phi^2 v]uv \, d\mu \\
&= \int_{M^n} 2p \nabla v \nabla (\Delta_\phi v)uv \, d\mu - \int_{M^n} (p-1)\nabla(uv^2)\nabla \Delta_\phi v \, d\mu \\
&= \int_{M^n} \nabla v \nabla (\Delta_\phi v)uv \, d\mu.
\end{aligned}$$

Hence

$$\begin{aligned}
(4-3) \quad \int_{M^n} (\Delta_\phi v)_t uv \, d\mu \\
&= \int_{M^n} \{(p-1)(\Delta_\phi v)^2 + \nabla v \nabla \Delta_\phi v + 2[|\nabla^2 v|^2 + \text{Ric}_\phi(\nabla v, \nabla v)]\}uv \, d\mu.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(4-4) \quad & \int_{M^n} \Delta_\phi v (uv)_t d\mu \\
&= \int_{M^n} \Delta_\phi v [v \Delta_\phi (u^p) + (p-1)uv \Delta_\phi v + u |\nabla v|^2] d\mu \\
&= \int_{M^n} [-\nabla(v \Delta_\phi v) \nabla(u^p) + (p-1)uv (\Delta_\phi v)^2 + u |\nabla v|^2 \Delta_\phi v] d\mu \\
&= \int_{M^n} [-\nabla(v \Delta_\phi v) u \nabla v + (p-1)uv (\Delta_\phi v)^2 + u |\nabla v|^2 \Delta_\phi v] d\mu \\
&= \int_{M^n} [-\nabla v \nabla \Delta_\phi v + (p-1)(\Delta_\phi v)^2] uv d\mu.
\end{aligned}$$

Inserting (4-3) and (4-4) into (4-2) concludes the proof of Lemma 4.2 □

Proof of Theorems 1.9 and 1.10. By Lemma 4.1, we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) &= -\tilde{a} t^{\tilde{a}-1} \int_{M^n} uv d\mu - (p-1) t^{\tilde{a}} \int_{M^n} (\Delta_\phi v) uv d\mu \\
&= -t^{\tilde{a}} \int_{M^n} \left((p-1) \Delta_\phi v + \frac{\tilde{a}}{t} \right) uv d\mu.
\end{aligned}$$

We obtain (1-29) and (1-32). On the other hand, from the definition of $\mathcal{W}_{p,m}^i(g, u, t)$ in (1-28), we have

$$\begin{aligned}
\mathcal{W}_{p,m}^i(g, u, t) &= \frac{d}{dt} [t \mathcal{N}_{p,m}(g, u, t)] \\
&= \mathcal{N}_{p,m}(g, u, t) + t \frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) \\
&= t^{\tilde{a}+1} \int_{M^n} \left(p \frac{|\nabla v|^2}{v} - \frac{\tilde{a}+1}{t} \right) uv d\mu,
\end{aligned}$$

where Lemma 4.1 was used in the last equality. Hence we derive (1-30) and (1-33).

Notice that the estimate (1-10) also holds for compact Riemannian manifolds. Taking $K = 0$ and then letting $\alpha \rightarrow 1$ in (1-10) yields

$$(p-1) \Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \geq 0,$$

which allows us to conclude that if $\text{Ric}_\phi^m \geq 0$, then $\mathcal{N}_{p,m}(g, u, t)$ is nonincreasing in t . When $p \in (1 - 2/m, 1)$ and $\text{Ric}_\phi^m \geq 0$, we also get from (1-12) that

$$(p-1) \Delta_\phi v + \frac{\tilde{a}}{t} = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \frac{\tilde{a}}{t} \leq 0,$$

which shows that $\mathcal{N}_{p,m}(g, u, t)$ is also nonincreasing in t .

Now we are in a position to prove (1-31). From (1-29), we have

$$\begin{aligned}
& \frac{d}{dt} \left(t \frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) \right) \\
&= \frac{d}{dt} \left(-t^{\tilde{a}+1} \int_{M^n} (p-1)(\Delta_\phi v)uv d\mu - \tilde{a}t^{\tilde{a}} \int_{M^n} uv d\mu \right) \\
&= \frac{d}{dt} \left(-t^{\tilde{a}+1} \int_{M^n} (p-1)(\Delta_\phi v)uv d\mu + \tilde{a}\mathcal{N}_{p,m}(g, u, t) \right) \\
&= -2t^{\tilde{a}+1} \int_{M^n} \left((p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\text{Ric}_\phi(\nabla v, \nabla v) \right) uv d\mu \\
&\quad - (\tilde{a}+1)t^{\tilde{a}} \int_{M^n} (p-1)(\Delta_\phi v)uv d\mu - \tilde{a}t^{\tilde{a}} \int_{M^n} \left((p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) uv d\mu,
\end{aligned}$$

where the last equality used Lemma 4.2. Hence

$$\begin{aligned}
(4-5) \quad & \frac{d}{dt} {}^o\mathcal{W}_{p,m}(g, u, t) \\
&= \frac{d}{dt} \left(t \frac{d}{dt} \mathcal{N}_{p,m}(g, u, t) + \mathcal{N}_{p,m}(g, u, t) \right) \\
&= -2t^{\tilde{a}+1} \int_{M^n} \left[(p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\text{Ric}_\phi(\nabla v, \nabla v) \right] uv d\mu \\
&\quad - (\tilde{a}+1)t^{\tilde{a}} \int_{M^n} (p-1)(\Delta_\phi v)uv d\mu - (\tilde{a}+1)t^{\tilde{a}} \int_{M^n} \left((p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) uv d\mu \\
&= -2t^{\tilde{a}+1} \int_{M^n} \left((p-1)^2(\Delta_\phi v)^2 + (p-1)|\nabla^2 v|^2 + (p-1)\text{Ric}_\phi(\nabla v, \nabla v) \right. \\
&\quad \left. + (p-1)\frac{\tilde{a}+1}{t}\Delta_\phi v + \frac{\tilde{a}^2+\tilde{a}}{2t^2} \right) uv d\mu.
\end{aligned}$$

Notice that

$$\begin{aligned}
& (p-1)^2(\Delta_\phi v)^2 + (p-1)\frac{\tilde{a}+1}{t}\Delta_\phi v + \frac{\tilde{a}^2+\tilde{a}}{2t^2} \\
&= \left| (p-1)\Delta_\phi v + \frac{m(p-1)}{[m(p-1)+2]t} \right|^2 + \frac{2(p-1)}{[m(p-1)+2]t}\Delta_\phi v + \frac{(p-1)m}{[m(p-1)+2]^2t^2},
\end{aligned}$$

and hence

$$\begin{aligned}
(4-6) \quad & (p-1)^2(\Delta_\phi v)^2 + (p-1)\frac{\tilde{a}+1}{t}\Delta_\phi v + \frac{\tilde{a}^2+\tilde{a}}{2t^2} + (p-1)|\nabla^2 v|^2 + \frac{p-1}{m-n}(\nabla\phi\nabla v)^2 \\
&= \left| (p-1)\Delta_\phi v + \frac{m(p-1)}{[m(p-1)+2]t} \right|^2 + (p-1) \left| \nabla^2 v + \frac{g}{[m(p-1)+2]t} \right|^2 \\
&\quad + \frac{p-1}{m-n} \left| \nabla\phi\nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2.
\end{aligned}$$

We complete the proof of (1-31) by putting (4-6) into (4-5).

When $p \in (0, 1)$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & -(p-1) \left| \nabla^2 v + \frac{g}{[m(p-1)+2]t} \right|^2 \\ & \geq -\frac{p-1}{n} \left| \Delta v + \frac{n}{[m(p-1)+2]t} \right|^2 \\ & = -\frac{1}{n(p-1)} \left| (p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 - \frac{p-1}{n} \left| \nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \\ & \quad - \frac{2}{n} \left((p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) \left(\nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right). \end{aligned}$$

Hence

$$\begin{aligned} (4-7) \quad & -(p-1) \left| \nabla^2 v + \frac{g}{[m(p-1)+2]t} \right|^2 - \frac{p-1}{m-n} \left| \nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \\ & \quad - \left| (p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 \\ & \geq \frac{1-n(1-p)}{n(1-p)} \left| (p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 + \frac{m(1-p)}{n(m-n)} \left| \nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \\ & \quad - \frac{2}{n} \left((p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right) \left(\nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right) \\ & \geq \left(\frac{1-n(1-p)}{n(1-p)} - \frac{\varepsilon}{n} \right) \left| (p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 \\ & \quad + \left(\frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon} \right) \left| \nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2, \end{aligned}$$

where $\varepsilon \geq m-n$ is a positive constant and satisfies

$$1 - 1/(n + \varepsilon) \leq p \leq 1 - (m - n)/(m\varepsilon).$$

Inserting (4-7) into (1-31) gives

$$\begin{aligned} & \frac{d}{dt} \mathcal{W}_{p,m}(g, u, t) \\ & \leq 2t^{a+1} \int_{M^n} \left((1-p) \operatorname{Ric}_\phi^m(\nabla v, \nabla v) + \left(\frac{1-n(1-p)}{n(1-p)} - \frac{\varepsilon}{n} \right) \left| (p-1)\Delta_\phi v + \frac{\tilde{a}}{t} \right|^2 \right. \\ & \quad \left. + \left(\frac{m(1-p)}{n(m-n)} - \frac{1}{n\varepsilon} \right) \left| \nabla\phi \nabla v - \frac{m-n}{[m(p-1)+2]t} \right|^2 \right) uv \, d\mu. \end{aligned}$$

This completes the proof of (1-34). \square

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GUANGYUE HUANG
COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
HENAN NORMAL UNIVERSITY
XINXIANG 453007
CHINA
hgy@henannu.edu.cn

HAIZHONG LI
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING 100084
CHINA
hli@math.tsinghua.edu.cn

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Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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
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